ON ALGEBRAICITY OF THE EXTERIOR CAUCHY TRANSFORM OF AN ALGEBRAIC OVAL

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To the memory of Vladimir Arnold on the occasion of his 90th birthday

ABSTRACT. Let $\Omega \subset \mathbb{C}$ be a bounded plane domain whose boundary is a (smooth) oval of a real algebraic curve, and let

$$\mathcal{C}_\Omega(z) = \frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{z-\zeta}, \qquad z \in \mathbb{C} \setminus \overline{\Omega},$$

be the exterior Cauchy transform of planar Lebesgue measure restricted to Ω. We explain how the algebraicity problem for $\mathcal{C}_\Omega$ is naturally reformulated in terms of Cauchy-type integrals of algebraic data on the normalization of the Schwarz correspondence $P(z,w) = 0$. This leads to a Picard–Lefschetz type obstruction to algebraicity parallel in spirit to the Arnold–Vassiliev theory of Newton's lemma. We also carry out a more detailed analysis of the cubic and quartic situations. For cubics, the normalization is elliptic in the smooth case and rational in the nodal degeneration; the elliptic case already produces infinite monodromy in generic families. For quartics, one has to distinguish the hyperelliptic biquadratic case from the generic plane quartic case, but in both settings the ambient Picard–Lefschetz group is large enough that finite branching of the exterior Cauchy transform should be exceptional. Ellipses remain the basic algebraic but non-rational examples.

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a bounded domain with sufficiently smooth boundary $\Gamma = \partial\Omega$. The exterior Cauchy transform

$$\mathcal{C}_\Omega(z) = \frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{z-\zeta}, \qquad z \in \mathbb{C} \setminus \overline{\Omega},$$

is holomorphic in the exterior domain and decays as $|z| \to \infty$. If Γ is an oval of a real algebraic curve, it is natural to ask whether one can characterize those ovals for which $\mathcal{C}_\Omega$ is algebraic.

For quadrature domains the answer is classical: $\mathcal{C}_\Omega$ is rational; see [1, 10, 5, 6, 4]. This phenomenon is tied to the meromorphic continuation of the Schwarz function and, in a more intrinsic language, to the fact that the Schottky double carries a meromorphic extension of the coordinate function; see [7, 8]. On the other hand, already for ellipses the transform is algebraic but non-rational. Thus the more natural problem is not rationality but algebraicity.

From the planar point of view one would like to write the boundary integral in terms of a Schwarz function $S(z)$ satisfying $S(z) = \bar{z}$ on Γ. The difficulty is that for a general algebraic oval the Schwarz function does not extend to the complement of the oval as a single-valued univalent meromorphic function. What persists globally is not a planar branch but the algebraic correspondence between z and $\bar{z}$. This suggests lifting the problem from the plane to the complex algebraic curve

$$P(z,w)=0,$$

obtained by complexifying the equation $P(z,\bar{z})=0$ of the oval. On the normalization of this curve the coordinate functions z and w are honest meromorphic functions, and the exterior Cauchy transform becomes a Cauchy-type integral of algebraic data on a compact Riemann surface.

This lifted formulation is very close in spirit to the framework of Pakovich, Roytvarf and Yomdin [9], where algebraicity of a Cauchy-type integral is governed by finite combinatorial monodromy and rationality corresponds to trivial monodromy. In the present setting the relevant monodromy is related to the covering $\pi : X \to \mathbb{CP}^1$ defined by the z-projection of the normalization X of $P(z,w)=0$. From that viewpoint one is led to a monodromy criterion for algebraicity, closely analogous to the Picard–Lefschetz obstructions that appear in Arnold's and Vassiliev's work on Newton's lemma and algebraically integrable bodies [3, 11, 2, 12].

The purpose of the note is modest. We do not claim a complete theorem characterizing all algebraic ovals with algebraic exterior Cauchy transform. Instead, we isolate the relevant geometric mechanism, formulate a natural vanishing-cycle obstruction, and explain how the low-degree examples fit this picture.

2. Basic constructions and results

2.1. **Lifting to the Schwarz correspondence and the projection.** Suppose that Γ is a smooth oval of a real algebraic curve, written in the form

$$P(z,\bar{z})=0,$$

with P a polynomial in two formally independent variables. Replacing $\bar{z}$ by a new complex variable w gives the complex algebraic curve

$$P(z,w)=0.$$

Let X be the normalization of its projective closure, and let

$$\pi : X \to \mathbb{CP}^1, \qquad \pi(p)=z(p),$$

$$\eta : X \to \mathbb{CP}^1, \qquad \eta(p)=w(p),$$

be the meromorphic functions induced by the coordinate projections. On the real oval one has the boundary relation $w=\bar{z}$, so the pair (π,η) should be viewed as the intrinsic version of the Schwarz correspondence.

The main advantage of passing to X is that the multivaluedness of the planar Schwarz function disappears upstairs. Even when there is no single-valued meromorphic function on $\mathbb{CP}^1 \setminus \overline{\Omega}$ whose boundary values equal $\bar{z}$ on Γ, there is always a globally defined meromorphic function η on X. Thus the failure of the Schwarz function to extend univalently to the complement is encoded not by a defect of analyticity, but by the branching and monodromy of the projection π.

More precisely, if there exists a connected component $U \subset X \setminus \pi^{-1}(\Gamma)$ lying over the exterior domain such that

$$\pi : U \to \mathbb{CP}^1 \setminus \overline{\Omega}$$

is biholomorphic, then η descends to a single-valued meromorphic function of z on the exterior, which is exactly the univalent exterior continuation of the Schwarz function. In general this need not happen: the exterior domain may meet several sheets of the covering, or contain branch values of π, and then continuation in the exterior permutes the branches. From this perspective, planar multivaluedness of $S(z)$ is simply the manifestation downstairs of nontrivial covering geometry upstairs.

This is also the point at which the Schottky double enters naturally. The double of Ω may be identified, in the algebraic case, with the real form carried by the normalization together with the antiholomorphic involution exchanging the two coordinate projections; see [7, 8]. The Schwarz function is then the transition map between the two sides of the double only when the projection to the z-sphere is single-sheeted over the exterior. In the general algebraic case one must work on X rather than on the plane.

After lifting the boundary oval Γ to a cycle $\gamma \subset X$, the exterior Cauchy transform becomes

$$\mathcal{C}_\Omega(z) = \frac{1}{2\pi i}\int_\gamma \frac{\eta\, d\pi}{\pi - z}, \qquad z \in \mathbb{CP}^1 \setminus \overline{\Omega}$$

This is a Cauchy-type integral of algebraic data on a compact curve. Accordingly, the analytic continuation of $\mathcal{C}_\Omega$ is controlled by the monodromy of the covering π together with the induced transport of the contour class. This is precisely the kind of mechanism isolated in [9]: finite monodromy leads to algebraicity, while trivial monodromy leads to rationality.

2.2. **The basic identity.**

Let $\Omega \subset \mathbb{C}$ be a bounded domain with piecewise smooth positively oriented boundary $\Gamma = \partial\Omega$. For $z \notin \overline{\Omega}$ one has

$$\mathcal{C}_\Omega(z) = \frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{z - \zeta} = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\zeta}\, d\zeta}{\zeta - z}.$$

Indeed,

$$\frac{\partial}{\partial\overline{\zeta}}\left(\frac{\overline{\zeta}}{\zeta - z}\right) = \frac{1}{\zeta - z},$$

and Green's formula gives the claimed boundary representation.

Thus the algebraicity question for $\mathcal{C}_\Omega$ is naturally a boundary Cauchy integral problem. If Γ is contained in a real algebraic curve, then $\overline{\zeta}$ is algebraically related to ζ along Γ, and it is natural to encode this relation by the Schwarz correspondence.

2.3. **The Schwarz correspondence of an algebraic oval.**

Suppose that Γ is a smooth oval of a real algebraic curve. Then one may write its defining equation in the form

$$P(z, \bar{z}) = 0,$$

where P is a polynomial in two formally independent variables. Replacing $\bar{z}$ by a new complex variable w gives the complex algebraic curve

$$P(z, w) = 0.$$

Let X be its normalization. Then the coordinate functions z, w become meromorphic on X.

On the real oval one has the boundary relation $w = \bar{z}$. Thus $P(z, w) = 0$ should be viewed as a multivalued Schwarz correspondence rather than, in general, as a single-valued Schwarz function in the plane. The projection

$$\pi : X \to \mathbb{CP}^1, \qquad \pi(p) = z(p),$$

may have nontrivial monodromy, and this monodromy is precisely the obstruction to choosing a global planar branch.

From this point of view the exterior Cauchy transform is more naturally interpreted on X than on the z-plane alone.

2.4. **Reformulation as a Cauchy-type integral of algebraic data.** Locally along the physical sheet of the Schwarz correspondence one may choose a branch S satisfying

$$S(\zeta)=\overline{\zeta}, \qquad \zeta\in\Gamma.$$

Then

$$\mathcal{C}_\Omega(z)=\frac{1}{2\pi i}\int_\Gamma \frac{S(\zeta)\,d\zeta}{\zeta-z}, \qquad z\in\mathbb{C}\setminus\overline{\Omega}.$$

Equivalently, after lifting the contour to the normalization X, one can write

$$\mathcal{C}_\Omega(z)=\frac{1}{2\pi i}\int_\gamma \omega_z, \qquad \omega_z:=\frac{\eta\,d\pi}{\pi-z},$$

where $\gamma\subset X$ is the lifted real oval, viewed under analytic continuation as a distinguished class in the corresponding local system.

This places the problem into the general framework of Cauchy-type integrals of algebraic functions considered in [9]. In that setting, rationality corresponds to trivial branching and algebraicity corresponds to finite branching. The same dichotomy strongly suggests the appropriate geometric mechanism for algebraic ovals.

2.5. **A monodromy criterion and the Arnold–Vassiliev analogy.** Let $\Sigma\subset\mathbb{CP}^1$ be the set consisting of the critical values of the projection π together with the projections of the poles of the differential ω_z. For $z\in\mathbb{CP}^1\setminus\Sigma$ consider the relative homology groups

$$\mathscr{H}_z:=H_1\big(X\setminus\pi^{-1}(z),D_z\big),$$

where D_z denotes the divisor of poles of ω_z. These groups form a local system over $\mathbb{CP}^1\setminus\Sigma$, endowed with the Gauss–Manin connection.

Let $\delta_1,\dots,\delta_m$ be vanishing cycles corresponding to the critical values in Σ. The Picard–Lefschetz transformations act on homology by

$$T_j(\gamma)=\gamma\pm\langle\gamma,\delta_j\rangle\,\delta_j.$$

Let $[\Gamma]\in\mathscr{H}_z$ denote the class represented by the physical boundary oval, and set

$$\mathcal{L}:=\langle[\Gamma],\delta_1,\dots,\delta_m\rangle.$$

The following statement records the expected obstruction mechanism.

Proposition 1 (Vanishing-cycle obstruction)**.** *Assume that the orbit of* $[\Gamma]$ *under the subgroup of* $\mathrm{Aut}(\mathcal{L})$ *generated by the Picard–Lefschetz transformations is infinite. Then the analytic continuation of* $\mathcal{C}_\Omega$ *has infinitely many branches. In particular,* $\mathcal{C}_\Omega$ *is not algebraic.*

Conversely, if $\mathcal{C}_\Omega$ *is algebraic, then the monodromy orbit of* $[\Gamma]$ *is finite. If this orbit is trivial, then* $\mathcal{C}_\Omega$ *is rational.*

Proof sketch. Analytic continuation in the parameter z along a loop $\ell \subset \mathbb{CP}^1 \setminus \Sigma$ transports the distinguished class $[\Gamma] \in \mathscr{H}_z$ by the Gauss–Manin monodromy operator associated with ℓ. Hence a continued branch is given by

$$\mathcal{C}_\Omega^\ell(z) = \frac{1}{2\pi i} \int_{\rho(\ell)([\Gamma])} \omega_z,$$

Thus the branching of $\mathcal{C}_\Omega$ is governed by the orbit of the homology class of the contour. If this orbit is infinite, there are infinitely many analytic branches, and the function cannot be algebraic. If the orbit is finite, one is in the finite-monodromy situation familiar from [9]; trivial monodromy corresponds to the rational case. □

Remark 1. *This proposition is meant as a geometric formulation of the mechanism rather than as a complete theorem with all analytic details spelled out. What is rigorous in the literature is the corresponding finite-monodromy criterion for general Cauchy-type integrals of algebraic functions in the sense of* [9]. *The present setting fits that framework through the Schwarz correspondence.*

The analogy with Arnold's and Vassiliev's work is immediate: in Newton's lemma and in the theory of algebraically integrable bodies, nontrivial Picard–Lefschetz monodromy produces infinite ramification and hence obstructs algebraicity [3, 11, 2]. Proposition 1 is the direct counterpart for the exterior Cauchy transform.

2.6. Relative homology and the oval class.

We now make precise the homological framework underlying the Cauchy transform.

Let $\pi : X \to \mathbb{CP}^1$ be the projection and fix $z \notin \Sigma$. Consider

$$\mathcal{H}_z := H_1(X \setminus \pi^{-1}(z), D_z),$$

where D_z is the divisor of poles of the differential ω_z.

2.6.1. *Exact sequence.* At the level of absolute homology one has the natural exact sequence

$$0 \to H_1(X) \to H_1(X \setminus \pi^{-1}(z)) \to \widetilde{H}_0(\pi^{-1}(z)) \to 0.$$

Thus removing the fiber introduces additional generators corresponding to small loops around the points of $\pi^{-1}(z)$. The relative group

$$\mathcal{H}_z := H_1(X \setminus \pi^{-1}(z), D_z)$$

is obtained from this absolute group by allowing chains with boundary on the pole divisor D_z.

2.6.2. *The oval class.* The oval Γ determines a fundamental class

$$[\Gamma] \in H_1(X).$$

By inclusion into $X \setminus \pi^{-1}(z)$, we obtain a distinguished class

$$\mathrm{Cap}_{\Omega,z} \in \mathcal{H}_z.$$

Remark 2.
- $\mathrm{Cap}_{\Omega,z}$ *is canonical up to Gauss–Manin transport (and is determined by* Ω*),*
- *it defines the integration cycle,*
- *its monodromy orbit controls the analytic continuation of* $\mathcal{C}_\Omega$.

2.6.3. *Gauss–Manin connection.* The groups $\mathcal{H}_z$ form a local system over $\mathbb{CP}^1 \setminus \Sigma$. The Gauss–Manin connection provides parallel transport

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \Sigma) \to \mathrm{Aut}(\mathcal{H}_z).$$

The Cauchy transform is a period:

$$\mathcal{C}_\Omega(z) = \frac{1}{2\pi i} \int_{\mathrm{Cap}_{\Omega,z}} \omega_z.$$

3. TYPE I AND TYPE II CURVES AND NON-ALGEBRAICITY OF THE CAUCHY TRANSFORM

Let X be a smooth projective complex algebraic curve defined over $\mathbb{R}$, and let $X(\mathbb{R})$ denote its real locus. A classical dichotomy distinguishes between *type I* (dividing) and *type II* (non-dividing) curves.

3.1. Type I vs. Type II.

We recall:

- X is of **type I** (dividing) if $X(\mathbb{R})$ separates X into two connected components:
$$X \setminus X(\mathbb{R}) = X^+ \sqcup X^-,$$
interchanged by complex conjugation.
- X is of **type II** (non-dividing) if $X \setminus X(\mathbb{R})$ is connected.

Let $\Gamma \subset X(\mathbb{R})$ be a real oval, and let $\Omega \subset \mathbb{C}$ denote the corresponding planar domain bounded by its projection.

3.2. Absolute Homology and the Type II Case.

In the type II case, the real oval Γ defines a nontrivial homology class:

$$[\Gamma] \neq 0 \in H_1(X, \mathbb{Z}).$$

This immediately yields a sufficient condition for non-algebraicity:

Proposition 2. *If X is of type II and Γ is a real oval, then the Cauchy transform of Ω is non-algebraic.*

Idea. Nontriviality of $[\Gamma]$ implies the existence of a holomorphic 1-form ω such that

$$\int_\Gamma \omega \neq 0.$$

This produces nontrivial monodromy of the Cauchy transform, ruling out algebraicity. □

Thus, for type II curves, non-algebraicity is essentially automatic.

3.3. Failure of the Absolute Criterion in Type I.

In contrast, if X is of type I, every real oval is null-homologous:

$$[\Gamma] = 0 \in H_1(X, \mathbb{Z}).$$

Therefore, the above argument breaks down completely. In particular, absolute homology does not detect the obstruction to algebraicity.

3.4. **Relative Homology and the Cap Class.** The correct invariant is instead the *cap class*

$$\mathrm{Cap}_{\Omega,z} \in H_2\big(X, \pi^{-1}(z)\big),$$

obtained by lifting Ω to the Riemann surface X and considering it as a relative 2-chain with boundary lying over the fiber $\pi^{-1}(z)$.

Even when $[\Gamma] = 0$, this class may be nontrivial.

3.5. Sufficient Condition in the Type I Case.

Proposition 3. *Let X be of type I and let Ω correspond to one of the two halves of $X \setminus X(\mathbb{R})$. If there exists a holomorphic 1-form ω on X such that*

$$\int_{\mathrm{Cap}_{\Omega,z}} \omega \neq 0,$$

then the Cauchy transform of Ω is non-algebraic.

Idea. Nonvanishing of this period implies that analytic continuation of the Cauchy transform produces nontrivial additive periods. Algebraic functions cannot exhibit such behavior, hence the transform is non-algebraic. ☐

3.6. **Abel–Jacobi Interpretation.** The above condition can be reformulated using the Abel–Jacobi map:

$$A : H_2\big(X, \pi^{-1}(z)\big) \longrightarrow \mathrm{Jac}(X).$$

Corollary 1. *If the Abel–Jacobi image of the cap class satisfies*

$$A(\mathrm{Cap}_{\Omega,z}) \neq 0,$$

then the Cauchy transform is non-algebraic.

3.7. **Geometric Interpretation.** In the type I case, although the boundary Γ cancels in homology, the domain Ω may still capture nontrivial periods of the Riemann surface. The obstruction to algebraicity is therefore encoded in how Ω sits inside X, rather than in its boundary alone.

3.8. **Generic Non-Algebraicity.** For curves of genus $g \geq 1$, the condition

$$A(\mathrm{Cap}_{\Omega,z}) \neq 0$$

is generically satisfied. Thus:

- Type II curves, non-algebraicity holds under very mild conditions and is generic.
- Type I curves: non-algebraicity holds generically, but requires analysis of the cap class.

3.9. **Exceptional Algebraic Cases.** Algebraicity of the Cauchy transform can occur only in highly constrained situations, corresponding to *quadrature domains.* These arise when all relevant periods vanish, which imposes strong symmetry and low-genus conditions on the curve (e.g. circles, ellipses, and certain special algebraic domains).

In summary, the distinction between type I and type II curves reflects two different mechanisms for non-algebraicity:

- in type II, it is detected by absolute homology,
- in type I, it is detected by relative (cap) homology and Abelian integrals.

4. Examples

4.1. Quadratic case: explicit computation. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In complex form, one obtains a quadratic relation between z and $\bar{z}$, leading to the Schwarz function

$$S(z) = \alpha z + \beta\sqrt{z^2 - c^2}.$$

The branch points are located at $\pm c$.

4.1.1. *Algebraicity.* For the ellipse with semiaxes $a > b > 0$ and focal distance

$$c^2 = a^2 - b^2,$$

the exterior Cauchy transform can in fact be written explicitly as

$$\mathcal{C}_\Omega(z) = \frac{2ab}{c^2}\Big(z - \sqrt{z^2 - c^2}\Big), \qquad z \in \mathbb{C} \setminus \overline{\Omega},$$

where the branch of $\sqrt{z^2 - c^2}$ is chosen by the condition

$$\sqrt{z^2 - c^2} \sim z \qquad (z \to \infty).$$

Equivalently,

$$\mathcal{C}_\Omega(z) = \frac{2ab}{z + \sqrt{z^2 - c^2}}.$$

In particular,

$$\mathcal{C}_\Omega(z) = \frac{ab}{z} + O(z^{-3}) \qquad (z \to \infty),$$

as required by the area asymptotic $\mathrm{Area}(\Omega)/\pi = ab$.

Thus $\mathcal{C}_\Omega$ is algebraic of degree 2 and is non-rational unless $c = 0$ (the circular case).

4.1.2. *Monodromy.* The only nontrivial monodromy arises from the square root, giving a $\mathbb{Z}/2\mathbb{Z}$ action. Thus the orbit of Cap_Ω is finite.

4.2. Cubic case: strengthened theorem-level consequences. We now reformulate the cubic case so that the monodromy input is derived from the geometry of the projection rather than imposed as a separate hypothesis. The essential point is that the smooth cubic is the first case in which the normalization has positive genus.

4.2.1. *Smooth cubics.* Let the real cubic be given, after birational changes of coordinates, by a Weierstrass equation

$$y^2 = (x - e_1)(x - e_2)(x - e_3), \qquad e_1 < e_2 < e_3 \in \mathbb{R}.$$

Its smooth projective model is an elliptic curve X, hence

$$g(X) = 1, \qquad H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^2.$$

The projection

$$\pi : X \to \mathbb{CP}^1, \qquad (x, y) \mapsto x,$$

is a double covering branched over the four points e_1, e_2, e_3, ∞. By Riemann–Hurwitz,

$$2g(X) - 2 = 2(-2) + 4,$$

so indeed $g(X) = 1$.

The point is that for a simply branched double cover of $\mathbb{CP}^1$ the local monodromies are automatically Picard–Lefschetz transvections in the associated vanishing cycles, and these vanishing cycles generate $H_1(X, \mathbb{Z})$. Thus the real geometric input needed for non-algebraicity is reduced to the nontriviality of the lifted physical contour class.

Lemma 1 (Nontriviality of the oval class)**.** *Let* Γ *be an oval of a smooth real cubic, and let* $[\Gamma] \in H_1(X, \mathbb{Z})$ *be its lift to the normalization of the Schwarz correspondence. Then* $[\Gamma] \neq 0$*.*

Proof. For a smooth real cubic with an oval, the lifted real component is a noncontractible simple closed curve on the elliptic normalization. Equivalently, it represents a primitive class in $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^2$. In particular $[\Gamma] \neq 0$. $\square$

Proposition 4 (Single-transvection criterion)**.** *Let* X *be a compact Riemann surface, let* $\delta \in H_1(X, \mathbb{Z})$ *be a vanishing cycle, and let* $[\Gamma] \in H_1(X, \mathbb{Z})$*. If*

$$\langle [\Gamma], \delta \rangle \neq 0,$$

then the orbit of $[\Gamma]$ *under the Picard–Lefschetz transformation* T_δ *is infinite.*

Proof. The Picard–Lefschetz formula gives

$$T_\delta(\gamma) = \gamma + \langle \gamma, \delta \rangle \, \delta$$

up to the usual orientation sign convention. Hence

$$T_\delta^n([\Gamma]) = [\Gamma] + n \langle [\Gamma], \delta \rangle \, \delta, \qquad n \in \mathbb{Z},$$

and these classes are pairwise distinct whenever $\langle [\Gamma], \delta \rangle \neq 0$. $\square$

Theorem 1 (Smooth cubic non-algebraicity)**.** *Let* Γ *be an oval on a smooth real cubic, and let* Ω *be the bounded domain enclosed by* Γ*. Assume that the projection*

$$\pi : X \to \mathbb{CP}^1$$

of the normalization of the Schwarz correspondence has only simple branching. Then:

- *(i)* X *has genus* 1*;*
- *(ii) the vanishing cycles of* π *generate* $H_1(X, \mathbb{Z})$*;*
- *(iii) there exists a vanishing cycle* δ *such that*

$$\langle [\Gamma], \delta \rangle \neq 0.$$

Consequently, the Picard–Lefschetz orbit of $[\Gamma]$ *is infinite, and the exterior Cauchy transform* $\mathcal{C}_\Omega$ *is not algebraic.*

Proof. The genus statement is the above Riemann–Hurwitz computation. For a simply branched degree-two map from an elliptic curve to $\mathbb{CP}^1$, the local monodromies are Picard–Lefschetz transvections in the vanishing cycles, and these vanishing cycles generate $H_1(X, \mathbb{Z})$. By Lemma 1, the physical contour class $[\Gamma]$ is nonzero. Since the intersection form on $H_1(X, \mathbb{Z})$ is nondegenerate and the vanishing cycles generate the whole homology, there exists a vanishing cycle δ with $\langle [\Gamma], \delta \rangle \neq 0$. Proposition 4 therefore gives an infinite monodromy orbit of $[\Gamma]$, and Proposition 1 implies that $\mathcal{C}_\Omega$ cannot be algebraic. $\square$

4.2.2. *A concrete type I example.*

Example 1 (A dividing cubic with non-algebraic Cauchy transform)**.** *Consider the smooth real cubic*

$$y^2 = x(x-1)(x-4).$$

Its smooth projective model is an elliptic curve, so $g(X) = 1$. Since the polynomial on the right-hand side has three distinct real roots, the real locus $X(\mathbb{R})$ has two connected components: one compact oval lying over $0 \leq x \leq 1$, and one noncompact component passing through the point at infinity. Hence X is of type I (dividing).

Let Ω be the bounded planar domain enclosed by the compact oval. Then the exterior Cauchy transform $\mathcal{C}_\Omega$ is not algebraic.

Indeed, by Theorem 1, it is enough to know that the lifted contour class is nonzero in $H_1(X, \mathbb{Z})$ and that the vanishing cycles of the projection generate $H_1(X, \mathbb{Z})$. Both properties hold for this cubic: the compact oval lifts to a nontrivial homology class on the elliptic normalization, and the simply branched double covering

$$\pi : X \to \mathbb{CP}^1, \qquad (x, y) \mapsto x,$$

has vanishing cycles generating $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^2$. Therefore some vanishing cycle δ satisfies

$$\langle [\Gamma], \delta \rangle \neq 0,$$

so the Picard–Lefschetz orbit of the corresponding cap class is infinite. Proposition 1 then implies that $\mathcal{C}_\Omega$ has infinitely many analytic branches and hence is non-algebraic.

This example is worth stressing because it shows that even for a type I curve, where the absolute class of a real oval may fail to be the right invariant in general, the Cauchy transform can still be forced to be non-algebraic by the monodromy of the lifted contour/cap class.

Example 2 (The compact oval $y^2 = x(x-1)(x-4)$)**.** *Let Ω be the compact oval of the smooth real cubic*

$$y^2 = x(x-1)(x-4) = x(1-x)(4-x),$$

so that the real oval is given by $0 \leq x \leq 1$ and

$$y = \pm\sqrt{x(x-1)(x-4)}.$$

For $z \in \mathbb{C} \setminus \overline{\Omega}$, writing the boundary as the union of the upper and lower arcs and differentiating under the integral sign gives

$$\mathcal{C}'_\Omega(z) = -\frac{2}{\pi} \int_0^1 \frac{\sqrt{x(x-1)(x-4)}\, dx}{(z-x)^2 + x(x-1)(x-4)}.$$

Since

$$(z-x)^2 + x(x-1)(x-4) = x^3 - 4x^2 + (4-2z)x + z^2 =: P_z(x),$$

and

$$y^2 = x(x-1)(x-4) = 4x(1-x)\Big(1 - \frac{x}{4}\Big),$$

this is naturally an elliptic integral with Legendre modulus

$$m = \frac{1}{4}.$$

Indeed, with the standard substitution

$$x = \operatorname{sn}^2(u \mid m), \qquad m = \frac{1}{4},$$

one has

$$dx = 2\operatorname{sn}(u)\operatorname{cn}(u)\operatorname{dn}(u)\,du, \qquad \sqrt{x(x-1)(x-4)} = 2\operatorname{sn}(u)\operatorname{cn}(u)\operatorname{dn}(u),$$

so that $dx = \sqrt{x(x-1)(x-4)}\,du$ *and therefore*

$$\mathcal{C}'_\Omega(z) = -\frac{2}{\pi}\int_0^{K(m)} \frac{x(u)(x(u)-1)(x(u)-4)}{P_z(x(u))}\,du, \qquad x(u) = \operatorname{sn}^2(u \mid m),$$

where $K(m)$ *is the complete elliptic integral of the first kind.*

Now let $r_1(z), r_2(z), r_3(z)$ *be the three roots of*

$$P_z(x) = x^3 - 4x^2 + (4-2z)x + z^2.$$

Since

$$\frac{x(x-1)(x-4)}{P_z(x)} = 1 - \frac{(x-z)^2}{P_z(x)} = 1 - \sum_{j=1}^{3} \frac{A_j(z)}{x - r_j(z)}, \qquad A_j(z) = \frac{(r_j(z)-z)^2}{P'_z(r_j(z))},$$

we obtain, using the standard identity

$$\int_0^{K(m)} \frac{du}{\operatorname{sn}^2(u \mid m) - r} = -\frac{1}{r}\,\Pi\!\left(\frac{1}{r}\,\middle|\, m\right),$$

that

$$\mathcal{C}'_\Omega(z) = -\frac{2}{\pi}\left[K\!\left(\frac{1}{4}\right) + \sum_{j=1}^{3} \frac{(r_j(z)-z)^2}{r_j(z)\,P'_z(r_j(z))}\,\Pi\!\left(\frac{1}{r_j(z)}\,\middle|\,\frac{1}{4}\right)\right].$$

Equivalently, since

$$P'_z(r_j) = 3r_j^2 - 8r_j + 4 - 2z,$$

one may write

$$\mathcal{C}'_\Omega(z) = -\frac{2}{\pi}\left[K\!\left(\frac{1}{4}\right) + \sum_{j=1}^{3} \frac{(r_j(z)-z)^2}{r_j(z)(3r_j(z)^2 - 8r_j(z) + 4 - 2z)}\,\Pi\!\left(\frac{1}{r_j(z)}\,\middle|\,\frac{1}{4}\right)\right].$$

Thus the Cauchy transform of the compact oval is not expected to be algebraic; already its derivative is a linear combination of complete elliptic integrals of the third kind with algebraic parameters. The transform itself is recovered from the condition $\mathcal{C}_\Omega(z) \to 0$ *as* $z \to \infty$*, namely*

$$\mathcal{C}_\Omega(z) = -\int_\infty^z \mathcal{C}'_\Omega(\tau)\,d\tau,$$

with the integration path taken in the exterior domain.

4.2.3. *Singular cubic degenerations.* The smooth cubic should be contrasted with the singular degenerations, where the normalization becomes rational.

Proposition 5 (Nodal and cuspidal cubics)**.** *Let X be the normalization of a nodal or cuspidal cubic. Then $g(X) = 0$ and hence*

$$H_1(X, \mathbb{Z}) = 0.$$

Therefore the absolute-homology obstruction of Proposition 1 disappears.

Proof. Both the nodal cubic and the cuspidal cubic have rational normalization. Thus $X \simeq \mathbb{CP}^1$ and $H_1(X, \mathbb{Z}) = 0$. The conclusion is immediate. □

Example 3 (The nodal loop $y^2 = x^2(1-x)$)**.** *Let Ω be the bounded loop of the real nodal cubic*

$$y^2 = x^2(1-x).$$

A convenient normalization parameter is $s \in \mathbb{CP}^1$, with real loop given by

$$x = 1 - s^2, \qquad y = s(1-s^2), \qquad -1 \le s \le 1.$$

Thus

$$\zeta(s) = x + iy = (1-s^2)(1+is) = 1 + is - s^2 - is^3,$$

while on the boundary

$$\overline{\zeta(s)} = (1-s^2)(1-is).$$

With the positively oriented parameter interval $s \in [-1, 1]$, the exterior Cauchy transform in the normalization used in the introduction is

$$\mathcal{C}_\Omega(z) = -\frac{1}{2\pi i}\int_{-1}^{1} \frac{(1-s^2)(1-is)\,\zeta'(s)}{\zeta(s) - z}\,ds, \qquad z \in \mathbb{C} \setminus \overline{\Omega},$$

where

$$\zeta'(s) = i - 2s - 3is^2.$$

Equivalently,

$$\mathcal{C}_\Omega(z) = -\frac{1}{2\pi i}\int_{-1}^{1} \frac{-3s^5 - is^4 + 2s^3 + 2is^2 + s - i}{is^3 + s^2 - is + z - 1}\,ds.$$

Let $r_1(z), r_2(z), r_3(z)$ be the three roots of

$$is^3 + s^2 - is + z - 1 = 0,$$

that is, the three preimages of z under the rational normalization map $s \mapsto \zeta(s)$. A partial-fraction decomposition then gives the explicit formula

$$\mathcal{C}_\Omega(z) = \frac{2}{\pi} - \frac{1}{2\pi i}\sum_{j=1}^{3} A_j(z) \log\left(\frac{1 - r_j(z)}{-1 - r_j(z)}\right),$$

with

$$A_j(z) = \frac{-3r_j(z)^5 - ir_j(z)^4 + 2r_j(z)^3 + 2ir_j(z)^2 + r_j(z) - i}{3ir_j(z)^2 + 2r_j(z) - i}.$$

This formula is single-valued on the exterior domain after choosing the branches continuously from $z = \infty$, and its expansion at infinity begins with

$$\mathcal{C}_\Omega(z) = \frac{\operatorname{Area}(\Omega)}{\pi z} + O(z^{-2}) = \frac{8}{15\pi}\,\frac{1}{z} + O(z^{-2}).$$

In particular, for this nodal cubic the transform is reduced completely to the rational normalization and to the three algebraic inverse branches $r_j(z)$.

4.3. **Quartic case.** For quartics one must distinguish several geometric regimes. The normalization of the Schwarz correspondence may have genus 1, 2, or 3, depending on singularities and on the way the correspondence is presented. The main point is that in positive genus the previous conditional transvection hypothesis can again be replaced by concrete geometric assumptions on the projection and on the lifted oval class.

4.3.1. *Genus bookkeeping.* There are two standard genus computations in the quartic range.

(i) If $X \subset \mathbb{P}^2$ is a smooth plane quartic, then

$$g(X) = \frac{(4-1)(4-2)}{2} = 3,$$

so

$$H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^6.$$

(ii) If the normalization is hyperelliptic of the form

$$y^2 = Q_d(x),$$

then Riemann–Hurwitz gives

$$g(X) = \begin{cases} 1, & d = 4, \\ 2, & d = 5 \text{ or } 6. \end{cases}$$

In particular, quartic Schwarz correspondences of biquadratic type may lead to genus 1 or genus 2 curves.

Thus every non-rational quartic regime already has enough homology to support nontrivial Picard–Lefschetz dynamics.

Lemma 2 (A sufficient condition for nontriviality)**.** *If the lifted oval* Γ *is noncontractible on the normalization* X*, then*

$$[\Gamma] \neq 0 \in H_1(X, \mathbb{Z}).$$

Proof. A noncontractible simple closed curve on a compact Riemann surface represents a nonzero homology class. Hence the lifted oval defines a nontrivial class in $H_1(X, \mathbb{Z})$. □

4.3.2. *Positive-genus quartics.*

Theorem 2 (Quartic non-algebraicity: simple-branching case)**.** *Let* Γ *be an oval on a quartic, and let* X *be the normalization of its Schwarz correspondence. Assume:*

(i) $g(X) \geq 1$*;*

(ii) the projection

$$\pi : X \to \mathbb{CP}^1$$

has only simple critical values;

(iii) the lifted oval class $[\Gamma] \in H_1(X, \mathbb{Z})$ *is nonzero.*

Then there exists a vanishing cycle δ *such that*

$$\langle [\Gamma], \delta \rangle \neq 0,$$

and hence the Picard–Lefschetz orbit of $[\Gamma]$ *is infinite. In particular, the exterior Cauchy transform* $\mathcal{C}_\Omega$ *is not algebraic.*

Proof. If π has only simple critical values, then the local monodromies are Picard–Lefschetz transvections in the corresponding vanishing cycles. In the quartic situations under consideration, these vanishing cycles generate the relevant positive-genus homology. Since $[\Gamma] \neq 0$ and the intersection form on $H_1(X, \mathbb{Z})$ is nondegenerate, there exists a vanishing cycle δ with $\langle [\Gamma], \delta \rangle \neq 0$. Proposition 4 then gives an infinite monodromy orbit of $[\Gamma]$, and Proposition 1 implies that $\mathcal{C}_\Omega$ is not algebraic. □

Remark 3. *Theorem 2 applies in particular to smooth plane quartics of genus* 3*, and also to the hyperelliptic quartic correspondences of genus* 1 *or* 2*, provided the* z*-projection is of Lefschetz type and the lifted oval class is nontrivial.*

4.3.3. *Degenerate quartics.* The remaining quartic possibilities are the degenerate ones with rational normalization.

Proposition 6 (Rational quartic degenerations)**.** *If the normalization of the quartic Schwarz correspondence is rational, then*

$$H_1(X, \mathbb{Z}) = 0.$$

Hence the non-algebraicity criterion of Theorem 2 does not apply.

Proof. If $X \simeq \mathbb{CP}^1$, then $H_1(X, \mathbb{Z}) = 0$. □

4.3.4. *A quartic trichotomy.* The quartic cases therefore split into three topological regimes.

Theorem 3 (Quartic trichotomy)**.** *Let* Γ *be an oval of a real quartic and let* Ω *be the enclosed domain.*

(i) If $g(X) \geq 1$*, the projection* π *has simple branching, and* $[\Gamma] \neq 0$*, then* $\mathcal{C}_\Omega$ *is not algebraic.*
(ii) If $g(X) = 0$*, then the absolute-homology obstruction does not apply.*
(iii) Thus algebraicity in degree 4 *can occur only in the rational case, or on the exceptional locus where either the projection is not simple-branched or the oval class is homologically trivial.*

Thus, just as in the cubic case, the decisive invariant is the topology of the normalized Schwarz correspondence rather than the bare planar degree. In positive genus, non-algebraicity is therefore the expected behavior for Lefschetz-type projections; any algebraic example must lie in a nongeneric locus where the Lefschetz property fails or the physical oval class becomes invisible in homology.

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