

**CORRIGENDUM: “HOMOGENIZED SPECTRAL PROBLEMS
FOR EXACTLY SOLVABLE OPERATORS: ASYMPTOTICS OF
POLYNOMIAL EIGENFUNCTIONS”**

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ABSTRACT. Here we provide a correct proof of Proposition 6 of [2]. No other results of the latter paper are affected.

1. NECESSARY RESULTS AND CORRECTED PROOF

To make this note self-contained we briefly recall the basic set-up of [2]. Given a $(k+1)$ -tuple of polynomials $(Q_k(z), Q_{k-1}(z), \dots, Q_0(z))$ with $\deg Q_i(z) \leq i$ consider the *homogenized spectral pencil* of differential operators given by:

$$T_\lambda = \sum_{i=0}^k Q_i(z) \lambda^{k-i} \frac{d^i}{dz^i}. \quad (1.1)$$

Introduce the algebraic curve Γ associated with T_λ and given by the equation

$$\sum_{i=0}^k Q_i(z) w^i = 0, \quad (1.2)$$

where the polynomials $Q_i(z) = \sum_{j=0}^i a_{i,j} z^j$ are the same as in (1.1).

The curve Γ and its associated pencil T_λ are called of *general type* if the following two nondegeneracy requirements are satisfied:

- (i) $\deg Q_k(z) = k$ (i.e., $a_{k,k} \neq 0$),
- (ii) no two roots of the (characteristic) equation

$$a_{k,k} + a_{k-1,k-1}t + \dots + a_{0,0}t^k = 0 \quad (1.3)$$

lie on a line through the origin (in particular, 0 is not a root of (1.3)).

The first statement of [2] we need is as follows.

Proposition 1. *If the characteristic equation (1.3) has k distinct solutions $\alpha_1, \alpha_2, \dots, \alpha_k$ and satisfies the preceding nondegeneracy assumptions (in particular, these imply that $a_{0,0} \neq 0$ and $a_{k,k} \neq 0$) then*

- (i) *for all sufficiently large n there exist exactly k distinct eigenvalues $\lambda_{n,j}$, $j = 1, \dots, k$, such that the associated spectral pencil T_λ has a polynomial eigenfunction $p_{n,j}(z)$ of degree exactly n ,*
- (ii) *the eigenvalues $\lambda_{n,j}$ split into k distinct families labeled by the roots of (1.3) such that the eigenvalues in the j -th family satisfy*

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n,j}}{n} = \alpha_j, \quad j = 1, \dots, k.$$

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[†] J. B. unexpectedly passed away on April 8, 2009 at the age of 40.

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The main result of [2] is given below.

Theorem 1. *In the notation of Proposition 1 for any pencil T_λ of general type and every $j = 1, \dots, k$ there exists a subsequence $\{n_{i,j}\}$, $i = 1, 2, \dots$, such that the limits*

$$\Psi_j(z) := \lim_{i \rightarrow \infty} \frac{p'_{n_{i,j}}(z)}{\lambda_{n_{i,j}} p_{n_{i,j}}(z)}, \quad j = 1, \dots, k,$$

exist almost everywhere in \mathbb{C} and are analytic functions in some neighborhood of ∞ . Each $\Psi_j(z)$ satisfies equation (1.2), i.e., $\sum_{i=0}^k Q_i(z) \Psi_j^i(z) = 0$ almost everywhere in \mathbb{C} , and the functions $\Psi_1(z), \dots, \Psi_k(z)$ are independent sections of Γ considered as a branched covering over $\mathbb{C}\mathbb{P}^1$ in a sufficiently small neighborhood of ∞ .

Its proof requires the next Lemma 1 and Proposition 2. (The proof of the latter Proposition suggested in [2] was erroneous and is corrected below.)

Lemma 1 (cf. Lemma 2 of [2]). *Let $\{q_m(z)\}$ be a sequence of polynomials with $\deg q_m(z) \rightarrow \infty$ as $m \rightarrow \infty$. Denote by μ_m and μ'_m the root-counting measures of $q_m(z)$ and $q'_m(z)$, respectively, and assume that there exists a compact set K containing the supports of all measures μ_m and therefore also the supports of all measures μ'_m . If $\mu_m \rightarrow \mu$ and $\mu'_m \rightarrow \mu'$ as $m \rightarrow \infty$ and u and u' are the logarithmic potentials of μ and μ' , respectively, then $u' \leq u$ in \mathbb{C} with equality in the unbounded component of $\mathbb{C} \setminus \text{supp}(\mu)$.*

Example 1. *Consider the polynomial sequence $\{z^m - 1\}$. The measure μ is then the uniform distribution on the unit circle of total mass 1. Its logarithmic potential $u(z)$ equals $\log |z|$ if $|z| \geq 1$ and 0 in the disk $|z| \leq 1$. On the other hand, the sequence of derivatives is given by $\{mz^{m-1}\}$ and the corresponding (limiting) logarithmic potential $u'(z)$ equals $\log |z|$ in $\mathbb{C} \setminus \{0\}$. Obviously, $u(z) = u'(z)$ in $|z| \geq 1$ and $u'(z) < u(z)$ in $|z| < 1$.*

In the notation of Theorem 1 consider the family of eigenpolynomials $\{p_{n,j}(z)\}$ for some arbitrarily fixed value of the index $j = 1, \dots, k$. Assume that N_j is a subsequence of the natural numbers such that

$$\mu_j^{(i)} := \lim_{n \rightarrow \infty, n \in N_j} \mu_{n,j}^{(i)} \quad (1.4)$$

exists for $i = 0, \dots, k$, where $\mu_{n,j}^{(i)}$ denotes the root-counting measure of $p_{n,j}^{(i)}(z)$. The existence of such N_j follows from the above remark on $M(K)$ (Helly's theorem). Notice that for each i the logarithmic potential $u_j^{(i)}$ of $\mu_j^{(i)}$ satisfies a.e. the identity

$$u_j^{(i)}(z) - u_j^{(0)}(z) = \lim_{n \rightarrow \infty, n \in N_j} \frac{1}{n} \log \left| \frac{p_{n,j}^{(i)}(z)}{n(n-1) \dots (n-i+1) p_{n,j}(z)} \right|.$$

The next proposition completes the proof of Theorem 1 and also shows the remarkable property that if one considers a sequence of eigenpolynomials for some spectral pencil then the situation $u'(z) < u(z)$ seen in Example 1 can never occur. In fact, for the validity of Proposition 2 one only needs two assumptions: a) $\deg Q_k(z) = k$ (i.e., $a_{k,k} \neq 0$, so that all $\alpha_j, j = 1, \dots, k$ are non-zero) and b) $Q_0 \neq 0$.

Proposition 2. *The measures $\mu_j^{(i)}$, $i = 0, \dots, k$, are all equal and the scalar multiple $\tilde{\Psi}_j = C_\mu / \alpha_j$ of the Cauchy transform of this common measure μ_j satisfies equation (1.2) almost everywhere.*

Proof. For $n \in N_j$ one has

$$\frac{p_{n,j}^{(i+1)}(z)}{(n-i)p_{n,j}^{(i)}(z)} \rightarrow C^{(i+1)}(z) := \int_{\mathbb{C}} \frac{d\mu_j^{(i)}(\zeta)}{z - \zeta} \text{ as } n \rightarrow \infty$$

with convergence in L^1_{loc} . The well-known property of convergence in L^1_{loc} implies that passing to a subsequence one can assume that the above convergence is actually the pointwise convergence almost everywhere in \mathbb{C} . It follows that

$$\frac{p_{n,j}^{(i)}(z)}{n^i p_{n,j}(z)} \rightarrow C^{(1)}(z)C^{(2)}(z)\dots C^{(i)}(z), \quad (1.5)$$

pointwise almost everywhere in \mathbb{C} . We claim that this limit is non-zero a.e. Given this, consider

$$u_j^{(k)}(z) - u_j^{(0)}(z) = \lim_{n \rightarrow \infty, n \in N_j} \frac{1}{n} \log \left| \frac{p_{n,j}^{(k)}(z)}{n(n-1)\dots(n-k+1)p_{n,j}(z)} \right| = 0$$

almost everywhere in \mathbb{C} . On the other hand, $u_j^0 \geq u_j^{(1)} \geq \dots \geq u_j^{(k)}$ by Lemma 1. Hence the potentials $u_j^{(i)}$ are all equal and the corresponding measures $\mu_j^{(i)} = \Delta u_j^{(i)}/2\pi$ are equal as well.

It remains to settle the above claim. Recall that $p_{n,j}(z)$ satisfies the differential equation $T_{\lambda_{n,j}} p_{n,j}(z) = 0$, i.e.,

$$Q_k(z)p_{n,j}^{(k)}(z) + \lambda_{n,j}Q_{k-1}(z)p_{n,j}^{(k-1)}(z) + \dots + \lambda_{n,j}^k Q_0(z)p_{n,j}(z) = 0. \quad (1.6)$$

Therefore,

$$Q_k(z)\frac{p_{n,j}^{(k)}(z)}{n^k p_{n,j}(z)} + \frac{\lambda_{n,j}}{n}Q_{k-1}(z)\frac{p_{n,j}^{(k-1)}(z)}{n^{k-1}p_{n,j}(z)} + \dots + \frac{\lambda_{n,j}^k}{n^k}Q_0(z) = 0.$$

Using the asymptotics $\lambda_{n,j} \sim \alpha_j n$ and the pointwise convergence a.e. in (1.5) we get

$$Q_k(z)C^{(1)}(z)\dots C^{(k)}(z) + \alpha_j Q_{k-1}(z)C^{(1)}(z)\dots C^{(k-1)}(z) + \dots + \alpha_j^k Q_0(z) = 0. \quad (1.7)$$

Using the assumption that $Q_0 \neq 0 \neq \alpha_j$, we conclude that $C^{(1)}(z) \neq 0$ a.e. To prove that $C^{(2)}(z)$ is also non-zero a.e., we consider the differential equation satisfied by $p'_{n,j}(z)$

$$Q_k(z)p_{n,j}^{(k+1)}(z) + (Q'_k(z) + \lambda_{n,j}Q_{k-1}(z))p_{n,j}^{(k)}(z) + \dots + (\lambda_{n,j}^{k-1}Q'_1(z) + \lambda_{n,j}^k Q_0(z))p'_{n,j}(z) = 0$$

which is obtained by differentiating (1.6). Repeating the previous analysis we get

$$Q_k(z)\frac{p_{n,j}^{(k+1)}(z)}{n^k p'_{n,j}(z)} + \frac{(Q'_k(z) + \lambda_{n,j}Q_{k-1}(z))}{n} \frac{p_{n,j}^{(k)}(z)}{n^{k-1}p'_{n,j}(z)} + \dots + \frac{(\lambda_{n,j}^{k-1}Q'_1(z) + \lambda_{n,j}^k Q_0(z))}{n^k} = 0.$$

Hence in the limit we obtain

$$Q_k(z)C^{(2)}(z)\dots C^{(k+1)}(z) + \alpha_j Q_{k-1}(z)C^{(2)}(z)\dots C^{(k)}(z) + \dots + \alpha_j^k Q_0(z) = 0,$$

which implies that $C^{(2)}(z)$ is non-zero a.e. as well. Similarly, $C^{(i)}(z)$, $i \geq 3$ is non-zero a.e. which proves the claim.

The fact that the multiple $\tilde{\Psi}_j = C_\mu/\alpha_j$ of the Cauchy transform of this common measure μ_j satisfies equation (1.2) almost everywhere follows by (1.7), since the equality of the measures implies that $C_\mu = C^{(1)} = C^{(2)} = \dots$, and thus

$$Q_k(z)(C_\mu(z))^k + \alpha_j Q_{k-1}(z)(C_\mu(z))^{k-1} + \dots + \alpha_j^k Q_0(z) = 0$$

which is equivalent to (1.2). \square

Note that in Example 1, the polynomials $p_n(z) := z^n - 1$ satisfy the differential equation $z p_n''(z) - (n-1)p_n'(z) = 0$, and that the corresponding homogenized pencil $z \frac{d^2}{dz^2} - n \frac{d}{dz}$ has $Q_0 = 0$.

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