SIMPLY-LACED COXETER GROUPS AND GROUPS GENERATED BY SYMPLECTIC TRANSVECTIONS

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To Bill Fulton on the occasion of his sixtieth birthday

Abstract. Let W be an arbitrary Coxeter group of simply-laced type (possibly infinite but of finite rank), let u and v be any two elements in W, and let $\mathbf{i}$ be a reduced word (of length m) for the pair (u, v) in the Coxeter group $W \times W$. Generalizing a construction in [10, 11], we associate to $\mathbf{i}$ a subgroup $\Gamma_{\mathbf{i}}$ in $GL_m(\mathbb{Z})$ generated by symplectic transvections. We also generalize the enumeration result of [10, 11] by showing that, under certain assumptions on u and v, the number of $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$ is equal to 3×2^s, where s is the number of simple reflections that appear in a reduced decomposition for u or v, and $\mathbb{F}_2$ is the two-element field. When W is the Weyl group of a simply-laced root system, we conjecture that the $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$ enumerate connected components of the real part of the reduced double Bruhat cell corresponding to (u, v).

1. Introduction

The point of departure for this paper is the following result obtained in [10, 11]. Let N_n^0 denote the semi-algebraic set of all unipotent upper-triangular $n \times n$ matrices x with real entries such that, for every $k = 1, \ldots, n-1$, the minor of x with rows $1, \ldots, k$ and columns $n-k+1, \ldots, n$ is non-zero. Then the number $\#_n$ of connected components of N_n^0 is given as follows: $\#_2 = 2, \#_3 = 6, \#_4 = 20, \#_5 = 52$, and $\#_n = 3 \cdot 2^{n-1}$ for $n \geq 6$.

An interesting feature of this answer is that every case which one can check by hand turns out to be exceptional. But the method of the proof seems to be even more interesting than the answer itself: it is shown that the connected components of N_n^0 are in a bijection with the orbits of a certain group Γ_n that acts in a vector space of dimension $n(n-1)/2$ over the two-element field $\mathbb{F}_2$, and is generated by symplectic transvections. Such groups appeared earlier in singularity theory, see e.g., [5] and references therein.

The construction of Γ_n given in [10, 11] uses the combinatorial machinery (developed in [1]) of pseudo-line arrangements associated with reduced expressions in the symmetric group. In this paper we present the following far-reaching generalization of this construction. Let W be an arbitrary Coxeter group of simply-laced

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type (possibly infinite but of finite rank). Let u and v be any two elements in W, and $\mathbf{i}$ be a reduced word (of length $m = \ell(u)+\ell(v)$) for the pair (u, v) in the Coxeter group $W \times W$ (see Section 2 for more details). We associate to $\mathbf{i}$ a subgroup $\Gamma_{\mathbf{i}}$ in $GL_m(\mathbb{Z})$ generated by symplectic transvections. We prove among other things that the subgroups corresponding to different reduced words for the same pair (u, v) are conjugate to each other inside $GL_m(\mathbb{Z})$. To recover the group Γ_n from this general construction, one needs several specializations and reductions: take W to be the symmetric group S_n; take $(u, v) = (w_0, e)$, where w_0 is the longest permutation in S_n, and e is the identity permutation; take $\mathbf{i}$ to be the lexicographically minimal reduced word $1, 2, 1, \ldots, n-1, n-2, \ldots, 1$ for w_0; and finally, take the group $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$ obtained from $\Gamma_{\mathbf{i}}$ by reducing the linear transformations from $\mathbb{Z}$ to $\mathbb{F}_2$.

We also generalize the enumeration result of [10, 11] by showing that, under certain assumptions on u and v, the number of $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$ is equal to $3 \cdot 2^s$, where s is the number of simple reflections in W that appear in a reduced decomposition for u or v. We deduce this from a description of orbits in an even more general situation which sharpens the results in [5, 11] (see Section 7 below).

Although the results and methods of this paper are purely algebraic and combinatorial, our motivation for the study of the groups $\Gamma_{\mathbf{i}}$ and their orbits comes from geometry. In the case when W is the (finite) Weyl group of a simply-laced root system, we expect (see Conjecture 4.1 below) that the $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$ enumerate connected components of the real part of the reduced double Bruhat cell corresponding to (u, v). Double Bruhat cells were introduced and studied in [4] as a natural framework for the study of total positivity in semisimple groups; as explained to us by N. Reshetikhin, they also appear naturally in the study of symplectic leaves in semisimple groups (see [6]). Let us briefly recall their definition.

Let G be an $\mathbb{R}$-split simply connected semisimple algebraic group with the Weyl group W; thus $W = \mathrm{Norm}_G(H)/H$, where H is an $\mathbb{R}$-split maximal torus in G. Let B and B_- be two (opposite) Borel subgroups in G such that $B \cap B_- = H$. The *double Bruhat cells* $G^{u,v}$ are defined as the intersections of ordinary Bruhat cells taken with respect to B and B_-:

$$G^{u,v} = BuB \cap B_-vB_- \ .$$

In view of the well-known Bruhat decomposition, the group G is the disjoint union of all $G^{u,v}$ for $(u, v) \in W \times W$.

The term "cell" might be misleading because the topology of $G^{u,v}$ can be quite complicated. The torus H acts freely on $G^{u,v}$ by left (as well as right) translations, and there is a natural section $L^{u,v}$ for this action which we call the *reduced double Bruhat cell.* These sections are introduced and studied in a forthcoming paper [3] (for the definition see Section 4 below).

The special case when $(u, v) = (e, w)$ for some element $w \in W$ is of particular geometric interest. In this case, $L^{u,v}$ is biregularly isomorphic to the so-called *opposite Schubert cell*

$$C_w^0 := C_w \cap w_0 C_{w_0} \ ,$$

where w_0 is the longest element of W, and $C_w = (BwB)/B \subset G/B$ is the *Schubert cell* corresponding to w. These opposite cells appeared in the literature in various contexts, and were studied (in various degrees of generality) in [1, 2, 8, 9, 10, 11].

In particular, the variety N_n^0 which was the main object of study in [10, 11] is naturally identified with the real part of the opposite cell $C_{w_0}^0$ for $G = SL_n$.

By the informal "complexification principle" of V. I. Arnold, if the group $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$ enumerates connected components of the real part of $L^{u,v}$, the group $\Gamma_{\mathbf{i}}$ itself (which acts in $\mathbb{Z}^m$ rather than in $\mathbb{F}_2^m$) should supposedly provide information about topology of the complex variety $L^{u,v}$. So far we did not find a totally satisfactory "complexification" along these lines.

The paper is organized as follows. Main definitions, notations and conventions are collected in Section 2. Our main results are formulated in Section 3 and proved in sections 5–7. The geometric connection outlined above is discussed in more detail in Section 4.

2. Definitions

2.1. **Simply-laced Coxeter groups.** Let Π be an arbitrary finite graph without loops and multiple edges. Throughout the paper, we use the following notation: write $i \in \Pi$ if i is a vertex of Π, and $\{i,j\} \in \Pi$ if the vertices i and j are adjacent in Π. The (simply-laced) Coxeter group $W = W(\Pi)$ associated with Π is generated by the elements s_i for $i \in \Pi$ subject to the relations

$$s_i^2 = e; \quad s_i s_j = s_j s_i \ (\{i,j\} \notin \Pi); \quad s_i s_j s_i = s_j s_i s_j \ (\{i,j\} \in \Pi) \ . \tag{2.1}$$

A word $\mathbf{i} = (i_1, \dots, i_m)$ in the alphabet Π is a *reduced word* for $w \in W$ if $w = s_{i_1} \cdots s_{i_m}$, and m is the smallest length of such a factorization. The length m of any reduced word for w is called the *length* of w and denoted by $m = \ell(w)$. Let $R(w)$ denote the set of all reduced words for w.

The "double" group $W \times W$ is also a Coxeter group; it corresponds to the graph $\tilde{\Pi}$ which is the union of two disconnected copies of Π. We identify the vertex set of $\tilde{\Pi}$ with $\{+1,-1\} \times \Pi$, and write a vertex $(\pm 1, i) \in \tilde{\Pi}$ simply as $\pm i$. For each $\pm i \in \tilde{\Pi}$, we set $\varepsilon(\pm i) = \pm 1$ and $|\pm i| = i \in \Pi$. Thus two vertices i and j of $\tilde{\Pi}$ are joined by an edge if and only if $\varepsilon(i) = \varepsilon(j)$ and $\{|i|,|j|\} \in \Pi$. In this notation, a reduced word for a pair $(u,v) \in W \times W$ is an arbitrary shuffle of a reduced word for u written in the alphabet $-\Pi$ and a reduced word for v written in the alphabet Π.

In view of the defining relations (2.1), the set of reduced words $R(u,v)$ is equipped with the following operations:

- *2-move.* Interchange two consecutive entries i_{k-1}, i_k in a reduced word $\mathbf{i} = (i_1, \dots, i_m)$ provided $\{i_{k-1}, i_k\} \notin \tilde{\Pi}$.
- *3-move.* Replace three consecutive entries i_{k-2}, i_{k-1}, i_k in $\mathbf{i}$ by $i_{k-1}, i_{k-2}, i_{k-1}$ if $i_k = i_{k-2}$ and $\{i_{k-1}, i_k\} \in \tilde{\Pi}$.

In each case, we will refer to the index $k \in [1,m]$ as the *position* of the corresponding move. Using these operations, we make $R(u,v)$ the set of vertices of a graph whose edges correspond to 2- and 3-moves. It is a well known result due to Tits that this graph is *connected*, i.e., any two reduced words in $R(u,v)$ can be obtained from each other by a sequence of 2- and 3-moves. We will say that a 2-move interchanging the entries i_{k-1} and i_k is *trivial* if $i_k \neq -i_{k-1}$; the remaining 2-moves and all 3-moves will be referred to as *non-trivial*.

2.2. **Groups generated by symplectic transvections.** Let Σ be a finite directed graph. As before, we shall write $k \in \Sigma$ if k is a vertex of Σ, and $\{k,l\} \in \Sigma$ if the vertices k and l are adjacent in the underlying graph obtained from Σ by forgetting directions of edges. We also write $(k \to l) \in \Sigma$ if $k \to l$ is a directed edge of Σ.

Let $V = \mathbb{Z}^\Sigma$ be the lattice with a fixed $\mathbb{Z}$-basis $(e_k)_{k\in\Sigma}$ labeled by vertices of Σ. Let $\xi_k \in V^*$ denote the corresponding coordinate functions, i.e., every vector $v \in V$ can be written as

$$v = \sum_{k\in\Sigma} \xi_k(v) e_k \ .$$

We define a skew-symmetric bilinear form Ω on V by

$$\Omega = \Omega_\Sigma = \sum_{(k\to l)\in\Sigma} \xi_k \wedge \xi_l \ . \tag{2.2}$$

For each $k \in \Sigma$, we define the symplectic transvection $\tau_k = \tau_{k,\Sigma} : V \to V$ by

$$\tau_k(v) = v - \Omega(v, e_k)e_k \ . \tag{2.3}$$

(The word "symplectic" might be misleading since Ω is allowed to be degenerate; still we prefer to keep this terminology from [5].) In the coordinate form, we have $\xi_l(\tau_k(v)) = \xi_l(v)$ for $l \neq k$, and

$$\xi_k(\tau_k(v)) = \xi_k(v) - \sum_{(a\to k)\in\Sigma} \xi_a(v) + \sum_{(k\to b)\in\Sigma} \xi_b(v) \ . \tag{2.4}$$

For any subset B of vertices of Σ, we denote by $\Gamma_{\Sigma,B}$ the group of linear transformations of $V = \mathbb{Z}^\Sigma$ generated by the transvections τ_k for $k \in B$.

Note that all transformations from $\Gamma_{\Sigma,B}$ are represented by integer matrices in the standard basis e_k. Let $\Gamma_{\Sigma,B}(\mathbb{F}_2)$ denote the group of linear transformations of the $\mathbb{F}_2$-vector space $V(\mathbb{F}_2) = \mathbb{F}_2^\Sigma$ obtained from $\Gamma_{\Sigma,B}$ by reduction modulo 2 (recall that $\mathbb{F}_2$ is the 2-element field).

3. Main results

3.1. **The graph $\Sigma(\mathbf{i})$.** We now present our main combinatorial construction that brings together simply-laced Coxeter groups and groups generated by symplectic transvections. Let $W = W(\Pi)$ be the simply-laced Coxeter group associated to a graph Π (see Section 2.1). Fix a pair $(u, v) \in W \times W$, and let $m = \ell(u) + \ell(v)$. Let $\mathbf{i} = (i_1, \dots, i_m) \in R(u, v)$ be any reduced word for (u, v). We shall construct a directed graph $\Sigma(\mathbf{i})$ and a subset $B(\mathbf{i})$ of its vertices, thus giving rise to a group $\Gamma_{\Sigma(\mathbf{i}),B(\mathbf{i})}$ generated by symplectic transvections.

First of all, the set of vertices of $\Sigma(\mathbf{i})$ is just the set $[1, m] = \{1, 2, \dots, m\}$. For $l \in [1, m]$, we denote by $l^- = l^-_{\mathbf{i}}$ the maximal index k such that $1 \leq k < l$ and $|i_k| = |i_l|$; if $|i_k| \neq |i_l|$ for $1 \leq k < l$ then we set $l^- = 0$. We define $B(\mathbf{i}) \subset [1, m]$ as the subset of indices $l \in [2, m]$ such that $l^- > 0$.

The indices $l \in B(\mathbf{i})$ will be called $\mathbf{i}$-*bounded.*

It remains to define the edges of $\Sigma(\mathbf{i})$.

Definition 3.1. A pair $\{k, l\} \subset [1, m]$ with $k < l$ is an edge of $\Sigma(\mathbf{i})$ if it satisfies one of the following three conditions:

(i) $k = l^-$;

(ii) $k^- < l^- < k$, $\{|i_k|, |i_l|\} \in \Pi$, and $\varepsilon(i_{l^-}) = \varepsilon(i_k)$;

(iii) $l^- < k^- < k$, $\{|i_k|, |i_l|\} \in \Pi$, and

$\varepsilon(i_{k^-}) = -\varepsilon(i_k)$.

The edges of type (i) are called *horizontal*, and those of types (ii) and (iii) *inclined.* A horizontal (resp. inclined) edge $\{k, l\}$ with $k < l$ is directed ¿from k to l if and only if $\varepsilon(i_k) = +1$ (resp. $\varepsilon(i_k) = -1$).

We will give a few examples in the end of Section 3.2.

3.2. **Properties of graphs $\Sigma(\mathbf{i})$.** We start with the following property of $\Sigma(\mathbf{i})$ and $B(\mathbf{i})$.

Proposition 3.2. *For any non-empty subset $S \subset B(\mathbf{i})$, there exists a vertex $a \in [1,m] \setminus S$ such that $\{a,b\} \in \Sigma(\mathbf{i})$ for a unique $b \in S$.*

For any edge $\{i,j\} \in \Pi$, let $\Sigma_{i,j}(\mathbf{i})$ denote the induced directed subgraph of $\Sigma(\mathbf{i})$ with vertices $k \in [1,m]$ such that $|i_k| = i$ or $|i_k| = j$.

We shall use the following planar realization of $\Sigma_{i,j}(\mathbf{i})$ which we call the (i,j)-*strip* of $\Sigma(\mathbf{i})$.

Consider the infinite horizontal strip $\mathbb{R} \times [-1,1] \subset \mathbb{R}^2$, and identify each vertex $k \in \Sigma_{i,j}(\mathbf{i})$ with the point $A = A_k = (k,y)$, where $y = -1$ for $|i_k| = i$, and $y = 1$ for $|i_k| = j$. We represent each (directed) edge $(k \to l)$ by a straight line segment ¿from A_k to A_l. (This justifies the terms "horizontal" and "inclined" edges in Definition 3.1.)

Note that every edge of $\Sigma(\mathbf{i})$ belongs to some (i,j)-strip, so we can think of $\Sigma(\mathbf{i})$ as the union of all its strips glued together along horizontal lines.

Theorem 3.3. *(a) The (i,j)-strip of $\Sigma(\mathbf{i})$ is a planar graph; equivalently, no two inclined edges cross each other inside the strip.*
(b) The boundary of any triangle or trapezoid formed by two consecutive inclined edges and horizontal segments between them is a directed cycle in $\Sigma_{i,j}(\mathbf{i})$.

Our next goal is to compare the directed graphs $\Sigma(\mathbf{i})$ and $\Sigma(\mathbf{i}')$ when two reduced words $\mathbf{i}$ and $\mathbf{i}'$ are related by a 2- or 3-move. To do this, we associate to $\mathbf{i}$ and $\mathbf{i}'$ a permutation $\sigma_{\mathbf{i}',\mathbf{i}}$ of $[1,m]$ defined as follows. If $\mathbf{i}$ and $\mathbf{i}'$ are related by a trivial 2-move in position k then $\sigma_{\mathbf{i}',\mathbf{i}} = (k-1,k)$, the transposition of $k-1$ and k; if $\mathbf{i}$ and $\mathbf{i}'$ are related by a non-trivial 2-move then $\sigma_{\mathbf{i}',\mathbf{i}} = e$, the identity permutation of $[1,m]$; finally, if $\mathbf{i}$ and $\mathbf{i}'$ are related by a 3-move in position k then $\sigma_{\mathbf{i}',\mathbf{i}} = (k-2,k-1)$. The following properties of $\sigma_{\mathbf{i}',\mathbf{i}}$ are immediate from the definitions.

Proposition 3.4. *The permutation $\sigma_{\mathbf{i}',\mathbf{i}}$ sends $\mathbf{i}$-bounded indices to $\mathbf{i}'$-bounded ones. If the move that relates $\mathbf{i}$ and $\mathbf{i}'$ is non-trivial then its position k is $\mathbf{i}$-bounded, and $\sigma_{\mathbf{i}',\mathbf{i}}(k) = k$.*

The relationship between the graphs $\Sigma(\mathbf{i})$ and $\Sigma(\mathbf{i}')$ is now given as follows.

Theorem 3.5. *Suppose two reduced words $\mathbf{i}$ and $\mathbf{i}'$ are related by a 2- or 3-move in position k, and $\sigma = \sigma_{\mathbf{i}',\mathbf{i}}$ is the corresponding permutation of $[1,m]$. Let a and b be two distinct elements of $[1,m]$ such that at least one of them is $\mathbf{i}$-bounded. Then*

$$(a \to b) \in \Sigma(\mathbf{i}) \Leftrightarrow (\sigma(a) \to \sigma(b)) \in \Sigma(\mathbf{i}') \ , \tag{3.1}$$

with the following two exceptions.

1. *If the move that relates $\mathbf{i}$ and $\mathbf{i}'$ is non-trivial then $(a \to k) \in \Sigma(\mathbf{i}) \Leftrightarrow (k \to \sigma(a)) \in \Sigma(\mathbf{i}')$.*
2. *If the move that relates $\mathbf{i}$ and $\mathbf{i}'$ is non-trivial, and $a \to k \to b$ in $\Sigma(\mathbf{i})$ then $\{a,b\} \in \Sigma(\mathbf{i}) \Leftrightarrow \{\sigma(a),\sigma(b)\} \notin \Sigma(\mathbf{i}')$; furthermore, the edge $\{a,b\} \in \Sigma(\mathbf{i})$ can only be directed as $b \to a$.*

The following example illustrates the above results.

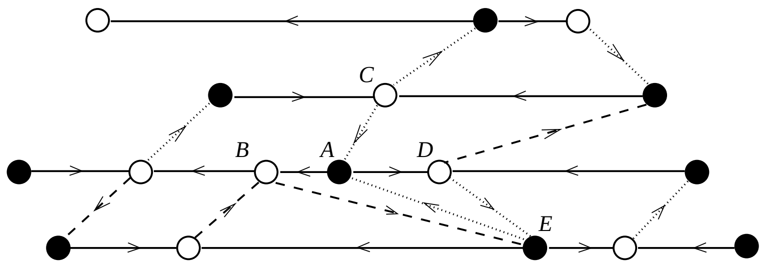

FIGURE 1. Graph $\Sigma(\mathbf{i})$ for type A_4.

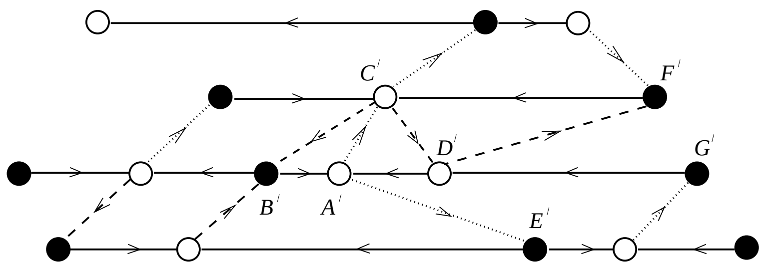

FIGURE 2. Graph transformation under a non-trivial 2-move.

Example 3.6. Let Π be the Dynkin graph A_4, i.e., the chain formed by vertices $1, 2, 3$, and 4. Let $u = s_4s_2s_1s_2s_3s_2s_4s_1$ and $v = s_2s_1s_3s_2s_4s_1s_3s_2s_1$ (in the standard realization of W as the symmetric group S_5, with the generators $s_i = (i, i+1)$ (adjacent transpositions), the permutations u and v can be written in the one-line notation as $u = 53241$ and $v = 54312$). The graph $\Sigma(\mathbf{i})$ corresponding to the reduced word $\mathbf{i} = (2, 1, -4, -2, -1, 3, -2, 2, -3, -2, 4, 1, -4, -1, 3, 2, 1)$ of (u, v) is shown on Fig. 1. Here white (resp. black) vertices of each horizontal level i correspond to entries of $\mathbf{i}$ that are equal to $-i$ (resp. to i). Horizontal edges are shown by solid lines, inclined edges of type (ii) in Definition 3.1 by dashed lines, and inclined edges of type (iii) by dotted lines.

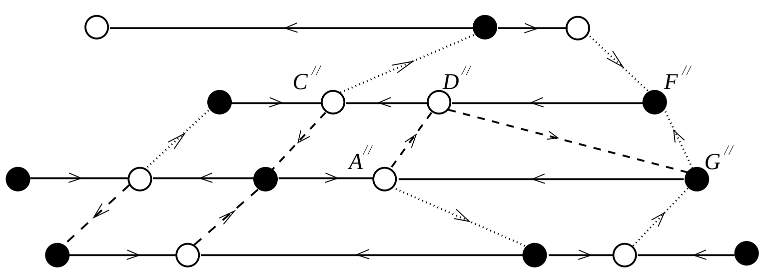

FIGURE 3. Graph transformation under a 3-move.

Now let $\mathbf{i}'$ be obtained from $\mathbf{i}$ by the (non-trivial) 2-move in position 8, i.e., by interchanging $i_7 = -2$ with $i_8 = 2$. The corresponding graph $\Sigma(\mathbf{i}')$ is shown on Fig. 2.

Notice that the edges of $\Sigma(\mathbf{i})$ that fall into the first exceptional case in Theorem 3.5 are $A \to B$, $C \to A$, and $A \to D$; by reversing their orientation, one obtains the edges $B' \to A'$, $A' \to C'$, and $D' \to A'$ of $\Sigma(\mathbf{i}')$. The second exceptional case in Theorem 3.5 applies to two edges $B \to E$ and $D \to E$ of $\Sigma(\mathbf{i})$ and two "non-edges" $\{C, B\}$ and $\{C, D\}$; the corresponding edges and non-edges of $\Sigma(\mathbf{i}')$ are $C' \to B'$, $C' \to D'$, $\{E', B'\}$, and $\{E', D'\}$.

Finally, consider the reduced word $\mathbf{i}''$ obtained from $\mathbf{i}'$ by the 3-move in position 10, i.e., by replacing $(i'_8, i'_9, i'_{10}) = (-2, -3, -2)$ with $(-3, -2, -3)$. The corresponding graph $\Sigma(\mathbf{i}'')$ is shown on Fig. 3.

Now the first exceptional case in Theorem 3.5 covers the edges $D' \to A'$, $C' \to D'$, $D' \to F'$, and $G' \to D'$ of $\Sigma(\mathbf{i}')$, and the corresponding edges $A'' \to D''$, $D'' \to C''$, $F'' \to D''$, and $D'' \to G''$ of $\Sigma(\mathbf{i}'')$. The second exceptional case covers the edges $F' \to C'$ and $A' \to C'$, and non-edges $\{G', F'\}$ and $\{G', A'\}$ of $\Sigma(\mathbf{i}'')$; the corresponding edges and non-edges of $\Sigma(\mathbf{i}'')$ are $G'' \to F''$, $G'' \to A''$, $\{C'', F''\}$, and $\{A'', C''\}$.

3.3. **The groups $\Gamma_{\mathbf{i}}$ and conjugacy theorems.** As before, let $\mathbf{i} = (i_1, \dots, i_m)$ be a reduced word for a pair (u, v) of elements in a simply-laced Coxeter group W. By the general construction in Section 2.2, the pair $(\Sigma(\mathbf{i}), B(\mathbf{i}))$ gives rise to a skew symmetric form $\Omega_{\Sigma(\mathbf{i})}$ on $\mathbb{Z}^m$, and to a subgroup $\Gamma_{\Sigma(\mathbf{i}), B(\mathbf{i})} \subset GL_m(\mathbb{Z})$ generated by symplectic transvections. We denote these symplectic transvections by $\tau_{k,\mathbf{i}}$, and also abbreviate $\Omega_{\mathbf{i}} = \Omega_{\Sigma(\mathbf{i})}$, and $\Gamma_{\mathbf{i}} = \Gamma_{\Sigma(\mathbf{i}), B(\mathbf{i})}$.

Theorem 3.7. *For any two reduced words $\mathbf{i}$ and $\mathbf{i}'$ for the same pair $(u, v) \in W \times W$, the groups $\Gamma_{\mathbf{i}}$ and $\Gamma_{\mathbf{i}'}$ are conjugate to each other inside $GL_m(\mathbb{Z})$.*

Our proof of Theorem 3.7 is constructive. In view of the Tits result quoted in Section 2.1, it is enough to prove Theorem 3.7 in the case when $\mathbf{i}$ and $\mathbf{i}'$ are related by a 2- or 3-move. We shall construct the corresponding conjugating linear transformations explicitly. To do this, let us define two linear maps $\varphi^{\pm}_{\mathbf{i}',\mathbf{i}} : \mathbb{Z}^m \to \mathbb{Z}^m$. For $v \in \mathbb{Z}^m$, the vectors $\varphi^{+}_{\mathbf{i}',\mathbf{i}}(v) = v^+$ and $\varphi^{-}_{\mathbf{i}',\mathbf{i}}(v) = v^-$ are defined as follows. If $\mathbf{i}$ and $\mathbf{i}'$ are related by a trivial 2-move and l is arbitrary, or if $\mathbf{i}$ and $\mathbf{i}'$ are related by a non-trivial move in position k and $l \neq k$, then we set

$$\xi_l(v^+) = \xi_l(v^-) = \xi_{\sigma_{\mathbf{i}',\mathbf{i}}(l)}(v) \ ; \tag{3.2}$$

for $l = k$ in the case of a non-trivial move, we set

$$\xi_k(v^+) = \sum_{(a \to k) \in \Sigma(\mathbf{i})} \xi_a(v) - \xi_k(v) \ ; \quad \xi_k(v^-) = \sum_{(k \to b) \in \Sigma(\mathbf{i})} \xi_b(v) - \xi_k(v) \ . \tag{3.3}$$

Theorem 3.8. *If two reduced words $\mathbf{i}$ and $\mathbf{i}'$ for the same pair $(u, v) \in W \times W$ are related by a 2- or 3-move then the corresponding linear maps $\varphi^{+}_{\mathbf{i}',\mathbf{i}}$ and $\varphi^{-}_{\mathbf{i}',\mathbf{i}}$ are invertible, and*

$$\Gamma_{\mathbf{i}'} = \varphi^{+}_{\mathbf{i}',\mathbf{i}} \circ \Gamma_{\mathbf{i}} \circ (\varphi^{+}_{\mathbf{i}',\mathbf{i}})^{-1} = \varphi^{-}_{\mathbf{i}',\mathbf{i}} \circ \Gamma_{\mathbf{i}} \circ (\varphi^{-}_{\mathbf{i}',\mathbf{i}})^{-1} \ . \tag{3.4}$$

Our proof of Theorem 3.8 is based on the following properties of the maps $\varphi^{\pm}_{\mathbf{i}',\mathbf{i}}$, which might be of independent interest.

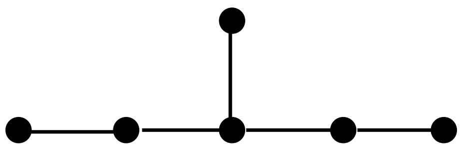

FIGURE 4. Dynkin graph E_6.

Theorem 3.9. *(a) The linear maps* $\varphi^{\pm}_{\mathbf{i}',\mathbf{i}}$ *satisfy:*

$$\varphi^{-}_{\mathbf{i},\mathbf{i}'} \circ \varphi^{+}_{\mathbf{i}',\mathbf{i}} = \varphi^{+}_{\mathbf{i},\mathbf{i}'} \circ \varphi^{-}_{\mathbf{i}',\mathbf{i}} = \mathrm{Id} \ . \tag{3.5}$$

(b) If the move that relates $\mathbf{i}$ *and* $\mathbf{i}'$ *is non-trivial in position* k *then*

$$\varphi^{+}_{\mathbf{i},\mathbf{i}'} \circ \varphi^{+}_{\mathbf{i}',\mathbf{i}} = \tau_{k,\mathbf{i}} \ . \tag{3.6}$$

(c) For any $\mathbf{i}$*-bounded index* $l \in [1, m]$*, we have*

$$\varphi^{+}_{\mathbf{i}',\mathbf{i}} \circ \tau_{l,\mathbf{i}} = \tau_{\sigma_{\mathbf{i}',\mathbf{i}}(l),\mathbf{i}'} \circ \varphi^{+}_{\mathbf{i}',\mathbf{i}} \tag{3.7}$$

unless the move that relates $\mathbf{i}$ *and* $\mathbf{i}'$ *is non-trivial in position* k*, and* $(l \to k) \in \Sigma_{\mathbf{i}}$.

3.4. **Enumerating $\Gamma_{\Sigma,B}(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^{\Sigma}$.** Let Σ and B have the same meaning as in Section 2.2, and let $\Gamma = \Gamma_{\Sigma,B}(\mathbb{F}_2)$ be the corresponding group of linear transformations of the vector space $\mathbb{F}_2^{\Sigma}$.

The following definition is motivated by the results in [5, 10, 11].

Definition 3.10. A finite (non-directed) graph is E_6-compatible if it is connected, and it contains an induced subgraph with 6 vertices isomorphic to the Dynkin graph E_6 (see Fig. 4).

Theorem 3.11. *Suppose that the induced subgraph of* Σ *with the set of vertices* B *is* E_6*-compatible. Then the number of* Γ*-orbits in* $\mathbb{F}_2^{\Sigma}$ *is equal to*

$$2^{\#(\Sigma \setminus B)} \cdot \left(2 + 2^{\dim(\mathbb{F}_2^B \cap \mathrm{Ker}\ \overline{\Omega})}\right) \ ,$$

where $\overline{\Omega}$ *denotes the* $\mathbb{F}_2$*-valued bilinear form on* $\mathbb{F}_2^{\Sigma}$ *obtained by reduction modulo* 2 *from the form* $\Omega = \Omega_{\Sigma}$ *in (2.2).*

Theorem 3.11 has the following corollary which generalizes the main enumeration result in [10, 11].

Corollary 3.12. *Let* u *and* v *be two elements of a simply-laced Coxeter group* $W(\Pi)$*, and suppose that for some reduced word* $\mathbf{i} \in R(u, v)$*, the induced subgraph of* $\Sigma(\mathbf{i})$ *with the set of vertices* $B(\mathbf{i})$ *is* E_6*-compatible. Then the number of* $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$*-orbits in* $\mathbb{F}_2^m$ *is equal to* $3 \cdot 2^s$*, where* s *is the number of indices* $i \in \Pi$ *such that some (equivalently, any) reduced word for* (u, v) *has an entry* $\pm i$.

4. Connected components of real double Bruhat cells

In this section we give a (conjectural) geometric application of the above constructions. We assume that Π is a Dynkin graph of simply-laced type, i.e., every connected component of Π is the Dynkin graph of type A_n, D_n, E_6, E_7, or E_8. Let G be a simply connected semisimple algebraic group with the Dynkin graph Π. We fix a pair of opposite Borel subgroups B_- and B in G; thus $H = B_- \cap B$ is a maximal torus in G. Let N and N_- be the unipotent radicals of B and B_-, respectively. Let $\{\alpha_i : i \in \Pi\}$ be the system of simple roots for which the corresponding root subgroups are contained in N. For every $i \in \Pi$, let $\varphi_i : SL_2 \to G$ be the canonical embedding corresponding to α_i. The (split) real part of G is defined as the subgroup $G(\mathbb{R})$ of G generated by all the subgroups $\varphi_i(SL_2(\mathbb{R}))$. For any subset $L \subset G$ we define its real part by $L(\mathbb{R}) = L \cap G(\mathbb{R})$.

The *Weyl group* W of G is defined by $W = \mathrm{Norm}_G(H)/H$. It is canonically identified with the Coxeter group $W(\Pi)$ (as defined in Section 2.1) via $s_i = \overline{s_i}H$, where

$$\overline{s_i} = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Norm}_G(H) \ .$$

The representatives $\overline{s_i} \in G$ satisfy the braid relations in W; thus the representative $\overline{w}$ can be unambiguously defined for any $w \in W$ by requiring that $\overline{uv} = \overline{u} \cdot \overline{v}$ whenever $\ell(uv) = \ell(u) + \ell(v)$.

The group G has two *Bruhat decompositions*, with respect to B and B_- :

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_-vB_- \ .$$

The *double Bruhat cells* $G^{u,v}$ are defined by $G^{u,v} = BuB \cap B_-vB_-$.

Following [3], we define the *reduced double Bruhat cell* $L^{u,v} \subset G^{u,v}$ as follows:

$$L^{u,v} = N\overline{u}N \cap B_-vB_- \ . \tag{4.1}$$

The maximal torus H acts freely on $G^{u,v}$ by left (or right) translations, and $L^{u,v}$ is a section of this action. Thus $G^{u,v}$ is biregularly isomorphic to $H \times L^{u,v}$, and all properties of $G^{u,v}$ can be translated in a straightforward way into the corresponding properties of $L^{u,v}$ (and vice versa). In particular, Theorem 1.1 in [4] implies that $L^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $\ell(u) + \ell(v)$.

Conjecture 4.1. For every two elements u and v in W, and every reduced word $\mathbf{i} \in R(u,v)$, the connected components of $L^{u,v}(\mathbb{R})$ are in a natural bijection with the $\Gamma_{\mathbf{i}}(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^{\ell(u)+\ell(v)}$.

The precise form of this conjecture comes from the "calculus of generalized minors" developed in [4] and in a forthcoming paper [3]. If u is the identity element $e \in W$ then $L^{e,v} = N \cap B_-vB_-$ is the variety N^v studied in [2]. When $G = SL_n$, and $v = w_0$, the longest element in W, the real part $N^{w_0}(\mathbb{R})$ is the semi-algebraic set N_n^0 discussed in the introduction; in this case, the conjecture was proved in [10, 11] (for a special reduced word $\mathbf{i} = (1, 2, 1, \ldots, n-1, n-2, \ldots, 2, 1) \in R(w_0)$).

5. Proofs of results in Section 3.2

5.1. **Proof of Proposition 3.2.** By the definition of $\mathbf{i}$-bounded indices, we have $k^- \in [1,m]$ for any $k \in S$. Now pick $b \in S$ with the smallest value of b^-, and set $a = b^-$. Clearly, $a \notin S$, and $\{a,b\}$ is a horizontal edge in $\Sigma(\mathbf{i})$. We claim that b is the only vertex in S such that $\{a,b\} \in \Sigma(\mathbf{i})$. Indeed, if $\{a,c\} \in \Sigma(\mathbf{i})$ for some $c \neq b$ then $c^- < a$, in view of Definition 3.1. Because of the way b was chosen, we have $c \notin S$, as required. $\square$

5.2. **Proof of Theorem 3.3.** In the course of the proof, we fix a reduced word $\mathbf{i} \in R(u,v)$, and an edge $\{i,j\} \in \Pi$; we shall refer to the (i,j)-strip of $\Sigma(\mathbf{i})$ as simply the strip. For any vertex $A = A_k = (k,y)$ in the strip, we set $y(A) = y$, and $\varepsilon(A) = \varepsilon(i_k)$; we call $y(A)$ the *level*, and $\varepsilon(A)$ the *sign* of A. We also set

$$c(A) = y(A)\varepsilon(A) \ ,$$

and call $c(A)$ the *charge* of a vertex A. Finally, we linearly order the vertices by setting $A_k \prec A_l$ if $k < l$, i.e., if the vertex A_k is to the left of A_l. In these terms, one can describe inclined edges in the strip as follows.

Lemma 5.1. *A vertex B is the left end of an inclined edge in the strip if and only if it satisfies the following two conditions:*
(1) B is not the leftmost vertex in the strip, and the preceding vertex A has opposite charge $c(A) = -c(B)$.
(2) there is a vertex C of opposite level $y(C) = -y(B)$ that lies to the right of B.
Under these conditions, an inclined edge with the left end B is unique, and its right end is the leftmost vertex C satisfying (2).

This is just a reformulation of conditions (ii) and (iii) in Definition 3.1. □

Lemma 5.2. *Suppose $A \prec C \prec C'$ are three vertices in the strip such that $c(A) = -c(C)$, and $y(C) = -y(C')$. Then there exists a vertex B such that $A \prec B \preceq C$, and B is the left end of an inclined edge in the strip.*

Proof. Let B be the leftmost vertex such that $A \prec B \preceq C$ and $c(B) = -c(A)$. Clearly, B satisfies condition (1) in Lemma 5.1. It remains to show that B also satisfies condition (2); that i we need to find a vertex of opposite level to B that lies to the right of B. Depending on the level of B, either C or C' is such a vertex, and we are done. □

Now everything is ready for the proof of Theorem 3.3. To prove part (a), assume that $\{B,C\}$ and $\{B',C'\}$ are two inclined edges that cross each other inside the strip. Without loss of generality, assume that $B \prec C$, $B' \prec C'$, and $C \prec C'$. Then we must have $B' \prec C$ (otherwise, our inclined edges would not cross). Since $\{B',C'\}$ is an inclined edge, and $B' \prec C \prec C'$, Lemma 5.1 implies that $y(C) = y(B')$. Therefore, $y(B) = -y(C) = -y(B')$. Again applying Lemma 5.1 to the inclined edge $\{B',C'\}$, we conclude that $B \prec B'$, i.e., we must have $B \prec B' \prec C \prec C'$. But then, by the same lemma, $\{B,C\}$ cannot be an inclined edge, providing a desired contradiction.

To prove part (b), consider two consecutive inclined edges $\{B,C\}$ and $\{B',C'\}$. Again we can assume without loss of generality that $B \prec C$, $B' \prec C'$, and $C \prec C'$. Let P be the boundary of the polygon with vertices B, C, B', and C'. By Lemma 5.1, the leftmost vertex of P is B, the rightmost vertex is C', and P does not contain a vertex D such that $B' \prec D \prec C'$; in particular, we have either $C \preceq B'$ or $C = C'$. Now we make the following crucial observation: all the vertices D on P such that $B \prec D \prec B'$ must have the same charge $c(D) = c(B)$. Indeed, assume that $c(D) = -c(B)$ for some D with $B \prec D \prec B'$. Then Lemma 5.2 implies that some B'' with $B \prec B'' \preceq D$ is the left end of an inclined edge; but this contradicts our assumption that $\{B,C\}$ and $\{B',C'\}$ are two *consecutive* inclined edges. We see that $c(D) = c(B)$ for any vertex $D \in P \setminus \{B',C'\}$. Combining this fact with condition (1) in Lemma 5.1 applied to the inclined edge $\{B',C'\}$ with the left end B', we conclude that $c(B') = -c(B)$. Remembering the definition of charge, the

above statements can be reformulated as follows: B' has the same (resp. opposite) sign with all vertices of opposite (resp. same) level in $P \setminus \{C'\}$. Using the definition of directions of edges in Definition 3.1, we obtain:
1. Horizontal edges on opposite sides of P are directed opposite way since their left ends have opposite signs.
2. Suppose B' is the right end of a horizontal edge $\{A, B'\}$ in P. Then exactly one of the edges $\{A, B'\}$ and $\{B', C'\}$ is directed towards B' since their left ends A and B' have opposite signs.
3. The same argument shows that if C' is the right end of a horizontal edge $\{A, C'\}$ in P then exactly one of the edges $\{A, C'\}$ and $\{B', C'\}$ is directed towards C'.
4. Finally, if B is the left end of a horizontal edge $\{B, D\}$ in P then exactly one of the edges $\{B, C\}$ and $\{B, D\}$ is directed towards B.

These facts imply that P is a directed cycle, which completes the proof of Theorem 3.3. □

5.3. **Proof of Theorem 3.5.** Let us call a pair of indices $\{a, b\}$ *exceptional* (for $\mathbf{i}$ and $\mathbf{i}'$) if it violates (3.5). We need to show that exceptional pairs are precisely those in two exceptional cases in Theorem 3.5; to do this, we shall examine the relationship between the corresponding strips in $\Sigma(\mathbf{i})$ and $\Sigma(\mathbf{i}')$. Let us consider the following three cases:

Case 1 (trivial 2-move). Suppose $i_k = i'_{k-1} = i_0$, $i_{k-1} = i'_k = j_0$, and $i_l = i'_l$ for $l \notin \{k-1, k\}$, where $i_0, j_0 \in \tilde{\Pi}$ are such that $|i_0| \neq |j_0|$ and $\{i_0, j_0\} \notin \tilde{\Pi}$.

If both i and j are different from $|i_0|$ and $|j_0|$ then the strip $\Sigma_{i,j}(\mathbf{i})$ is identical to $\Sigma_{i,j}(\mathbf{i}')$, and so does not contain exceptional pairs.

If say $i = |i_0|$ but $j \neq |j_0|$ then the only vertex in $\Sigma_{i,j}(\mathbf{i})$ but not in $\Sigma_{i,j}(\mathbf{i}')$ is A_k (in the notation of Section 5.2), while the only vertex in $\Sigma_{i,j}(\mathbf{i}')$ but not in $\Sigma_{i,j}(\mathbf{i})$ is $A'_{k-1} = A'_{\sigma(k)}$. The vertex A_k has the same level and sign and so the same charge as the vertex $A'_{\sigma(k)}$ in $\Sigma_{i,j}(\mathbf{i}')$; by Lemma 5.1, there are no exceptional pairs in the strip $\Sigma_{i,j}(\mathbf{i})$.

Finally, suppose that $\{i, j\} = \{|i_0|, |j_0|\}$; in particular, in this case we have $\{|i_0|, |j_0|\} \in \Pi$, hence $\varepsilon(i_0) = -\varepsilon(j_0)$. Now the only vertices in $\Sigma_{i,j}(\mathbf{i})$ but not in $\Sigma_{i,j}(\mathbf{i}')$ are A_k and A_{k-1}, while the only vertices in $\Sigma_{i,j}(\mathbf{i}')$ but not in $\Sigma_{i,j}(\mathbf{i})$ are $A'_{k-1} = A'_{\sigma(k)}$ and $A'_k = A'_{\sigma(k-1)}$. Since A_k and A_{k-1} are of opposite level and opposite sign, they have the same charge, which is also equal to the charge of $A'_{\sigma(k-1)}$ and $A'_{\sigma(k)}$. Again using Lemma 5.1, we see that the strip in question also does not contain exceptional pairs.

Case 2 (non-trivial 2-move). Suppose $i_k = i'_{k-1} = i_0 \in \tilde{\Pi}$, $i_{k-1} = i'_k = -i_0$, and $i_l = i'_l$ for $l \notin \{k-1, k\}$. Interchanging if necessary $\mathbf{i}$ and $\mathbf{i}'$, we can and will assume that $i_0 \in \Pi$. Clearly, an exceptional pair can only belong to an (i, j)-strip with $i = i_0$. In our case, the location of all vertices in $\Sigma_{i,j}(\mathbf{i})$ and $\Sigma_{i,j}(\mathbf{i}')$ is the same; the only difference between the two strips is that the vertices A_{k-1} and A_k in $\Sigma_{i,j}(\mathbf{i})$ have opposite signs and hence opposite charges to their counterparts in $\Sigma_{i,j}(\mathbf{i}')$. It follows that exceptional pairs of vertices of the same level are precisely horizontal edges containing A_k, i.e., $\{A_{k-1}, A_k\}$ and $\{A_k, C\}$, where C is the right neighbor of A_k of the same level (note that C does not necessarily exist). Since $\varepsilon(i_k) = \varepsilon(i'_{k-1}) = +1$, and $\varepsilon(i_{k-1}) = \varepsilon(i'_k) = -1$, we have

$$(A_k \to A_{k-1}) \in \Sigma(\mathbf{i}),\ (A_k \to C) \in \Sigma(\mathbf{i}),$$

$$(A'_{k-1} \to A'_k) \in \Sigma(\mathbf{i}'),\ (C' \to A'_k) \in \Sigma(\mathbf{i}'),$$

so both pairs $\{A_{k-1}, A_k\}$ and $\{A_k, C\}$ fall into the first exceptional case in Theorem 3.5.

Let us now describe exceptional pairs corresponding to inclined edges. Let B be the vertex of the opposite level to A_k and closest to A_k ¿from the right (as the vertex C above, B does not necessarily exist). By Lemma 5.1, the left end of an exceptional inclined pair can only be A_{k-1}, A_k, or the leftmost of B and C; furthermore, the corresponding inclined edges can only be $\{A_{k-1}, B\}$, $\{A_k, B\}$, or $\{B, C\}$. We claim that all these three pairs are indeed exceptional, and each of them falls into one of the exceptional cases in Theorem 3.5.

Let us start with $\{B, C\}$. Since A_k is the preceding vertex to the leftmost member of $\{B, C\}$, and it has opposite charges in the two strips, Lemma 5.1 implies that $\{B, C\}$ is an edge in precisely one of the strips. By Theorem 3.3 (b), the triangle with vertices A_k, B, and C is a directed cycle in the corresponding strip. Thus the pair $\{B, C\}$ falls into the second exceptional case in Theorem 3.5.

The same argument shows that $\{A_{k-1}, B\}$ falls into the second exceptional case in Theorem 3.5 provided one of A_{k-1} and B is $\mathbf{i}$-bounded, i.e., A_{k-1} is not the leftmost vertex in the strip. As for $\{A_k, B\}$, it is an edge in both strips, and it has opposite directions in them because its left end A_k has opposite signs there. Thus $\{A_k, B\}$ falls into the first exceptional case in Theorem 3.5.

It remains to show that the exceptional pairs (horizontal and inclined) just discussed exhaust all possibilities for the two exceptional cases in Theorem 3.5. This is clear because by the above analysis, the only possible edges through A_k in $\Sigma(\mathbf{i})$ are $(A_k \to A_{k-1})$, $(A_k \to C)$, and $(B \to A_k)$ with B of the kind described above.

Case 3 (3-move). Suppose $i_k = i_{k-2} = i'_{k-1} = i_0$, $i_{k-1} = i'_k = i'_{k-2} = j_0$ for some $\{i_0, j_0\} \in \Pi$, and $i_l = i'_l$ for $l \notin \{k-2, k-1, k\}$ (the case when $\{i_0, j_0\} \in -\Pi$ is totally similar). As in the previous case, we need to describe all exceptional pairs.

First an exceptional pair can only belong to an (i, j)-strip with at least one of i and j equal to i_0 or j_0. Next let us compare the (i_0, j_0)-strips in $\Sigma(\mathbf{i})$ and $\Sigma(\mathbf{i}')$. The location of all vertices in these two strips is the same with the exception of A_{k-2}, A_{k-1}, and A_k in the former strip, and their counterparts $A'_{k-2} = A'_{\sigma(k-1)}, A'_{k-1} = A'_{\sigma(k-2)}$, and A'_k in the latter strip. Each of the six exceptional vertices has sign $+1$; so its level is equal to its charge. These charges (or levels) are given as follows:

$$c(A_{k-2}) = c(A'_{\sigma(k-2)}) = c(A_k) = -1,\ c(A_{k-1}) = c(A'_{\sigma(k-1)}) = c(A'_k) = 1\ .$$

Let B (resp. B') denote the vertex in both strips which is the closest ¿from the right to A_k on the same (resp. opposite) level; note that B or B' may not exist. Since the trapezoid T with vertices A_{k-2}, A_{k-1}, B', and B in $\Sigma_{i_0,j_0}(\mathbf{i})$ is in the same relative position to all outside vertices as the trapezoid T' with vertices $A'_{\sigma(k-2)}, A'_{\sigma(k-1)}, B'$, and B in $\Sigma_{i_0,j_0}(\mathbf{i}')$, it follows that every exceptional pair is contained in T.

An inspection using Lemma 5.1 shows that T contains the directed edges

$$A_{k-2} \to A_k \to A_{k-1} \to B' \to A_k \to B$$

and does not contain any of the edges $\{A_{k-2}, B\}$, $\{A_{k-2}, B'\}$, or $\{A_{k-1}, B\}$. Similarly (or by interchanging $\mathbf{i}$ and $\mathbf{i}'$), we conclude that T' contains the directed edges

$$A'_{\sigma(k-1)} \to A'_k \to A'_{\sigma(k-2)} \to B \to A'_k \to B'$$

and does not contain any of the edges $\{A'_{\sigma(k-1)}, B'\}$, $\{A'_{\sigma(k-1)}, B'\}$, or $\{A'_{\sigma(k-2)}, B'\}$. Furthermore, $\{B, B'\}$ is an edge in precisely one of the strips (since the preceding vertices A_k and A'_k have opposite charges); and precisely one of the pairs $\{A_{k-2}, A_{k-1}\}$ and $\{A'_{\sigma(k-1)}, A'_{\sigma(k-2)}\}$ is an edge in its strip provided A_{k-2} is not the leftmost vertex (since their left ends A_{k-2} and $A'_{\sigma(k-1)}$ have opposite charges).

Comparing this information for the two trapezoids, we see that the exceptional pairs in T are all pairs of vertices in T with the exception of two diagonals $\{A_{k-2}, B'\}$ and $\{A_{k-1}, B\}$ (and also of $\{A_{k-2}, A_{k-1}\}$ if A_{k-2} is the leftmost vertex in the strip). By inspection based on Theorem 3.3 (b), all these exceptional pairs fall into the two exceptional cases in Theorem 3.5.

A similar (but much simpler) analysis shows that any (i, j)-strip with precisely one of i and j belonging to $\{i_0, j_0\}$ does not contain extra exceptional pairs, and also has no inclined edges through A_k or A'_k. We conclude that all the exceptional pairs are contained in the above trapezoid T. The fact that these exceptional pairs exhaust all possibilities for the two exceptional cases in Theorem 3.5 is clear because by the above analysis, the only edges through A_k in $\Sigma(\mathbf{i})$ are those connecting A_k with the vertices of T. Theorem 3.5 is proved. □

6. Proofs of results in Section 3.3

We have already noticed that Theorem 3.7 follows from Theorem 3.8. Let us first prove Theorem 3.9 and then deduce Theorem 3.8 from it.

6.1. **Proof of Theorem 3.9.** We fix reduced words $\mathbf{i}$ and $\mathbf{i'}$ related by a 2- or 3-move, and abbreviate $\sigma = \sigma_{\mathbf{i'},\mathbf{i}} = \sigma_{\mathbf{i},\mathbf{i'}}$ and $\varphi^+ = \varphi^+_{\mathbf{i'},\mathbf{i}}$. Let us first prove parts (a) and (b). We shall only prove the first equality in (3.4); the proof of the second one and of (3.5) is completely similar. Let $v \in \mathbb{Z}^m$, $v^+ = \varphi^+(v)$, and $v' = \varphi^-_{\mathbf{i},\mathbf{i'}}(v^+)$; thus we need to show that $v = v'$, i.e., that $\xi_l(v) = \xi_l(v')$ for all $l \in [1, m]$. Note that the permutation σ is an involution. In view of (3.1), this implies the desired equality $\xi_l(v) = \xi_l(v')$ in all the cases except the following one: the move that relates $\mathbf{i}$ and $\mathbf{i'}$ is non-trivial in position k, and $l = k$. To deal with this case, we use the first exceptional case in Theorem 3.5 which we can write as

$$(k \to b) \in \Sigma(\mathbf{i'}) \Leftrightarrow (\sigma(b) \to k) \in \Sigma(\mathbf{i}) \ .$$

Combining this with the definitions (3.1) and (3.2), we obtain

$$\xi_k(v') = \sum_{(k \to b) \in \Sigma(\mathbf{i'})} \xi_b(v^+) - \xi_k(v^+)$$

$$= \sum_{(\sigma(b) \to k) \in \Sigma(\mathbf{i})} \xi_{\sigma(b)}(v) - \Big(\sum_{(a \to k) \in \Sigma(\mathbf{i})} \xi_a(v) - \xi_k(v) \Big) = \xi_k(v) \ ,$$

as required.

We deduce part (c) from the following lemma which says that the maps $(\varphi^{\pm}_{\mathbf{i'},\mathbf{i}})^*$ induced by $\varphi^{\pm}_{\mathbf{i'},\mathbf{i}}$ "almost" transform the form $\Omega_{\mathbf{i'}}$ into $\Omega_{\mathbf{i}}$.

Lemma 6.1. *If the move that relates* $\mathbf{i}$ *and* $\mathbf{i'}$ *is trivial then*

$$(\varphi^+_{\mathbf{i'},\mathbf{i}})^*(\Omega_{\mathbf{i'}}) = (\varphi^-_{\mathbf{i'},\mathbf{i}})^*(\Omega_{\mathbf{i'}}) = \Omega_{\mathbf{i}} \ .$$

If the move that relates $\mathbf{i}$ *and* $\mathbf{i}'$ *is non-trivial in position* k *then*

$$(\varphi^+_{\mathbf{i}',\mathbf{i}})^*(\Omega_{\mathbf{i}'}) = (\varphi^-_{\mathbf{i}',\mathbf{i}})^*(\Omega_{\mathbf{i}'}) = \Omega_{\mathbf{i}} - \sum_{\substack{(a\to k\to b)\in\Sigma(\mathbf{i})\\ a,b\notin B(\mathbf{i})}} \xi_a \wedge \xi_b \ . \tag{6.1}$$

Proof. We will only deal with $(\varphi^+)^*(\Omega_{\mathbf{i}'}) = (\varphi^+_{\mathbf{i}',\mathbf{i}})^*(\Omega_{\mathbf{i}'})$; the form $(\varphi^-_{\mathbf{i}',\mathbf{i}})^*(\Omega_{\mathbf{i}'})$ can be treated in the same way. By the definition,

$$(\varphi^+)^*(\Omega_{\mathbf{i}'}) = \sum_{(a'\to b')\in\Sigma(\mathbf{i}')} (\varphi^+)^*\xi_{a'} \wedge (\varphi^+)^*\xi_{b'} \ .$$

The forms $(\varphi^+)^*\xi_{a'}$ are given by (3.1) and (3.2). In particular, if $\mathbf{i}$ and $\mathbf{i}'$ are related by a trivial move then $(\varphi^+)^*\xi_{a'} = \xi_{\sigma(a')}$ for any $a' \in [1,m]$; by Theorem 3.5, in this case we have

$$(\varphi^+)^*(\Omega_{\mathbf{i}'}) = \sum_{(a\to b)\in\Sigma(\mathbf{i})} \xi_a \wedge \xi_b$$

as claimed.

Now suppose that $\mathbf{i}$ and $\mathbf{i}'$ are related by a non-trivial move in position k. Then we have

$$\begin{aligned}(\varphi^+)^*(\Omega_{\mathbf{i}'}) \quad &= \sum_{\substack{(\sigma(a)\to\sigma(b))\in\Sigma(\mathbf{i}')\\ a,b\neq k}} \xi_a \wedge \xi_b \\ &+ \sum_{(k\to\sigma(a'))\in\Sigma(\mathbf{i}')} \Big(\sum_{(a\to k)\in\Sigma(\mathbf{i})} \xi_a - \xi_k\Big) \wedge \xi_{a'} \\ &+ \sum_{(\sigma(b)\to k)\in\Sigma(\mathbf{i}')} \xi_b \wedge \Big(\sum_{(a\to k)\in\Sigma(\mathbf{i})} \xi_a - \xi_k\Big) \ .\end{aligned}$$

Using the second exceptional case in Theorem 3.5, we can rewrite the first summand as

$$\sum_{\substack{(a\to b)\in\Sigma(\mathbf{i})\\ a,b\neq k}} \xi_a \wedge \xi_b + \sum_{\substack{(a\to k\to b)\in\Sigma(\mathbf{i})\\ \{a,b\}\cap B(\mathbf{i})\neq\emptyset}} \xi_a \wedge \xi_b \ .$$

Similarly, using the first exceptional case in Theorem 3.5, we can rewrite the last two summands as

$$\sum_{(a\to k)\in\Sigma(\mathbf{i})} \xi_a \wedge \xi_k + \sum_{(k\to b)\in\Sigma(\mathbf{i})} \xi_k \wedge \xi_b - \sum_{(a\to k\to b)\in\Sigma(\mathbf{i})} \xi_a \wedge \xi_b$$

(note that the missing term

$$\sum_{\substack{(a\to k)\in\Sigma(\mathbf{i})\\ (a'\to k)\in\Sigma(\mathbf{i})}} \xi_a \wedge \xi_{a'}$$

is equal to 0). Adding up the last two sums, we obtain (6.1). □

Now everything is ready for the proof of Theorem 3.9 (c). Since l is assumed to be $\mathbf{i}$-bounded, Lemma 6.1 implies that $\Omega_{\mathbf{i}}(v, e_l) = \Omega_{\mathbf{i}'}(\varphi^+(v), \varphi^+(e_l))$ for any $v \in \mathbb{Z}^m$. On the other hand, since the case when the move that relates $\mathbf{i}$ and $\mathbf{i}'$ is non-trivial in position k, and $(l \to k) \in \Sigma_{\mathbf{i}}$, is excluded, we have $\varphi^+(e_l) = \pm e_{\sigma(l)}$ (with the minus sign for $l = k$ only). Therefore, our assumptions on l imply that

$$\Omega_{\mathbf{i}}(v, e_l)\varphi^+(e_l) = \Omega_{\mathbf{i}'}(\varphi^+(v), e_{\sigma(l)})e_{\sigma(l)} \ .$$

Remembering the definition (2.3) of symplectic transvections, we conclude that

$$(\tau_{\sigma(l),\mathbf{i}'} \circ \varphi^+)(v) = \varphi^+(v) - \Omega_{\mathbf{i}'}(\varphi^+(v), e_{\sigma(l)})e_{\sigma(l)}$$
$$= \varphi^+(v) - \Omega_{\mathbf{i}}(v, e_l)\varphi^+(e_l) = (\varphi^+ \circ \tau_{l,\mathbf{i}})(v) \ ,$$

as required. This completes the proof of Theorem 3.9. □

Remark 6.2. It is possible to modify all skew symmetric forms $\Omega_{\mathbf{i}}$ without changing the corresponding groups $\Gamma_{\mathbf{i}}$ in such a way that the modified forms will be preserved by the maps $(\varphi^{\pm}_{\mathbf{i}',\mathbf{i}})^*$. There are several ways to do it. Here is one "canonical" solution: replace each $\Omega_{\mathbf{i}}$ by the form

$$\tilde{\Omega}_{\mathbf{i}} = \Omega_{\mathbf{i}} - \frac{1}{2}\sum \varepsilon(i_k)\xi_k \wedge \xi_l \ ,$$

where the sum is over all pairs of $\mathbf{i}$-unbounded indices $k < l$ such that $\{|i_k|, |i_l|\} \in \Pi$. It follows easily from Lemma 6.1 that $(\varphi^{\pm}_{\mathbf{i}',\mathbf{i}})^*(\tilde{\Omega}_{\mathbf{i}'}) = \tilde{\Omega}_{\mathbf{i}}$. Unfortunately, the forms $\tilde{\Omega}_{\mathbf{i}}$ are not defined over $\mathbb{Z}$; in particular, they cannot be reduced to bilinear forms over $\mathbb{F}_2$.

6.2. **Proof of Theorem 3.8.** The fact that $\varphi^+_{\mathbf{i}',\mathbf{i}}$ and $\varphi^-_{\mathbf{i}',\mathbf{i}}$ are invertible follows from (3.4). To prove (3.3), it remains to show that $\varphi^+_{\mathbf{i}',\mathbf{i}} \circ \tau_{l,\mathbf{i}} \circ (\varphi^+_{\mathbf{i}',\mathbf{i}})^{-1} \in \Gamma_{\mathbf{i}'}$ for any $\mathbf{i}$-bounded index $l \in [1, m]$. This follows from (3.6) unless the move that relates $\mathbf{i}$ and $\mathbf{i}'$ is non-trivial in position k, and $(l \to k) \in \Sigma_{\mathbf{i}}$. In this exceptional case, we conclude by interchanging $\mathbf{i}$ and $\mathbf{i}'$ in (3.6) that

$$\varphi^+_{\mathbf{i},\mathbf{i}'} \circ \tau_{\sigma_{\mathbf{i}',\mathbf{i}}(l),\mathbf{i}'} = \tau_{l,\mathbf{i}} \circ \varphi^+_{\mathbf{i},\mathbf{i}'} \ .$$

Using (3.5), we obtain that

$$\varphi^+_{\mathbf{i}',\mathbf{i}}\circ\tau_{l,\mathbf{i}}\circ(\varphi^+_{\mathbf{i}',\mathbf{i}})^{-1} = (\varphi^+_{\mathbf{i}',\mathbf{i}}\circ\varphi^+_{\mathbf{i},\mathbf{i}'})\circ\tau_{\sigma_{\mathbf{i}',\mathbf{i}}(l),\mathbf{i}'}\circ(\varphi^+_{\mathbf{i}',\mathbf{i}}\circ\varphi^+_{\mathbf{i},\mathbf{i}'})^{-1} = \tau_{k,\mathbf{i}'}\circ\tau_{\sigma_{\mathbf{i}',\mathbf{i}}(l),\mathbf{i}'}\circ\tau^{-1}_{k,\mathbf{i}'} \in \Gamma_{\mathbf{i}'},$$

as required. This completes the proofs of Theorems 3.8 and 3.7. □

7. Proof of Theorem 3.11

7.1. **Description of Γ-orbits.** In this section we shall only work over the field $\mathbb{F}_2$. Therefore we find it convenient to change our notation a little bit. Let V be a finite-dimensional vector space over $\mathbb{F}_2$ with a skew-symmetric $\mathbb{F}_2$-valued form Ω (i.e., $\Omega(v, v) = 0$ for any $v \in V$). For any $v \in V$, let $\tau_v : V \to V$ denote the corresponding symplectic transvection acting by $\tau_v(x) = x - \Omega(x, v)v$. Fix a linearly independent subset $B \subset V$, and let Γ be the subgroup of $GL(V)$ generated by the transvections τ_b for $b \in B$. We make B the set of vertices of a graph with $\{b, b'\}$ an edge whenever $\Omega(b, b') = 1$.

We shall deduce Theorem 3.11 from the following description of the Γ-orbits in V in the case when the graph B is E_6-compatible (see Definition 3.10).

Let $U \subset V$ be the linear span of B. The group Γ preserves each parallel translate $(v + U) \in V/U$ of U in V, so we only need to describe Γ-orbits in each $v + U$.

Let us first describe one-element orbits, i.e., Γ-fixed points in each "slice" $v + U$. Let $V^\Gamma \subset V$ denote the subspace of Γ-invariant vectors, and $K \subset U$ denote the kernel of the restriction $\Omega|_U$.

Proposition 7.1. *If $\Omega(K, v + U) = 0$ then $(v + U) \cap V^\Gamma$ is a parallel translate of K; otherwise, this intersection is empty.*

Proof. Suppose the intersection $(v+U)\cap V^\Gamma$ is non-empty; without loss of generality, we can assume that v is Γ-invariant. By the definition, $v \in V^\Gamma$ if and only if $\Omega(u,v) = 0$ for all $u \in U$. In particular, $\Omega(K,v) = 0$, hence $\Omega(K, v+U) = 0$. Furthermore, an element $v+u$ of $v+U$ is Γ-invariant if and only if $u \in K$, and we are done. □

Following [5], we choose a function $Q : V \to \mathbb{F}_2$ satisfying the following properties:

$$Q(u+v) = Q(u) + Q(v) + \Omega(u,v)\ (u,v \in V), \quad Q(b) = 1\ (b \in B)\ . \tag{7.1}$$

(Clearly, these properties uniquely determine the restriction of Q to U.) An easy check shows that $Q(\tau_v(x)) = Q(x)$ whenever $Q(v) = 1$; in particular, the function Q is Γ-invariant.

Now everything is ready for a description of Γ-orbits in V.

Theorem 7.2. *If the graph B is E_6-compatible then Γ has precisely two orbits in each set $(v+U)\setminus V^\Gamma$. These two orbits are intersections of $(v+U)\setminus V^\Gamma$ with the level sets $Q^{-1}(0)$ and $Q^{-1}(1)$ of Q.*

The proof will be given in the next section. Let us show that this theorem implies Theorem 3.11 and Corollary 3.12.

Corollary 7.3. *If the graph B is E_6-compatible then the number of Γ-orbits in V is equal to $2^{\dim(V/U)}\cdot(2+2^{\dim(U\cap \mathrm{Ker}\ \Omega)})$; in particular, if $U\cap \mathrm{Ker}\ \Omega = \{0\}$ then this number is $3\cdot 2^{\dim(V/U)}$.*

Proof. By Proposition 7.1 and Theorem 7.2, each slice $v+U$ with $\Omega(K, v+U) = 0$ splits into $2^{\dim K}+2$ Γ-orbits, while each of the remaining slices splits into 2 orbits. There are $2^{\dim(V^\Gamma/K)}$ slices of the first kind and $2^{\dim(V/U)} - 2^{\dim(V^\Gamma/K)}$ slices of the second kind. Thus the number of Γ-orbits in V is equal to

$$2^{\dim(V^\Gamma/K)}\cdot(2^{\dim K}+2) + (2^{\dim(V/U)} - 2^{\dim(V^\Gamma/K)})\cdot 2\ .$$

Our statement follows by simplifying this answer. □

Now Theorem 3.11 is just a reformulation of this Corollary. As for Corollary 3.12, one only needs to show that its assumptions imply that $U\cap \mathrm{Ker}\ \Omega = \{0\}$. But this follows at once from Proposition 3.2.

7.2. **Proof of Theorem 7.2.** We split the proof into several lemmas. Let $E \subset U$ be the linear span of 6 vectors from B that form an induced subgraph isomorphic to E_6. The restriction of Ω to E is nondegenerate; in particular, $E\cap K = \{0\}$.

Lemma 7.4. *(a) Every 4-dimensional vector subspace of E contains at least two non-zero vectors with $Q = 0$.*
(b) Every 5-dimensional vector subspace of E contains at least two vectors with $Q = 1$.

Proof. (a) It suffices to show that every 3-dimensional subspace of E contains a non-zero vector with $Q = 0$. Let e_1, e_2, and e_3 be three linearly independent vectors. If we assume that $Q = 1$ on each of the 6 vectors $e_1, e_2, e_3, e_1+e_2, e_1+e_3$, and e_2+e_3 then, in view of (7.1), we must have $\Omega(e_1,e_2) = \Omega(e_1,e_3) = \Omega(e_2,e_3) = 1$. But then $Q(e_1+e_2+e_3) = 0$, as required.

(b) It follows from the results in [5] (or by direct counting) that E consists of 28 vectors with $Q = 0$ and 36 vectors with $Q = 1$. Since the cardinality of every 5-dimensional subspace of E is 32, our claim follows. □

Lemma 7.5. *The function Q is nonconstant on each set $(v + U) \setminus V^\Gamma$.*

Proof. Suppose $v \in V \setminus V^\Gamma$. By Lemma 7.4 (b), there exist two vectors $e \neq e'$ in E such that

$$\Omega(v, e) = \Omega(v, e') = 0,\ Q(e) = Q(e') = 1\ .$$

In view of (7.1), we have $Q(v+e) = Q(v+e') = Q(v)+1$, and it is clear that at least one of the vectors $v+e$ and $v+e'$ is not Γ-invariant (otherwise we would have $\Omega(e-e', u) = 0$ for all $u \in U$, which contradicts the fact that $\Omega|_E$ is nondegenerate). □

To prove Theorem 7.2, it remains to show that Γ acts transitively on each level set of Q in $(v + U) \setminus V^\Gamma$. To do this, we shall need the following important result due to Janssen [5, Theorem 3.5].

Lemma 7.6. *If u is a vector in $U \setminus K$ such that $Q(u) = 1$ then the symplectic transvection τ_u belongs to Γ.*

We also need the following result from [11, Lemma 4.3].

Lemma 7.7. *If the graph B is E_6-compatible then Γ acts transitively on each of the level sets of Q in $U \setminus K$.*

To continue the proof, let us introduce some terminology. For a linear form $\xi \in U^*$, denote

$$T_\xi = \{u \in U \setminus K : Q(u) = \xi(u) = 1\}\ .$$

We shall call a family of vectors $(u_1, u_2, \dots, u_s)$ *weakly orthogonal* if $\Omega(u_1 + \cdots + u_{i-1}, u_i) = 0$ for $i = 2, \cdots, s$.

Lemma 7.8. *Let $\xi \in U^*$ be a linear form on U such that $\xi|_K \neq 0$. Then every nonzero vector $u \in U$ such that $Q(u) = \xi(u)$ can be expressed as the sum $u = u_1 + \cdots + u_s$ of some weakly orthogonal family of vectors $(u_1, u_2, \dots, u_s)$ from T_ξ.*

Proof. We need to construct a required weakly orthogonal family $(u_1, u_2, \dots, u_s)$ in each of the following three cases.

Case 1. Let $0 \neq u = k \in K$ be such that $Q(k) = \xi(k) = 0$. Since $\xi \neq 0$, we have $\xi(b) = 1$ for some $b \in B$. By (7.1), we also have $Q(b) = 1$. Since $b \notin K$, we can take $(u_1, u_2) = (b, k - b)$ as a desired weakly orthogonal family.

Case 2. Let $u = k \in K$ be such that $Q(k) = \xi(k) = 1$. By Lemma 7.4 (a), there exist distinct nonzero vectors e and e' in E such that $Q(e) = \xi(e) = Q(e') = \xi(e') = \Omega(e, e') = 0$. Then we can take $(u_1, u_2, u_3) = (k - e, k - e', e + e' - k)$ as a desired weakly orthogonal family.

Case 3. Let $u \in U \setminus K$ be such that $Q(u) = \xi(u) = 0$. Since $\xi|_K \neq 0$, we can choose $k \in K$ so that $\xi(k) = 1$. If $Q(k) = 1$ then a desired weakly orthogonal family for u can be chosen as (u_1, u_2, u_3, u_4), where (u_1, u_2, u_3) is a weakly orthogonal family for k constructed in Case 2 above, and $u_4 = u - k$. If $Q(k) = 0$, choose $e \in E$ such that $Q(e) = 1, \Omega(u, e) = 0$, and $u - e \notin K$ (the existence of such a vector e follows from Lemma 7.4 (b)). If $\xi(e) = 1$ then a desired weakly orthogonal family for u can be chosen as $(u_1, u_2) = (e, u - e)$. Finally, if $\xi(e) = 0$ then a desired weakly orthogonal family for u can be chosen as $(u_1, u_2) = (e + k, u - e - k)$. □

Now everything is ready for completing the proof of Theorem 7.2. Take any slice $v + U \in V/U$; we need to show that Γ acts transitively on each of the level sets of Q in $(v + U) \setminus V^\Gamma$. First suppose that $(v + U) \cap V^\Gamma \neq \emptyset$; by Proposition 7.1, this means that $\Omega(K, v + U) = 0$. Without loss of generality, we can assume that v is Γ-invariant. Then $\Omega(u, v) = 0$ for any $u \in U$, so we have $Q(v + u) = Q(v) + Q(u)$. On the other hand, we have $g(v + u) = v + g(u)$ for any $g \in \Gamma$ and $u \in U$. Thus the correspondence $u \mapsto v + u$ is a Γ-equivariant bijection between U and $v + U$ preserving partitions into the level sets of Q. Therefore our statement follows from Lemma 7.7.

It remains to treat the case when $\Omega(K, v + U) \neq 0$. In other words, if we choose any representative v and define the linear form $\xi \in U^*$ by $\xi(u) = \Omega(u, v)$ then $\xi|_K \neq 0$. Let $u \in U$ be such that $Q(v) = Q(v + u)$; we need to show that $v + u$ belongs to the Γ-orbit $\Gamma(v)$. In view of (7.1), we have $Q(u) = \xi(u)$. In view of Lemma 7.8, it suffices to show that $\Gamma(v)$ contains $v + u_1 + \cdots + u_s$ for any weakly orthogonal family of vectors $(u_1, u_2, \ldots, u_s)$ from T_ξ. We proceed by induction on s. The statement is true for $s = 1$ because $v + u_1 = \tau_{u_1}(v)$, and $\tau_{u_1} \in \Gamma$ by Lemma 7.6. Now let $s \geq 2$, and assume that $v' = v + u_1 + \cdots + u_{s-1} \in \Gamma(v)$. The definition of a weakly orthogonal family implies that

$$v + u_1 + \cdots + u_s = v' + u_s = \tau_{u_s}(v') \in \Gamma(v) \ ,$$

and we are done. This completes the proof of Theorem 7.2. $\square$

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