$\bar\partial$-FREE MAPS SATISFY THE HOMOTOPY PRINCIPLE

B. Shapiro

Department of Mathematics, University of Stockholm,
S-10691, Sweden, shapiro@matematik.su.se

Abstract. In this short note we show that the space of all $\bar\partial$-free maps of any complex manifold V_c into $\mathbf{C}^n$ is always homotopically equivalent to the space of all sections of the corresponding bundle in the space of jets of smooth maps $V_c \to \mathbf{C}^n$. In particular, the space of all linear ordinary differential equations with complex-valued coefficients on an elliptic curve with the identical monodromy (defined as the conjugacy class in the corresponding loop group) is weakly homotopically equivalent to the space of all based maps of the curve in $GL_n(\mathbf{C})$. The proof is based on Gromov's theory of convex integration, see e.g. [Gr,McD].

§0. Preliminaries and results

A smooth map $f : V \to \mathbf{R}^q$ is called *free of order* k if at any point x of a manifold V the vectors $\frac{\partial f}{\partial u_i}(v); \frac{\partial^2 f}{\partial u_{i_1}\partial u_{i_2}}(v); \dots; \frac{\partial^k f}{\partial u_{i_1}\dots\partial u_{i_k}}(v)$ are linearly independent in $\mathbf{R}^q$, where u_i are some local coordinates in a neighborhood of x and $v = f(x)$, see [Gr]. Obviously, $q \ge \binom{n+k}{k} - 1 = n + \binom{n}{2} + \dots + \binom{n+k-1}{k}$.

Let $X = V \times \mathbf{R}^q \to V$ and $X^{(k)}$ denote the space of k-jets of maps $V \to \mathbf{R}^q$. If $q \ge \binom{n+k}{k} - 1$ then the differential relation $\mathcal{F}^k \subset X^{(k)}$ of freedom of order k is an open dense subset in $X^{(k)}$. The relation $\mathcal{F}^k$ is fibered over $\mathcal{F}^{k-1}$ with the fibers canonically equivalent to the Stiefel variety $St_{\binom{n+k-1}{k}}\mathbf{R}^q$.

The notion of freedom of order k generalizes straightforwardly to maps into affine, projective and Riemannian spaces by taking local charts and/or covariant derivatives.

The following result is formulated in [Gr].

Theorem. If $q > \binom{n+k}{k} - 1$ or V is an open manifold then the inclusion of the space Φ of all maps $V \to \mathbf{R}^q$ which are free of order k into the space Ψ of all sections of the bundle $\mathcal{F}^k \to V$ induces the homotopy equivalence of Φ and Ψ.

Remark. If $V$is parallelizable then the bundle $\mathcal{F}^k \to V$ is trivial and the space Ψ of all its sections is homotopically equivalent to the space of all maps $V \to St_{\binom{n+k}{k}-1}\mathbf{R}^q$.

However, in the most interesting case when V is closed and $q = \binom{n+k}{k} - 1$ the situation is far more complicated, comp. [MSh-ShSh]. Surprisingly little is known (to the best of the author's knowledge) even about the space NC_n of all closed curves free of order n in $\mathbf{R}^n$ (also called nondegenerate curves). Namely, if one fixes the orientation of $\mathbf{R}^n$ and the orientation

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of the osculating frame then the space NC_n consists of 3 connected components if n is odd and consists of 2 connected components if n is even, see [MSh]. One should mention that in this case every connected component of the space Ψ (which is homotopically equivalent to the space of all maps $\mathbf{S}^1 \to GL_n^+(\mathbf{R})$) contains at least one connected component of the space $\Phi = NC_n$. No information about the cohomology or the homotopy groups of NC_n is available, apparently, due to the lack of the covering homotopy, comp. [KSh2].

THE MAIN DEFINITION. A smooth map $f : V_c \to \mathbf{C}^q$ of a complex manifold V_c is called *$\bar\partial$-free of order k* if the vectors $\frac{\partial f}{\partial \bar z_i}(v); \frac{\partial^2 f}{\partial \bar z_{i_1}\partial \bar z_{i_2}}(v); \dots ; \frac{\partial^k f}{\partial \bar z_{i_1}\dots\partial \bar z_{i_k}}(v)$ are linearly independent in $\mathbf{C}^q$ at any $v \in V_c$.

REMARK. Obviously, $\bar\partial$-freedom does not depend on the choice of local complex coordinates on V_c. Although $\bar\partial$-freedom is very similar to the usual freedom they have essential distinctions. In particular, a $\bar\partial$-free map is not necessarily an imbedding.

We will also need the following obvious extention of the notion of $\bar\partial$-freedom. A smooth map $f : V_c \to \mathbf{C}^q$ is called extended $\bar\partial$-free of order k if the vectors $f(v)$ and $\frac{\partial f}{\partial \bar z_i}(v); \frac{\partial^2 f}{\partial \bar z_{i_1}\partial \bar z_{i_2}}(v); \dots ;$ $\frac{\partial^k f}{\partial \bar z_{i_1}\dots\partial \bar z_{i_k}}(v)$ are linearly independent in $\mathbf{C}^q$ at any $v \in V_c$.

Let $X_c = V_c \times \mathbf{C}^q$ and let $\mathcal{F}_c^k \subset X_c^{(k)}$ denote the differential relation of $\bar\partial$-freedom of order k. Analogously, let $\mathcal{EF}_c^k \subset X_c^{(k)}$ denote the differential relation of extended $\bar\partial$-freedom of order k.

The result of this note is the following simple

THEOREM 1. If $q \geq \binom{n+k}{k} - 1$ (for the extended freedom $q \geq \binom{n+k}{k}$) then for any n-dimensional complex manifold V_c the inclusion of the space Φ_c of all its (extended) $\bar\partial$-free maps in $\mathbf{C}^q$ of order k into the space Ψ_c of all sections of $\mathcal{F}_c^k \to V_c$ ($\mathcal{EF}_c^k \to V_c$ resp.) induces the weak homotopy equivalence of Φ_c and Ψ_c.

Fixing a point $x_0 \in V$ and the k-jet of the considered $\bar\partial$-free maps at x_0 one gets the obvious based modifications of theorem 1.

APPLICATION. Let $\mathcal{L}_{S^1}$ denote the space of all linear ordinary differential equations (l.o.d.e) of some fixed order n with smooth real-valued periodic coefficients and $\mathcal{L}_\Gamma$ denote the space of all l.o.d.e on an elliptic curve Γ of the form $\bar\partial^n + \sum_{j=0}^{n-1} u_{j+1}\bar\partial^j$, where $u_j \in C^\infty(\Gamma, \mathbf{C})$. Both of these spaces carry a natural Poisson structure called the Gelfand-Dickey Poisson structure, see [GD].

The symplectic leaves of this structure in the space $\mathcal{L}_{S^1}$ are locally in $1-1$-correspondence with the usual monodromy operator, see [OK]. The notion of the monodromy can be generalized to the case of $\mathcal{L}_\Gamma$ as well, see [EK]. This monodromy is the conjugacy class of the action of the group $LGL(\mathbf{C})$ of all $GL_n(\mathbf{C})$-valued functions on the cylinder $\Sigma = \mathbf{C}/\mathbf{Z}$ on the semidirect product $\Sigma \ltimes LGL_n(\mathbf{C})_0$, where $LGL_n(\mathbf{C})_0$ is the connected component of identity in $LGL_n(\mathbf{C})$. For a given matrix $\mathcal{M}(z) \in LGL_n(\mathbf{C})_0$ the following statement is true. A map $f : \Sigma \to \mathbf{C}^n$ with prescribed monodromy $\mathcal{M}(z)$ (i.e. $f(z+\tau) = f(z)\mathcal{M}(z)$) and nonvanishing $\bar\partial$-Wronskian is called *a quasiperioodic nondegenerate tube.*

PROPOSITION, SEE [EK]. Symplectic leaves of the Gelfand-Dikii bracket whose monodromy is the conjugacy class of $\mathcal{M}(z) \in LGL_n(\mathbf{C})_0$ are in 1-1-correspondence with the homotopy classes of quasiperiodic nondegenerate tubes with the monodromy $\mathcal{M}(z)$.

REMARK. If the monodromy is identity then the symplectic leaves of the GD-bracket are in 1-1-correspondence with the homotopy classes of extended $\bar\partial$-free maps $f : \Gamma \to \mathbf{C}^n$ of order $n-1$.

COROLLARY OF THEOREM 1. The subspace $\mathcal{L}_\Gamma^{id} \subset \mathcal{L}_\Gamma$ consisting of all l.o.d.e with the identical monodromy is weakly homotopically equivalent to the space Ψ_Γ^b of all based maps

$f : \Gamma \to GL_n(\mathbf{C})$. Moreover, two l.o.d.e with identical monodromy belong to the same connected symplectic leave of the Gelfand-Dikii bracket if any two maps $f_1 : \Gamma \to GL_n(\mathbf{C})$ and $f_2 : \Gamma \to GL_n(\mathbf{C})$ defined by these l.o.d.e induce the same homomorphism $\pi_1(\Gamma) \to \pi_1(GL_n(\mathbf{C})$. (Note that $\pi_1(\Gamma) = \mathbf{Z}^2$ and $\pi_1(GL_n(\mathbf{C})) = \mathbf{Z}$.)

Both theorem 1 and its corollary follow directly from Gromov's theory of convex integration. The author is very grateful to IHES for their hospitality and especially to M. Gromov who (apart from formulating numerous problems) was patient enough to clarify the basic ideas of convex integration. Sincere thanks for many useful discussions are due to P. Etingof and B. Khesin whose question posed in [EK] was the starting point of this note.

Proofs

Proof of Theorem 1. Let us recall some basic notions of the method of convex integration. Let $p : X \to V$ be a smooth fibration with q-dimensional fibers (in our case the fibration is $p : V_c \times \mathbf{C}^q \to V_c$) and let $X^{(k)}$ denote the space of k-jets of smooth sections of this bundle. $X^{(k)}$ is fibered over $X^{(k-1)}$ with the $\binom{n+k-1}{k}$-dimensional affine fibers. We define the class of the *main* or the *coordinate* q-dimensional affine subspaces in the fibers of $X^{(k)} \to X^{(k-1)}$ by the following condition. Two k-jets belong to the same fiber of $X^{(k)} \to X^{(k-1)}$ if there exists a germ of a smooth real hypersurface $H : \{u_1 = 0\}$ in a neighborhood of the considered point such that the restrictions of both k-jets to H coincide. Now a subset in the affine space is called *ample* if the convex hull of any its connected component coincides with the whole affine space or this subset is empty. A differential relation $\mathcal{R} \subset X^{(k)}$ is called *ample* if its restriction to any main subspace is ample. In particular, if any such restriction is empty or its complement has the codimension at least 2 then the relation $\mathcal{R}$ is ample. By the results of the sections 2.4.1-2.4.3 in [Gr] the ample relations satisfy all the forms of the homotopy principle and, in particular, the inclusion of the space of all solutions of the ample differential relation $\mathcal{R}$ into the space of all sections of the bundle $\mathcal{R} \to V$ is the weak homotopy equivalence.

Let us show that (extended) $\bar\partial$-freedom is the ample differential relation. Choosing a point $x \in V_c$ and a germ of a smooth real hypersurface H passing through x one can always find local complex coordinates $z_1, \dots, z_n$ on V_c such that H is given by $\{\mathrm{Im}\ z_1 = 0\}$. Using these coordinates one can present main affine subspaces in $X^{(k)}$ in the following way. Any main subspace consists of the k-jets of maps $f : V_c \to \mathbf{C}^q$ such that all the vectors $\frac{\partial f}{\partial \bar z_i}(v); \frac{\partial^2 f}{\partial \bar z_{i_1}\partial \bar z_{i_2}}(v); \dots; \frac{\partial^k f}{\partial \bar z_{i_1}\dots\partial \bar z_{i_k}}(v)$ except for the $\frac{\partial^k f}{(\partial \bar z_1)^k}$ are fixed while the coordinates of $\frac{\partial^k f}{(\partial \bar z_1)^k}$ form the complex q-dimensional subspace of independent variables. (In the extended case we fix the value of $f(v)$ as well.) If the fixed vectors are linearly dependent then the restriction of the differential relation $\mathcal{F}_c^k$ ($\mathcal{EF}_c^k$ resp.) onto such a main subspace is empty and therefore ample. If they are linearly independent then the linear dependence of coordinates of $\frac{\partial^k f}{(\partial \bar z_1)^k}$ with the rest of the vectors determines a complex subspace of positive codimension in the complex q-dimensional space of variables since $q > \binom{n+k}{k} - 1$ and thus its real codimension is at least 2. $\square$

Proof of Corollary. According to [EK] each equation $L \in \mathcal{L}_\Gamma^{id}$ determines an extended map $f_L : \Gamma \to \mathbf{C}^n$ $\bar\partial$-free of order $n-1$. The map f_l is defined uniquely up to a multiplication by an arbitrary matrix from $GL_n(\mathbf{C})$. Therefore, if we fix a point x_0 on Γ and require that $f_L^b(x_0)$, $\frac{\partial f_L^b}{\partial \bar z}(x_0)$; $\frac{\partial^2 f_L^b}{\partial \bar z^2}(x_0); \dots; \frac{\partial^{n-1} f_L^b}{\partial \bar z^{n-1}}(x_0)$ is the identity matrix then f_L^b is determined uniquely. Thus the space $\mathcal{L}_\Gamma^{id}$ coincides with the space Φ_Γ^b of extended $\bar\partial$-free maps $\Gamma \to \mathbf{C}^n$ with the identical $(n-1)$-jet at the given point x_0. By theorem 1 the inclusion of the space Φ^b_Γ into the space Ψ^b_Γ of all based maps $(\Gamma, x_0) \to (GL_c(\mathbf{C}), id)$ is the weak homotopy equivalence. Now the homotopy class of a map $f : \Gamma \to GL_n(\mathbf{C}))$ is uniquely defined by the induced homomorphism $\pi_1(\Gamma) \to \pi_1(GL_n(\mathbf{C}))$ since $\pi_2(GL_n(\mathbf{C})) = 0$.

$\square$

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