How to Run a Centipede: a Topological Perspective

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To Andrey

Abstract In this paper we study the topology of the configuration space of a device with d legs ("centipede") under some constraints, such as the impossibility to have more than k legs off the ground. We construct feedback controls stabilizing the system on a periodic gait and defined on a 'maximal' subset of the configuration space.

Key words: stable control, attractor, configuration of tori

A centipede was happy quite!
Until a toad in fun
Said, "Pray, which leg moves after which?"
This raised her doubts to such a pitch,
She fell exhausted in the ditch
Not knowing how to run.

Katherine Craster

1 Introduction

How the centipedes move? This question becomes nontrivial once one starts to think about it, or when one is designing a multi-legged robotic device [2]. Indeed, the motivation for this work comes from a class of agile robotic devices, *RHex* [3]. Our take on the centipede's quandary is that it is caused by essentially *topological reasons*, preventing continuous feedback controls.

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In this note we consider a caricature of an automotive robot moving around using rotating "legs", making the configuration space a torus $\mathbb{T}^d$, i.e. a d-fold product of the circle, $\mathbb{T}^1$. The similarities with the wheeled vehicles end here: for obvious reasons, there exist regions in the configuration space, a "forbidden" subset, where the system should avoid at any cost. The picture below, taken from `http://kodlab.seas.upenn.edu/RHex/Home` illustrate the kind of systems we are dealing here.

Fig. 1 A specimen of the *RHex* family of legged robots, designed in University of Pennsylvania.

As an example, the configuration where all the legs point up should be forbidden. Of course the forbidden configurations are design specific: thus in RHex, the forbidden configurations also include those with all legs up on one side of the robot, or those with just two legs (out of six) pointing down.

Excluding the forbidden regions makes the topology of the configuration space interesting, and the control problems (even in the fully actuated setting) nontrivial. Typically, the control design problem aims at a closed-loop feedback control that stabilizes the system on a (say, periodic) trajectory, a *gait*. As the homotopy type of the configuration space differs from that of the limiting attractor, a continuous feedback control is impossible, and a locus of discontinuity emerges. This locus of discontinuity is *not* canonical, and depends on the realization of the feedback control, but its topology is, as it turns out, more or less fixed by the mismatch of the homotopy types of the configuration space and the attractor.

This motivates our attention to the topology of the configuration space and constructions of the minimal, in a suitable sense, discontinuity loci.

In this paper our objective is to analyze from this viewpoint the topology of the configuration spaces of *RHex*-like robots which we will be referring to as the *centipedes*. To do this we

- describe the topology of the discontinuity loci;
- present an explicit construction of the discontinuity loci for a large class of robots (and their corresponding forbidden regions), and
- find a feedback control for rotation of centipede's legs stabilizing the system on a prespecified (diagonal) gait.

1.1 Setup

Let us fix the notation. We denote the total number of legs as d, which are fully actuated and can (apriori) take all possible position. The space of legs positions, the d-dimensional torus $\mathbb{T}^d$ is coordinatized by the *angles* $\phi_i, i = 1, \ldots, d, \phi_i \in \mathbb{T}^1 = [0, 2\pi]/\langle 0 = 2\pi \rangle$. We will assume that $\phi_i = 0$ corresponds to the position of the i-th leg pointing vertically up.

To describe the class of forbidden configurations, we will need the notion of *coordinate toric arrangements*. Let I be an *ideal* in the Boolean lattice B_d of subsets of $\{1, \ldots, d\}$ (i.e. if $A \in I$, and $B \subset A$ then $B \in I$).

The coordinate toric arrangement $\mathscr{A}_I$ is the union of all coordinate tori $T_A, A \in I$:

$$\mathscr{A}_I = \bigcup_{A \in I} \mathbb{T}_A,$$

where $\mathbb{T}_A = \{\phi_i = 0 \text{ for } i \notin A\}$ (the size of A is the dimension of $\mathbb{T}_A$). We remark that the tori $\mathbb{T}_A$ provide a natural stratification of the arrangement $\mathscr{A}_I$. The inclusion $I_1 \subset I_2$ implies $\mathscr{A}_{I_1} \subset \mathscr{A}_{I_2}$.

One typical example is $I = \{A : |A| \geq k\}$, the configurations with at least $k \leq d$ legs are pointing up. In this case, the corresponding toric arrangement is just the k-skeleton of the torus.

A toric arrangement is good approximation for a forbidden region: the fact that a whole coordinate torus is forbidden is equivalent to the natural assumption, that if having some collection of legs up causes failure of the device when the rest of the legs point down, then bringing these remaining legs into *any* configuration still will result in a failure. Thus, for the original *RHex*, having three right legs up, and three left legs down is a failure, and any other position of the left legs will still be a failure.

Of course, having the forbidden set a toric arrangement is merely a caricature of the *physical* set of forbidden configurations: clearly, the stability of a robotic device cannot fail exactly when some collection of legs is pointing upwards, and not in nearby points. However, from the topological perspective, this assumption is rather reasonable, if one adopts its softer version.

1.2 Conventions

We will be assuming (relying on the intuition outlined above, and developed in the literature on RHex, see, e.g. [4, 3]) that *set of failure positions* $\mathtt{Fb}_I \subset \mathbb{T}^d$ is an open domain containing $\mathscr{A}_I$ with *smooth boundary*, such that $\mathscr{A}_I \subset \mathtt{Fb}_I$ is a *deformation retract*. Its complement $\mathtt{Fr}_I = \mathbb{T}^d \setminus \mathtt{Fb}_I$ is the *set of safe configurations*.

Further, we assume that $\mathtt{Fb}_I$ is an open and $\mathtt{Fr}_I$ is a closed manifold with a smooth boundary $\partial\mathtt{Fr}_k$.

We are interested in *closed loop feedback stabilization*, that is in vector fields $\mathbf{v}$ defined at least in $\mathtt{Fr}_I$ (including its boundary $\partial\mathtt{Fr}_I$) and such that the field $\mathbf{v}$ points into $\mathtt{Fr}_I$ on $\partial\mathtt{Fr}_I$. The vector field $\mathbf{v}$ should have as an attractor a periodic trajectory (gait) γ.

If (as is typical) $\mathtt{Fb}_I$ does not have the homotopy type of a circle, it is impossible to have $\mathtt{Fb}_I$ as the basin of attraction for γ. Hence, we need to find a subset $\mathtt{Bs}_I \subset \mathtt{Fr}_I$ which contains the attractor γ, is as large as possible and is a basin of attraction for γ. (We are deliberately vague here about the meaning of the expression "as large as possible" which will be clarified below.)

The complement to such a basin will be called a *cut*. The fact that a continuous feedback stabilization is impossible if the topologies of the configuration space and the attractor do not match has been noticed long ago (see, e.g. [6]). What we emphasize here is the nontrivial topology of the cuts (implying that it has to be non-empty), and some useful criteria for its minimality.

1.3

The general theory of the topologically forced cuts in the closed loop feedback stabilization will be addressed elsewhere; this note serves as an extended example of the stabilization in nontrivial configuration spaces, rich and relevant to applications yet simple to be analyzed completely.

The structure of the paper is as follows. In Section 2 we describe some relevant topological preliminaries. In Section 3 we introduce a construction of a cut that is optimal for all ideals I. In Section 4 we describe a vector field stabilizing the system to a periodic trajectory on the optimal $\mathtt{Bs}_I$. Finally, in Appendix we describe an intriguing discrete dynamical system associated with our choice of the basin and cut.

1.4 Acknowledgements.

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2 Topology of $\mathscr{A}_I$ and $\mathtt{Fb}_I$

2.1 Topology of forbidden set

By assumption, the set $\mathtt{Fb}_I$ of forbidden configurations is retractable to the toric arrangement $\mathscr{A}_I$ so that the embedding of its complement $\mathtt{Fr}_I$ to $T^d \setminus \mathscr{A}_I$ is a homotopy equivalence.

The space $T^d \setminus \mathscr{A}_I$ is in its turn is retractable to a certain toric arrangement. We refer for the detailed exposition to, e.g. [1], and present here just the result.

An ideal I (of the partition lattice) can be considered as a non-increasing Boolean function f_I: of the vector of $0,1$'s is the indicator function of A, then $f_I(A) = 1$ iff $A \in I$. The function

$$f_{I^\circ} : (x_1,\dots,x_d) \mapsto 1 - f_I(1-x_1,\dots,1-x_d)$$

is also non-increasing and therefore defines a Boolean ideal I°; we call it the *dual ideal* to I.

The toric arrangement corresponding to I° on which $T^d \setminus \mathscr{A}_I$ retracts can be described as

$$\mathscr{A}_I^\circ = \bigcup_{B \in I^\circ} \mathbb{T}_B^\circ,$$

where $\mathbb{T}_B^\circ = \{\phi_j = \pi \text{ for } j \notin B\}$.

In particular, if I° contains all singletons (or, equivalently, if *each* leg can make a full turn avoiding forbidden configurations, with the remaining legs in some fixed positions), then the first homology of $\mathtt{Fr}_I$ coincides with that of the torus. More generally, if I° contains the all subsets of size k (or I does not contain subset of size $(d-k)$ or more), then the integer (co)homology groups of $\mathtt{Fr}_I$ coincide with these of T^d up to the dimension $d-k-1$ and the isomorphism of (co)homology groups is induces by the inclusion $\mathtt{Fr}_I \subset T^d$.

Also the fundamental group $\pi_1(\mathtt{Fr}_I)$ of $\mathtt{Fr}_I$ is isomorphic to that of T^d and thus coincides with $\mathbb{Z}^d$ for if I does not contain subsets of size $(d-2)$.

2.2 Feedback stabilization

2.2.1 Attractors

We are concerned primarily with the stabilization on a specific gait, a periodic trajectory representing the diagonal homology class in $H_1(\mathbb{T}^d,\mathbb{Z})$. (Note that in principle other classes are possible, for example a multiple of the diagonal class, corresponding to a periodic gait.)

Remark 1 *Knotted attractors present a potential complicating twist. If the number of legs is three, there are infinitely many nonequivalent (under an ambient isotopy) trajectories representing the same (free) homotopy class in the space* $\mathtt{Fr}_I$. *We will be ignoring this problem - there are few plausible engineering designs with mere three legs, and in* $d \geq 4$ *piece-wise smoothly embedded closed curves are isotopic when they represent the same homotopy class.*

However, it would be interesting to try to construct a knotted gait for three-legged robots, and a feedback control stabilizing on such a gait.

We fix this closed simple oriented curve γ in $\mathtt{Fr}_I$, the attractor convergence to which we are seeking, representing the diagonal homology class in $H_1(\mathbb{T}^d,\mathbb{Z})$ (which means, in words, that over the trajectory, each leg makes exactly one turn around).

If $\mathtt{Fb}_I$ is a sufficiently small neighborhood of $\mathscr{A}_I$ we can choose γ among the geodesics of the flat metrics on $\mathbb{T}^d$, i.e. among $\gamma_\phi = \phi + t(1,...,1)$ on $\mathbb{T}^d$ where $t \in \mathbb{R}$ and γ is a sufficiently generic point in $\mathbb{T}^d$. (This does not reduce generality as by assumption, one can always find a diffeomorphism – fixing $\mathscr{A}_I$ – that would shrink $\mathtt{Fb}_I$ to a small vicinity of $\mathscr{A}_I$.)

2.3 Vector fields and their basins

As we mentioned above, the closed loop feedback stabilization of $\mathtt{Fr}_I$ on γ is impossible in nontrivial situations: there is no vector field $\mathbf{v}$ on $\mathtt{Fr}_I$, pointing inward $\mathtt{Fr}_I$ on the boundary, such that all solutions tend to the attractor γ. This means one need to reduce the domain where the vector field is defined.

Definition 1 *We will be calling an open subset* $\mathtt{Bs} \subset \mathtt{Fr}_I$ *an* admissible basin*, if there exists a smooth vector field on* $\mathtt{Fr}$ *such that*

- *the gait* γ *is an attractor of the positive time flow* $g^t_{\mathbf{v}}$ *defined by* $\mathbf{v}$*, and*
- *the negative trajectories* $g^t_{\mathbf{v}}x, t < 0$ *starting outside* γ *leave* $\mathtt{Bs}$ *in finite time (depending on the starting point* $x \notin \gamma$*).*

The complement $\mathtt{Ct}_I$ *to the admissible basin* $\mathtt{Bs}_I$ *will be called an* admissible cut*, or simply a cut.*

We will call an admissible basin $\mathtt{Bs}_I$ set maximal *in* $\mathtt{Fr}_I$ *if no proper superset of* $\mathtt{Bs}_I$ *in* $\mathtt{Fr}_I$ *has the same homotopy type as* $\mathtt{Bs}_I$.

The set-maximality property of $\mathtt{Bs}_I$ is rather basic and departs from the natural geometric characteristics like volume of $\mathtt{Ct}$ or its dimension, Hausdorff measure and suchlike. The reason is obvious: the definition is universal, and independent of any extraneous data save the topological ones.

3 Universal cut

One of the main contributions of this paper is the construction of a universal cut, that is one that serves *all* arrangements $\mathscr{A}_I$.

3.1 Main construction.

Represent $\mathbb{T}^d$ as the d-dimensional cube $K_d = [-\pi,\pi]^d$ with its parallel sides identified in the standard way. We use the system of coordinates ψ, with $\psi_i = \pi - \phi_i$, so that the origin $O = (0,0,...,0)$ corresponds now to the position 'all legs down'.

The tori $\mathbb{T}_A, A \subset \{1,\dots,d\}$ introduced above define a stratification of $\mathbb{T}^d$. Its open strata are cells of different dimensions, again indexed by the subsets A. We will be referring to these open cells as the *cubes* $\mathtt{Cb}_A$. The union of the cells of the stratification of dimensions $\le k$ - the k-skeleton - is denoted as $\mathtt{Sk}_k$.

Consider the cone $\mathtt{Co}_d$ in $\mathbb{T}^d$ over the $(d-2)$-skeleton $\mathtt{Sk}_{d-2}$ with the vertex at O. This cone is a singular hypersurface in $\mathbb{T}^d$ stratified by the cones over different coordinate subtori contained in $\mathtt{Sk}_{d-2}$. Notice that $\mathtt{Co}_d - \mathtt{Sk}_{d-2}$ contains $2^k\binom{d}{k}$ strata of codimension $(k-1)$ (the factor 2^k comes from various ways to connect the torus $\mathbb{T}_A, |A| = d-k$ with O), so that the total number of cones over $(d-2)$-dimensional cubes in $\mathtt{Co}_d$, that is flats of codimension 1 is $2d(d-1)$.

The complement $\mathbb{T}^d \setminus \mathtt{Co}_d$ consists of d open polyhedra, each being the union of two pyramids over the $(d-1)$-dimensional open cube $\mathtt{Cb}_{-i}$, the open cell in $\mathbb{T}_{-i} = \{\phi_i = 0\}$.

Let us denote these polytopes as $\mathtt{Pyr}_i, i = 1,...,d$: here i is the coordinate missing in the $(d-1)$-dimensional cube $\mathtt{Cb}_{-i}$ which is coned. The gait γ intercepts the boundary of each $\mathtt{Pyr}_i$ at two points belonging to some faces $\mathtt{Fc}_i^+, \mathtt{Fc}_i^-$ on its boundary. Each such face is the interior of a cone (one of 4 possible), still with apex at O, over some $(d-2)$-dimensional cube $\mathtt{Cb}_{-i-j}$.

The face where the trajectory γ enters (resp. leaves) $\mathtt{Pyr}_i$ is called the *i-th entrance face* $\mathtt{Fc}_i^+$ (resp. the *i*th exit face $\mathtt{Fc}_i^-$) and the corresponding points are called the entrance/exit points. Another important point within $\mathtt{Pyr}_i$ besides the entrance and exit points is the point where γ intersects the base of $\mathtt{Pyr}_i$, i.e. the corresponding $(d-1)$-dimensional cube, see Fig.2.

We remark that γ defines a *cyclic order* on the set of all $\mathtt{Pyr}_i$ according to the order in which the trajectory hits them, see Fig.1. Note that the exit face for any pyramid is at the same time the entrance face of the next one in this cyclic order.

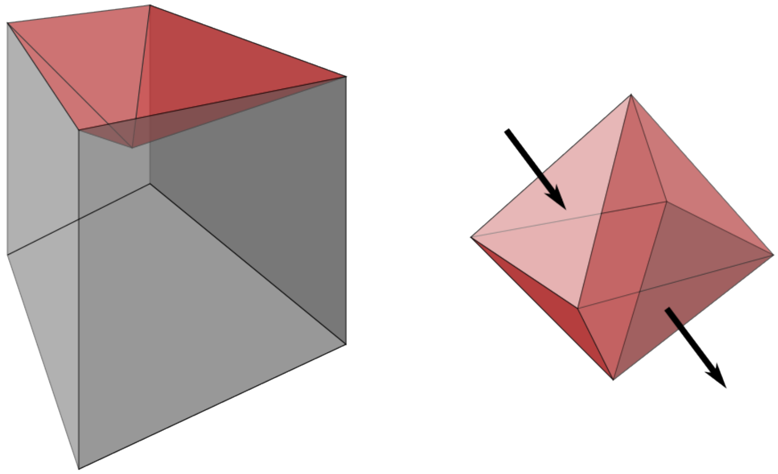

Fig. 2 Left: some of the strata of the cone $\mathtt{Co}_3$. Right: a pyramid.

Without loss of generality, we can assume that this cyclic order is $1 < 2 < 3 < \ldots < d < 1$.

In the configuration space, the exit face $\mathtt{Fc}_i^-$ of the pyramid $\mathtt{Pyr}_i$ is identified with the entrance face of $\mathtt{Pyr}_{i+1}$. We will be calling this face, which is, again, a cone over $\mathtt{Cb}_{-i-(i+1)}$, the *i-th door*.

Finally, we define

$$\mathtt{Ct}_d = \mathtt{Co}_d \setminus \bigcup_{i=1}^{d} \mathtt{Fc}_i^+ = \mathtt{Co}_d \setminus \bigcup_{j=1}^{d} \mathtt{Fc}_j^-$$

to be the union of the cones (with the apex O) over all codimension 2 cubes in the $(d-2)$ skeleton of $\mathbb{T}^d$ *with exception of the doors*. Equivalently, it is the cone over the full $(d-2)$-skeleton with the doors removed.

Theorem 1 *The stratified hypersurface $\mathtt{Ct}_d$ is a set-minimal cut for any $\mathtt{Fb}_I$, as long as the boundary of $\mathtt{Fb}_I$ is transversal to $\mathtt{Ct}_d$.*

The transversality required in the theorem is automatic if, for example, $\mathtt{Fb}_I$ is a small enough tubular neighborhood of $\mathscr{A}_I$.

Before moving to the proof of the Theorem 1, we will describe the cut in more "engineering" terms.

3.2 Forbidden leg positions

For the sake of clarity let us present a simple description of $\mathtt{Ct}_d$ in terms of configurations of legs. Let $(\psi_1,...,\psi_d),\ -\pi \le \psi_i \le \pi$ be the usual angular coordinates on the torus $\mathbb{T}^d$, $\psi = 0$ corresponding to the "leg down" position.

The i-th open pyramid $\mathtt{Pyr}_i$ consists then of exactly those leg positions, for which the i-th leg has the maximal height, i.e. $1-\cos\psi_i > 1-\cos\psi_j, \forall j \ne i$.

Its entrance face is the set of all leg positions when exactly the $(i-1)$-st leg and the i-th leg are at the maximal height among all legs. Additionally, their positions are not allowed to coincide ($\psi \neq \psi_j$) and the corresponding angles are in the correct cyclic position (i.e. $\psi_i > \psi_{i+1}$).

The Figure 3 illustrates the positions in the cut and outside it.

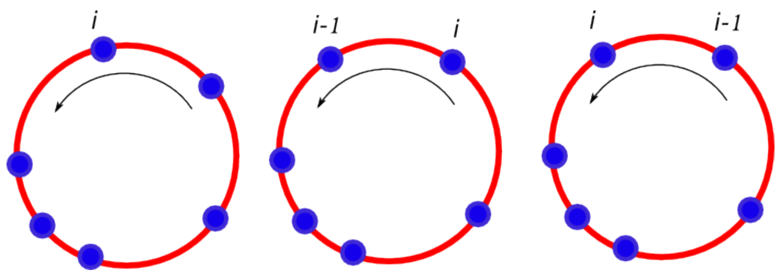

Fig. 3 Left: a typical configuration inside $\mathtt{Pyr}_i$; middle and right: configurations on the entrance and exit faces of $\mathtt{Pyr}_i$.

Proof (Proof of the Theorem 1). The torus $\mathbb{T}^d$ with the cut $\mathtt{Co}_d$ deleted can be constructed by the identifying the d pyramids $\mathtt{Pyr}_i, i = 1,\dots,d$ along the pairs of exit-entrance faces: the exit face of the pyramid $\mathtt{Pyr}_i$ is identified with the entrance face of $\mathtt{Pyr}_{i+1}$. This immediately implies that the admissible basin $\mathtt{Bs}_d = \mathbb{T}^d - \mathtt{Co}_d$ is homeomorphic to the d-dimensional solid torus, the product of $(d-1)$-dimensional (open) ball and $\mathbb{T}^1$. The trajectory γ is embedded into the *basin* and, again by construction, generates $H_1(\mathtt{Bs}_d,\mathbb{Z})$. Now, the assumption of unknottedness implies immediately that γ is a deformation retract of $\mathtt{Bs}_d$. (In fact, we will construct an explicit flow on $\mathtt{Bs}_d$ realizing such a deformation.)

Now, it remains to show that the cut is set-minimal. Assume that a superset S of $\mathtt{Bs}_d$ contains a point $x \in \mathtt{Co}_d \cap \mathtt{Fr}_I$. As the intersection of a small ball around x in $\mathbb{T}_d$ intersected with $\mathtt{Bs}_d$ contains more than one connected components (corresponding to different pyramids), one can choose (piecewise-linear) curves that connects x to the some point x_1, x_2 on the segments of the gait γ in the corresponding pyramids. Combining these curves with a segment of γ connecting x_1 and x_2 one obtains a closed curve that represents a class β in $H_1(\mathbb{T}^d,\mathbb{Z})$ different from the diagonal class $\delta = [\gamma]$. Hence, $H_1(S,\mathbb{Z})$ has rank at least two, and the homotopy type of S cannot be that of $\mathbb{T}^1$.

4 Feedback stabilization on γ

In this section we will construct two explicit vector fields on $\mathtt{Bs}_I$ for $\mathtt{Fb}_I = \mathscr{A}_I$, such that applying one for a short period of time (one full rotation of a leg) and then switching on the other, all trajectories will converge to a prespecified gait (for example, the equispaced gait γ_s with the phases ϕ_i of the d legs uniformly spaced over the circle and moving with constant speed. This particular trajectory is not necessarily a realistic one and is chosen just to simplify the presentation.

We remark that any control mechanism that stabilizes on the equispaced ordered gait γ_s can be considered as a continuous-time *sorting algorithm*: starting with any leg configuration, we align them, after some time, in a prearranged cyclic order. In fact, this is precisely the task that the first vector field will perform: we will show that after one period, all the legs are cyclically ordered (say, in the standard order assumed above). The second stage is then a straightforward synchronization, locking the gait on the exponentially stable period trajectory γ_s.

Not to overload the exposition, we consider just the case where $\mathtt{Fb} = \mathscr{A}_I$, although quite general sets of forbidden configurations (tubular neighborhoods of $\mathscr{A}_I$ can be handled in a similar fashion.

4.1 Rearranging the legs

It is piece-wise smooth and analytic in each of the open pyramids where the single leg is the highest one. (In principle, the idea behind this dynamics is very similar to that of the time-dependent dynamics described in the next section.) Take a pyramid $\mathtt{Pyr}_i$ where the i-th leg has the strictly largest height among all legs, i.e. $h_i = 1 - \cos\psi_i$ is greater than all the other h_j's. (Recall that $\psi_j,\ j = 1,...,d$ are the angle coordinates on our torus $\mathbb{T}^d$ normalized so that $\psi = 0$ corresponds to the "leg down" position.)

Define $\mathbf{v}$ on $\mathtt{Pyr}_i$ as

$$\begin{cases} \dot{\psi}_j = 1 \text{ for } j \neq i+1, \\ \dot{\psi}_{i+1} = (h_i - \max_{j \neq i, i+1} h_j)^{-1/2}. \end{cases}$$

This vector field is well defined outside of the "diagonals" $\Delta_{kl} = \{h_k = h_l\}, 1 \leq k < l \leq d$ (in fact, it is real-analytic on the complement to the union of these diagonals).

Conceptually, on $\mathtt{Pyr}_i$, where the leg i is at the highest position, the $(i+1)$-th coordinate accelerates so that it overtakes all other coordinates while i is still the highest height leg - that is while still in $\mathtt{Pyr}_i$.

The structure of the trajectories on $\mathtt{Bs}_I$ is given by the following

Proposition 1 *The vector field* $\mathbf{v}$ *defines a continuous flow on each pyramid* $\mathtt{Pyr}_i$. *Furthermore,*

- *for any point inside* $\texttt{Pyr}_i$*, the forward trajectory reaches the exit door (a point on the exit face* $\{h_i = h_{i+1}, \phi_i < 0 < \phi_{i+1}\}$*) of the pyramid* $\texttt{Pyr}_i$ *in finite time;*
- *moreover, for any point on the entrance door of* $\texttt{Pyr}_i$ *(that is a point with* $h_i = h_{i-1}, h_i > h_j, j \neq i, i+1, \phi_{i-1} < 0 < \phi_i$*) there exists a unique trajectory of* $\mathbf{v}$ *on* $\texttt{Pyr}_i$ *having that point as its initial value;*

Proof. The proof of these claims is pretty straightforward. The first statement follows from the evident fact that $\mathbf{v}$ is smooth as long as $(i+1)$-th leg is not the second in height, and near the diagonal $h_i > h_{i+1} = h_k > h_l, l \neq i, i+1, k$ (where $\mathbf{v}$ loses smoothness - but not continuity), the flow can be constructed explicitly.

The second statement follows from the fact as long as the $(i+1)$-st leg is not the second in height after i-st leg, the velocity of ψ_{i+1} behaves like $(t_* - t)^{-1}$ (where t_* is the instant when the the height of i-th leg equals to the height of some of the other legs with index $\neq i+1$ - recall that on $\texttt{Pyr}_i$, all legs but $(i+1)$-st have constant velocity). It follows that $(i+1)$-st leg becomes the closest competitor to the leader i overtaking all other legs.

Once the $(i+1)$-st leg become the competitor to i-th one, it remains second in height, eventually taking over the leadership, as can be computed explicitly, again.

The sorting to which we alluded above is achieved after just one full rotation (of the initial leader leg).

4.2 Asymptotic stability

Once we know that the legs are in a required cyclic order, it is a routine matter to stabilize them on a desired trajectory γ_s: as an example, one can consider the following vector field,

$$\dot{\psi}_i = 1 - (\phi_{i-1} - \psi_i)^{-2} + (\psi_i - \psi_{i+1})^{-2}.$$

Note that the phase differences are well defined as the phases are cyclically ordered.

This system can be interpreted as d particles constrained to the circle, under the Coulomb's repulsive force between nearby particles and constant drift. It is immediate to see that the flow preserves the cyclic order, and has the gait γ_s as the global asymptotically stable attractor.

Remark 2 *The above dynamics consists of two phases: the 1st turn of the legs and the remaining motion. During the first turn all the legs are placed in the clockwise order coinciding with their cyclic order. This is done within a rather small time interval and might be difficult to technically realize in practice since it requires quick motions of legs and quick stops. One observes that small measurement mistakes can result in the instability of the motion since the order of leading legs can experience big changes. The second phase, on the other hand, presents no difficulties, and the motion quickly converges to the rotation of the equally spaced legs with the unit speed.*

5 Further remarks and speculations

In the present note we introduced and discussed the notion of a set-theoretical maximality of the set $\mathtt{Bs}_I$. Obviously, this is a rather weak notion: there are many set maximal basins (just act by a diffeomorphism of the torus identical near $\mathtt{Fb}_I$), and our definition does not single out any of them. To do so one needs some alternative notions of minimality for the cuts (on top of set-minimality). As an example of another notion of maximality that makes sense one can suggest the $(d-1)$-*volume* of the cut $\mathtt{Ct}_I \subset \mathbb{T}^d$.

While in our situation, the cut is always a (singular) hypersurface, there are similar models, where the cut has higher co-dimension. In such cases one should consider the volume form of the appropriate dimension.

We remark that the set $\mathtt{Bs}_I$ which was constructed above is *not* volume minimal in the above sense: the easiest way to see it is to remember that in the minimal soap films, the codimension 1 sheets come together at a codimension 2 strata in triples, at the angle of $120°$. The problem of finding of the set $\mathtt{Bs}_k$ of the minimal volume is interesting even in the standard case $\mathtt{Fb}_k$ of the configuration "no more than k legs up"...

6 Appendix. Discrete autonomous control

6.1 Entrance-Base-Exit Flows

Below we describe an interesting discrete dynamical system associated with our construction above. It addresses a somewhat different problem - not the stabilization on a single attractor, but rather generating a simple flow with piece-wise linear trajectories, but its nice mathematical features compelled us to present it here.

We construct a flow through the union of the pyramids $\mathtt{Pyr}_i$ such that on each of them this flow enters only through its entrance face, $F := \mathtt{Fc}_i^+$ and leaves through the exit face, $G := \mathtt{Fc}_i^-$.

Both faces are cones over certain $(d-2)$-dimensional cubes (corresponding to the legs $i,(i-1)$ and $i,(i+1)$ being simultaneously leaders, in the proper order). The flow we are looking for should move from the entrance face F through the $(d-1)$-cube $B := \mathtt{Cb}_{-i}$ of the whole pyramid and then further to the exit face G.

6.2 Birational mappings

Let us define two natural maps from the (open) entrance face F to the (open) base cube B and then from B to the (open) exit face G. Each such map can be transformed into a (continuous) flow by connecting the preimage and its image by a straight line within the pyramid. (Thus each trajectory of such a flow within $\mathtt{Pyr}_i$ will be the union of two straight segments.)

The most natural way to do it is by using the so-called blow-up/blow down rational transformations [5]. We present these transformations explicitly below for the cases $d = 3$ and $d \geq 4$. (The essential distinction of these two cases is explained by the fact that for $d = 3$ the entrance/exit faces are the usual triangles and, therefore, they allow additional symmetry transformations unavailable for $d \geq 4$.)

Case $d = 3$

The entrance/exit faces F and G are usual triangles and the base cube B is a usual square. Let us identify the entrance triangle F with the triangle with the vertices $(0,0),(1,0),(1,1)$ in $\mathbb{R}^2$; the base square B with the square whose vertices are $(0,0),(1,0),(0,1)$, $(1,1)$ and, finally, the exit triangle G with the triangle with the vertices $(0,0),(0,1),(1,1)$.

The *blow-up* map $\Phi : (x,y) \to (x, \frac{y}{x})$ sends F to B. (It sends the pencil of lines through the origin to the pencil of horizontal lines.) Its inverse *blow-down* map $\Psi : (s,t) \to (st,t)$ maps B to G. It sends the pencil of vertical lines to the pencil of lines through the origin. Their composition $\chi = \Psi \circ \Phi : (x,y) \to (y, \frac{y}{x})$ sends F to G, see Fig.4

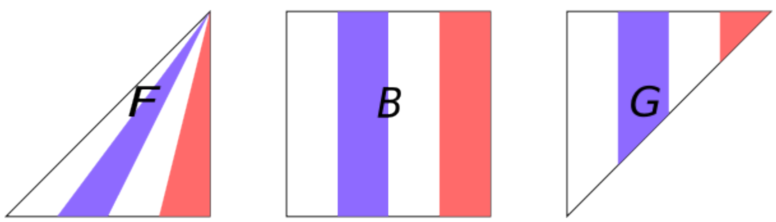

Fig. 4 Birational transformation from the entrance face to base to exit face.

To get the whole discrete dynamical system assume that the three (since $d = 3$) pyramids $\mathtt{Pyr}_1, \mathtt{Pyr}_2, \mathtt{Pyr}_3$ are cyclically ordered as $1 < 2 < 3 < 1$ by the choice of Γ_γ. Denote their entrance faces as F_1, F_2, F_3 and their exit faces as G_1, G_2, G_3. Notice that $F_1 = G_2, F_2 = G_3, F_3 = G_1$. Assume now that we apply our transformation χ three times consecutively, i.e first from F_1 to $G_1 = F_2$, then from F_2 to $G_2 = F_3$, and, finally back to $G_3 = F_1$. The resulting self-map $\Theta : F_1 \to F_1$ is classically referred to as the *Poincare return map* of the dynamical system. To calculate it explicitly we need to find a suitable affine transformation A sending G back to F in the above example. Then we get the self-map Θ by composing χ with A and taking the 3-rd power of the resulting composition. As such a map A one can choose $A : (u,v) \to (1-u, 1-v)$ which implies that the required Poincare return map is the third power of $\Theta = A \circ \chi$ where:

$$\Theta : (x,y) \to \left(1-y, 1-\frac{y}{x}\right).$$

Lemma 1 *The above map Θ has and unique fixed point within the triangle F_1 and its fifth power is identity.*

Proof. The system of equations defining fixed points reads as

$$\begin{cases} x = 1 - y, \\ y = 1 - \frac{y}{x} \end{cases}$$

and its two solutions are $\psi_1 = \frac{2\sqrt{2}-1}{2}$, $y_1 = \frac{3-2\sqrt{2}}{2}$ and $\psi_2 = -\frac{1+2\sqrt{2}}{2}$, $y_2 = \frac{3+2\sqrt{2}}{2}$. One can easily check that only the first solution belongs to F_1. Direct calculations show that

$$\Theta^2 : (x,y) \to \left(\frac{y}{x}, \frac{y(1-x)}{x(1-y)}\right), \quad \Theta^3 : (x,y) \to \left(\frac{x-y}{x(1-y)}, \frac{x-y}{(1-x)}\right)$$

$$\Theta^4 : (x,y) \to \left(\frac{1-x}{1-y}, 1-x\right), \quad \Theta^5 : (x,y) \to (x,y).$$

The Poincare return map is thus equals to $\Theta^3 : (x,y) \to \left(\frac{x-y}{x(1-y)}, \frac{x-y}{(1-x)}\right)$.

Case $d \geq 4$

Analogously, we have d pyramids each being a cone over a $(d-1)$-cube. Their entrance and exit faces are cones over a square respectively. The map Φ sends the open entrance face F to the open base $(d-1)$-cube B and the map Ψ sends the open base cube B to the open exit face G. They can be given explicitly as follows. Let us identify F with the domain $\{0 < \psi_2 < \psi_1 < 1; 0 < \psi_3 < \psi_1 < 1; \ldots 0 < \psi_{d-1} < \psi_1 < 1\}$, i.e. with the cone over the square $\{0 < \psi_2 < 1,\ 0 < \psi_{d-1} < 1\}$ with the vertex at the origin. The base B will be identified with the cube $\{0 < \psi_1 < 1,\ 0 < \psi_2 < 1, 0 < \psi_{d-1} < 1\}$, and, finally, the exit face G with $\{0 < \psi_1 < \psi_2 < 1;\ 0 < \psi_3 < \psi_2 < 1, \ldots, 0 < \psi_{d-1} < \psi_2 < 1\}$. Then the *blow-up map* Φ and the *blow-down map* Ψ can be chosen as follows:

$$\Phi : (\psi_1, \psi_2, \ldots, \psi_{d-1}) \to \left(\psi_1, \frac{\psi_2}{\psi_1}, \frac{\psi_3}{\psi_1}, \ldots, \frac{\psi_{d-1}}{\psi_1}\right)$$

$$\Psi : (y_1, y_2, \ldots, y_{d-1}) \to (y_1 y_2, y_2, \ldots, y_{d-1} y_2).$$

Their composition $\chi : F \to G$ coincides with

$$\chi : (\psi_1, \psi_2, \ldots, \psi_{d-1}) \to \left(\psi_2, \frac{\psi_2}{\psi_1}, \frac{\psi_2 \psi_3}{\psi_1^2}, \ldots, \frac{\psi_2 \psi_{d-1}}{\psi_1^2}\right).$$

An appropriate linear map A sending G back to F is just a cyclic permutation of coordinates:

$$A : (z_1, z_2, \ldots, z_{d-1}) \to (z_2, z_3, \ldots, z_1).$$

Thus we get the composition $\Theta = A \circ \chi : F \to F$ (whose d-th power is the Poincare return map) given by:

$$\Theta : (\psi_1, \psi_2, ..., \psi_{d-1}) \to \left(\frac{\psi_2}{\psi_1}, \frac{\psi_2 \psi_3}{\psi_1^2}, ..., \frac{\psi_2 \psi_{d-1}}{\psi_1^2}, \psi_2 \right).$$

Proposition 2 *The above map Θ has a curve of fixed points parameterized by $(t, t^2, t^2, ..., t^2)$, $t \in \mathbb{R}$. Moreover, for any $d \geq 3$ one has that $\Theta^{d-1} = id$.*

Proof. Indeed, the system of equations defining fixed points reads as

$$\psi_1 = \frac{\psi_2}{\psi_1},\ \psi_2 = \frac{\psi_2 \psi_3}{\psi_1^2},\ \psi_3 = \frac{\psi_2 \psi_4}{\psi_1^2},\ \cdots \psi_{d-2} = \frac{\psi_2 \psi_{d-1}}{\psi_1^2},\ \psi_{d-1} = \psi_2.$$

which immediately implies $\psi_1^2 = \psi_2 = \psi_3 = ... = \psi_{d-1}$. To show that $\Theta^{d-1} = id$ notice that since Θ is a monomial map it suffices to show that $M_d^{d-1} = id_{d-1}$ where M_d is the matrix of exponents of the map Θ and id_{d-1} is the identity matrix of size $d-1$. (Indeed, the matrix of exponents for Θ^i coincides with M_d^i.) This is done in the following lemma.

Lemma 2 *The characteristic polynomial of the $(d-1) \times (d-1)$-matrix M_d equals $(-1)^d(1-t^{d-1})$. Therefore, by the Hamilton-Cayley theorem $M_d^{d-1} = id_{d-1}$.*

Proof. Looking at the exponents of Θ we see that the matrix M_d has the form

$$M_d = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ -2 & 1 & 1 & 0 & \dots & 0 \\ -2 & 1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2 & 1 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

To make our calculations easy we introduce two families of $(k \times k)$-matrices D_k and E_k given by:

$$D_k = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -t & 1 & 0 & \dots & 0 \\ 1 & 0 & -t & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \cdots & -t & 1 \\ 1 & 0 & 0 & \cdots & \cdots & -t \end{pmatrix}, \quad E_k = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 2 & -t & 1 & 0 & \dots & 0 \\ 2 & 0 & -t & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 0 & 0 & \cdots & -t & 1 \\ 0 & 0 & 0 & \cdots & 0 & -t \end{pmatrix}.$$

Expanding by the first row one obtains the following recurrences

$$Det(D_k) = (-t)^{k-1} - Det(D_{k-1}) \quad Det(E_k) = 2(-t)^{k-1} - Det(E_{k-1})$$

resulting in the formulas

$$Det(D_k) = (-1)^{k-1}(t^{k-1} + t^{k-2} + ... + 1), \quad Det(E_k) = (-1)^{k-1} 2(t^{k-1} + t^{k-2} + ... + t).$$

Expanding now the characteristic polynomial $Ch_d(t)$ of M_d by the first row (after the sign change in the first row) we get the relation

$$-Ch_d(t) = (t+1)[(1-t)(-t)^{d-3} - Det(D_{d-3})] - Det(E_{d-2}).$$

Substituting of the expressions for $Det(D_{d-3})$ and $Det(E_{d-2})$ in the latter formula one gets $Ch_d(t) = (-1)^d(1-t^{d-1})$.

This completes the proof.

Corollary 1 *The Poincare return map equals $\Theta^d = \Theta$.*

References

1. G. Denham, A. Suciu, *Moment-angle complexes, monomial ideals and Massey products.* Pure Appl. Math. Q. 3 (2007), no. 1, part 3, 25–60.
2. G. Carbone, M. Ceccarelli, *Legged robotic systems*, Cutting Edge Robotics, (2005), 553–576.
3. R. Altendorfer, N. Moore, H. Komsuoglu, H. B. Brown Jr., D. McMordie, U. Saranli, R. Full, and D. E. Koditschek, *RHex: A Biologically Inspired Hexapod Runner*, Autonomous Robots, 2001.
4. E. Klavins, D.E. Koditschek, *Phase Regulation of Decentralized Cyclic Robotic Systems*, International Journal of Robotics Research, 21 (2002), no. 3, 257–275.
5. D.Mumford, *Algebraic geometry. I. Complex projective varieties. Reprint of the 1976 edition. Classics in Mathematics*. (1995), Springer.
6. E. Sontag, *Stability and stabilization: discontinuities and the effect of disturbances.* In: *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, (1999), 551-598.