PERIODIC DE BRUIJN TRIANGLES: EXACT AND ASYMPTOTIC RESULTS

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Abstract. We study the distribution of the number of permutations with a given periodic up-down sequence w.r.t. the last entry, find exponential generating functions and prove asymptotic formulas for this distribution.

§1. Introduction and results

Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be a permutation of length n. We associate with σ its *up-down sequence* (sometimes called the *shape* of σ, or the *signature* of σ) $\mathcal{P}(\sigma) = (p_1, \dots, p_{n-1})$, which is a binary vector of length $n-1$ such that $p_i = 1$ if $\sigma_i < \sigma_{i+1}$ and $p_i = 0$ otherwise. During the last 120 years, many authors have studied the number $\sharp_n^{\mathcal{P}}$ of all permutations of length n with a given up-down sequence $\mathcal{P}$. Apparently, for the first time this problem was investigated by D. Andrè [An1, An2], who considered the so-called alternating (or up-down) permutations corresponding to the sequence $\mathcal{P} = (1,0,1,0,\dots) = (10)^*$ and proved that the exponential generating function for the number of such permutations is equal to $\tan x + \sec x$. In [An3] he proved that this number grows asymptotically as $2n!(2/\pi)^{n+1}$.

A general approach to this problem was suggested by MacMahon (see [MM]). This approach leads to determinantal formulas for $\sharp_n^{\mathcal{P}}$, rediscovered later by Niven [Ni] from very basic combinatorial considerations. For the relations of this approach to the representation theory of the symmetric group, and for its generalizations, see [Fo, St1, BW].

Another, purely combinatorial approach to the same problem was suggested by Carlitz [Ca1]. His general recursive formula for $\sharp_n^{\mathcal{P}}$ is rather difficult to use. However, he managed to obtain explicit expressions for the corresponding exponential generating functions for certain *periodic* cases, that is for up-down sequences of the form $\mathcal{P} = (p)^*$, where p is a binary vector of a fixed length called the *period* of $\mathcal{P}$. In [Ca1, Ca2] he considered the case $\mathcal{P} = (1^k0)^*$ and expressed the corresponding generating function via the *Olivier functions of the kth order*

$$\varphi_{k,i}(x) = \sum_{j=0}^{\infty} \frac{x^{jk+i}}{(jk+i)!}, \qquad 0 \leqslant i \leqslant k-1.$$

Another case, $\mathcal{P} = (1^2 0^2)^*$, was considered in [CS1,CS2] and solved via Olivier functions of the fourth order. It follows that asymptotically $\sharp_n^{\mathcal{P}}$ in this case grows as $4n!(2/\gamma)^{n+1}$, where $\gamma = 3.7502\dots$ is the smallest positive solution of the equation $\cos t \cosh t + 1 = 0$.

The general periodic problem was solved completely in [CGJN]. As in the two particular cases mentioned above, the answer is expressed via Olivier functions. The techniques used involves matrix Riccati equations, and is rather complicated. For a different solution based on Möbius functions see [St2, Ch. 3.16, and Ex. 3.80].

An additional dimension in the problem was introduced by Entringer [En] who studied the distribution of the alternating permutations by the last entry. He observed that the number $\sharp_{i,j}$ of alternating permutations of length i whose last entry equals j satisfy the following equations:

$$\begin{aligned} &\sharp_{i,j} = \sharp_{i,j-1} + \sharp_{i-1,j-1}, \quad \sharp_{i,1} = 0, \qquad i = 2k, k > 0, \\ &\sharp_{i,j} = \sharp_{i,j+1} + \sharp_{i-1,j}, \qquad \sharp_{i,i} = 0, \qquad i = 2k+1, k > 0, \end{aligned} \tag{1.1}$$

with $\sharp_{1,1} = 1$. These equations can be represented graphically as the following triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 1 & & & \\ & & 1 & & 1 & & 0 & & \\ & 0 & & 1 & & 2 & & 2 & \\ 5 & & 5 & & 4 & & 2 & & 0 \end{array}$$

FIG. 1. THE ENTRINGER TRIANGLE

Each even row of the triangle starts with 0, and an entry in such a row is equal to the sum of its *left* neighbors in the current and in the previous rows. Similarly, each odd row (except for the first one) ends with 0, and an entry in such a row is equal to the sum of its *right* neighbors in the current and in the previous rows.

The Entringer triangle was studied by many authors. In particular, Arnold [Ar1, Ar2] gave an interpretation of the entries of this triangle in terms of real polynomials with real critical values. Besides, he considered the exponential generating function

$$A(x,y) = \sum_{i\geqslant 1}\sum_{j=1}^{i}(-1)^{(i-1)(i-2)/2}\sharp_{i,j}\frac{x^{i-j}y^{j-1}}{(i-j)!(j-1)!}$$

and proved that $A(x,y) = e^y/\cosh(x+y)$. In fact, $A(x,y)$ is the generating function of the *signed Entringer triangle*, which is obtained from the ordinary one by reversing signs in each ith row, where i equals 0 or 3 modulo 4. Observe that the entries $\tilde\sharp_{i,j}$ of the signed Entringer triangle satisfy relations

$$\tilde\sharp_{i,j} = \tilde\sharp_{i,j-1} + \tilde\sharp_{i-1,j-1} \tag{1.2}$$

with boundary conditions $\tilde\sharp_{i,1} = 0$ for $i = 2k$, $\tilde\sharp_{i,i} = 0$ for $i = 2k+1$, $k > 0$, $\tilde\sharp_{1,1} = 1$. General triangles satisfying relation (1.2) with arbitrary boundary conditions were first studied more than 120 years ago by Seidel [Se]. In particular, he proved that the ratio of exponential generating functions for the numbers on the right and on the left sides of such a triangle equals e^x. More recently such triangles where studied, from the combinatorial point of view, in [DV, Du1, Du2]. In particular, it is proved in [DV] that the exponential generating function for a Seidel triangle is equal to $e^yF(x+y)$, where $F(x)$ is the corresponding function for the left side of the triangle.

The case of general up-down sequences was addressed by de Bruijn [dB] (see also [Vi] for another version of the same result). Let $\sharp^{\mathcal P}_{i,j}$ be the number of permutations of length i whose last entry equals j and whose up-down sequence equals $\mathcal P = (p_1, p_2, \dots)$. He proved that these numbers satisfy the following equations:

$$\begin{aligned} &\sharp^{\mathcal P}_{i,j} = \sharp^{\mathcal P}_{i,j-1} + \sharp^{\mathcal P}_{i-1,j-1}, \quad \sharp^{\mathcal P}_{i,1} = 0, \qquad \text{if } p_{i-1} = 1 \\ &\sharp^{\mathcal P}_{i,j} = \sharp^{\mathcal P}_{i,j+1} + \sharp^{\mathcal P}_{i-1,j}, \qquad \sharp^{\mathcal P}_{i,i} = 0, \qquad \text{if } p_{i-1} = 0. \end{aligned}$$

with $\sharp^{\mathcal P}_{1,1} = 1$. Evidently, for $\mathcal P = (10)^*$ one gets the Entringer relations (1.1). As before, these equations can be represented graphically as a triangle, and the direction in which one has to advance along the rows of the triangle is governed by the sequence $\mathcal{P}$. We call this triangle the *de Bruijn triangle* corresponding to the up-down sequence $\mathcal{P}$. A de Bruijn triangle is said to be *periodic* if the corresponding up-down sequence is periodic.

Let $\mathcal{P}$ be a periodic up-down sequence with period p of length $m > 1$, and let $i_1 < i_2 < \cdots < i_r$ be the locations of zeros in p. Without loss of generality we assume that $i_r = m$ (otherwise we consider instead of $\mathcal{P}$ the up-down sequence $\bar{\mathcal{P}} = (\bar{p}_i)^*$, where $\bar{p}_i = 1 - p_i$ for $1 \leqslant i \leqslant m$; evidently, the de Bruijn triangle for $\bar{\mathcal{P}}$ is obtained from that for $\mathcal{P}$ by the reflection in the vertical axis).

The *signed de Bruijn triangle* is obtained from the ordinary de Bruijn triangle by multiplying its ith row by

$$\varepsilon_i = (-1)^{\bar{p}_1 + \bar{p}_2 + \cdots + \bar{p}_{i-1}}, \qquad i \geqslant 1. \tag{1.3}$$

The corresponding exponential generating function is defined by

$$F^{\mathcal{P}}(x,y) = \sum_{i\geqslant 1} \sum_{j=1}^{i} \varepsilon_i \sharp_{ij}^{\mathcal{P}} \frac{x^{i-j} y^{j-1}}{(i-j)!(j-1)!}. \tag{1.4}$$

Theorem 1. *The exponential generating function of the signed periodic de Bruijn triangle corresponding to the up-down sequence $\mathcal{P}$ is given by $F^{\mathcal{P}}(x,y) = e^y f^{\mathcal{P}}(x+y)$, where*

$$f^{\mathcal{P}}(t) = \frac{\det \bar{M}^{\mathcal{P}}(t)}{\det M^{\mathcal{P}}(t)}$$

and $M^{\mathcal{P}}(t)$ and $\bar{M}^{\mathcal{P}}(t)$ are $r \times r$ matrices

$$M^{\mathcal{P}}(t) = \begin{pmatrix} \varphi_{m,0} & \varphi_{m,m-i_1} & \varphi_{m,m-i_2} & \cdots & \varphi_{m,m-i_{r-1}} \\ \varphi_{m,i_1} & \varphi_{m,0} & \varphi_{m,m+i_1-i_2} & \cdots & \varphi_{m,m+i_1-i_{r-1}} \\ \varphi_{m,i_2} & \varphi_{m,i_2-i_1} & \varphi_{m,0} & \cdots & \varphi_{m,m+i_2-i_{r-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{m,i_{r-1}} & \varphi_{m,i_{r-1}-i_1} & \varphi_{m,i_{r-1}-i_2} & \cdots & \varphi_{m,0} \end{pmatrix}$$

and

$$\bar{M}^{\mathcal{P}}(t) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \varphi_{m,i_1} & \varphi_{m,0} & \varphi_{m,m+i_1-i_2} & \cdots & \varphi_{m,m+i_1-i_{r-1}} \\ \varphi_{m,i_2} & \varphi_{m,i_2-i_1} & \varphi_{m,0} & \cdots & \varphi_{m,m+i_2-i_{r-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{m,i_{r-1}} & \varphi_{m,i_{r-1}-i_1} & \varphi_{m,i_{r-1}-i_2} & \cdots & \varphi_{m,0} \end{pmatrix}$$

with $\varphi_{m,j} = \varphi_{m,j}(t)$.

In particular, for signed Entringer numbers one has $m = 2$, $r = 1$, and hence $f^{\{10\}^*}(t) = \varphi_{2,0}^{-1}(t) = 1/\cosh t$, thus recovering the Arnold formula for $A(x,y)$. Moreover, the same techniques allows to obtain generating functions for other Seidel triangles with periodic boundary conditions, such as the triangle for Genocchi numbers (see [DV]). It can be also extended to pairs of Seidel triangles with periodic boundary conditions, such as Arnold triangles $L(\beta)$ and $R(\beta)$ for Springer numbers (see [Ar2, Du2]), thus recovering several combinatorial results obtained in [Sp, Ar2]; see §2 for details.

As an immediate corollary of Theorem 1 we get the above mentioned results of [CGJN] and [St2] concerning the generating functions for the number of permutations with a given up-down sequence.

Corollary 1. *Let r be even, then the exponential generating function for the numbers $\sharp_n^{\mathcal{P}}$ is equal*

$$1 + \frac{\det \widetilde{M}^{\mathcal{P}}(t)}{\det M^{\mathcal{P}}(t)},$$

where $\widetilde{M}^{\mathcal{P}}(t)$ *is an* $r\times r$ *matrix*

$$\widetilde{M}^{\mathcal{P}}(t)=\begin{pmatrix} \psi_{m,0} & \psi_{m,1} & \psi_{m,2} & \cdots & \psi_{m,r-1}\\ \varphi_{m,i_1} & \varphi_{m,0} & \varphi_{m,m+i_1-i_2} & \cdots & \varphi_{m,m+i_1-i_{r-1}}\\ \varphi_{m,i_2} & \varphi_{m,i_2-i_1} & \varphi_{m,0} & \cdots & \varphi_{m,m+i_2-i_{r-1}}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{m,i_{r-1}} & \varphi_{m,i_{r-1}-i_1} & \varphi_{m,i_{r-1}-i_2} & \cdots & \varphi_{m,0} \end{pmatrix}$$

with

$$\psi_{m,j}=\varepsilon_{i_j+1}+\sum_{i\neq i_j}\varepsilon_{i+1}\varphi_{m,m+i-i_j}$$

and $\varphi_{m,j}=\varphi_{m,j}(t)$.

To get a similar result for odd r, it suffices to consider $\mathcal{P}$ as a periodic sequence with a period of length $2m$.

Let us now consider the asymptotic behavior of the numbers $\sharp^{\mathcal{P}}_{i,j}$. It was observed without a proof in [Ar2, p. 18] that the even rows of the ordinary Entringer triangle approximate, after an appropriate normalization, the function $\sin x$ on the interval $[0,\pi/2]$, while the odd rows approximate $\cos x$. Exact statements with the first two correction terms can be found in [SY].

We generalize this result to arbitrary periodic de Bruijn triangles.

Theorem 2. *For any* l, $0\leqslant l\leqslant m-1$, *one has*

$$\lim_{n\to\infty,\frac{j}{mn+l}\to t}\frac{\sharp^{\mathcal{P}}_{mn+l,j}\lambda^{mn+l}}{(mn+l-1)!}=c_{m,l}u^{\mathcal{P}}_l(t),$$

where $c_{m,l}$ *is a constant depending only on* m *and* l, *and* $u^{\mathcal{P}}_l$ *is the normalized first eigenfunction of the two-point spectral problem*

$$u^{(m)}=(-1)^r\lambda^m u$$

with m *homogeneous boundary conditions*

$$\begin{aligned} u^{(i)}(0)&=0 \qquad \textit{if } p_{m+l-i-1}=1,\\ u^{(i)}(1)&=0 \qquad \textit{if } p_{m+l-i-1}=0, \end{aligned}$$

where r *is the number of zeros in the period* p.

The first eigenvalue of the above spectral problem is the minimal absolute value among the solutions of the equation

$$\det M^{\mathcal{P}}(t)=0,$$

and the eigenfunction $u^{\mathcal{P}}_l$ *is normalized by the condition* $\max_{0\leqslant t\leqslant 1}u^{\mathcal{P}}_l(t)=1$.

In particular, for the Entringer numbers $\sharp_{i,j}$ one gets the sine law of [SY]:

$$\lim_{k\to\infty,\ \frac{j}{2k+1}\to t}\frac{\sharp_{2k+1,j}\left(\frac{\pi}{2}\right)^{2k+1}}{(2k)!}=c_{2,1}\cos\frac{\pi t}{2},$$

$$\lim_{k\to\infty,\ \frac{j}{2k}\to t}\frac{\sharp_{2k,j}\left(\frac{\pi}{2}\right)^{2k}}{(2k-1)!}=c_{2,0}\sin\frac{\pi t}{2};$$

it is shown in [SY] that $c_{2,0}=c_{2,1}=2$.

The starting point of this research was the result by the first and the third author that the numbers $\sharp^{\mathcal{P}}_{i,j}$ for $\mathcal{P}=(1^2 0^2)^*$ arise naturally in counting real rational functions of a certain type, see [SV]. The authors are grateful to Max–Planck–Institut für Mathematik, Bonn for its hospitality in September 2000 and to the Royal Institute of Technology, Stockholm for the financial support of the visit of A. V. to Stockholm in July-August 2001. Sincere thanks goes to R. Ehrenborg, A. Laptev, H. Shapiro, P. Yuditski and A.Volberg for useful discussions.

§2. Proofs

Proof of Theorem 1. Define $\tilde\sharp^{\mathcal P}_{i,j} = \varepsilon_i \sharp^{\mathcal P}_{i,j}$. Since by (1.3), ε_i equals ε_{i-1} if $p_{i-1} = 1$ and $-\varepsilon_{i-1}$ otherwise, we immediately get that $\tilde\sharp^{\mathcal P}_{i,j} = \tilde\sharp^{\mathcal P}_{i,j-1} + \tilde\sharp^{\mathcal P}_{i-1,j-1}$. In terms of the generating function $F^{\mathcal P}$ defined by (1.4) this relation translates to $F^{\mathcal P} = \partial F^{\mathcal P}/\partial y - \partial F^{\mathcal P}/\partial x$. The general solution of this equation is given by $F^{\mathcal P}(x,y) = e^y f^{\mathcal P}(x+y)$, where $f^{\mathcal P}$ is a function of one variable to be defined from the boundary conditions. Let $F^{\mathcal P}_L$ and $F^{\mathcal P}_R$ be the restrictions of $F^{\mathcal P}$ to the left and the right side of the signed de Bruijn triangle. It follows from the above discussion that $F^{\mathcal P}_L(t) = f^{\mathcal P}(t)$ and $F^{\mathcal P}_R(t) = e^t f^{\mathcal P}(t)$. Evidently, there exist unique representations

$$f^{\mathcal P}(t) = \sum_{i=0}^{m-1} t^i f_i^{\mathcal P}(t^m), \qquad e^t f^{\mathcal P}(t) = \sum_{i=0}^{m-1} t^i g_i^{\mathcal P}(t^m);$$

besides, $e^t = \sum_{i=0}^{m-1} \varphi_{m,i}(t)$. Combining these expressions together we get

$$\begin{pmatrix} g_0^{\mathcal P}(t^m) \\ t g_1^{\mathcal P}(t^m) \\ t^2 g_2^{\mathcal P}(t^m) \\ \vdots \\ t^{m-1} g_{m-1}^{\mathcal P}(t^m) \end{pmatrix} = \tag{2.1}$$

$$\begin{pmatrix} \varphi_{m,0} & \varphi_{m,m-1} & \varphi_{m,m-2} & \cdots & \varphi_{m,1} \\ \varphi_{m,1} & \varphi_{m,0} & \varphi_{m,m-1} & \cdots & \varphi_{m,2} \\ \varphi_{m,2} & \varphi_{m,1} & \varphi_{m,0} & \cdots & \varphi_{m,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{m,m-1} & \varphi_{m,m-2} & \varphi_{m,m-3} & \cdots & \varphi_{m,0} \end{pmatrix} \begin{pmatrix} f_0^{\mathcal P}(t^m) \\ t f_1^{\mathcal P}(t^m) \\ t^2 f_2^{\mathcal P}(t^m) \\ \vdots \\ t^{m-1} f_{m-1}^{\mathcal P}(t^m) \end{pmatrix},$$

where $\varphi_{m,i} = \varphi_{m,i}(t)$. Besides, relations $p_i = 0$ for $i = i_1, \dots, i_{r-1}$ imply $\tilde\sharp^{\mathcal P}_{km+i+1,km+i+1} = 0$ for $k = 0, 1, \dots$, and hence $g_i^{\mathcal P} = 0$ for $i = i_1, \dots, i_{r-1}$. Similarly, $p_m = 0$ together with $\tilde\sharp^{\mathcal P}_{1,1} = 1$ imply $g_0^{\mathcal P} = 1$, and $p_i = 1$ for $i \ne i_1, \dots, i_r$ imply $\tilde\sharp^{\mathcal P}_{km+i+1,1} = 0$ for $k = 0, 1, \dots$, and hence $f_i^{\mathcal P} = 0$ for $i \ne 0, i_1, \dots, i_{r-1}$. Therefore (2.1) is reduced to

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{m,0} & \varphi_{m,m-i_1} & \varphi_{m,m-i_2} & \cdots & \varphi_{m,m-i_{r-1}} \\ \varphi_{m,i_1} & \varphi_{m,0} & \varphi_{m,m+i_1-i_2} & \cdots & \varphi_{m,m+i_1-i_{r-1}} \\ \varphi_{m,i_2} & \varphi_{m,i_2-i_1} & \varphi_{m,0} & \cdots & \varphi_{m,m+i_2-i_{r-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{m,i_{r-1}} & \varphi_{m,i_{r-1}-i_1} & \varphi_{m,i_{r-1}-i_2} & \cdots & \varphi_{m,0} \end{pmatrix} \begin{pmatrix} f_0^{\mathcal P}(t^m) \\ t^{i_1} f_{i_1}^{\mathcal P}(t^m) \\ t^{i_2} f_{i_2}^{\mathcal P}(t^m) \\ \vdots \\ t^{i_{r-1}} f_{i_{r-1}}^{\mathcal P}(t^m) \end{pmatrix},$$

and the result follows. $\square$

The Seidel triangle $\{\gamma_{i,j}\}$ for signed Genocchi numbers presented in [DV] is satisfies periodic boundary conditions $\gamma_{2k,2k} = 0$ for $k = 1, 2, \dots$, $\gamma_{2,2} = 1$ and $\gamma_{2k+1,1} + \gamma_{2k+1,2k+1} = 0$ for $k = 0, 1, \dots$. Thus we arrive to equations

$$\begin{pmatrix} -f_0(t^2) \\ t \end{pmatrix} = \begin{pmatrix} \varphi_{2,0}(t) & \varphi_{2,1}(t) \\ \varphi_{2,1}(t) & \varphi_{2,0}(t) \end{pmatrix} \begin{pmatrix} f_0(t^2) \\ t f_1(t^2) \end{pmatrix},$$

which mean that the generating function for this triangle is $2(x+y)e^y/(e^{x+y}+1)$.

In a slightly different way one can treat signed versions of Arnold's pairs of triangles $\{L(\beta), R(\beta)\}$, $\{L(b), R(b)\}$, and $\{L(d), R(d)\}$ involving Euler and Springer numbers (see [Ar2, Du2]). The first pair consists of triangles $\{\beta^L_{i,j}\}$ and $\{\beta^R_{i,j}\}$ satisfying periodic boundary conditions

$$\begin{aligned} &\beta^L_{2k+1,1} = 0, \quad k = 1, 2, \dots, \qquad \beta^L_{1,1} = 1, \\ &\beta^L_{k,k} = \beta^R_{k,1}, \quad k = 0, 1, \dots, \\ &\beta^R_{2k,2k} = 0, \quad k = 1, 2, \dots. \end{aligned}$$

Their signed versions obtained by multiplying the ith row by $(-1)^{(i-1)(i-2)/2}$ are Seidel triangles.

Similarly, the second pair consists of triangles $\{b^L_{i,j}\}$ and $\{b^R_{i,j}\}$ satisfying periodic boundary conditions

$$\begin{aligned} &b^L_{2k+1,1} = 0, \quad k = 0, 1, \dots, \\ &b^L_{k,k} = b^R_{k,1}, \quad k = 2, 3, \dots, \qquad b^R_{1,1} = 1, \\ &b^R_{2k,2k} = 0, \quad k = 1, 2, \dots. \end{aligned}$$

To get a pair of Seidel triangles one has to multiply the ith row by $(-1)^{i(i-1)/2}$.

Finally, the third pair consists of triangles $\{d^L_{i,j}\}$ and $\{d^R_{i,j}\}$ satisfying periodic boundary conditions

$$\begin{aligned} &d^L_{2k,1} = 0, \quad k = 1, 2, \dots, \\ &d^L_{k,k} = d^R_{k,1}, \quad k = 2, 3, \dots, \\ &d^R_{2k+1,2k+1} = 0, \quad k = 1, 2, \dots, \qquad d^R_{1,1} = 1. \end{aligned}$$

To get a pair of Seidel triangles one has to multiply the ith row by $(-1)^{(i-1)(i-2)/2}$.

Let $f^\beta(t) = f^\beta_0(t^2) + tf^\beta_1(t^2)$ and $g^\beta(t) = g^\beta_0(t^2) + tg^\beta_1(t^2)$ be the restrictions of the generating functions for the signed versions of $L(\beta)$ and $R(\beta)$ to the left and right sides, respectively. Then we get the following equations:

$$\begin{pmatrix} g^\beta_0(t^2) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{2,0}(t) & \varphi_{2,1}(t) \\ \varphi_{2,1}(t) & \varphi_{2,0}(t) \end{pmatrix}^2 \begin{pmatrix} 1 \\ tf^\beta_1(t^2) \end{pmatrix},$$

and hence the generating functions for the triangles $L(\beta)$ and $R(\beta)$ are

$$F^\beta_L(x,y) = \frac{e^{-2x-y}}{\cosh 2(x+y)}, \qquad F^\beta_R(x,y) = \frac{e^{-x}}{\cosh 2(x+y)}.$$

For similar restrictions $f^b(t)$ and $g^b(t)$ one has

$$\begin{pmatrix} g^b_0(t^2) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{2,0}(t) & \varphi_{2,1}(t) \\ \varphi_{2,1}(t) & \varphi_{2,0}(t) \end{pmatrix}^2 \begin{pmatrix} 1 \\ tf^b_1(t^2) \end{pmatrix} + \begin{pmatrix} \cosh t \\ t^{-1}\sinh t \end{pmatrix},$$

and hence the generating functions for the for the signed versions of triangles $L(b)$ and $R(b)$ are

$$F^b_L(x,y) = -\frac{e^{y}\sinh(x+y)}{\cosh 2(x+y)}, \qquad F^b_R(x,y) = \frac{e^{-x}\cosh(x+y)}{\cosh 2(x+y)}.$$

Finally, for $f^d(t)$ and $g^d(t)$ one has

$$\begin{pmatrix} 1 \\ tg^d_1(t^2) \end{pmatrix} = \begin{pmatrix} \varphi_{2,0}(t) & \varphi_{2,1}(t) \\ \varphi_{2,1}(t) & \varphi_{2,0}(t) \end{pmatrix}^2 \begin{pmatrix} f^d_0(t^2) \\ 0 \end{pmatrix} + \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix},$$

and hence the generating functions for the for the signed versions of triangles $L(d)$ and $R(d)$ are

$$F^d_L(x,y) = \frac{e^{y}(1-\cosh(x+y))}{\cosh 2(x+y)}, \qquad F^d_R(x,y) = \frac{e^{x+2y}(1-\cosh(x+y))}{\cosh 2(x+y)}.$$

Proof of Corollary 1. It is enough to observe that for r even, the exponential generating function for $\sharp^{\mathcal{P}}_n$ is equal to $1 + \sum_{i=0}^{m-1} \varepsilon_{i+1} t^i (f^{\mathcal{P}}_i + g^{\mathcal{P}}_i)$, and to evaluate this sum according to the proof of Theorem 1. □

Proof of Theorem 2. Consider the space $PC[0,1]$ of piecewise continuous functions on the segment $[0,1]$ with the norm $||f|| = \max_{0\leqslant t\leqslant 1}|f(t)|$. Define two operators, S^0 and S^1, taking $PC[0,1]$ to itself:

$$S^0 f(x) = \int_0^x f(t)dt, \qquad S^1 f(x) = \int_x^1 f(t)dt.$$

It is easy to see that both S^0 and S^1 can be written as integral operators

$$S^i f(x) = \int_0^1 K^i(x,y)f(y)dy, \quad i = 0,1,$$

with the kernels

$$K^0(x,y) = \begin{cases} 1, & x \geqslant y, \\ 0, & x < y, \end{cases}$$

and $K^1(x,y) = 1 - K^0(x,y)$. Besides, consider two families of integral operators S^i_n, $i = 0,1$, with the kernels $K^i_n(x,y)$ given by

$$K^0_n(x,y) = \begin{cases} 1, & \lfloor (n+1)x \rfloor \geqslant \lfloor ny \rfloor, \\ 0, & \text{otherwise}, \end{cases}$$

and $K^1_n(x,y) = 1 - K^0_n(x,y)$. It is easy to see that

$$||S^i - S^i_n|| \leqslant \frac{c}{n} \tag{2.2}$$

for some constant $c > 0$.

Consider the operator $S^{\mathcal{P}} = S^{\bar{p}_m} S^{\bar{p}_{m-1}} \cdots S^{\bar{p}_1}$. Evidently, $S^{\mathcal{P}}$ is compact, as a product of compact operators. To study the spectral properties of $S^{\mathcal{P}}$, we make use of the infinite-dimensional Perron-Frobenius theory, as presented in [KR].

Lemma 1. *For any* $f \in PC[0,1]$,

$$S^{\mathcal{P}} f = \mu\psi(f)u^{\mathcal{P}} + Af, \tag{2.3}$$

where $\mu > 0$, ψ *is a strictly positive functional,* $S^{\mathcal{P}}u^{\mathcal{P}} = \mu u^{\mathcal{P}}$, $||u^{\mathcal{P}}|| = 1$, $Au^{\mathcal{P}} = 0$, $\psi(Af) = 0$ *and* $\lim_{n\to\infty} ||A^n||^{1/n} < \mu$.

Proof. Recall that we have assumed without loss of generality that $p_m = 0$. Let s be the largest index satisfying relations $p_{m+1-s'} = 0$ for $1 \leqslant s' \leqslant s$. Define a linear operator $R\colon PC[0,1] \to PC[0,1]$ by $Rf(x) = (1-x)^s f(x)$, and let $\mathcal{R}$ be the image of $PC[0,1]$ under R. We consider $\mathcal{R}$ as a Banach space in the induced norm, that is, $||f||_{\mathcal{R}} = \max_{0\leqslant x\leqslant 1} \frac{f(x)}{(1-x)^s}$.

Let K be the cone of nonnegative functions in $\mathcal{R}$, that is, $f \in \mathcal{R}$ belongs to K if $f(x) \geqslant 0$ for $x \in [0,1]$. Evidently, the closure of the linear span of K coincides with $\mathcal{R}$. We say that a linear operator $A\colon \mathcal{R} \to \mathcal{R}$ is *strongly positive* if it preserves K and takes any boundary point of K to an inner point. It is easy to see that $f \in K$ is an inner point of K if $\liminf_{x\to 1} f(x)/(1-x)^s > 0$.

Consider the restriction of $S^{\mathcal{P}}$ to $\mathcal{R}$, which we denote by $S^{\mathcal{P}}_{\mathcal{R}}$. Evidently, $S^{\mathcal{P}}_{\mathcal{R}}$ is compact and strongly positive. Hence, by Theorem 6.3 of [KR], the maximal eigenvalue μ of $S^{\mathcal{P}}_{\mathcal{R}}$ is positive, the corresponding eigenspace is one-dimensional, and all the other eigenvalues lie strictly inside the circle of radius μ. Since $S^{\mathcal{P}}$ takes the whole $PC[0,1]$ to $\mathcal{R}$, the same μ is the maximal eigenvalue of $S^{\mathcal{P}}$, and all the above properties of $S^{\mathcal{P}}_{\mathcal{R}}$ remain valid for $S^{\mathcal{P}}$. It follows from the proof of Theorem 6.3 in [KR] that (2.3) holds. $\square$

By differentiating equation $S^{\mathcal{P}}u^{\mathcal{P}} = \mu u^{\mathcal{P}}$ m times we see that $u^{\mathcal{P}}$ satisfies equation $\mu u^{(m)} = (-1)^r u$ with the boundary conditions $u^{(i)}(0) = 0$ if $p_{m-i} = 0$ and $u^{(i)}(1) = 0$ if $p_{m-i} = 1$. Put

$\lambda = \mu^{-1/m}$; then λ is the minimal positive solution of the equation $\det \widehat{M}^{\mathcal{P}}(\xi t) = 0$ with $\xi^m = (-1)^r$ and

$$\widehat{M}^{\mathcal{P}}(t) = \begin{pmatrix} \varphi_{m,0} & \varphi_{m,m-i_{r-1}} & \varphi_{m,m-i_{r-2}} & \cdots & \varphi_{m,m-i_1} \\ \varphi_{m,i_{r-1}} & \varphi_{m,0} & \varphi_{m,m+i_{r-1}-i_{r-2}} & \cdots & \varphi_{m,i_{r-1}-i_1} \\ \varphi_{m,i_{r-2}} & \varphi_{m,m+i_{r-2}-i_{r-1}} & \varphi_{m,0} & \cdots & \varphi_{m,i_{r-2}-i_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{m,i_1} & \varphi_{m,m+i_1-i_{r-1}} & \varphi_{m,m+i_1-i_{r-2}} & \cdots & \varphi_{m,0} \end{pmatrix}$$

with $\varphi_{m,j} = \varphi_{m,j}(t)$. It is easy to see that $M^{\mathcal{P}}$ can be obtained from $\widehat{M}^{\mathcal{P}}$ by the transformation $(k,l) \mapsto (r+2-k, r+2-l)$, and hence λ is the minimum absolute value of the solutions of the equation $\det M^{P}(t) = 0$.

Observe that the condition on the norms of $||A^n||$ in (2.3) means that there exists k such that $||A^k|| < \mu^k$. Denote $\alpha = \frac{1}{\mu^k}||A^k|| < 1$ and $T = (\frac{1}{\mu}S^{\mathcal{P}})^k$.

We are approximating the operator $S^{\mathcal{P}}$ with the help of operators $S^{\mathcal{P}}_n$ defined by

$$S^{\mathcal{P}}_{n+1} = S^{\bar{p}_m}_{(n+1)m} S^{\bar{p}_{m-1}}_{(n+1)m-1} \cdots S^{\bar{p}_1}_{nm+1}.$$

It follows immediately from (2.2) that

$$||S^{\mathcal{P}} - S^{\mathcal{P}}_n|| \leqslant \frac{c'}{n} + O\left(\frac{1}{n^2}\right). \tag{2.4}$$

Finally, to approximate T define operators $T_{n+1} = \frac{1}{\mu^k} S^{\mathcal{P}}_{(n+1)k} S^{\mathcal{P}}_{(n+1)k-1} \cdots S^{\mathcal{P}}_{kn+1}$; it follows from the above inequality that

$$||T_n - T|| \leqslant \frac{c''}{n} + O\left(\frac{1}{n^2}\right).$$

For any function f define a sequence $\{f_n\}$ by $f_n = T_n f_{n-1}$ with $f_0 = f$.

Lemma 2. *Let $\{||f_n||\}$ be bounded, then the sequence $f_n = f_n/||f_n||$ converges to $u^{\mathcal{P}}$ as $n \to \infty$.*

Proof. By (2.3), each of f_i can be uniquely represented as $f_i = f_i^u + f_i^A$, where f_i^u is a multiple of $u^{\mathcal{P}}$ and f_i^A belongs to the maximal subspace invariant under A and not containing $u^{\mathcal{P}}$. Therefore,

$$f_n = T_n f_{n-1} = (T + (T_n - T))(f^u_{n-1} + f^A_{n-1}) = f^u_{n-1} + A^k f^A_{n-1} + (T_n - T)(f^u_{n-1} + f^A_{n-1}),$$

which together with $||u^{\mathcal{P}}|| = 1$ gives

$$||f^u_n|| \geqslant ||f^u_{n-1}|| - ||(T_n - T) f_{n-1}||,$$
$$||f^A_n|| \leqslant ||A^k f^A_{n-1}|| + ||f^u_{n-1}|| \cdot ||(T_n - T)u^{\mathcal{P}}|| + ||(T_n - T) f^A_{n-1}||.$$

Recall that $\{||f_n||\}$ is bounded, and hence

$$||f^u_n|| \geqslant ||f^u_{n-1}|| - \left(\frac{c'''}{n} + O\left(\frac{1}{n^2}\right)\right) \geqslant \beta ||f^u_{n-1}||,$$

where $\beta < 1$ can be chosen arbitrary close to 1 for n big enough. Therefore,

$$\delta_n = \frac{||f^A_n||}{||f^u_n||} \leqslant \frac{||f^u_{n-1}||}{\beta ||f^u_{n-1}||}\left(\frac{c''}{n} + O\left(\frac{1}{n^2}\right)\right) + \frac{||f^A_{n-1}||}{\beta ||f^u_{n-1}||}\left(\alpha + \frac{c''}{n} + O\left(\frac{1}{n^2}\right)\right) \leqslant$$
$$\frac{c''}{\beta n} + \left(\frac{\alpha}{\beta} + \frac{c''}{\beta n}\right)\delta_{n-1} + O\left(\frac{1}{n^2}\right).$$

Therefore, either $\delta_{n-1} \leqslant n^{-1/2}$, or $\delta_{n-1} > n^{-1/2}$ and

$$\delta_n \leqslant \left(\frac{c''}{\beta\sqrt{n}} + \frac{\alpha}{\beta} + \frac{c''}{\beta n}\right)\delta_{n-1} + O\left(\frac{1}{n^2}\right) \leqslant \alpha' \delta_{n-1}$$

for some constant $\alpha' < 1$. In any case $\delta_n \to 0$ as $n \to \infty$, and hence the sequence $\{f_n/||f_n^u||\}$ converges to a multiple of $u^{\mathcal{P}}$, which implies the convergence of $\{f_n/||f_n||\}$ to $u^{\mathcal{P}}$. $\square$

For an arbitrary finite sequence $a = \{a_1, \dots, a_k\}$ define a piecewise constant function $\hat{a} \in PC[0,1]$ whose value equals a_i on the interval $\left[\frac{i-1}{k}, \frac{i}{k}\right)$ for $i = 1, \dots, k-1$ and a_k on $\left[\frac{k-1}{k}, 1\right]$. Let $\sharp_k = \{\sharp_{k,1}, \dots, \sharp_{k,k}\}$. It is easy to see that $\hat{\sharp}_{k+1} = kS_k^{\bar{p}_k}\hat{\sharp}_k$, which means that

$$S_{n+1}^{\mathcal{P}}\hat{\sharp}_{mn+1} = \frac{(mn)!}{(m(n+1))!}\hat{\sharp}_{m(n+1)+1}. \tag{2.5}$$

Observe that the sequence of functions $g_n = \frac{\hat{\sharp}_{mn+1}\lambda^{mn+1}}{(mn)!}$ is bounded. Indeed, $||\hat{\sharp}_{mn+1}|| = \sharp_{mn+1,1}$. The exponential generating function for the numbers $\sharp_{mn+1,1}$ is calculated in Corollary 1. Since the numerator of the corresponding expression is a polynomial in Olivier functions, which converge in the whole plain, the numbers $\frac{\sharp_{mn+1,1}}{(mn)!}$ grow asymptotically as $\frac{\gamma}{\lambda^{mn+1}}$, hence for n big enough one has $\frac{\sharp_{mn+1,1}\lambda^{mn+1}}{(mn)!} < \gamma'$, and therefore the sequence $\{g_n\}$ is bounded. Moreover, (2.5) can be rewritten as $\lambda^m S_{n+1}^{\mathcal{P}} g_{mn+1} = g_{m(n+1)+1}$. Therefore, Lemma 2 applies, and $g_{nm+1} \to u^{\mathcal{P}}$ as $n \to \infty$.

Combining the above results we get

$$\lim_{n\to\infty,\frac{j}{mn+1}\to t} \frac{\sharp^{\mathcal{P}}_{mn+1,j}\lambda^{mn+1}}{(mn)!} = c_{m,1}u_1^{\mathcal{P}}(t),$$

with $c_{m,1} = \gamma'$, and hence Theorem 2 is proved for $l = 1$.

To get the proof for the other values of l one has to consider, instead of $S^{\mathcal{P}}$, a different operator: $S^{\bar{p}_{m+l-1}}S^{\bar{p}_{m+l-2}}\dots S^{\bar{p}_l}$. Its properties are identical to those of $S^{\mathcal{P}}$; to prove this one has to use operators R and L: $f(x) \mapsto x^s f(x)$, depending on the value of p_{m+l-1}. $\square$

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