Asymptotic distribution of isodynamic points for classical orthogonal polynomials

Christian Hägg and Boris Shapiro

Department of Mathematics, Stockholm University

Abstract

We determine the asymptotic distribution of isodynamic points (in the sense of [3]) for the classical orthogonal polynomials. A single Bochner-type computation covers Jacobi, Laguerre, and Hermite, including varying parameters. The asymptotic step rests on a general one-cut theorem: if the normalized logarithmic derivatives $\frac{1}{n}P_n'/P_n$ converge to a Cauchy transform C_ν with boundary convergence away from the endpoints of a real interval, then the limiting isodynamic measure is the averaged pushforward of ν by the boundary values of $z \mapsto z - 1/C_\nu(z)$. We also isolate the affine-arcsine case, which yields the uniform circle law on $|u-b| = 2a$, and formulate a conjectural higher-order extension for eigenpolynomials of exactly solvable differential operators.

1 Introduction

Let P be a squarefree polynomial of degree $n \geq 3$. The rational map

$$R_P(z) := z - \frac{nP(z)}{P'(z)}$$

has degree $n-1$; by the Riemann–Hurwitz formula its ramification divisor has total multiplicity $2n-4$. Following [3], we call the finite critical values of R_P, counted with multiplicity, the *isodynamic points* of P, write $\mathcal{I}(P)$ for the corresponding divisor, and set

$$\mu_P := \frac{1}{2n-4} \sum_{u \in \mathcal{I}(P)} \delta_u$$

for the normalized counting measure. The isodynamic divisor encodes global information about the root-derivative interaction: it is the branch locus of R_P viewed as a branched cover, and for structured polynomial families the natural question is the weak limit of μ_{P_n} as $\deg P_n \to \infty$.

This paper has two goals. The first is computational: for each classical family (Jacobi, Laguerre, Hermite), Bochner's classification [4] reduces the isodynamic condition to an algebraic relation between critical points and critical values that is *linear* in the critical point. This reduction is stated once for the general eigenvalue equation $\sigma Y'' + \tau Y' + \lambda_n Y = 0$ with $\deg \sigma \leq 2$, $\deg \tau \leq 1$ (Propositions 1 and 3), and the three families reappear only as specializations. The varying-parameter Jacobi and Laguerre formulas are likewise explicit.

The second goal is asymptotic. We prove a one-cut theorem (Theorem 14): for degree-n polynomials with simple real zeros in a fixed compact interval, if their zero counting measures converge to a probability measure ν supported on $[A,B]$, $\frac{1}{n}P_n'/P_n$ converges to the Cauchy transform C_ν locally on $\widehat{\mathbb{C}} \setminus [A,B]$, and $U(z) := z - \frac{1}{C_\nu(z)}$ has boundary values away from the endpoints, then

$$\mu^{\mathrm{iso}}_{P_n} \xrightarrow[n\to\infty]{w^*} \tfrac{1}{2}(U_+)_*\nu + \tfrac{1}{2}(U_-)_*\nu, \qquad U(z) := z - \frac{1}{C_\nu(z)},$$

where $U_\pm$ are the non-tangential boundary values on (A, B). This requires no differential equation and applies, in particular, to orthogonal polynomials with varying exponential weights e^{-nV} (Corollary 17). Applied to the classical families it yields the unit-circle law for fixed-parameter Jacobi, as well as explicit nonuniform circle measures for Laguerre, Hermite, and the varying-parameter cases (Theorems 7, 9, and 11). We also record the affine-arcsine specialization of Theorem 14, which gives the uniform circle law on $|u - b| = 2a$, and formulate a Bergkvist–Rullgård-type conjecture for higher-order exactly solvable operators, where the candidate limiting isodynamic set is obtained from the support of the limiting zero measure by the multivalued map $z \mapsto z - 1/C(z)$. We also emphasize that the normalized projective isodynamic measure is exactly Möbius-equivariant (Proposition 18), and in Section 6 we illustrate the resulting measure-to-measure transform for quadratic algebraic Cauchy transforms.

2 General set-up and Bochner reduction

In the real-rooted simple-zero families treated below, ∞ is not an isodynamic point, so $\mathcal{I}(P)$ has total multiplicity $2n - 4$.

With: In the real-rooted simple-zero families treated below, ∞ is not a critical point of R_P, because the coefficient of z^{-1} in its Laurent expansion at infinity is the variance of the normalized zero counting measure of P, which is strictly positive. Hence every isodynamic point comes from a finite critical point, and $\mathcal{I}(P)$ has total multiplicity $2n - 4$.

Write

$$\sigma(z) = a_2 z^2 + a_1 z + a_0, \qquad \tau(z) = b_1 z + b_0.$$

If P_n is a polynomial solution of degree n of

$$\sigma(z)Y''(z) + \tau(z)Y'(z) + \lambda_n Y(z) = 0, \tag{1}$$

then comparison of the leading terms gives

$$\lambda_n = -n((n-1)a_2 + b_1). \tag{2}$$

Proposition 1. *Let P_n be a squarefree degree-n polynomial satisfying (1). Let $z \in \mathbb{C}$ satisfy*

$$R'_{P_n}(z) = 0, \qquad u := R_{P_n}(z) \in \mathbb{C}.$$

Then

$$((2(n-1)a_2 + b_1)u + (n-1)a_1 + b_0)z = ((n-1)a_2 + b_1)u^2 + b_0 u - (n-1)a_0. \tag{3}$$

If the coefficient of z in (3) is nonzero, then

$$z = J_n(u) := \frac{((n-1)a_2 + b_1)u^2 + b_0 u - (n-1)a_0}{(2(n-1)a_2 + b_1)u + (n-1)a_1 + b_0}. \tag{4}$$

Proof. Let $Y(z) = P_n(z)$. Since $u = R_{P_n}(z)$, one gets

$$nY(z) = (z - u)Y'(z). \tag{5}$$

Differentiating R_{P_n} and using $R'_{P_n}(z) = 0$ gives

$$nY(z)Y''(z) = (n-1)(Y'(z))^2.$$

Substituting these relations into (1) yields

$$(n-1)\sigma(z) + (z-u)\tau(z) + \frac{\lambda_n}{n}(z-u)^2 = 0.$$

Using (2), this becomes

$$(n-1)\sigma(z) + (z-u)\tau(z) - ((n-1)a_2 + b_1)(z-u)^2 = 0.$$

The z^2-terms cancel, giving (3). If the coefficient of z is nonzero, solving for z gives (4). ☐

Remark 2. The three Bochner normal forms already force the three natural critical-value scales. For $\sigma(z) = 1 - z^2$ the relation (3) stays nontrivial at $u = O(1)$. For $\sigma(z) = z$ it becomes

$$(n + O(1) - u)z = u(O(1) - u),$$

so the first nontrivial scale is $u, z = O(n)$. For $\sigma(z) = 1$ it becomes

$$2uz = 2u^2 + n - 1,$$

so the first nontrivial scale is $u, z = O(\sqrt{n})$. Thus the Laguerre and Hermite rescalings are already forced by the two confluences (here $\rightsquigarrow$ denotes the limiting form obtained by sending one or both roots of σ to ∞)

$$1 - z^2 \rightsquigarrow z \rightsquigarrow 1.$$

Proposition 3. *Assume in addition that P_n satisfies a* lowering identity *(expressing $\sigma(z)P_n'(z)$ as a linear combination of P_n and P_{n-1}) of the form*

$$\sigma(z)P_n'(z) = (\gamma_n z + \delta_n)P_n(z) + \varepsilon_n P_{n-1}(z), \tag{6}$$

with scalars $\gamma_n, \delta_n, \varepsilon_n$. Let $z \in \mathbb{C}$ satisfy

$$R_{P_n}'(z) = 0, \qquad u := R_{P_n}(z) \in \mathbb{C}.$$

If the coefficient of z in (3) is nonzero, then $z = J_n(u) \neq u$, and u is a zero of

$$F_n(u) := \varepsilon_n P_{n-1}(J_n(u)) - \left(\frac{n\sigma(J_n(u))}{J_n(u) - u} - \gamma_n J_n(u) - \delta_n\right) P_n(J_n(u)), \tag{7}$$

where J_n is given by (4).

Proof. By Proposition 1, one has $z = J_n(u)$. If $z = u$, then (5) gives $P_n(z) = 0$. Since P_n is squarefree, z is a simple zero, and therefore $R_{P_n}'(z) = 1 - n \neq 0$, a contradiction. Hence $z - u \neq 0$. Combining (5) with (6) gives

$$\frac{n\sigma(z)}{z - u}P_n(z) = (\gamma_n z + \delta_n)P_n(z) + \varepsilon_n P_{n-1}(z),$$

which is exactly (7). □

For the three Bochner families we use the differential equations

$$(1 - z^2)Y''(z) + (\beta - \alpha - (\alpha + \beta + 2)z)Y'(z) + n(n + \alpha + \beta + 1)Y(z) = 0, \tag{8}$$

$$zY''(z) + (\alpha + 1 - z)Y'(z) + nY(z) = 0, \tag{9}$$

$$Y''(z) - 2zY'(z) + 2nY(z) = 0, \tag{10}$$

and the standard lowering identities

$$(2n + \alpha + \beta)(1 - z^2)(P_n^{(\alpha,\beta)})'(z) = n(\alpha - \beta - (2n + \alpha + \beta)z)P_n^{(\alpha,\beta)}(z) + 2(n + \alpha)(n + \beta)P_{n-1}^{(\alpha,\beta)}(z), \tag{11}$$

$$z(L_n^{(\alpha)})'(z) = nL_n^{(\alpha)}(z) - (n + \alpha)L_{n-1}^{(\alpha)}(z), \tag{12}$$

$$H_n'(z) = 2nH_{n-1}(z). \tag{13}$$

Propositions 1 and 3 therefore give the exact reductions below.

Corollary 4 (Jacobi)**.** *Fix $\alpha, \beta > -1$. If u is an isodynamic point of $P_n^{(\alpha,\beta)}$, then the corresponding critical point is*

$$J_n^{(\alpha,\beta)}(u) := \frac{(n+\alpha+\beta+1)u^2 - (\beta-\alpha)u + (n-1)}{(2n+\alpha+\beta)u - (\beta-\alpha)}, \tag{14}$$

and every isodynamic point is a zero of

$$\begin{aligned} F_n^{(\alpha,\beta)}(u) := 2(n-1)(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(J_n^{(\alpha,\beta)}(u)) \\ - n(n+\alpha+\beta+1)\big((2n+\alpha+\beta)u+\alpha-\beta\big)P_n^{(\alpha,\beta)}(J_n^{(\alpha,\beta)}(u)). \end{aligned} \tag{15}$$

If

$$D_n(u) := (2n+\alpha+\beta)u - (\beta-\alpha),$$

then $D_n(u)^n F_n^{(\alpha,\beta)}(u)$ is a polynomial in u, and every isodynamic point is a zero of that polynomial.

Proof. Insert (8) and (11) into Propositions 1 and 3, and simplify. The denominator in (14) cannot vanish at an isodynamic point, because at $D_n(u) = 0$ the right-hand side of (3) equals

$$-\frac{4(n-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2} \neq 0.$$

□

Corollary 5 (Laguerre)**.** *Fix $\alpha > -1$. If u is an isodynamic point of $L_n^{(\alpha)}$, then the corresponding critical point is*

$$J_{n,\alpha}^{\mathrm{Lag}}(u) := \frac{u(\alpha+1-u)}{n+\alpha-u}, \tag{16}$$

and every isodynamic point is a zero of

$$F_{n,\alpha}^{\mathrm{Lag}}(u) := n(n+\alpha-u)L_n^{(\alpha)}(J_{n,\alpha}^{\mathrm{Lag}}(u)) - (n+\alpha)(n-1)L_{n-1}^{(\alpha)}(J_{n,\alpha}^{\mathrm{Lag}}(u)). \tag{17}$$

After multiplication by $(n+\alpha-u)^{n-1}$, this becomes a polynomial in u, and every isodynamic point is a zero of that polynomial.

Proof. Insert (9) and (12) into Propositions 1 and 3, and simplify. The denominator in (16) cannot vanish at an isodynamic point, because at $u = n+\alpha$ the right-hand side of (3) equals

$$(n+\alpha)(1-n) \neq 0.$$

□

Corollary 6 (Hermite)**.** *If u is an isodynamic point of H_n, then the corresponding critical point is*

$$J_n^{\mathrm{Her}}(u) := u + \frac{n-1}{2u}, \tag{18}$$

and every isodynamic point is a zero of

$$F_n^{\mathrm{Her}}(u) := uH_n(J_n^{\mathrm{Her}}(u)) - (n-1)H_{n-1}(J_n^{\mathrm{Her}}(u)). \tag{19}$$

After multiplication by u^{n-2}, this becomes a polynomial in u, and every isodynamic point is a zero of that polynomial.

Proof. Insert (10) and (13) into Propositions 1 and 3, and simplify. The denominator in (18) cannot vanish at an isodynamic point, because at $u = 0$ the relation (3) becomes $0 = n - 1$. □

Theorem 7. *Fix Jacobi parameters $\alpha, \beta > -1$ and a Laguerre parameter $\gamma > -1$. Let*

$$\mu_n^{(\alpha,\beta)} := \frac{1}{2n-4} \sum_{u \in \mathcal{I}(P_n^{(\alpha,\beta)})} \delta_u, \qquad \nu_{n,\gamma}^{\mathrm{Lag}} := \frac{1}{2n-4} \sum_{u \in \mathcal{I}(L_n^{(\gamma)})} \delta_{u/n},$$

and

$$\nu_n^{\mathrm{Her}} := \frac{1}{2n-4} \sum_{u \in \mathcal{I}(H_n)} \delta_{u/\sqrt{2n}}.$$

Then (all limits in the weak-∗ topology on measures)

$$\mu_n^{(\alpha,\beta)} \xrightarrow[n\to\infty]{\mathrm{w}^*} \frac{\mathrm{d}\theta}{2\pi} \qquad \text{on} \qquad u = e^{i\theta},$$

$$\nu_{n,\gamma}^{\mathrm{Lag}} \xrightarrow[n\to\infty]{w^*} \frac{1-\cos\theta}{2\pi}\,\mathrm{d}\theta \qquad \text{on} \qquad \xi = 1 + e^{i\theta},$$

and

$$\nu_n^{\mathrm{Her}} \xrightarrow[n\to\infty]{w^*} \frac{\sin^2\theta}{\pi}\,\mathrm{d}\theta \qquad \text{on} \qquad \xi = \tfrac{1}{2}e^{i\theta}.$$

Proof. Apply Theorem 14 of Section 5 to the naturally scaled sequence in each case. The required strip convergence follows from Lemma 13 and the classical Plancherel–Rotach formulas on the slit plane away from the endpoints; see [5, Section 8.21].

For Jacobi, the limiting zero measure is the arcsine law

$$\frac{\mathrm{d}x}{\pi\sqrt{1-x^2}} \qquad \text{on} \qquad [-1,1].$$

The limit of (14) is

$$x = \frac{1}{2}(\rho + \rho^{-1}), \qquad u = \rho.$$

Thus $x = \cos\theta$ and $u = e^{i\theta}$. Averaging the upper and lower branches gives $\mathrm{d}\theta/(2\pi)$ on the unit circle.

For Laguerre, apply the theorem to $P_n(\eta) := L_n^{(\gamma)}(n\eta)$. The limiting zero measure is the Marchenko–Pastur law

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,\mathrm{d}x \qquad \text{on} \qquad [0,4].$$

The limit of (16) after the scaling $u = n\xi$ is

$$x = \frac{\xi^2}{\xi - 1}.$$

Writing $\xi = 1+\rho$ gives $x = 2+\rho+\rho^{-1}$. Hence $x = 2+2\cos\theta$ and $\xi = 1+e^{i\theta}$. Pushing forward the Marchenko–Pastur density and averaging the two branches gives

$$\frac{1-\cos\theta}{2\pi}\,\mathrm{d}\theta.$$

For Hermite, apply the theorem to $P_n(\eta) := H_n(\sqrt{2n}\,\eta)$. The limiting zero measure is the semicircle law

$$\frac{2}{\pi}\sqrt{1-x^2}\,\mathrm{d}x \qquad \text{on} \qquad [-1,1].$$

The limit of (18) after the scaling $u = \sqrt{2n}\,\xi$ is

$$x = \xi + \frac{1}{4\xi}.$$

Writing $\xi = \rho/2$ gives $x = \frac{1}{2}(\rho + \rho^{-1})$. Hence $x = \cos\theta$ and $\xi = \frac{1}{2}e^{i\theta}$. Pushing forward the semicircle density and averaging the two branches gives

$$\frac{\sin^2\theta}{\pi}\,\mathrm{d}\theta.$$

□

Remark 8. The Jacobi part of Theorem 7 contains the Legendre, Gegenbauer, and both Chebyshev families as special cases. The Laguerre and Hermite parts are its two Bochner confluences, and the scales 1, n, and $\sqrt{n}$ are exactly the scales predicted by Remark 2.

3 Varying-parameter Jacobi and Laguerre

We now give the varying-parameter results explicitly. The exact mechanism is the same as above, but the outer asymptotics (i.e. asymptotics on the complement of the support of the limiting zero measure) have to be written row by row in the triangular array, tracking the moving parameters.

3.1 Jacobi polynomials with linearly varying parameters

Let $\alpha_n, \beta_n > -1$ satisfy

$$\frac{\alpha_n}{n} \to a \geq 0, \qquad \frac{\beta_n}{n} \to b \geq 0.$$

For the Jacobi polynomial $P_n^{(\alpha_n,\beta_n)}$, the exact reduction of Corollary 4 gives

$$J_n^{(\alpha_n,\beta_n)}(u) = \frac{(n+\alpha_n+\beta_n+1)u^2 - (\beta_n-\alpha_n)u + (n-1)}{(2n+\alpha_n+\beta_n)u - (\beta_n-\alpha_n)}.$$

Its large-n limit is the rational map

$$J_{a,b}(u) := \frac{(1+a+b)u^2 - (b-a)u + 1}{(2+a+b)u - (b-a)}.$$

Introduce the constants

$$c_{a,b} := \frac{b-a}{a+b+2}, \qquad r_{a,b} := \frac{2\sqrt{(a+1)(b+1)}}{(a+b+2)\sqrt{a+b+1}},$$

and

$$m_{a,b} := \frac{b^2-a^2}{(a+b+2)^2}, \qquad k_{a,b} := \frac{2\sqrt{(a+1)(b+1)(a+b+1)}}{(a+b+2)^2}.$$

A direct calculation shows that

$$J_{a,b}(c_{a,b} + r_{a,b}\rho) = m_{a,b} + k_{a,b}(\rho + \rho^{-1}). \tag{20}$$

Thus the limiting Jacobi support is the interval

$$I_{a,b} := [m_{a,b} - 2k_{a,b},\, m_{a,b} + 2k_{a,b}],$$

while the corresponding locus in the u-plane is the circle

$$\Gamma_{a,b} := \{u \in \mathbb{C} :\ |u - c_{a,b}| = r_{a,b}\}.$$

Theorem 9. *Let $\alpha_n/n \to a \geq 0$ and $\beta_n/n \to b \geq 0$. Let $\mu_n^{(\alpha_n,\beta_n)}$ be the normalized counting measure of the isodynamic points of $P_n^{(\alpha_n,\beta_n)}$. Then*

$$\mu_n^{(\alpha_n,\beta_n)} \xrightarrow[n\to\infty]{w^*} w_{a,b}(\theta)\,\mathrm{d}\theta \qquad \text{on the circle} \qquad u = c_{a,b} + r_{a,b}e^{i\theta},$$

where

$$w_{a,b}(\theta) := \frac{(a+b+2)k_{a,b}^2 \sin^2\theta}{\pi\big(1-(m_{a,b}+2k_{a,b}\cos\theta)^2\big)}.$$

In particular, the limiting support is the circle $\Gamma_{a,b}$.

Proof. Apply Theorem 14 to $P_n^{(\alpha_n,\beta_n)}$. The required strip convergence follows from Lemma 13 and the strong slit-plane asymptotics of Bosbach–Gawronski [7, Theorem 4.1]. The limiting zero measure is the weighted equilibrium measure on $I_{a,b}$,

$$\frac{a+b+2}{2\pi(1-x^2)}\sqrt{\big(x-(m_{a,b}-2k_{a,b})\big)\big((m_{a,b}+2k_{a,b})-x\big)}\;\mathrm{d}x.$$

Equation (20) gives the limiting critical-value map

$$x = m_{a,b} + k_{a,b}(\rho+\rho^{-1}), \qquad u = c_{a,b} + r_{a,b}\rho.$$

Thus $x = m_{a,b} + 2k_{a,b}\cos\theta$ and $u = c_{a,b} + r_{a,b}e^{i\theta}$. Pushing forward the interval density and averaging the two branches yields

$$\frac{(a+b+2)k_{a,b}^2\sin^2\theta}{\pi\big(1-(m_{a,b}+2k_{a,b}\cos\theta)^2\big)}\;\mathrm{d}\theta,$$

which is the stated limit measure. □

Remark 10. For $a = b = 0$ one has $c_{0,0} = 0$ and $r_{0,0} = 1$, so Theorem 9 reduces to the Jacobi part of Theorem 7.

3.2 Laguerre polynomials with linearly varying parameter

Now let $\alpha_n > -1$ satisfy

$$\frac{\alpha_n}{n} \to c \geq 0.$$

As in Corollary 5, after the natural scaling $u = n\xi$ the exact critical-value map converges to

$$L_c(\xi) := \frac{\xi(c-\xi)}{1+c-\xi}.$$

Introduce the change of variables

$$\xi = 1 + c + \sqrt{1+c}\,\rho.$$

A direct calculation gives

$$L_c(1+c+\sqrt{1+c}\,\rho) = c + 2 + \sqrt{1+c}(\rho+\rho^{-1}). \tag{21}$$

Hence the limiting Laguerre support is the interval

$$J_c := \big[c+2-2\sqrt{1+c},\; c+2+2\sqrt{1+c}\big],$$

while the corresponding locus in the ξ-plane is the circle

$$\Lambda_c := \{\xi \in \mathbb{C} :\; |\xi-(1+c)| = \sqrt{1+c}\}.$$

Theorem 11. *Let $\alpha_n/n \to c \geq 0$, and let $\nu_{n,\alpha_n}^{\mathrm{Lag}}$ be the normalized counting measure of the scaled isodynamic points u/n of $L_n^{(\alpha_n)}$. Then*

$$\nu_{n,\alpha_n}^{\mathrm{Lag}} \xrightarrow[n\to\infty]{w^*} w_c(\theta)\,\mathrm{d}\theta \qquad \text{on the circle} \qquad \xi = 1 + c + \sqrt{1+c}\,e^{i\theta},$$

where

$$w_c(\theta) := \frac{(1+c)\sin^2\theta}{\pi\,(c+2+2\sqrt{1+c}\cos\theta)}.$$

In particular, the limiting support is the circle Λ_c.

Proof. Apply Theorem 14 to $P_n(\eta) := L_n^{(\alpha_n)}(n\eta)$. The required strip convergence follows from Lemma 13 and the strong slit-plane asymptotics of Bosbach–Gawronski [6, Theorem 3.1]. The limiting zero measure is the Marchenko–Pastur law on $J_c = [a_c, b_c]$,

$$\frac{1}{2\pi x}\sqrt{(b_c - x)(x - a_c)}\,\mathrm{d}x, \qquad a_c := c + 2 - 2\sqrt{1+c}, \quad b_c := c + 2 + 2\sqrt{1+c}.$$

Equation (21) gives the limiting critical-value map

$$x = c + 2 + \sqrt{1+c}(\rho + \rho^{-1}), \qquad \xi = 1 + c + \sqrt{1+c}\,\rho.$$

Thus $x = c + 2 + 2\sqrt{1+c}\cos\theta$ and $\xi = 1 + c + \sqrt{1+c}\,e^{i\theta}$. Pushing forward the interval density and averaging the two branches yields

$$\frac{(1+c)\sin^2\theta}{\pi\,(c+2+2\sqrt{1+c}\cos\theta)}\,\mathrm{d}\theta,$$

which is the stated limit measure. □

Remark 12. For $c = 0$, Theorem 11 reduces to the Laguerre part of Theorem 7.

4 Illustrative figures

The following plots show the zero distributions of several representative classical families together with the corresponding isodynamic point configurations. In each right-hand panel, the dashed curve indicates the limiting support predicted by the asymptotic theorems proved above.

5 One-cut regular asymptotics on a real interval

The asymptotic step does not require a Bochner equation once the limiting zero measure is supported on a single real interval. This is the setting relevant for *one-cut regular* orthogonal polynomials with varying exponential weights; that is, the case where the equilibrium measure is supported on a single compact interval and has square-root vanishing at both endpoints.

Let P_n be degree-n polynomials with real coefficients and simple real zeros, all contained in a fixed compact interval. Since R_{P_n} is unchanged by multiplication of P_n by a nonzero constant, we normalize P_n to be monic. Write

$$\nu_n := \frac{1}{n}\sum_{P_n(x)=0}\delta_x, \qquad G_n(z) := \frac{1}{n}\frac{P_n'(z)}{P_n(z)}, \qquad R_n(z) := z - \frac{1}{G_n(z)}.$$

Assume that $\nu_n \xrightarrow{w^*} \nu$, where ν is a probability measure supported on $[A, B] \subset \mathbb{R}$, absolutely continuous with respect to Lebesgue measure, and with continuous positive density on (A, B). Set

$$\Omega := \widehat{\mathbb{C}} \setminus [A, B].$$

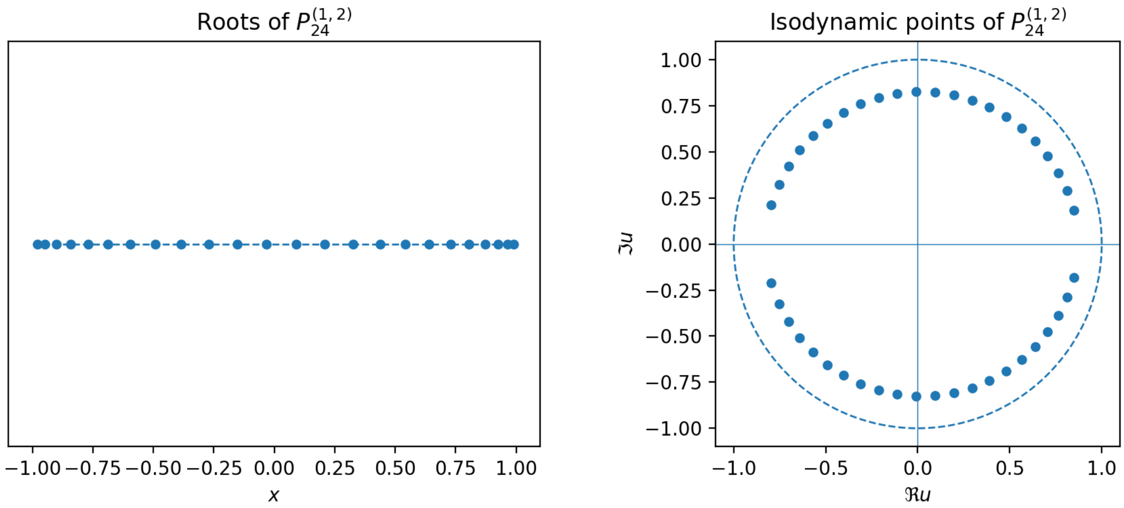

Figure 1: Jacobi case: roots of $P_{24}^{(1,2)}$ and the corresponding isodynamic points. The dashed circle in the right-hand panel is the unit circle from Theorem 7.

Assume that

$$G_n(z) \longrightarrow C_\nu(z) := \int_A^B \frac{\mathrm{d}\nu(x)}{z-x} \tag{22}$$

locally uniformly on Ω and, for every $\varepsilon, \eta > 0$, uniformly on

$$S^{\pm}_{\varepsilon,\eta} := \{x \pm iy : \ x \in [A+\varepsilon, B-\varepsilon], \ 0 < y \le \eta\}.$$

In the applications below, both the exterior convergence in (22) and the strip convergence are obtained by logarithmic differentiation of strong asymptotic formulas. For the fixed classical families we use [5, Section 8.21], for varying Jacobi and Laguerre parameters [7, 6], and for one-cut varying exponential weights on $\mathbb{R}$ [8]. The next lemma records the elementary differentiation step.

Lemma 13. *Let $K \subset \mathbb{C}$ be compact and let N be an open neighborhood of K. Assume that g, a_n, and r_n are analytic on N, that a_n is nonvanishing on N, and that*

$$P_n(z) = a_n(z) e^{ng(z)} (1 + r_n(z))$$

on N, with $\sup_N |r_n| \to 0$ and

$$\sup_{z\in N} \left| \frac{1}{n} \frac{a_n'(z)}{a_n(z)} \right| \to 0.$$

Then

$$\frac{1}{n} \frac{P_n'(z)}{P_n(z)} \longrightarrow g'(z)$$

uniformly on K.

Proof. Choose N' with $K \subset N' \Subset N$. For all large n, the function $1 + r_n$ is nonvanishing on N', and Cauchy estimates give $r_n' \to 0$ uniformly on N'. Hence on N',

$$\frac{1}{n}\frac{P_n'}{P_n} - g' = \frac{1}{n}\frac{a_n'}{a_n} + \frac{1}{n}\frac{r_n'}{1+r_n},$$

and the right-hand side tends uniformly to 0. $\square$

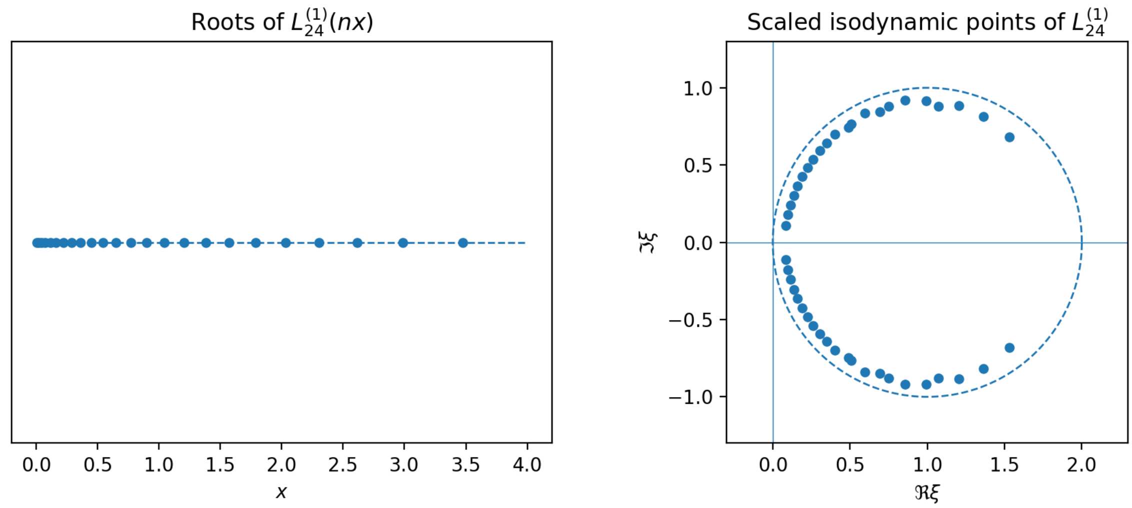

Figure 2: Laguerre case: roots of $L_{24}^{(1)}(24x)$ and scaled isodynamic points $\xi = u/24$. The dashed circle in the right-hand panel is $|\xi - 1| = 1$, in agreement with Theorem 7.

Since $\Im C_\nu(z) \neq 0$ for $z \in \Omega \setminus \mathbb{R}$ and C_ν has fixed sign on $\mathbb{R} \setminus [A, B]$, it does not vanish on Ω. Define

$$U(z) := z - \frac{1}{C_\nu(z)}, \qquad z \in \Omega.$$

Assume that the non-tangential boundary values $U_\pm$ of U on (A, B) exist and satisfy, for every $\varepsilon > 0$,

$$\lim_{y \downarrow 0} \sup_{x \in [A+\varepsilon, B-\varepsilon]} |U(x \pm iy) - U_\pm(x)| = 0. \tag{23}$$

(For a measurable map f and a measure μ, we write $f_*\mu$ for the pushforward measure defined by $f_*\mu(E) := \mu(f^{-1}(E))$.) Finally, set

$$W_n(z) := (n-1)(P_n'(z))^2 - nP_n(z)P_n''(z).$$

Since P_n has simple real zeros, the zeros of P_n' are also real and simple. A direct calculation gives

$$R_n'(z) = -\frac{W_n(z)}{(P_n'(z))^2}.$$

If z_0 is a zero of P_n', then $P_n(z_0) \neq 0$ and $P_n''(z_0) \neq 0$, so $W_n(z_0) = -nP_n(z_0)P_n''(z_0) \neq 0$; hence all poles of R_n are simple. If

$$m_{k,n} := \int x^k \, \mathrm{d}\nu_n(x),$$

then

$$G_n(z) = \frac{1}{z} + \frac{m_{1,n}}{z^2} + \frac{m_{2,n}}{z^3} + O(z^{-4}), \qquad G_n'(z) + G_n(z)^2 = -\frac{m_{2,n} - m_{1,n}^2}{z^4} + O(z^{-5}),$$

and therefore

$$R_n(z) = m_{1,n} + \frac{m_{2,n} - m_{1,n}^2}{z} + O(z^{-2}) \qquad (z \to \infty).$$

Since ν_n is not a point mass, $m_{2,n} - m_{1,n}^2 > 0$. Hence $\deg W_n = 2n - 4$, ∞ is not a critical point of R_n, and the critical divisor of R_n is the zero divisor of W_n. Set

$$\omega_n := \frac{1}{2n-4} \sum_{W_n(\zeta)=0} \delta_\zeta \quad \text{(with multiplicity)}, \qquad \mu_n^{\mathrm{iso}} := (R_n)_*\omega_n = \frac{1}{2n-4} \sum_{u \in \mathcal{I}(P_n)} \delta_u.$$

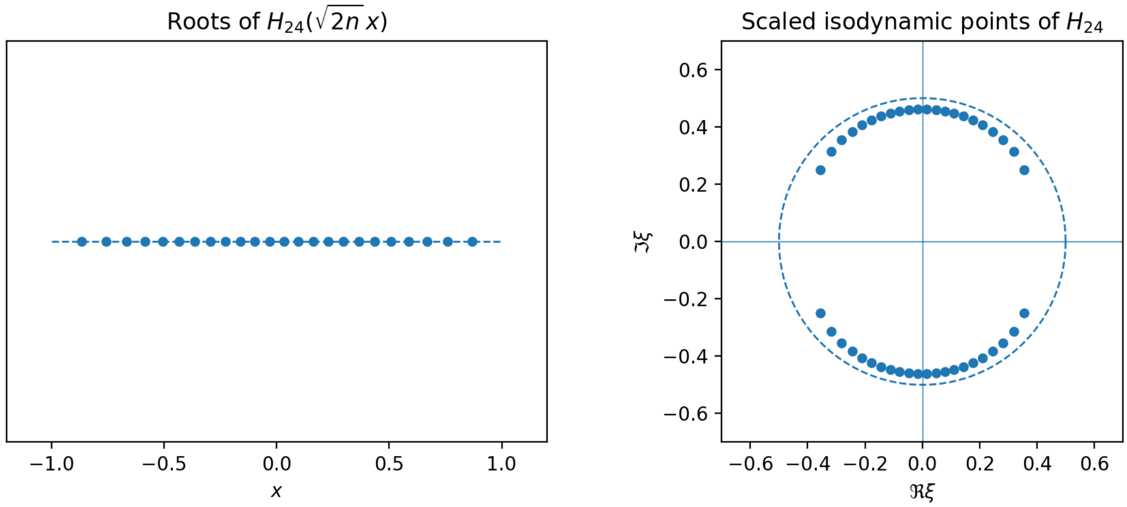

Figure 3: Hermite case: roots of $H_{24}(\sqrt{48}\,x)$ and scaled isodynamic points $\xi = u/\sqrt{48}$. The dashed circle in the right-hand panel is $|\xi| = \frac{1}{2}$, in agreement with Theorem 7.

Theorem 14. *Under the assumptions above,*

$$\mu_n^{\mathrm{iso}} \xrightarrow[n\to\infty]{w^*} \mu_\nu^{\mathrm{iso}} := \frac{1}{2}(U_+)_*\nu + \frac{1}{2}(U_-)_*\nu.$$

In particular, the limiting support is contained in

$$\Gamma_\nu := \overline{U_+((A,B)) \cup U_-((A,B))}.$$

Proof. Because $\nu_n \xrightarrow{w^*} \nu$ and the zeros stay in a fixed compact interval,

$$\frac{1}{n}\log|P_n(z)| = \int \log|z-x|\,\mathrm{d}\nu_n(x) \longrightarrow \int \log|z-x|\,\mathrm{d}\nu(x)$$

locally uniformly on Ω. Since $G_n \to C_\nu$ locally uniformly on Ω, Cauchy estimates also give $G_n' \to C_\nu'$ there. Using

$$W_n(z) = -n^2(G_n'(z) + G_n(z)^2)P_n(z)^2,$$

set

$$q_n(z) := \frac{1}{2n}\log|W_n(z)|, \qquad p_\nu(z) := \int_A^B \log|z-x|\,\mathrm{d}\nu(x).$$

Each q_n is subharmonic on $\mathbb{C}$. Moreover,

$$q_n(z) = \frac{1}{n}\log|P_n(z)| + o(1)$$

locally uniformly on compact subsets of

$$\Omega \setminus Z, \qquad Z := \{z \in \Omega :\ C_\nu'(z) + C_\nu(z)^2 = 0\}.$$

The factor $C_\nu' + C_\nu^2$ is not identically zero: its coefficient at z^{-4} is $-\mathrm{Var}(\nu)$. Hence Z is discrete, in particular of planar measure zero.

Because the zeros of P_n remain in a fixed compact interval and P_n is monic, Cauchy estimates give a local exponential bound on P_n, P_n', and P_n'', so the functions q_n are locally uniformly bounded above. Fix $z_0 \in \Omega \setminus Z$. Then $q_n(z_0) \to p_\nu(z_0)$, so the sequence does not collapse to $-\infty$. By the compactness theorem for subharmonic functions, every subsequence therefore has a further L^1_{loc}-convergent subsequence. Any such limit agrees with p_ν on $\Omega \setminus Z$, hence almost everywhere on $\mathbb{C}$, since $[A, B] \cup Z$ has planar measure zero. Thus

$$q_n \longrightarrow p_\nu \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}).$$

Since $q_n = (2n)^{-1} \log |W_n|$ and W_n is a polynomial, applying $\frac{1}{2\pi}\Delta$ in the sense of distributions gives

$$\left(1 - \frac{2}{n}\right) \omega_n = \frac{1}{2\pi} \Delta q_n \longrightarrow \frac{1}{2\pi} \Delta p_\nu = \nu,$$

so $\omega_n \xrightarrow{w^*} \nu$.

Moreover, W_n has no real zeros. If $x_{1,n}, \dots, x_{n,n}$ are the zeros of P_n, then for $x \in \mathbb{R} \setminus \{x_{1,n}, \dots, x_{n,n}\}$ one has

$$\frac{W_n(x)}{P_n(x)^2} = n \sum_{j=1}^{n} \frac{1}{(x - x_{j,n})^2} - \left(\sum_{j=1}^{n} \frac{1}{x - x_{j,n}} \right)^2 > 0$$

by strict Cauchy–Schwarz: the vectors

$$(1, \dots, 1), \qquad \left(\frac{1}{x - x_{1,n}}, \dots, \frac{1}{x - x_{n,n}} \right)$$

are not proportional because the zeros $x_{j,n}$ are distinct. At a zero $x_{j,n}$ of P_n,

$$W_n(x_{j,n}) = (n - 1)(P_n'(x_{j,n}))^2 > 0.$$

Hence W_n has no real zeros, so $\omega_n(\mathbb{R}) = 0$.

Since $R_n(z) = z - 1/G_n(z)$, (22) yields $R_n \to U$ locally uniformly on Ω. Fix a bounded continuous function f on $\mathbb{C}$. Because W_n has real coefficients, ω_n is invariant under complex conjugation, and therefore

$$\int f \; \mathrm{d}\mu_n^{\mathrm{iso}} = \int \psi_n(z) \; \mathrm{d}\omega_n(z), \qquad \psi_n(z) := \frac{1}{2} f(R_n(z)) + \frac{1}{2} f(R_n(\bar{z})).$$

Fix $\varepsilon > 0$. Define

$$\psi_\varepsilon(x) := \frac{1}{2} f(U_+(x)) + \frac{1}{2} f(U_-(x)), \qquad x \in [A + \varepsilon, B - \varepsilon],$$

and extend it to a bounded continuous function on $\mathbb{C}$ by

$$h_\varepsilon(z) := \psi_\varepsilon\big(\min\{B - \varepsilon, \max\{A + \varepsilon, \Re z\}\}\big).$$

By (23), choose $\eta > 0$ so small that $U(x \pm iy)$ is uniformly close to $U_\pm(x)$ for $x \in [A + \varepsilon, B - \varepsilon]$ and $0 < y \le \eta$. Since ν has positive continuous density on $[A + \varepsilon, B - \varepsilon]$, C_ν is bounded away from 0 on $S^\pm_{\varepsilon,\eta}$; hence the strip convergence of G_n implies uniform convergence of R_n to U there. The images of $S^+_{\varepsilon,\eta} \cup S^-_{\varepsilon,\eta}$ under R_n therefore lie, for all large n, in a fixed compact set, so f is uniformly continuous there. Consequently

$$\sup_{z \in S^+_{\varepsilon,\eta} \cup S^-_{\varepsilon,\eta}} |\psi_n(z) - h_\varepsilon(z)| < 2\varepsilon$$

for all sufficiently large n.

Let

$$T_{\varepsilon,\eta} := \{x + iy : \; x \in [A + \varepsilon, B - \varepsilon], \; |y| < \eta\}.$$

Since $\omega_n \xrightarrow{w^*} \nu$, ν is supported on $[A,B]$, and $\partial T_{\varepsilon,\eta} \cap [A,B]$ has ν-measure zero,

$$\omega_n(\mathbb{C} \setminus T_{\varepsilon,\eta}) \longrightarrow \nu([A, A+\varepsilon] \cup [B-\varepsilon, B]).$$

Therefore, for all sufficiently large n,

$$\left| \int f \, \mathrm{d}\mu_n^{\mathrm{iso}} - \int h_\varepsilon \, \mathrm{d}\omega_n \right| \le 2\varepsilon + 2\|f\|_\infty \, \omega_n(\mathbb{C} \setminus T_{\varepsilon,\eta}).$$

Since h_ε is bounded and continuous, $\int h_\varepsilon \, \mathrm{d}\omega_n \to \int h_\varepsilon \, \mathrm{d}\nu$. Letting first $n \to \infty$ and then $\varepsilon \to 0$, and using dominated convergence together with

$$h_\varepsilon(x) \to \frac{1}{2} f(U_+(x)) + \frac{1}{2} f(U_-(x)) \qquad \text{for } \nu\text{-a.e. } x \in [A,B],$$

we obtain

$$\int f \, \mathrm{d}\mu_n^{\mathrm{iso}} \longrightarrow \frac{1}{2} \int_A^B f(U_+(x)) \, \mathrm{d}\nu(x) + \frac{1}{2} \int_A^B f(U_-(x)) \, \mathrm{d}\nu(x).$$

This is the stated weak convergence. The support claim is immediate. □

Corollary 15 (Affine arcsine case)**.** *Under the assumptions of Theorem 14, assume in addition that*

$$\mathrm{d}\nu(x) = \frac{\mathbf{1}_{[b-2a,b+2a]}(x)}{\pi\sqrt{4a^2 - (x-b)^2}} \, \mathrm{d}x, \qquad a > 0.$$

Then

$$\mu_n^{\mathrm{iso}} \xrightarrow[n\to\infty]{w^*} \frac{\mathrm{d}\theta}{2\pi} \qquad \textit{on} \qquad u = b + 2ae^{i\theta}.$$

Proof. For the affine arcsine law,

$$C_\nu(z) = \frac{1}{\sqrt{(z-b)^2 - 4a^2}},$$

with the branch normalized by $C_\nu(z) \sim z^{-1}$ at infinity. Hence on $x \in (b-2a, b+2a)$,

$$(C_\nu)_\pm(x) = \mp \frac{i}{\sqrt{4a^2 - (x-b)^2}},$$

so, after relabelling the two side limits if necessary,

$$U_\pm(x) = x \pm i\sqrt{4a^2 - (x-b)^2}.$$

Writing $x = b + 2a\cos\theta$, $0 < \theta < \pi$, the measure ν becomes $\mathrm{d}\theta/\pi$, while U_+ and U_- parametrize the upper and lower semicircles of $u = b + 2ae^{i\theta}$. Averaging the two pushforwards gives the uniform measure $\mathrm{d}\theta/(2\pi)$ on the full circle. □

Remark 16. Corollary 15 is the natural one-cut/Nevai-type specialization of Theorem 14: whenever a real-rooted orthogonal family has the affine arcsine limit law on $[b-2a, b+2a]$ and the strip convergence required there, its isodynamic points are asymptotically uniform on the circle $|u-b| = 2a$.

Corollary 17. *Let p_n be the degree-n orthogonal polynomial with respect to $e^{-nV(x)}\,dx$ on $\mathbb{R}$. Assume that V is real analytic with sufficient growth at infinity and that the corresponding equilibrium measure ν_V is one-cut regular (in the sense of Section 5), supported on $[A,B]$ with density ρ_V. Then Theorem 14 applies, and the limiting isodynamic measure is*

$$\mu_V^{\mathrm{iso}} = \frac{1}{2}(u_+)_*\nu_V + \frac{1}{2}(u_-)_*\nu_V, \qquad u_\pm(x) = x - \frac{1}{\frac{1}{2}V'(x) \mp i\pi\rho_V(x)}.$$

Proof. In this setting the one-cut regular Deift–Zhou asymptotics of [8], combined with Lemma 13, give (22) with $\nu = \nu_V$, the required strip convergence away from the endpoints, and confinement of the zeros of p_n to a fixed compact interval for all large n. On (A, B), the boundary values of the Cauchy transform satisfy the Sokhotski–Plemelj formula (see, e.g., [9])

$$(C_{\nu_V})_\pm(x) = \frac{1}{2}V'(x) \mp i\pi\rho_V(x).$$

Hence the boundary maps $u_\pm$ are exactly the boundary values of $U(z) = z - 1/C_{\nu_V}(z)$, and (23) holds on every compact subinterval of (A, B). Apply Theorem 14. □

6 Beyond the one-cut case

Section 5 shows that the correct limiting object is the boundary map

$$U(z) = z - \frac{1}{C_\nu(z)},$$

where C_ν is the limiting Cauchy transform of the zeros. In the one-cut real case, the limiting isodynamic measure is obtained from the limiting zero measure by the rule

$$\nu \longmapsto \mu_\nu^{\text{iso}} := \frac{1}{2}(U_+)_*\nu + \frac{1}{2}(U_-)_*\nu.$$

Only in the arcsine case (Corollary 15) does this reduce to the uniform (arc-length) measure on a single circle.

Proposition 18 (Möbius equivariance)**.** *Let F be a squarefree binary form of degree at least 3, and let $\mathcal{I}(F)$ be its projective isodynamic divisor on $\widehat{\mathbb{C}}$. Define*

$$\mu_F^{\text{iso}} := \frac{1}{\deg \mathcal{I}(F)} \sum_{w \in \operatorname{supp} \mathcal{I}(F)} m_w\, \delta_w,$$

where m_w is the multiplicity of w in $\mathcal{I}(F)$. Then for every $M \in \mathrm{PGL}_2(\mathbb{C})$ one has

$$\mu_{M\cdot F}^{\text{iso}} = M_*\mu_F^{\text{iso}}.$$

Consequently, if $\{F_n\}$ is a sequence of squarefree binary forms such that

$$\mu_{F_n}^{\text{iso}} \xrightarrow{w^*} \mu^{\text{iso}} \qquad \text{on } \widehat{\mathbb{C}},$$

then

$$\mu_{M\cdot F_n}^{\text{iso}} \xrightarrow{w^*} M_*\mu^{\text{iso}}.$$

Proof. The projective isodynamic divisor is $\mathrm{PGL}_2(\mathbb{C})$-equivariant; see [3]. Thus

$$\mathcal{I}(M \cdot F) = M \cdot \mathcal{I}(F)$$

as divisors on $\widehat{\mathbb{C}}$. After normalization by the total degree this gives $\mu_{M\cdot F}^{\text{iso}} = M_*\mu_F^{\text{iso}}$, and the weak-convergence statement follows by testing against continuous functions. □

6.1 Quadratic algebraic Cauchy transforms

The rule $\nu \mapsto \mu_\nu^{\mathrm{iso}}$ becomes particularly explicit when the Cauchy transform satisfies a quadratic algebraic equation.

Proposition 19. *Let ν be a compactly supported probability measure on $\mathbb{C}$, and assume that on some domain $\Omega \subset \widehat{\mathbb{C}} \setminus \mathrm{supp}(\nu)$ its Cauchy transform satisfies*

$$A(z)C_\nu(z)^2 + B(z)C_\nu(z) + D(z) = 0,$$

where A, B, and D are rational functions and $A(z)D(z) \neq 0$ on Ω. Set

$$\Delta(z) := B(z)^2 - 4A(z)D(z).$$

Then on Ω the map

$$U(z) := z - \frac{1}{C_\nu(z)}$$

has the two algebraic branches

$$U_\pm(z) = z + \frac{B(z) \pm \sqrt{\Delta(z)}}{2D(z)}.$$

Proof. Solving the quadratic equation for C_ν gives

$$C_\nu(z) = \frac{-B(z) \pm \sqrt{\Delta(z)}}{2A(z)}.$$

Hence

$$\frac{1}{C_\nu(z)} = \frac{2A(z)}{-B(z) \pm \sqrt{\Delta(z)}} = -\frac{B(z) \pm \sqrt{\Delta(z)}}{2D(z)},$$

after rationalizing the denominator and, if necessary, relabelling the two branches. Substituting into $U(z) = z - 1/C_\nu(z)$ gives the formula. □

Remark 20. Proposition 19 is only an algebraic identity for the map U. Whenever a theorem of the type of Theorem 14 is available, the corresponding limiting isodynamic measure is then obtained by averaging the pushforwards of ν by the relevant boundary branches.

Example 21 (Joukowski ellipse)**.** Fix $c \in \mathbb{C}$ and $\alpha \geq \beta > 0$, and let $\nu_{\alpha,\beta,c}$ be the pushforward of $\mathrm{d}\theta/(2\pi)$ under

$$z(\theta) = c + \alpha e^{i\theta} + \beta e^{-i\theta}.$$

For $\alpha > \beta$ the support is an ellipse, while for $\alpha = \beta = a$ it degenerates to the line segment joining $c - 2a$ and $c + 2a$. A residue computation gives

$$C_{\nu_{\alpha,\beta,c}}(z) = \frac{1}{\sqrt{(z-c)^2 - 4\alpha\beta}},$$

with the branch normalized by $C_{\nu_{\alpha,\beta,c}}(z) \sim z^{-1}$ at infinity. Thus

$$U_\pm(z) = z \mp \sqrt{(z-c)^2 - 4\alpha\beta}.$$

On the support, write

$$z - c = \alpha w + \beta w^{-1}, \qquad |w| = 1.$$

Then

$$\sqrt{(z-c)^2 - 4\alpha\beta} = \pm(\alpha w - \beta w^{-1}),$$

and therefore

$$U_+(z) = c + 2\beta w^{-1}, \qquad U_-(z) = c + 2\alpha w.$$

Hence the branch-averaged output of the rule $\nu \mapsto \frac{1}{2}(U_+)_*\nu + \frac{1}{2}(U_-)_*\nu$ is

$$\frac{1}{2}(U_+)_*\nu_{\alpha,\beta,c} + \frac{1}{2}(U_-)_*\nu_{\alpha,\beta,c} = \frac{1}{2}\,\sigma_{c,2\alpha} + \frac{1}{2}\,\sigma_{c,2\beta},$$

where $\sigma_{c,R}$ denotes the uniform probability measure on the circle $|u-c| = R$. In the degenerate case $\alpha = \beta = a$, the two circles coincide; for $c = b \in \mathbb{R}$ this is Corollary 15.

Conjecture 22. Let P_n be degree-n polynomials with regular n-th root asymptotics on $\widehat{\mathbb{C}} \setminus S$, where $S := \operatorname{supp}(\nu)$ is the support of the limiting zero measure ν. Assume that on each connected component of $\widehat{\mathbb{C}} \setminus S$ one has

$$\frac{1}{n}\frac{P_n'(z)}{P_n(z)} \longrightarrow C_\nu(z)$$

locally uniformly, and set

$$U(z) := z - \frac{1}{C_\nu(z)}.$$

Then the normalized critical-point measures of R_{P_n} should converge to ν, and the normalized isodynamic measures should converge to the pushforwards of ν by the boundary values of U on each arc of S, averaged over all branches. In the one-cut real case this is Theorem 14. In multi-cut situations one should expect several branches, and for quasi-periodic families a periodic limit family, rather than a single stationary curve.

6.2 Higher-order exactly solvable operators

A concrete testing ground for Conjecture 22 is provided by the exactly solvable operators of Bergkvist and Rullgård [1],

$$T_Q = \sum_{j=0}^{k} Q_j(z)\frac{d^j}{dz^j}, \qquad \deg Q_j \le j, \qquad \deg Q_k = k,$$

with Q_k monic. For all sufficiently large n there exists a unique monic degree-n eigenpolynomial p_n and a scalar δ_n such that

$$T_Q(p_n) = \delta_n p_n.$$

If μ_{Q_k} denotes the limiting zero measure of $\{p_n\}$, then its Cauchy transform C satisfies

$$C(z)^k = \frac{1}{Q_k(z)}$$

almost everywhere [1].

Conjecture 23. Let ν_n^{iso} be the normalized isodynamic measures of the monic eigenpolynomials p_n. Then every weak limit of $\{\nu_n^{\mathrm{iso}}\}$ is supported on the union of the boundary-value images of $\operatorname{supp}(\mu_{Q_k})$ under

$$z \mapsto z - \frac{1}{C(z)}.$$

More precisely, along each smooth support arc one should obtain the limiting measure by averaging the pushforwards of μ_{Q_k} by the two boundary maps $z \mapsto z - 1/C_\pm(z)$.

Remark 24. Equivalently, since $C^k = 1/Q_k$, the candidate support is obtained by applying the local branches

$$z \mapsto z - \omega Q_k(z)^{1/k}, \qquad \omega^k = 1,$$

to the support arcs of μ_{Q_k}. The WKB expansion of Borrego-Morell [2] is consistent with this picture on each branch domain, and a plausible route to a proof is through the discriminants

$$D_n(u) := \mathrm{Disc}_z\big(np_n(z) - (z-u)p_n'(z)\big),$$

whose zeros are exactly the isodynamic points of p_n. In the quadratic case $k = 2$ with $Q_2(z) = z^2 - 1$, the two boundary maps reduce to $u = x \pm i\sqrt{1-x^2}$ and one recovers the unit circle.

A second natural direction is provided by semiclassical orthogonal polynomials. In the Pearson-type situations (i.e. when the weight w satisfies $(\sigma w)' = \tau w$ with $\deg \sigma > 2$) where a second-order equation with higher-degree σ is available, the substitution

$$nP = (z-u)P', \qquad nPP'' = (n-1)(P')^2,$$

still reduces the isodynamic condition to an algebraic relation between z and u. Once $\deg \sigma > 2$, that relation is no longer linear in z. One should then expect multivalued critical-point maps and supports given by higher-degree algebraic curves rather than by the circles of the Bochner and one-cut cases.

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