The homotopy type of spaces of locally convex curves in spheres

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1 Introduction

For a fixed positive integer $n \geq 2$, consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. A smooth curve $\gamma : [0, 1] \to S^n$ is locally convex if, for all $t \in [0, 1]$, we have

$$\det(\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t)) > 0.$$ 

In particular, the vectors $\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t)$ are L.I.; factor

$$(\gamma(t) \quad \gamma'(t) \quad \cdots \quad \gamma^{(n)}(t) ) = \mathfrak{F}_\gamma(t) R(t),$$

$R(t)$ being an upper triangular matrix with positive diagonal, to define $\mathfrak{F}_\gamma : [0, 1] \to SO_{n+1}$. Given $Q \in SO_{n+1}$, let $\mathcal{L}_n(Q)$ be the space of locally convex curves $\gamma$ with $\mathfrak{F}_\gamma(0) = I$, $\mathfrak{F}_\gamma(1) = Q$. We want to understand the spaces $\mathcal{L}_n(Q)$, particularly their homotopy types.

Recall that $\pi_1(SO_{n+1}) = \mathbb{Z}/(2)$ (for $n \geq 2$) with double cover $\Pi : Spin_{n+1} \to SO_{n+1}$. Let 1 and $-1$ be the elements of $\Pi^{-1}[\{I\}]$ so that $-1$ is an element of order 2 in the center of $Spin_{n+1}$. Given $\gamma$, the path $\mathfrak{F}_\gamma : [0, 1] \to SO_{n+1}$ can be uniquely lifted to a continuous path $\tilde{\mathfrak{F}}_\gamma : [0, 1] \to Spin_{n+1}$ with $\tilde{\mathfrak{F}}_\gamma(0) = 1$. For $z \in Spin_{n+1}$, let $\mathcal{L}_n(z) \subset \mathcal{L}_n(\Pi(z))$ be the set of curves $\gamma$ for which $\tilde{\mathfrak{F}}_\gamma(1) = z$; thus, if $\Pi(z) = Q$ we have $\mathcal{L}_n(Q) = \mathcal{L}_n(z) \sqcup \mathcal{L}_n(-z)$. If there exists a convex curve in $\mathcal{L}_n(z)$ then this set has two connected components: a contractible one $\mathcal{L}_{n,\text{convex}}(z)$ consisting of convex curves and another one $\mathcal{L}_{n,\text{non-convex}}(z)$ consisting of non-convex curves (see [1], [2], [3], [6]).

Let $\Omega_z(\text{Spin}_{n+1})$ be the set of smooth curves $\Gamma : [0, 1] \to \text{Spin}_{n+1}$ with $\Gamma(0) = 1$ and $\Gamma(1) = z$; recall that the homotopy type of the spaces $\Omega_z(\text{Spin}_{n+1})$ does not depend on the choice of $z \in \text{Spin}_{n+1}$. The map $\mathfrak{F} : \mathcal{L}_n(z) \to \Omega_z(\text{Spin}_{n+1})$ is injective and homotopically surjective; in some but not all cases it is a homotopy equivalence (see [4], [5]).
Let \( \text{Diag}_{n+1}^+ \subset B_{n+1}^+ \subset \text{SO}_{n+1} \) be the subgroups of diagonal matrices and signed permutation matrices, respectively. Notice that \( \text{Diag}_{n+1}^+ \) is isomorphic to \( \{\pm 1\}^n \). Let \( \text{Quat}_{n+1} = \Pi^{-1}[\text{Diag}_{n+1}^+] \subset \text{Spin}_{n+1} \). It follows from [5] that if \( \mathcal{L}_n(z) \) is understood for all \( z \in \text{Quat}_{n+1} \) then it is also understood for all \( z \in \text{Spin}_{n+1} \) (see also Section ??). Recall that the center \( Z(\text{SO}_{n+1}) \) of \( \text{SO}_{n+1} \) equals \( \{I\} \) if \( n \) is even and \( \{\pm I\} \) if \( n \) is odd (here \( I \) is the identity matrix). We have \( Z(\text{Quat}_{n+1}) = \Pi^{-1}[Z(\text{SO}_{n+1})] \).

We now state some of our main results. They all follow from Theorem ?? below, which requires a longer explanation before it can be stated.

The main result in [4] given the homotopy types for \( n = 2 \):

\[
\mathcal{L}_2(1) \approx \Omega S^3 \lor S^2 \lor S^6 \lor S^{10} \lor \ldots, \quad \mathcal{L}_2(-1) \approx \Omega S^3 \lor S^9 \lor S^4 \lor S^8 \lor \ldots.
\]

The following theorem obtains a similar answer for \( n = 3 \).

Recall that \( \text{Spin}_4 \) is isomorphic to \( S^3 \times S^3 \) (where \( S^3 \) is the group of unit quaternions): an explicit identification is given by taking \((z_L, z_R) \in S^3 \times S^3\) to the linear transformation taking \( w \) to \( z_L w z_R^{-1} \) (again with quaternion multiplication). We then have

\[
\Pi^{-1}[\{J\}] = \{(+1, +1), (-1, -1)\}, \quad \Pi^{-1}[\{-I\}] = \{(+1, -1), (-1, +1)\}
\]

and therefore

\[
Z(\text{Quat}_4) = \{(+1, +1), (-1, -1), (+1, -1), (-1, +1)\}.
\]

It turns out that there exist convex curves in \( \mathcal{L}_3((+1, -1)) \); the other three spaces are connected (see [?] for more on these spaces).

**Theorem 1.** Consider the space \( \mathcal{L}_3 \) of locally convex curves in \( S^3 \) and \( z \in \text{Quat}_4 \subset \text{Spin}_4 = S^3 \times S^3 \).

If \( z \notin Z(\text{Quat}_4) \) then the inclusion \( \mathcal{F} : \mathcal{L}_3(z) \to \Omega_z(S^3 \times S^3) \) is a weak homotopy equivalence. For \( z \in Z(\text{Quat}_4) \), we have the following weak homotopy equivalences:

\[
\mathcal{L}_3((+1, +1)) \approx \Omega(S^3 \times S^3) \lor S^4 \lor S^6 \lor S^8 \lor S^{12} \lor S^{12} \lor \ldots, \\
\mathcal{L}_3((-1, -1)) \approx \Omega(S^3 \times S^3) \lor S^2 \lor S^6 \lor S^8 \lor S^{10} \lor S^{10} \lor \ldots, \\
\mathcal{L}_3((+1, -1)) \approx \Omega(S^3 \times S^3) \lor S^0 \lor S^4 \lor S^4 \lor S^8 \lor S^8 \lor \ldots, \\
\mathcal{L}_3((-1, +1)) \approx \Omega(S^3 \times S^3) \lor S^2 \lor S^6 \lor S^6 \lor S^{10} \lor S^{10} \lor \ldots.
\]

The above bouquets include one copy of \( S^k \), two copies of \( S^{(k+4)} \), \ldots, \( j+1 \) copies of \( S^{(k+4j)} \), \ldots, and so on.

We do not know a similar statement for \( \mathcal{L}_n(z) \), \( n > 3 \). The following theorem gives some partial results.
Theorem 2. Consider $n > 3$ and $z \in \text{Quat}_{n+1}$.

(i) If $z \notin Z(\text{Quat}_{n+1})$ then the inclusion $\tilde{\mathcal{F}} : \mathcal{L}_n(z) \to \Omega_z(\text{Spin}_{n+1})$ is a weak homotopy equivalence.

(ii) The space $\mathcal{L}_n(z)$ has one or two connected components. If there are two, exactly one of them is contractible: the set of convex curves.

(iii) Every connected component of $\mathcal{L}_n(z)$ is simply connected.

(iv) We have either $H_2(\mathcal{L}_n(z)) = \mathbb{Z}$ or $H_2(\mathcal{L}_n(z)) = \mathbb{Z}^2$. The last case happens if and only if $\mathcal{L}_n(-z)$ is disconnected.

Item (i) should be compared with the results in [5]: the results there are similar, but the new result is stronger. Item (ii) is well known and follows from the remarks above; it is stated for completeness and clarity. Items (iii) and (iv) appear to be new.

Our third main theorem is harder to state and is the main ingredient in the proof of the first three. In Section 2 we define a partition $\mathcal{L}_n(z) = \mathcal{M}_n(z) \sqcup \mathcal{Y}_n(z)$, where the subsets $\mathcal{M}_n(z) \subset \mathcal{L}_n(z)$ and $\mathcal{Y}_n(z) \subset \mathcal{L}_n(z)$ are closed and open, respectively. The subset $\mathcal{M}_n(z)$ is non-empty if and only if $z \in Z(\text{Quat}_{n+1})$. If there are convex curves in $\mathcal{L}_n(z)$ then $\mathcal{L}_{n,\text{convex}}(z) \subset \mathcal{M}_n(z)$ and the open set $\mathcal{Y}_n(z) \subset \mathcal{L}_{n,\text{non-convex}}(z)$ is dense; otherwise, $\mathcal{Y}_n(z) \subset \mathcal{L}_n(z)$ is dense.

Theorem 3. Consider $n \geq 2$ and $z \in \text{Quat}_{n+1}$. The inclusion $\tilde{\mathcal{F}} : \mathcal{Y}_n(z) \to \Omega \text{Spin}_{n+1}$ is a weak homotopy equivalence.

Notice that item (i) of Theorem 1 follows directly from Theorem 2.

We now provide a quick sketch of the definition of $\mathcal{M}_n(z)$. In Section 2 we recall the decomposition of $\text{SO}_{n+1}$ and $\text{Spin}_{n+1}$ into Bruhat or Schubert cells. In Section 4 we use this decomposition to define the itinerary of a curve, a word $w \in \mathcal{W}_n$ in the alphabet $\mathcal{A}_n = S_{n+1} \setminus \{e\}$. A permutation $\alpha \in S_{n+1}$ is parity alternating (or PA) if $i^\alpha \not\equiv (i + 1)^\alpha \pmod{2}$ for all $i$, $1 \leq i \leq n$. A curve $\gamma \in \mathcal{L}_n(z)$ belongs to $\mathcal{M}_n(z)$ if and only if every letter in its itinerary is parity alternating.

For $n = 2$ (the case discussed in [4]), $\mathcal{M}_2(z)$ is the set of multiconvex curves and the case $n = 2$ of Theorem 2 above is Proposition 1.4 in [4]. For $n > 2$ and $z \in Z(\text{Quat}_{n+1})$, the sets $\mathcal{M}_n(z)$ are more complicated. Understanding them and their inclusion in $\mathcal{L}_n(z)$ is what is required to determine the homotopy type of $\mathcal{L}_n(z)$, as in Theorem 2 (for $n = 3$).
For $\Gamma(t) = \mathfrak{F}_\gamma(t)$ as above we have $\Gamma'(t) = \Gamma(t)\Lambda(t)$ for $\Lambda : [0, 1] \to \mathfrak{J} \subset \mathfrak{s} \mathfrak{o}_{n+1}$. Here $\mathfrak{J}$ is the cone of skew-Jacobi matrices, i.e., tridiagonal skew matrices with positive subdiagonal entries. Conversely, given $\Lambda : [0, 1] \to \mathfrak{J}$, for a solution $\Gamma : [0, 1] \to \mathfrak{s} \mathfrak{o}_{n+1}$ of the differential equation $\Gamma'(t) = \Gamma(t)\Lambda(t)$, the curve $\gamma(t) = \Gamma(t)e_1$ is locally convex. Given $\Lambda : [0, 1] \to \mathfrak{J}$, let $\Gamma(t_0; \cdot) : [0, 1] \to \mathfrak{s} \mathfrak{o}_{n+1}$ be the solution of the ivp

$$\Gamma'(t_0; t) = \Gamma(t_0; t)\Lambda(t), \quad \Gamma(t_0; t_0) = I.$$  

(The derivative here is of course with respect to the $t$ coordinate; $t_0$ is regarded as a parameter.) Thus, for an arbitrary solution $\Gamma$ of the differential equation we have $\Gamma(t_0; t_1) = (\Gamma(t_0))^{-1}\Gamma(t_1)$; also, $\Gamma(t_0, t_2) = \Gamma(t_0; t_1)\Gamma(t_1; t_2)$.

2 Bruhat cells

Let $\text{Diag}^+_n \subset B^+_n \subset \mathfrak{s} \mathfrak{o}_{n+1}$ be the subgroups of diagonal matrices and signed permutation matrices, respectively. Notice that $\text{Diag}^+_n$ is isomorphic to $\{\pm 1\}^n$. Let $\text{Quat}_n = \Pi^{-1}[\text{Diag}^+_n] \subset \text{Spin}_{n+1}$ and $\tilde{B}^+_n = \Pi^{-1}[B^+_n]$. The group $\text{Quat}_n$ has $2^{(n+1)}$ elements and is a normal subgroup of $\tilde{B}^+_n$; the quotient is the symmetric group $S_{n+1}$.

Following [5] and [2], if $Q_0 \in B^+_n$ set

$$\text{Bru}_{Q_0} = \{Q \in \mathfrak{s} \mathfrak{o}_{n+1} \mid \exists U_0, U_1 \in \text{Up}^+, Q = U_0Q_0U_1\}$$

where $\text{Up}^+$ is the group of upper triangular matrices with positive diagonal; signed Bruhat cells are contractible and disjoint. Unsigned Bruhat cells are defined similarly, but using the group $\text{Up}$ of all invertible upper triangular matrices. Each unsigned Bruhat is a disjoint union of $2^n$ signed Bruhat cells; unsigned Bruhat cells are naturally labeled by permutations. The preimage of each cell by $\Pi$ is a disjoint union of two contractible components: we call these connected components the lifted Bruhat cells in $\text{Spin}_{n+1}$: for $z \in \tilde{B}^+_n$, let $\text{Bru}_z$ be the connected component of $\Pi^{-1}[\text{Bru}_{\Pi(z)}]$ containing $z$.

We recall the definition of chopping and present the dual notion of advancing. For $z \in \text{Spin}_{n+1}$, let $\gamma : (-\epsilon, \epsilon) \to \mathfrak{S}^n$ be a locally convex curve such that $\mathfrak{F}_\gamma(0) = \Pi(z)$; lift this to define $\tilde{\mathfrak{F}}_\gamma : (-\epsilon, \epsilon) \to \text{Spin}_{n+1}$ with $\tilde{\mathfrak{F}}_\gamma(0) = z$. There exists $\epsilon_1 > 0$ and a unique element $\text{chop}(z) \in \tilde{B}^+_n$ such that, for all $t \in (-\epsilon_1, 0)$, $\tilde{\mathfrak{F}}_\gamma(t)$ and $\text{chop}(z)$ are in the same cell. Similarly, there exists $\epsilon_2 > 0$ and a unique element $\text{adv}(z) \in \tilde{B}^+_n$ such that, for all $t \in (0, \epsilon_1)$, $\tilde{\mathfrak{F}}_\gamma(t)$ and $\text{adv}(z)$ are in the
same cell. Define \( a = \text{chop}(1) \): we have \( \text{adv}(1) = a^{-1} \);

\[
A = \Pi(a) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
-1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 1
\end{pmatrix}
\]

is the *Arnold matrix* ([5]). Thus, if \( \gamma : [0,1] \to \mathbb{S}^n \) is locally convex, \( \mathcal{F}_\gamma(t) \) belongs to an open cell for all but finitely many values of \( t \).

Define

\[
\mathcal{L}_n = \bigsqcup_{z \in \text{Quat}_{n+1}} \mathcal{L}_n(z).
\]

In [5] it is proved that given \( z \in \text{Spin}_{n+1} \) there exists a unique \( z_0 \in \text{Quat}_{n+1} \) such that \( \text{chop}(z) = \text{chop}(z_0) \) and that \( \mathcal{L}_n(z) \) and \( \mathcal{L}_n(z_0) \) are homeomorphic. Thus, understanding \( \mathcal{L}_n \) is sufficient to understand all spaces \( \mathcal{L}_n(z) \) and \( \mathcal{L}_n(Q) \). Moreover, the particularly interesting cases \( \mathcal{L}_n(I) \) and, for \( n \) odd, \( \mathcal{L}_n(-I) \), are explicitly contained in \( \mathcal{L}_n \). Notice also that there is a unique \( z \in \text{Quat}_{n+1} \) for which \( \mathcal{L}_n(z) \) contains convex curves: the space \( \mathcal{L}_n \) has precisely \( 1 + 2^{(n+1)} \) connected components.

### 3 Itineraries

Consider the symmetric group \( S_{n+1} \) (acting on \( \{1, 2, \ldots, n+1\} \)) as the Coxeter-Weyl group \( A_n \), i.e., use the \( n \) generators \( a = a_1 = (12), b = a_2 = (23), \ldots, a_n = (n, n+1) \). For \( \pi \in A_n \) and \( k \in \{1, 2, \ldots, n+1\} \) we use the notation \( k^\pi \) (rather than \( \pi(k) \)) so that \( (k^\pi_1)^{\pi_2} = k^{(\pi_1 \pi_2)} \). A permutation can be denoted either as a list of values \( [1^\pi 2^\pi \cdots n^\pi (n+1)^\pi] \) or as a product of the generators above; for instance, \( [ab] = [312] \in S_3 \). For \( \pi \in S_{n+1} \), let \( P_{\pi} \) be the permutation matrix defined by \( e_k^\top P_{\pi} = e_k^\top \pi; \) for instance, for \( n = 2 \) we have:

\[
P_a = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad P_b = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

\[
P_{[ab]} = P_a P_b = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad P_{[ba]} = P_b P_a = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

For \( \pi \in S_{n+1} \), let \( \text{Bru}_\pi \) be the unsigned Bruhat cell

\[
\text{Bru}_\pi = \{Q \in \text{SO}_{n+1} \mid \exists U_0, U_1 \in \text{Up}, Q = U_0 P_{\pi} U_1 \}.
\]
Let \( \text{inv}(\pi) \) be the length of \( \pi \) with the above generators, i.e., the number of inversions of \( \pi \) (we reserve the symbol \( \ell \) for lengths of words, to be introduced soon). Recall that there exists a unique \( \eta \in S_{n+1} \) with \( \text{inv}(\eta) = n(n+1)/2 \), the Coxeter element (this is sometimes called \( w_0 \), but we reserve the letter \( w \) for words in \( \mathcal{W}_n \), to be introduced below); write

\[
H = P_\eta = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}.
\]

Consider the alphabet \( \mathcal{A}_n = A_n \setminus \{e\} \); we use square brackets to indicate that a permutation, even if written as a product of generators, is an individual letter; for instance:

\[
\mathcal{A}_2 = \{a = [213], b = [132], [ab] = [312], [ba] = [231], [aba] = [321]\};
\]

\[
\mathcal{A}_3 = \{a = [2134], b = [1324], c = [1243], [ab], [ba], [bc], [cb], [ac], \ldots\}.
\]

For \( \alpha \in \mathcal{A}_n \), define \( \dim(\alpha) = \text{inv}(\alpha) - 1 \); for instance, \( \dim(a) = 0 \) and \( \dim([ab]) = 1 \). Let \( \mathcal{W}_n \) be the set of words in the alphabet \( \mathcal{A}_n \) (including the empty word); set

\[
\dim(\alpha_1 \ldots \alpha_k) = \sum_j \dim(\alpha_j).
\]

We shall construct a (CW? polyhedral?) complex \( D_n \) with one cell \( c_w \) for each word \( w \in \mathcal{W}_n \); we shall have \( \dim(c_w) = \dim(w) \). The space \( D_n \) admits an abstract construction but can also be realized as a subset \( D_n \subset L_n \): the inclusion is a weak homotopy equivalence.

We now define the itinerary of \( \gamma \), a word \( w_\gamma \in \mathcal{W}_n \) associated to a curve \( \gamma \in L_n \). For each \( t \), write

\[
\mathcal{F}_\gamma(t) = U_1 HP_{\pi(\gamma;t)}U_2
\]

where \( U_1 \) and \( U_2 \) are upper triangular and \( P_{\pi(\gamma;t)} \) is a permutation matrix (unsigned Bruhat decomposition). There exist

\[
0 = t_0 < t_1 < \cdots < t_M < t_{M+1} = 1
\]

such that:

- if \( t_i < t < t_{i+1} \) then \( \pi(\gamma; t) = e \);
- \( \pi(\gamma; t_i) \neq e \) for \( 1 \leq i \leq M \).

Define \( \gamma = \alpha_1 \alpha_2 \ldots \alpha_M \in \mathcal{W}_n \) with \( \alpha_i = \pi(\gamma; t_i) \in \mathcal{A}_n \). For each \( w \in \mathcal{W}_n \), the set \( \mathcal{P}_n(w) \) of curves \( \gamma \in L_n \) with itinerary \( w \) (i.e., for which \( w_\gamma = w \)) is a nonempty contractible submanifold \( \mathcal{P}_n(w) \subset L_n \) of codimension \( \dim(w) \) ([2]).
Example 3.1. Consider again the case \( n = 2 \). A letter \( a \) corresponds to the curve \( \gamma \) transversally crossing the equator (i.e., the great circle \( z = 0 \)) at a point different from \( \pm e_1 \). A \( b \) occurs when the tangent geodesic (great circle) to \( \gamma \) at \( t \) includes the points \( \pm e_1 \) but the \( z \)-coordinate of \( \gamma(t) \) is non-zero. A \( [ab] \) indicates that the curve is tangent to the equator, but not at \( \pm e_1 \). A \( [ba] \) declares that the curve crosses the equator transversally and not at \( \pm e_1 \). Finally, \( [aba] \) proclaims that the curve is tangent to the equator at \( \pm e_1 \). Figure 1 shows a 2 parameter family of (portions of) curves in \( L_2 \) illustrating all these cases.

Figure 1: A family of curves in \( L_2 \). The equator is dashed and the fat dot indicates \( e_1 \). The vector \( e_2 \) is at the right.

4 Itineraries and components of \( L_n \) and \( D_n \)

Given \( \pi \in S_{n+1} \), take \( z \in \tilde{B}_{n+1}^+ \) such that \( \Pi(z) \) and \( P_\pi \) have non-zero entries in the same positions; in other words, \( P_\pi = E\Pi(z) \) for some \( E \in \text{Diag}_{n+1} \). Consider \( \hat{\pi} = (\text{chop}(z))^{-1}\text{adv}(z) \in \text{Quat}_{n+1} \); this is independent of the choice of \( z \).

We construct \( \hat{a}_k \), \( 1 \leq k \leq n \), which are generators for Quat\(_{n+1} \). Consider the path \( \alpha_k : [0, 1] \to \text{SO}_{n+1} \):

\[
\alpha_k(t)e_k = \cos(\pi t)e_k + \sin(\pi t)e_{k+1}; \quad \alpha_k(t)e_{k+1} = -\sin(\pi t)e_k + \cos(\pi t)e_{k+1};
\]

and \( \alpha_k(t)e_j = e_j \) for \( j < k \) or \( j > k + 1 \). Lift \( \alpha_k \) to obtain \( \hat{\alpha}_k : [0, 1] \to \text{Spin}_{n+1} \) with \( \hat{\alpha}_k(0) = 1 \) and \( \hat{\alpha}_k(1) = \hat{a}_k \). We have \( \hat{a}_{k+1}\hat{a}_k = -\hat{a}_k\hat{a}_{k+1} \) and \( \hat{a}_l\hat{a}_k = \hat{a}_k\hat{a}_l \) if \( |k - l| > 1 \).

Consider \( \gamma \in L_n \) with itinerary \( w_\gamma = \alpha_1\alpha_2 \cdots \alpha_M \). Then \( \gamma \in L_n(z) \) for

\[
z = a^{-1}\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_M a^{-1}.
\]

Notice that \( a \in \tilde{B}_{n+1}^+ \setminus \text{Quat}_{n+1} \) but \( a^{\pm 2} \in \text{Quat}_{n+1} \); also, if \( \alpha \in \text{Quat}_{n+1} \) then \( a^{\pm 1}\alpha a^{\pm 1} \in \text{Quat}_{n+1} \).

For a word \( w = \alpha_1\alpha_2 \cdots \alpha_M \in \mathcal{W}_n \), let \( \hat{w} = \hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_M \in \text{Quat}_{n+1} \). We therefore have \( \mathcal{P}_n(w) \subset L_n(a^{-1}\hat{w}a^{-1}) \). For \( z \in \text{Quat}_{n+1} \), we define \( D_n(z) \) as the union of the cells \( c_w \) for which \( \hat{w} = z \).
5 The poset $\mathcal{W}_n$

Recall that the Bruhat order in the symmetric group $S_{n+1}$ can be defined as follows: $\pi_0 \leq \pi_1$ if some reduced word for $\pi_0$ (in the generators $a_k$) is a substring of some reduced word for $\pi_1$ (a substring here need not have consecutive letters). Equivalently, $\pi_0 \leq \pi_1$ if and only if $\text{Bru}_\pi(\pi_0) \subseteq \text{Bru}_\pi(\pi_1)$. We define a similar order for words in $W_n$, but first we establish the equivalence of a few conditions.

**Lemma 5.1.** Let $w_0, w_1 \in W_n$. The following conditions are equivalent:

1. $P_n(w_0) \supseteq P_n(w_1)$;
2. $P_n(w_0) \supseteq P_n(w_1)$;
3. $P_n(w_0) \cap P_n(w_1) \neq \emptyset$;
4. there exists a sequence $(\gamma_k)$ of curves in $P_n(w_0)$ converging to $\gamma \in P_n(w_1)$;
5. given $\gamma_1 \in P_n(w_1)$ and an open neighborhood $U \subset L_n$ of $\gamma_1$ there exists $\gamma_0 \in U \cap P_n(w_0)$.

**Proof.** Easy. [TO BE WRITTEN] \hfill \Box

Write $w_0 \leq w_1$ if the conditions in Lemma 6.1 hold. We thus have

$$P_n(w_0) = \bigcup_{w_0 \leq w_1} P_n(w_1).$$

In order to understand how the strata $P_n(w)$ fit together we need to discuss this relation, which will be shown to be a partial order.

Consider a matrix $Q \in SO_{n+1}$ and $k \leq n$. Let $Q_k \in \mathbb{R}^{k \times k}$ be the southwest $k \times k$ minor, i.e., the submatrix so that $(Q_k)_{i,j} = Q_{i-k+n+1,j}$. For a locally convex curve $\gamma \in L_n$, let $m_k = m_{\gamma;k} : [0, 1] \to \mathbb{R}, 1 \leq k \leq n$, be the smooth function defined by the determinant of the southwest $k \times k$ minor of $F_\gamma$:

$$m_{\gamma;k}(t) = \det((F_\gamma(t))_k).$$

Recall that $\pi(\gamma; t_1) = e$ if and only if there exist upper triangular matrices $U_1$ and $U_2$ such that $\mathcal{F}_\gamma(t_1) = U_1HU_2$. It is a basic fact of linear algebra that this happens if and only if $m_{\gamma;k}(t_1) \neq 0$ for all $k$. In other words, roots of the functions $m_k$ indicate times $t$ for which $\pi(\gamma; t) \neq e$. As we shall now see, more information can be deduced from the functions $m_k$.

Write $\text{mult}_k(\gamma; t_1) = d$ if $t_1$ is a zero of multiplicity $d$ of the function $m_{\gamma;k}$, that is, if $(t-t_1)^{-d}m_{\gamma;k}(t)$ is smooth and non-zero at $t = t_1$. Let the **multiplicity vector** be

$$\text{mult}(\gamma; t_1) = (\text{mult}_1(\gamma; t_1), \text{mult}_2(\gamma; t_1), \ldots, \text{mult}_n(\gamma; t_1)).$$
For $\alpha \in A_n = S_{n+1} \setminus \{e\}$ and $1 \leq k \leq n$, let
\[ \text{mult}_k(\alpha) = \sum_{1 \leq j \leq k} (j\alpha - j), \quad \text{mult}(\alpha) = (\text{mult}_1(\alpha), \text{mult}_2(\alpha), \ldots, \text{mult}_n(\alpha)) .\]

With the convention mult_0(\alpha) = mult_{n+1}(\alpha) = 0, we have $k^\alpha = k + \text{mult}_k(\alpha) - \text{mult}_{k-1}(\alpha)$, so that mult(\(\alpha\)) easily determines \(\alpha\). If $d, \tilde{d} \in \mathbb{N}^n$ we write $d \leq \tilde{d}$ if, for all $k$, $d_k \leq \tilde{d}_k$. If $\alpha_0 \leq \alpha_1$ (in the Bruhat order) then mult(\(\alpha_0\)) $\leq$ mult(\(\alpha_1\)) and dim(\(\alpha_0\)) $\leq$ dim(\(\alpha_1\)).

Example 5.2. For $n = 5$, let $\alpha_0 = [432156]$ and $\alpha_1 = [612345]$. We have mult(\(\alpha_0\)) = (3, 4, 3, 0, 0) $\leq$ mult(\(\alpha_1\)) = (5, 4, 3, 2, 1) but inv(\(\alpha_0\)) = 6 $>$ inv(\(\alpha_1\)) = 5 and therefore dim(\(\alpha_0\)) $>$ dim(\(\alpha_1\)).

For $n = 6$, let $\alpha_2 = [4321567]$ and $\alpha_3 = [7654321]$. We have dim(\(\alpha_2\)) = dim(\(\alpha_3\)) and mult(\(\alpha_2\)) = (3, 4, 3, 0, 0, 0) $<$ mult(\(\alpha_3\)) = (6, 5, 4, 3, 2, 1).

Lemma 5.3. For $\gamma \in L_n$ and $t \in [0, 1]$, $\deg(\gamma; t) = \deg(\pi(\gamma; t))$.

Proof. Easy (to be written). \qed

Let $\mathbb{N}[q]$ be the set of polynomials with non-negative integer coefficients in the (formal) variable $q$; $\mathbb{N}[q]$ is ordered by
\[ p_0 < p_1 \iff \exists C \in \mathbb{R}, \forall x \in (C, +\infty), \ p_0(x) < p_1(x) .\]

In particular $\mathbb{N}[q]$ is well ordered of type $\omega^\omega$.

For $w = \alpha_1 \alpha_2 \cdots \alpha_M \in W_n$, set its length, degree and qdim to be
\[ \ell(w) = M \in \mathbb{N}, \quad \text{mult}(w) = \sum_j \text{mult}(\alpha_j) \in \mathbb{N}^n, \quad \text{qdim}(w) = \sum_j q^{\dim(\alpha_j)} \in \mathbb{N}[q] .\]

In particular, $w \in W_n$ implies qdim(\(w\)) $< q^{\frac{n(n+1)}{2}}$. Notice that if qdim(\(w\)) = $p$ then $\ell(w) = p(1)$ and dim(\(w\)) = $p'(1)$.

Lemma 5.4. For $\alpha_0 \in A_n$, $w \in W_n$, the following conditions are equivalent:

(i) $w \preceq \alpha_0$;

(ii) given $\gamma_1 \in P_n(\alpha_0)$ and an open neighborhood $U \subset L_n$ of $\gamma_1$ there exists $\gamma_0 \in U \cap P_n(w)$;

(iii) given $\gamma_1 \in P_n(\alpha_0)$, $\epsilon > 0$, $t_1 \in (0, 1)$ with $\pi(\gamma_1; t_1) = \alpha_0$ and an open neighborhood $U \subset L_n$ of $\gamma_1$ there exists $\gamma_0 \in U \cap P_n(w)$ with $\gamma_0$ and $\gamma_1$ coinciding outside $(t_1 - \epsilon, t_1 + \epsilon)$.
Lemma 5.7. The set $U_B \gamma Q_Q$ More precisely, given any $u$ neighborhood $U_B \gamma t$ and $\gamma \gamma$ Q are contained in $B_-$; moreover, there is an open neighborhood $U_\gamma \subset B_-$ of $\gamma \gamma(t_1 - \frac{\epsilon}{2})$ which is filled by $\gamma \gamma(t_1 - \frac{\epsilon}{2})$. More precisely, given any $Q_\gamma U_\gamma$ there exists a convex arc $\gamma : [t_1 - \epsilon, t_1] \to S^n$ with the same frames as $\gamma_1$ at $t_1 - \epsilon$ and $t_1$ and $\gamma_\gamma(t_1 - \frac{\epsilon}{2}) = Q_-$. Similarly, there exists an open neighborhood $U_+ \subset B_+$ of $\gamma \gamma(t_1 + \frac{\epsilon}{2})$ such that, given any $Q_+ \in U_+$ there exists a convex arc $\gamma : [t_1, t_1 + \epsilon] \to S^n$ with the same frames as $\gamma_1$ at $t_1$ and $t_1 + \epsilon$ and $\gamma(t_1 + \frac{\epsilon}{2}) = Q_+.

Take $U_2 \subset U_3$ so that $\gamma_0 \in U_2$ implies the following conditions: $\gamma_\gamma(t_1 - \frac{\epsilon}{2}) \in U_-, \gamma_\gamma(t_1 + \frac{\epsilon}{2}) \in U_+, \gamma_0|_{(t_1 - \frac{\epsilon}{2}) \cup (t_1 + \frac{\epsilon}{2})}$ is convex. Notice that the last two conditions imply that $\pi(\gamma_0; t) = \epsilon$ for $t \in (0, t_1 - \frac{\epsilon}{2}) \cup (t_1 + \frac{\epsilon}{2}, 1)$. Define $\gamma_0$ by

$$
\tilde{\gamma}_0(t) = \begin{cases} 
\gamma_1(t), & t \notin (t_1 - \epsilon, t_1 + \epsilon), \\
\gamma_0(t), & t \in [t_1 - \frac{\epsilon}{2}, t_1 + \frac{\epsilon}{2}], \\
\gamma_\gamma(t), & t \in [t_1 - \epsilon, t_1 - \frac{\epsilon}{2}], \\
\gamma(t), & t \in [t_1 + \frac{\epsilon}{2}, t_1 + \epsilon]. 
\end{cases}
$$

Here $\gamma_\gamma$ and $\gamma_+$ are chosen based on $Q_- = \gamma_\gamma(t_1 - \frac{\epsilon}{2}) \in U_-$ and $Q_+ = \gamma(t_1 + \frac{\epsilon}{2}) \in U_+$, respectively. The curve $\tilde{\gamma}_0$ has the same itinerary as $\gamma_0$. If $U_2$ is taken sufficiently small so that $\gamma_0$ is very near $\gamma_1$ and $\gamma_\gamma$ and $\gamma_+$ are also taken to be very near $\gamma_1$ then we have that $\tilde{\gamma}_0 \in U_3$ satisfies the conditions in item (iii).

Lemma 5.5. Consider $\alpha_0 \in A_n, w = \alpha_0 \alpha_2 \cdots \alpha_M \in W_n$, $w \leq \alpha_0$, $w \neq \alpha_0$. Then $M > 0$, $\operatorname{mult}(w) \leq \operatorname{mult}(\alpha_0)$ and $\alpha_i < \alpha_0$ for $1 \leq i \leq M$. In particular, $\operatorname{mult}(\alpha_i) < \operatorname{mult}(\alpha_0)$ and $\dim(\alpha_i) < \dim(\alpha_0)$.

Proof. Notice first of all that the empty word is known to be isolated in the poset $W_n$: this is equivalent to the fact that convex curves form a connected component of $\mathcal{L}_n$. This implies $M > 0$. By definition of the Bruhat order, $\alpha_i \leq \alpha_0$. Proximity guarantees that the real roots of $m_{k;\gamma_0}$ are obtained from the real roots of $m_{k;\gamma_0}$. (but pairs of multiple roots of $m_{k;\gamma_0}$ may cancel out and not show up in $m_{k;\gamma_1}$; they can be considered to have gone complex). This implies $\operatorname{mult}(w) \leq \operatorname{mult}(\alpha)$. Finally, $\alpha_i = \alpha_0$ together with $\operatorname{mult}(w) \leq \operatorname{mult}(\alpha)$ imply $w = \alpha_0$.

Question 5.6. Is it true that $w \leq \alpha_0$ implies $\dim(w) \leq \dim(\alpha_0)$?

Lemma 5.7. The set $W_n$ is a poset. Consider $w_0, w_1 \in W_n$, $w_1 = \alpha_1\alpha_2 \cdots \alpha_M$ with $\alpha_j \in A_n$ (for all $j$).

(i) $w_0 \leq w_1$ if and only if there exist non-empty words $\omega_1, \ldots, \omega_M \in W_n$ such that $w_0 = \omega_1\omega_2 \cdots \omega_M$ and, for all $j$, $\omega_j \leq \alpha_j$;
(ii) if $w_0 \preceq w_1$ then
\[ \hat{w}_0 = \hat{w}_1; \quad \ell(w_0) \geq \ell(w_1); \quad \text{mult}(w_0) \leq \text{mult}(w_1); \quad \text{qdim}(w_0) \leq \text{qdim}(w_1); \]

(iii) if $w_0 \preceq w_1$ and $\text{qdim}(w_0) = \text{qdim}(w_1)$ then $w_0 = w_1$;

(iv) $W_n$ is well-founded, and given $w_1$ there are only finitely many $w \in W_n$ such that $w \preceq w_1$;

(v) $W_n$ is converse well-founded, and given $w_0$ there are only finitely many $w \in W_n$ such that $w_0 \preceq w$.

Proof. From item (i) of Lemma 6.1, the relation $\preceq$ is reflexive and transitive. Item (i) follows from Lemmas 6.4 and 6.5. It now follows that $w_0 \preceq w_1 \preceq w_0$ implies $w_0 = w_1$, completing the proof the $W_n$ is a poset. The other items follow from item (i), Lemma 6.5 and the finiteness of $A_n$. \qed

6 Predecessors

Write $\pi_0 < \pi_1$ if $\pi_0$ is an immediate predecessor of $\pi_1$ (in the Bruhat order). Recall that $\pi_0 < \pi_1$ if $\text{inv}(\pi_1) = \text{inv}(\pi_0) + 1$ and $\pi_1 = \pi_0(j_0j_1)$ or, equivalently, $\pi_1 = (i_0i_1)\pi_0$; here $i_0 < i_1$, $j_0 < j_1$, $i_0^{\pi_0} = j_0$, $i_1^{\pi_0} = j_1$, $i_0^{\pi_1} = j_1$, $i_1^{\pi_1} = j_0$. If $\pi_1$ is written as $[1^{\pi_1} \cdots (n+1)^{\pi_1}]$, it is easy to find its immediate predecessors: look for integers $j_1 > j_0$ appearing in the list, $j_1$ to the left of $j_0$, such that the integers which appear in the list between $j_1$ and $j_0$ are either larger than $j_1$ or smaller than $j_0$; the permutation $\pi_0 < \pi_1$ is then obtained by transposing the entries $j_1$ and $j_0$. In the matrix $P_{\pi_1}$, we must look for positive entries $(i_0, j_1), (i_1, j_0)$ such that the interior of the rectangle with these vertices includes no positive entry.

Furthermore, $\pi_0 < \pi_k$ (with $k = \text{inv}(\pi_k) - \text{inv}(\pi_0)$) if and only if there exist $\pi_1, \ldots, \pi_{k-1}$ with
\[ \pi_0 < \pi_1 < \cdots < \pi_{k-1} < \pi_k. \]
If $\pi_0 < \pi_1$ and $P_{\pi_i}$ are the corresponding permutation matrices then the two matrices differ only in the four entries $(i_*, j_*)$; furthermore, the interior of the rectangular box with these four corners includes no non-zero entry of either matrix.

**Lemma 6.1.** Let $\pi_0 < \pi_i$ with $\pi_1 = (i_0i_1)\pi_0 = \pi_0(j_0j_1)$. Then
\[ \text{mult}_k(\pi_1) = \text{mult}_k(\pi_0) + (j_1 - j_0) [i_0 \leq k < i_1]. \]

Here we use Iverson notation:
\[ [i_0 \leq k < i_1] = \begin{cases} 1, & i_0 \leq k < i_1, \\ 0, & \text{otherwise}. \end{cases} \]
Proof. Easy. [TO BE WRITTEN]

Write \( \pi_0 \prec \pi_1 \) if \( \pi_0 \prec \pi_1 \) and \( j_0 \equiv j_1 \) (mod 2) (where \( \pi_1 = \pi_0(j_0j_1) \)). Thus, if \( \pi_0 \prec \pi_1 \), \( \pi_0 \preccurlyeq \pi_1 \) if and only if \( \text{mult}(\pi_0) \equiv \text{mult}(\pi_1) \) (mod 2).

Lemma 6.2. Let \( \alpha_0 \prec \alpha_1 \in \mathcal{A}_n \) with \( \pi_1 = (i_0i_1)\pi_0 = \pi_0(j_0j_1) \). Then there exist words \( w_0^+, w_0^-, w_1^+, w_1^- \in \mathcal{W}_n \) such that

\[
\begin{align*}
    w_0^+ \alpha_0 w_1^+ & \preccurlyeq \alpha_1, \\
    w_0^- \alpha_0 w_1^- & \preccurlyeq \alpha_1.
\end{align*}
\]

If \( \alpha_0 \preccurlyeq \alpha_1 \) then \( w_0^+ \) and \( w_1^- \) are both empty and

\[
\text{mult}_k(w_0^-) = \text{mult}_k(w_1^-) = [i_0 \leq k < i_1].
\]

If \( \alpha_0 \prec \alpha_1 \) then

\[
\begin{align*}
    \text{mult}_k(w_0^-) & = \text{mult}_k(w_1^+), \\
    \text{mult}_k(w_0^+) & = \text{mult}_k(w_1^-), \\
    \text{mult}_k(w_0^-) + \text{mult}_k(w_1^+) & = [i_0 \leq k < i_1].
\end{align*}
\]

In particular,

\[
\alpha_0 \preccurlyeq \alpha_1 \implies \alpha_0 \preccurlyeq \alpha_1.
\]

First, an easy result in linear algebra.

Lemma 6.3. Let \( k_1, k_2, \ldots, k_n \) be non-negative integers. Let \( M \) be the \( n \times n \) matrix with entries

\[
M_{i,1} = t^{k_i}, \quad M_{i,j+1} = \frac{d}{dt} M_{i,j}.
\]

Then

\[
\det(M) = Ct^d, \quad C = \prod_{i_0 < i_1} (k_{i_1} - k_{i_0}); \quad d = -\frac{n(n-1)}{2} + \sum_i k_i.
\]

Proof. First notice that all monomials have degree \( d \). Set therefore \( \tilde{M}_{i,j} = M_{i,j}t^{-(k_{i+1-j})} \) to obtain a matrix \( \tilde{M} \) with constant entries. The first column of \( \tilde{M} \) consists of ones; the second column has \( i \)-th entry equal to \( k_i \). The third column has \( i \)-th entry equal to \( k_i(k_i-1) = k_i^2 - k_i \); an operation on columns leaves the determinant unchanged but now makes the third column have entries \( k_i^2 \). Perform similar operations on columns to obtain a Vandermonde matrix, implying \( \det(M) = C \), the desired result.

Proof of Lemma 6.9: Consider \( \alpha_0 \prec \alpha_1 \) and let \( i_*, j_* \) be as in the previous discussion. For \( s \in \mathbb{R} \) (small), take \( M_s : \mathbb{R} \to GL(n+1, \mathbb{R}) \) with

\[
(M_s)_{n+2-i,1} = \begin{cases} 
    t^{(j_1-1)} + st^{(j_0-1)}, & i = i_0; \\
    t^{(i_1-1)}, & i \neq i_0;
\end{cases} \quad (M_s)_{i,j+1} = \frac{d}{dt}(M_s)_{i,j}.
\]

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This matrix $M_0$ has the form considered in Lemma 6.10 so its determinant is constant (in $t$), with the sign given by the parity of $\eta \alpha_1$ ($\eta$ is the Coxeter element). The matrix $M_s$ is obtained from $M_0$ by row operations so it has the same determinant. If necessary, change the sign of the first row of $M_s$ so that $M_s : \mathbb{R} \to GL^+(n+1, \mathbb{R})$. Perform a QR factorization $M_s(t) = \Gamma_s(t) \hat{R}_s(t)$ to obtain $\Gamma_s : \mathbb{R} \to SO_{n+1}$ and $\gamma_s(t) = \Gamma_s(t)e_1$: the curve $\gamma_s$ is locally convex.

Let $(M_s(t))_k$ and $(\Gamma_s(t))_k$ be the $k \times k$ southwest minors of $M_s(t)$ and $\Gamma_s(t)$, respectively. Consider the smooth functions

$$\tilde{m}_{s,k}(t) = \det((M_s(t))_k), \quad m_{\gamma_s,k}(t) = \det((\Gamma_s(t))_k);$$

the QR factorization shows that one is a positive multiple of the other. In order to study the multiplicities of zeroes we may as well use $\tilde{m}_{s,k}(t)$.

Let

$$C_{j,k} = \prod_{i_a < i_b \leq k} (i_a^{\alpha_j} - i_b^{\alpha_j}), \quad \tilde{C}_k = C_{1,k}, \quad d_{j,k} = -\frac{k(k-1)}{2} + \sum_{i \leq k} i^{\alpha_j}.$$

Notice that for $k < i_0$ and $k \geq i_1$ we have $d_{1,k} = d_{0,k}$ but for $i_0 \leq k < i_1$ we have $d_{1,k} = d_{0,k} + (j_1 - j_0)$. Also, for $k < i_0$ we have $\tilde{C}_k = 1$; for $i_0 \leq k < i_1$, $\tilde{C}_k$ has the same sign as $C_{i_0}$ (this is where we use the condition $\alpha_0 \prec \alpha_1$).

For $k < i_0$ Lemma 6.10 gives $\tilde{m}_{s,k}(t) = C_{1,k}t^{d_{1,k}}$. For $k \geq i_1$, $(M_s(t))_k$ is obtained from $(M_0(t))_k$ by row operations and Lemma 6.10 again implies $\tilde{m}_{s,k}(t) = C_{1,k}t^{d_{1,k}}$. Finally, for $i_0 \leq k < i_1$, we use linearity of the determinant on row $n + 2 - i_0$; more precisely, let $\hat{M}$ be defined by

$$(\hat{M})_{n+2-i_1} = t^{i_{\alpha_0}}, \quad (\hat{M})_{i_1,i_1+1} = \frac{d}{dt}(\hat{M})_{i_1,i_1},$$

with southwest $k \times k$ minor $(\hat{M})_k$ and let $\hat{m}_k(t) = \det((\hat{M}(t))_k)$. By linearity and Lemma 6.10 we have

$$\tilde{m}_{s,k}(t) = \det((M_s(t))_k) = \det((M_0(t))_k) + s \det((\hat{M}(t))_k) = \hat{m}_{0,k}(t) + s\hat{m}_k(t) = C_{1,k}t^{d_{1,k}} + C_{0,k}st^{d_{0,k}} = C_{0,k}t^{d_{0,k}}(\tilde{C}_k t^{(j_1 - j_0)} + s).$$

The multiplicities imply that $\pi(\gamma_0, 0) = \alpha_1$ and $\pi(\gamma_s, 0) = \alpha_0$ for $s \neq 0$. If $j_1 - j_0$ is odd there are extra roots with $t \neq 0$; if $j_1 - j_0$ is even and $s$ has the same sign as $C_{i_0}$ then there are no other real roots. This completes the proof. \hfill \Box

It would be nice to understand the relation $\leq$ better. The following rather special result will be useful later.

**Lemma 6.4.** Let $n > 1$, $\alpha_0, \alpha_1, \beta \in \mathcal{A}_n$, $w_0, w_1, w_2 \in \mathcal{W}_n$. If $\alpha_0 \triangleleft \beta$, $\alpha_1 \triangleleft \beta$ and $w_0a_0w_1\alpha_1w_2 \leq \beta$ then either $\dim(\beta) = 1$ or

$$\dim(\beta) = 2, \quad \beta = [a_k a_{k+2} a_{k+1}], \quad \alpha_0 = \alpha_1 = [a_k a_{k+2}].$$
For \(\dim(\beta) = 1\) we have the following examples:

\[
aba \preceq [ab], \ a, b \prec [ab]; \quad bab \preceq [ba], \ a, b \prec [ba]; \quad ac, ca \preceq [ac], \ a, c \prec [ac].
\]

The example with \(\dim(\beta) = 2\) is indeed possible:

\[
[ac]b[ac] \preceq [acb], \ [ac] \prec [acb];
\]

see Section ??, particularly in Figure 10. We are concerned with ruling out other examples.

**Proof.** Write \(\beta = (i_0i_1)\alpha_0 = \alpha_0(j_0j_1) = (i'_0i'_1)\alpha_1 = \alpha_1(j'_0j'_1), \ d = j_1 - j_0\) and \(d' = j'_1 - j'_0\). We have

\[
\mult_k(\beta) = \mult_k(\alpha_0) - d \ [i_0 \leq k < i_1] = \mult_k(\alpha_1) - d' \ [i'_0 \leq k < i'_1]
\]

and, since \(\mult_k(\beta) \geq \mult_k(\alpha_0) + \mult_k(\alpha_1),
\]

\[
\mult_k(\beta) \leq d \ [i_0 \leq k < i_1] + d' \ [i'_0 \leq k < i'_1].
\]

Assume without loss of generality that \(1^\beta > 1\) and \((n + 1)^\beta < (n + 1)\) so that \(\mult_1(\beta) > 0\) and \(\mult_n(\beta) > 0\). If \(i_0 > 1\) we have \(\mult_1(\alpha_0) = \mult_1(\beta)\) and therefore \(\mult_1(\alpha_1) = 0 < \mult_1(\beta)\) and therefore \(i'_0 = 1\). Similarly, \(i_1 < n + 1\) implies \(i'_1 = n + 1\).

Consider first the case \(i_0 = 1\) and \(i_1 = n + 1\). We must have \(j_1 = j_0 + 1\) (otherwise the entry \(j_0 + 1\) obstructs \(\alpha_0 \prec \beta\)) and therefore \(d = 1\). If \(i'_0 = 1\) and \(i'_1 = n + 1\) we have \(d' = 1\) and therefore \(\mult_k(\beta) \leq 2\) for all \(k\). In particular \(\mult_1(\beta) = j_1 - 1 \leq 2\) and \(\mult_n(\beta) = (n + 1) - j_0 \leq 2\) whence, adding, \(n \leq 3\).

For \(n = 2\) we have examples with \(\dim(\beta) = 1\). For \(n = 3\), \(\mult_1(\beta) \leq 2\) implies \(1^\beta \leq 3\) and \(\mult_3(\beta) \leq 2\) implies \(4^{\beta} \geq 2\): the only example is therefore the one in the statement.

Still in the case \(i_0 = 1\) and \(i_1 = n + 1\), if \(i'_0 > 1\) we have \(\mult_1(\beta) \leq 1\) and therefore \(j_1 = 2, j_0 = 1\). If \(i'_0 > 2\) we have \(\mult_2(\beta) \leq 1\) and therefore \(2^{\beta} \leq 2\), a contradiction; thus \(i'_0\) equals 1 or 2. Similarly, \(i'_1 < n + 1\) implies \(j_0 = n\) and \(j_1 = n + 1\); we must have \(i'_1\) equal to \(n\) or \(n + 1\). The case \(i'_0 = 2\) and \(i'_1 = n\) implies both \(j_0 = 1\) and \(j_1 = n\) and is therefore impossible. The case \(i_0 = 1, i'_0 = 2, i_1 = i'_1 = n + 1\) implies \(j'_0 = j_0\) and \(j'_1 = j'_0 + 2\) (otherwise \(j'_0 + 2\) is an obstacle) and therefore \(d' = 2\), which implies \(\mult_4(\beta) \leq 3\) (for all \(k\)). In particular we have \(\mult_n(\beta) = (n + 1) - (n + 1)^\beta = n \leq 3\). The case \(n = 2\) obtains the example \(\beta = [231] = [ba], \ \alpha_1 = [132] = b, \ \alpha_1 = [213] = a\) with \(\dim(\beta) = 1\). The case \(n = 3\) obtains \(\beta = [2341] = [cba], \ \alpha_1 = [1342] = [cb], \ \alpha_2 = [2143] = [ac]\). Multiplicities require one of the \(w\)'s to equal \(b\) and the other two to be empty; multiplicities then work but it can be verified that in no order the value in Quat\(_4\) matches (this
also follows from the computation of $\partial[cba]$; see Figure 6). The case $i_0 = i'_0 = 1$, $i'_1 = n$, $i_1 = n + 1$ is of course similar.

We are left with the case $i_0 = 1 < i'_0$, $i_1 < i'_1 = n + 1$. We have $1^β = j_1$, $\text{mult}_1(β) = j_1 - 1 \leq d = j_1 - j_0$ and therefore $j_0 = 1$. Similarly, we have $j'_1 = n + 1$. If $i'_0 > 2$ we must have $\text{mult}_2(β) = 1^β + 2^β - 1 - 2 = (j_1 - 1) + (2^β - 2) \leq d = j_1 - 1$ and therefore $2^β \leq 2$. We can not have $2^β = 2$: this implies both $i_1 > 2$ and $j_1 > 2$: $2^β = 2$ is then an obstacle to $α_0 \triangleleft β$. We thus have $2^β = 1$ and therefore $i_1 = 2$. Since $n > 1$, this implies $i_1 < i'_0 < i'_1 = n + 1$; a similar reasoning implies $i'_0 = n$ and therefore $β = [a_1a_n]$, $α_0 = a_n$ and $α_1 = a_1$, with $\dim(β) = 1$.

We are thus left with the case $i_0 = 1$, $i'_0 = 2$, $i_1 = n$, $i'_1 = n + 1$, $n \geq 3$. We then have $j'_0 \in \{n - 1, n\}$, (otherwise $n - 1$ or $n$ is an obstacle), $d = j_1 - 1$ and $d' = (n + 1) - j'_0$. We have $\text{mult}_2(β) \leq d + d'$ so that $(n + 1) + j_1 - 1 - 2 \leq (j_1 - 1) + ((n + 1) - j'_0)$ and therefore $j'_0 \leq 2$, so that $j'_0 = 2$. Therefore $n = 3$, $β = [3412] = [bacb]$, $α_0 = [1432] = [bcb]$, $α_1 = [3214] = [aba]$. This is consistent with multiplicities, with $\text{mult}(α_0) + \text{mult}(α_1) = \text{mult}(β)$, but $β = 1\, α_0 = 1\, α_1$ implies that $α_0α_1 \not\triangleleft β$, $α_1α_0 \not\triangleleft β$. This completes the proof. 

\begin{question}
Is the following true?

Let $α_0, α_1 \in A_n$. If $α_0 \preceq α_1$ then there exists a sequence $(β_j)_{0 \leq j \leq k}$ with

$\alpha_0 = β_0 \triangleright β_1 \triangleright β_2 \triangleright \cdots \triangleright β_k = α_1.$

\end{question}

\section{About spin}

In this section we give a simple formula for the map taking $π \in S_{n+1}$ to $\hat{π} \in \text{Quat}_{n+1}$. Recall that $\text{Quat}_{n+1}$ has elements of the form $±\hat{a}_{i_1}^{ε_1} \hat{a}_{i_2}^{ε_2} \cdots \hat{a}_{i_n}^{ε_n}$ with $ε_i \in \{0, 1\}$; we also have

$$\hat{a}_i^2 = -1,$$

$$\hat{a}_j \hat{a}_i = (-1)^{[i-j]=1} \hat{a}_i \hat{a}_j.$$ 

More generally,

$$g = ±a_1^{ε_1} \hat{a}_{i_2}^{ε_2} \cdots \hat{a}_{i_n}^{ε_n} \implies g\hat{a}_i = (-1)^{ε_{i-1}+ε_{i+1}} \hat{a}_i g$$

where we adopt the convention $ε_0 = ε_{n+1} = 0$.

The following lemma gives a simple formula for $\hat{π}$ up to sign. For $π_0, π_1 \in S_{n+1}$, write $π_0 \equiv π_1 \pmod{2}$ if $i^{π_0} \equiv i^{π_1} \pmod{2}$ for all $i \in \{1, 2, \ldots, n + 1\}$. A permutation $π$ is \textit{parity preserving} if $i^{π} \equiv i \pmod{2}$ for all $i$. The set $PP$ of parity preserving permutations is a subgroup: $PP < S_{n+1}$; $PP$ is isomorphic to $S_{\lfloor \frac{n+1}{2} \rfloor} \times S_{\lceil \frac{n+1}{2} \rceil}$. We have $π_0 \equiv π_1 \pmod{2}$ if and only if $π_1^{-1} π_0 \in PP$, or, equivalently, if $π_0 PP = π_1 PP$. 

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Lemma 7.1. For $\pi \in A_n$,
\[ \hat{\pi} = \pm \hat{a}_1^{\mult_1(\pi)} \hat{a}_2^{\mult_2(\pi)} \cdots \hat{a}_n^{\mult_n(\pi)}. \]
In particular, $\pi_0 \equiv \pi_1 \pmod{2}$ if and only if $\hat{\pi}_0 = \pm \hat{\pi}_1$.

Proof. Let $M : \mathbb{R} \rightarrow GL_{n+1}^+$ be such that entries are monomials in $t$ and
\[ M_{i,1}(t) = \pm t^{(i-1)}, \quad M'_{i,j}(t) = M_{i,j+1}(t), \]
with signs adjusted so that $\det(M(t)) = c_{n+1} > 0$. Let $\Gamma(t)$ be obtained from
$M(t)$ by $QR$ factorization: $\Gamma$ is holonomic with the desired itinerary. Let $m_k(t)$ be
the determinant of the southwest $k \times k$ minor of $M(t)$ so that $m_k(t) = c_k t^{\mult_k(\pi)}$
(where $c_k$ is a nonzero real constant). Let $P_- , P_+ \in B_{n+1}^+$ so that $\Gamma(t)$ is Bruhat
equivalent to $P_\pm$ (according to the sign of $t$): we have
\[ P_+ = P_- \operatorname{diag}((-1)^{\mult_1(\pi)}, \ldots, (-1)^{\mult_{i-1}(\pi) + \mult_i(\pi)}, \ldots, (-1)^{\mult_n(\pi)}), \]
which is what we need to prove. \qed

Lemma 7.2. Let $\pi_0, \pi_1 \in S_{n+1}$ with $\pi_0 \triangleright \pi_1$, $\pi_0 = a_{i_0} \pi_1 = \pi_1(j_0 j_1)$, $j_1 - j_0 = \delta > 0$. Then
\[ \hat{\pi}_{i_0} \hat{\pi}_1 = (-1)^{(\delta+1)} \hat{\pi}_1 \hat{\pi}_{i_0}. \]

Proof. We have
\[ \hat{\pi}_{i_0} \hat{\pi}_1 = (-1)^{\mult_{i_0} - \mult_{i_1} + \mult_{i_0+1} - \mult_{i_1+1}} \hat{\pi}_1 \hat{\pi}_{i_0} = (-1)^{\mult_{i_0} - \mult_{i_1} + \mult_{i_0+1} - \mult_{i_1+1}} \hat{\pi}_1 \hat{\pi}_{i_0} = (-1)^{(i_1 + i_0 - i_0 - 1)} \hat{\pi}_1 \hat{\pi}_{i_0} = (-1)^{(\delta+1)} \hat{\pi}_1 \hat{\pi}_{i_0}. \]

Lemma 7.3. Let $\pi_0, \pi_1 \in S_{n+1}$ with $\pi_0 \triangleright \pi_1$, $\pi_0 = a_{i_0} \pi_1 = \pi_1(j_0 j_1)$, $j_1 - j_0 = \delta > 0$. If $\delta$ is odd then
\[ a_{i_0} \pi_1 \preceq \pi_0, \quad \pi_1 a_{i_0} \preceq \pi_0, \quad \hat{\pi}_0 = \hat{\pi}_{i_0} \hat{\pi}_1 = \hat{\pi}_1 \hat{\pi}_{i_0}. \]
If $\delta$ is even then
\[ \pi_1 \preceq \pi_0, \quad a_{i_0} \pi_1 a_{i_0} \preceq \pi_0, \quad \hat{\pi}_0 = \hat{\pi}_1 = \hat{\pi}_{i_0} \hat{\pi}_1 \hat{\pi}_{i_0}. \]
In particular, $\pi_1 \preceq \pi_0$ implies $\hat{\pi}_1 = \hat{\pi}_0$ (as we had already seen).

Before presenting the proof, we show how using this result it is easy to compute
$\hat{\pi}_0$ given $\pi_0$. As an example, take $\pi_0 = [7245136]$. Take $\pi_0 = a_1 \pi_1 \triangleright \pi_1 = [2745136],$
$\pi_1 = a_2 \pi_2 \triangleright \pi_2 = [2475136], \pi_2 = a_3 \pi_3 \triangleright \pi_3 = [2457136], \pi_3 = a_4 \pi_4 \triangleright \pi_4 = [2457136],$
$\pi_4 = a_5 \pi_5 \triangleright \pi_5 = [2145736], \pi_5 = a_6 \pi_6 \triangleright \pi_6 = [2145736], \pi_6 = a_7 \pi_7 \triangleright \pi_7 = [1245736],$
$\pi_7 = a_8 \pi_8 \triangleright \pi_8 = [1245376], \pi_8 = a_9 \pi_9 \triangleright \pi_9 = [1243576], \pi_9 = a_{10} \pi_{10} \triangleright \pi_{10} = [1234576], \pi_{10} = a_6 \pi_{11} \triangleright \pi_{11} = [1234567] = e$. We therefore have $\hat{\pi}_{10} = \hat{\pi}_6,$
$\hat{\pi}_9 = \hat{\pi}_8 = \hat{\pi}_7 = \hat{\pi}_5 = \hat{\pi}_4 = \hat{\pi}_3 = \hat{\pi}_2 = \hat{\pi}_1 = -\hat{\pi}_1 \hat{\pi}_6 = \hat{\pi}_6.$

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Proof of Lemma 7.4: As in the proof of Lemma 6.9, take $M_s : \mathbb{R} \to GL_{n+1}^+$ with
\[
(M_s)_{n+2-i,1} = \begin{cases} 
t((j_1-1) + st(j_0-1), & i = i_0; \\
\pm t(i-1), & i \neq i_0;
\end{cases}
\]
with signs adjusted so that $\det(M_s(t))$ is a positive constant. Let $m_k(s; t)$ be the determinant of the $k \times k$ southwest minor of $M_s(t)$. For $k \neq i_0$, $m_k(s; t) = c_k t^{\text{mult}_k(\pi_1)}$ has a single root of multiplicity $\text{mult}_k(\pi_1)$ at $t = 0$ (for any value of $s$). For $k = i_0$,
\[
m_k(s; t) = c_k t^{\text{mult}_k(\pi_0)} + \tilde{c}_k s t^{\text{mult}_k(\pi_1)} = c_k t^{\text{mult}_k(\pi_1)} \left(t^\delta + \frac{\tilde{c}_k s}{c_k}\right).
\]
For $s = 0$, this polynomial as a single root of multiplicity $\text{mult}_k(\pi_0)$ at $t = 0$ and therefore has itinerary $\pi_0$. If $s \neq 0$, $t = 0$ is a root of multiplicity $\text{mult}_k(\pi_1)$; we have to check for other real roots. If $\delta$ is odd, there is exactly one other real root, which is simple and whose sign depends on the sign of $s$, obtaining the itineraries $a_{i_0} \pi_1$ and $\pi_1 a_{i_0}$, as desired. If $\delta$ is even, there are either zero or two other real roots, again depending on the sign of $s$; if there are two roots, they have opposite signs. This obtains the itineraries $\pi_1$ and $a_{i_0} \pi_1 a_{i_0}$.

The equations in Quat$_{n+1}$ follow from the fact that $w_1 \preceq w_0$ implies $\hat{w}_1 = \hat{w}_0$. \hfill \Box

Recall that for $\pi \in S_{n+1}$, an inversion is a pair $(i, j) \in \{1, 2, \ldots, n + 1\}^2$ with $i < j$ and $i^\pi > j^\pi$. An inversion $(i, j)$ is of type $(i^\pi \mod 2, j^\pi \mod 2)$: there are therefore four possible types of inversions: 00, 01, 10 and 11. Let $\text{inv}_{uv}(\pi)$ be the number of inversions of type $(u, v)$ so that
\[
\text{inv}(\pi) = \text{inv}_{00}(\pi) + \text{inv}_{01}(\pi) + \text{inv}_{10}(\pi) + \text{inv}_{11}(\pi).
\]

Lemma 7.4. Let $\pi_0 \geq \pi_1$ be permutations.

(i) If $\pi_0 \equiv \pi_1 \pmod{2}$ then
\[
\text{inv}_{10}(\pi_0) - \text{inv}_{01}(\pi_0) = \text{inv}_{10}(\pi_1) - \text{inv}_{01}(\pi_1).
\]

(ii) Assume that $\pi_0 = \pi_1(j_0j_1) = (i_0i_1)\pi_1$, $i_0 < i_1$, $j_0 < j_1$, $j_0 \equiv j_1 \pmod{2}$. We have
\[
\text{inv}_{01}(\pi_0) = \text{inv}_{01}(\pi_1) + k, \quad \text{inv}_{10}(\pi_0) = \text{inv}_{10}(\pi_1) + k
\]
where $k$ is the number of integers $i$ with
\[
i_0 < i < i_1, \quad j_0 < i^\pi_0 < j_1, \quad i^\pi_0 \not\equiv j_0 \pmod{2}.
\]
Proof. We first prove that the first item follows from the second one. The group $PP$ is generated by elements of the form $(j_0 j_1), j_0 \equiv j_1 \pmod{2}$. The lateral class $\pi_0 PP$ is therefore connected by moves from $\pi$ to $\pi(j_0 j_1), j_0 \equiv j_1 \pmod{2}$. We may therefore assume without loss of generality the conditions on the second item.

We now proceed to prove the second item: we provide a bijective proof, in the sense that we match inversions, leaving $k$ unmatched as per the statement. The pair $(i_0, i_1)$ is an inversion of $\pi_0$ but not of $\pi_1$ but it is of type 00 or 11, so does not need to be counted. If $\{i, i'\}$ is disjoint from $\{i_0, i_1\}$ then $(i, i')$ is an inversion of $\pi_0$ if and only if it is an inversion of $\pi_1$, with the same type; this also does not need to be counted. If $i < i_0, (i, i_0)$ is an inversion of $\pi_0$ if and only if $(i, i_1)$ is an inversion of $\pi_1$; conversely, $(i, i_1)$ is an inversion of $\pi_0$ if and only if $(i, i_0)$ is an inversion of $\pi_1$; these also do not need to be counted. A similar argument shows that the case $i > i_1$ also does not need to be counted. We are therefore left with counting inversions of the forms $(i_0, i)$ and $(i, i_1)$ for $i_0 < i < i_1$.

If $i_{\pi_0} < j_0$ then $(i_0, i)$ is an inversion of both $\pi_0$ and $\pi_1$; also $(i, i_1)$ is not an inversion for either permutation: that case also does not need to be counted. A similar argument takes care of the case $i_{\pi_0} > j_1$. We are therefore left with counting inversions of the forms $(i_0, i)$ and $(i, i_1)$ for $i_0 < i < i_1, j_0 < i_{\pi_0} < j_1$.

For each such $i$, $(i_0, i)$ and $(i, i_1)$ are both inversions of $\pi_0$ but neither is an inversion of $\pi_1$. If $i_{\pi_0} \equiv j_0 \pmod{2}$, however, both inversions are of type 00 or 11, so do not need to be counted. Finally, if $i_{\pi_0} \not\equiv j_0 \pmod{2}$ the two inversions are of types 01 and 10 (in some order): since there are $k$ such values of $i$, we are done.

\begin{lemma}
Let $\pi_0, \pi_1 \in S_{n+1}$. If $\pi_0 \equiv \pi_1 \pmod{2}$ then

\[ (-1)^{inv_{01}(\pi_0)} \pi_0 = (-1)^{inv_{01}(\pi_1)} \pi_1, \quad (-1)^{inv_{10}(\pi_0)} \pi_0 = (-1)^{inv_{10}(\pi_1)} \pi_1. \]

\end{lemma}

\begin{proof}
As in Lemma ??, we may assume without loss of generality that $\pi_0 \geq \pi_1$, $\pi_0 = \pi_1(j_0 j_1) = (i_0 i_1)\pi_1, i_0 < i_1, j_0 < j_1, j_0 \equiv j_1 \pmod{2}$. From the structure of $PP$ we may furthermore assume that $i_0 < i < i_1$ implies $i_{\pi_0} \not\equiv j_0 \pmod{2}$. Let $\delta = i_1 - i_0$: we prove by induction on $\delta$.

If $\delta = 1$ we have $\pi_1 \triangleright \pi_0$ and therefore $\pi_1 = \pi_0$. Use the second item of Lemma ??: we have $k = 0$ and therefore

\[ inv_{01}(\pi_0) = inv_{01}(\pi_1), \quad inv_{10}(\pi_0) = inv_{10}(\pi_1); \]

this completes the base of induction.

If $\delta > 1$, let $i_2 = i_0, i_3 = i_1 - 1, j_2 = j_0, j_3 = j_1, \pi_2 = a_{i_3} \pi_0$ and $\pi_3 = a_{i_3} \pi_1$. Notice that $\pi_2 \geq \pi_3$, $\pi_2 = \pi_3(j_2 j_3) = (i_2 i_3)\pi_3, i_2 < i_3, j_2 < j_3, j_2 \equiv j_3 \pmod{2}$ and $i_3 - i_2 < \delta$. We may therefore assume by induction hypothesis that

\[ (-1)^{inv_{01}(\pi_2)} \pi_2 = (-1)^{inv_{01}(\pi_3)} \pi_3, \quad (-1)^{inv_{10}(\pi_2)} \pi_2 = (-1)^{inv_{10}(\pi_3)} \pi_3. \]
We must now break the proof into cases. Assume first that \( j_0 \) is even.

Assume first \( i_3^{\pi_0} < j_0 \): we have \( \pi_0 \triangleright \pi_0, \pi_3 \triangleright \pi_1 \). Also, \( \pi_2 = a_{i_3} \pi_0 = \pi_0 (i_3^{\pi_0} j_0) \) and \( \pi_3 = a_{i_3} \pi_1 = \pi_1 (i_3^{\pi_0} j_1) \). From Lemma ??, \( \hat{a}_{i_3} \hat{\pi}_0 = \hat{\pi}_0 \hat{a}_{i_3} \) and \( \hat{a}_{i_3} \hat{\pi}_1 = \hat{\pi}_1 \hat{a}_{i_3} \). From Lemma ??, \( \hat{\pi}_2 = a_{i_3} \hat{\pi}_0, \hat{\pi}_3 = a_{i_3} \hat{\pi}_1 \). We also have \( \text{inv}_{\pi_0}(\pi_2) = 1 + \text{inv}_{\pi_0}(\pi_0) \), \( \text{inv}_{\pi_0}(\pi_2) = \text{inv}_{\pi_0}(\pi_0) \), \( \text{inv}_{\pi_0}(\pi_3) = 1 + \text{inv}_{\pi_0}(\pi_1) \), \( \text{inv}_{\pi_0}(\pi_3) = \text{inv}_{\pi_0}(\pi_1) \): using these and the induction step we have the desired conclusion.

Assume next \( j_0 < i_3^{\pi_0} < j_1 \): we have \( \pi_2 \triangleright \pi_0, \pi_3 \triangleright \pi_1 \). Also, \( \pi_2 = a_{i_3} \pi_2 = \pi_2 (j_0 i_3^{\pi_0}) \) and \( \pi_3 = a_{i_3} \pi_1 = \pi_1 (i_3^{\pi_0} j_1) \). From Lemma ??, \( \hat{a}_{i_3} \hat{\pi}_0 = \hat{\pi}_0 \hat{a}_{i_3} \) and \( \hat{a}_{i_3} \hat{\pi}_1 = \hat{\pi}_1 \hat{a}_{i_3} \). From Lemma ??, \( \hat{\pi}_0 = a_{i_3} \hat{\pi}_2 \), \( \hat{\pi}_2 = -\hat{a}_{i_3} \hat{\pi}_0 \) and \( \hat{\pi}_3 = \hat{a}_{i_3} \hat{\pi}_1 \). We also have \( \text{inv}_{\pi_0}(\pi_2) = \text{inv}_{\pi_0}(\pi_0) \), \( \text{inv}_{\pi_0}(\pi_2) = -1 + \text{inv}_{\pi_0}(\pi_0) \), \( \text{inv}_{\pi_0}(\pi_3) = 1 + \text{inv}_{\pi_0}(\pi_1) \), \( \text{inv}_{\pi_0}(\pi_3) = \text{inv}_{\pi_0}(\pi_1) \): we again have the desired conclusion.

The case \( i_3^{\pi_0} > j_1 \) is similar to the case \( i_3^{\pi_0} < j_0 \). Finally, the case \( j_0 \) odd is also similar, completing the proof. \( \square \)

8 Lower and upper sets

A subset \( I \subseteq W_n \) is a lower set if, for all \( w_1 \in I \) and \( w_0 \in W_n, w_0 \preceq w_1 \) implies \( w_0 \in I \). Thus, if \( I \) is a lower set then

\[
\mathcal{L}_n[I] = \bigcup_{w \in I} \mathcal{P}_n(w) \subseteq \mathcal{L}_n
\]

is an open subset. In particular, \( \emptyset \) and \( W_n \) are lower sets. Given \( w_0 \in W_n \), let \( I_{w_0} = \{ w \in W_n, w \preceq w_0 \} \) and \( I_{w_0}^* = I_{w_0} \setminus \{ w_0 \} \); both are finite lower sets. See Figure 2 for a diagram of \( I_{[aba]} = \{ aa, abab, bb, baba, [ba]a, a[ba], [ab]b, b[ab], [aba] \} \).

![Figure 2: The lower set I_{[aba]}](image)

Let \( I^{(0)} \subseteq W_n \) be the set of words of degree 0, i.e., words whose letters are generators \( a_k \). Clearly, \( I^{(0)} \) is a lower set. The set \( \mathcal{L}_n^{(0)} = \mathcal{L}_n[I^{(0)}] \) is a disjoint union of contractible open sets \( \mathcal{P}_n(w), w \in I^{(0)} \).

Similarly, \( U \subseteq W_n \) is an upper set if, for all \( w_0 \in U \) and \( w_1 \in W_n, w_0 \preceq w_1 \) implies \( w_1 \in U \). If \( U \) is an upper set then

\[
\mathcal{L}_n[U] = \bigcup_{w \in U} \mathcal{P}_n(w) \subseteq \mathcal{L}_n
\]

is a closed subset. Given \( w_0 \in W_n \), let \( U_{w_0} = \{ w \in W_n, w_0 \preceq w \} \) and \( U_{w_0}^* = U_{w_0} \setminus \{ w_0 \} \), both finite upper sets.

A few examples are in order.
Proposition 8.1. The following are examples of upper sets:
\[ U_{2,2} = \{ [aba] \} \subset \mathcal{W}_2, \quad U_{2,3} = \{ [aba], [bacb], [bcb] \} \subset \mathcal{W}_3, \]
\[ U_{x,3} = \{ [cba] \} \subset \mathcal{W}_3. \]

**Proof.** The example \( U_{2,2} \) follows directly from Lemma 6.7: there is no letter in \( A_2 \) of greater dimension than \([aba]\). For \( U_{2,3} \), first notice that \([aba] \preceq [bacb], [bcb] \preceq [bacb]\); next, verify that and that there are no other letters \( \alpha \in A_3 \) with \( \dim(\alpha) \geq 2 \) and \( \hat{\alpha} = [aba] \). For \( U_{x,3} \), check that \([cba] \not\preceq [abacba]\) and therefore \([cba] \not\in [abacba]\). \( \square \)

**Question 8.2.** Let
\[ U_{2,n} = \{ [aba], [bacb], \ldots, [a_k a_{k+1} a_k], [a_{k+1} a_k a_{k+2} a_{k+1}], \ldots, [a_{n-1} a_n a_{n-1}] \} \subset \mathcal{W}_n; \]
is it an upper set?

**Corollary 8.3.** The set \( \mathcal{M}_{2,n} = \mathcal{L}_n [U_{2,n}] \subset \mathcal{L}_n \) is a closed set and a topological submanifold of codimension 2.

It turns out that \( \mathcal{M}_{2,n} \) is not a smooth manifold for \( n > 2 \).

### 9 Valid complexes \( \mathcal{D}_n [\mathcal{I}] \)

Here \( \mathbb{D}^k \) is the closed disk of dimension \( k \). Let \( \mathcal{I} \subseteq \mathcal{W}_n \) be a lower set. A **valid complex** \( \mathcal{D}_n [\mathcal{I}] \) (for \( \mathcal{I} \)) is a family \((c_w)_{w \in \mathcal{I}}\) of cells (continuous maps) \( c_w : \mathbb{D}^{\dim(w)} \to \mathcal{L}_n [\mathcal{I}_w] \subseteq \mathcal{L}_n [\mathcal{I}] \) for which the following conditions hold:

(i) For each \( w \in \mathcal{I} \), the boundary \( \partial c_w = c_w \mid_{\partial \mathbb{D}^{\dim(w)}} \) has image contained in \( \bigcup_{\tilde{w} \in \mathcal{I}_w} c_{\tilde{w}} [\mathbb{D}^{\dim(\tilde{w})}] \).

(ii) For any \( w \in \mathcal{I} \), the cell \( c_w \) is a smooth embedding in the interior of \( \mathbb{D}^{\dim(w)} \) and intersects \( \mathcal{P}_n(w) \) transversally at \( c_w(0) \in \mathcal{P}_n(w) \).

(iii) If \( s \in \mathbb{D}^{\dim(w)} \), \( s \neq 0 \), then \( c_w(s) \in \mathcal{L}_n [\mathcal{I}_w^s] \) (and therefore \( c_w(s) \not\in \mathcal{P}_n(w) \)).

(iv) The images of the cells \( c_w \) are disjoint in their interiors.

We often abuse notation by confusing the family \( \mathcal{D}_n [\mathcal{I}] = (c_w)_{w \in \mathcal{I}} \) with its image
\[ \bigcup_{w \in \mathcal{I}} c_w [\mathbb{D}^{\dim(w)}] \subset \mathcal{L}_n [\mathcal{I}]. \]
A valid complex $D_n$ is just a valid complex for $I = W_n$. A valid complex is a CW complex if furthermore, for each $w$, 

$$c_w([D]^{\text{dim}(w)}) \subseteq \bigcup_{\tilde{w} \in I_n, \dim(\tilde{w}) < \dim(w)} c_{\tilde{w}}([D]^{\text{dim}(\tilde{w})}).$$

Notice that our generalization of the concept of CW complex allows for boundaries of cells to use previous cells (in the order of the poset) even if the dimension is not smaller; this variation is mostly harmless and occasionally helpful. Given a valid complex $D_n[I] = (c_w)_{w \in I}$ and a lower set $\tilde{I} \subseteq I$, the subcomplex $(c_w)_{w \in \tilde{I}}$ is also valid (for $\tilde{I}$).

**Lemma 9.1.** (i) Let $\tilde{I} \subset I \subseteq W_n$ be finite lower sets and $D_n[\tilde{I}]$ be a valid complex (for $\tilde{I}$). Then this complex can be extended to a valid complex $D_n[I]$ (for $I$). Furthermore, if $D_n[\tilde{I}]$ is a CW complex then $D_n[I]$ can also be taken to be a CW complex.

(ii) Let $I \subseteq W_n$ be a finite lower set and $D_n[I]$ be a valid complex. The inclusion $D_n[I] \subseteq L_n[I]$ is a homotopy equivalence.

**Proof.** The two items are proved together by induction on $|I|$. The case $I = \emptyset$ is trivial. For the case $|I| = 1$, we must have $I = \{w\}$ with $\dim(w) = 0$: we then know that $L_n[I] = P_n(w)$ is open and contractible, implying the lemma in this case. The cases $|I| \leq 3$ also follow easily from the previous discussion.

For the first item, we may assume $\tilde{I} = I \setminus \{w_0\}$ where $w_0 \in I$ is maximal. In other words, assume $D_n[I]$ given (and valid): we must construct the cell $c_{w_0}$ so that $D_n[I]$ is also valid. Let $k = \dim(w_0)$ and $\gamma_0 \in P_n(w_0)$ be an arbitrary curve. For a small ball $B \subset \mathbb{R}^k$ around the origin, construct an arbitrary smooth map $\tilde{c} : B \to L_n[I_{w_0}] \subseteq L_n$ with $\tilde{c}(0) = \gamma_0$ and transversal to $P_n(w_0)$ at this point. By taking $r > 0$ sufficiently small, the image under $\tilde{c}$ of a ball $B(2r) \subset B$ of radius $2r$ around the origin satisfies the following conditions:

- $s \in B(2r)$ and $s \neq 0$ imply $\tilde{c}(s) \in L_n[I_{w_0}^s]$;
- $\tilde{c}[B(2r)]$ is disjoint from $D_n[\tilde{I}]$.

Let $S(r) \subset B(2r)$ be the sphere of radius $r$ around the origin so that $\tilde{c}|_{S(r)} : S(r) \to L_n[I_{w_0}^s]$. By induction hypothesis, the inclusion $D_n[I_{w_0}] \subseteq L_n[I_{w_0}^s]$ is a homotopy equivalence. There exists therefore a homotopy taking values in $L_n[I_{w_0}^s]$ from $\tilde{c}|_{S(r)}$ to a map from $S^{k-1}$ to $D_n[I_{w_0}]$. In other words, there exists $c_{w_0} : \mathbb{D}^k \to L_n[I_{w_0}]$ coinciding with $\tilde{c}$ in $B(r)$ with boundary $\partial c_{w_0} : S^{k-1} \to D_n[I_{w_0}]$, since we are in infinite dimension, we may assume the image of $c_{w_0}$ to be disjoint from that of previously created cells. This completes the construction of a valid complex $D_n[I]$. Furthermore, we may assume by transversality that the
image of $\partial c_{w_0}$ does not include the center of any (previously constructed) cell $c_w$ with $w \leq w_0$, $\dim(w) \geq \dim(w_0)$: radial projection then causes $\partial c_{w_0}$ to avoid the interior of such cells. Provided $D_n[I]$ is a CW complex, this guarantees that the valid complex $D_n[I]$ is also a CW complex.

For the second item, we need to construct a map $\phi_I : L_n[I] \rightarrow D_n[I]$ such that both $\phi_I \circ i$ and $i \circ \phi_I$ are homotopic to the identity (in their respective spaces). By induction, we may assume given both $\phi : L_n[I] \rightarrow D_n[I]$ and the homotopy $H_\phi : [0, 1] \times L_n[I] \rightarrow L_n[I]$ from the identity to $i \circ \phi$, where $I = I \setminus \{w_0\}$ and $w_0 \in I$ is maximal. Notice that $\phi_I$ does not extend continuously to $P_n(w_0)$, i.e., to $L_n[I]$.

Recall that $P_n(w_0) \subseteq L_n[I]$ is a closed subset (of $L_n[I]$) and a contractible submanifold of codimension $k = \dim(w_0)$, hence diffeomorphic to the Hilbert space $\mathbb{H}$. Construct a thin tubular neighborhood of $P_n(w_0)$ and a diffeomorphism $\psi$ from $B(2r) \times \mathbb{H}$ to this neighborhood taking $\{0\} \times \mathbb{H}$ to $P_n(w_0)$, $(s, 0)$ to $c_{w_0}(s)$, and $(0, 0)$ to $c_{w_0}(0)$ ($r > 0$ is small and $s \in B(2r)$). We start by squeezing $\psi[B(r) \times \mathbb{H}]$ onto $c_{w_0}[B(r)]$, taking $\psi(s, *)$ to $\psi(s, 0) = c_{w_0}(s)$. More precisely, let $\beta : [0, 2r) \rightarrow [0, 1]$ be a smooth function with $\beta(x) = 0$ for $x < r$ and $\beta(x) = 1$ for $x > 3r/2$. Consider the homotopy $H : [0, 1] \times L_n[I] \rightarrow L_n[I]$, $H(t, \gamma) = \gamma$ if $\gamma \notin \psi[B(2r) \times \mathbb{H}]$, $H(t, \psi(s, v)) = \psi(((1-t)+t\beta(|s|))s, v)$. Next, for $\gamma \notin \psi[B(r) \times \mathbb{H}]$ apply the homotopy $H_\phi$ so that the complement of this tubular neighborhood is taken to $D_n[I]$: points of the form $\psi(s, *)$ do not move if $|s| < r/2$. It remains to define our homotopy in the region $|s| \in [r/2, r]$. The sphere of radius $r$ is moved from a small sphere in $c_{w_0}$ to a sphere in $D_n[I]$ homotopic in $L_n[I]$ and therefore in $D_n[I]$ to the boundary map $\partial c_{w_0}$. The sphere of radius $r/2$ is also homotopic to the boundary map in $D_n[I] \setminus \{\gamma_0\}$. The homotopy between these two maps fill in the gap. We thus have a map $\phi_I : L_n[I] \rightarrow D_n[I]$ and a homotopy $H_I$ as desired. Also, $\phi_I|_{D_n[I]} : D_n[I] \rightarrow D_n[I]$ is homotopic to the identity, completing the proof.

Following the above construction for $I_{[aba]}$ we have the cell shown in Figure 3; the transversal map $\tilde{c}$ in the proof above is shown in Figure 1. Notice that $c_{[bab]}$ is very similar. The immediate predecessors of $[aba]$ are $[ba]a$, $b[ab]$, $[ab]b$ and $a[ba]$ (see Figure 2) and we have

$$\partial[aba] = [ba]a + b[ab] - [ab]b - a[ba], \quad \partial[bcb] = [cb]b + c[bc] - [bc]c - b[cb].$$

This and other examples shall be discussed in more detail in the following sections.

Figure 3: The cells $c_{[aba]}$ and $c_{[bab]}$.

**Corollary 9.2.** For any lower set $I \subseteq W_n$ there exists a valid complex $D_n[I]$ for $I$. For any valid complex $D_n[I]$, the inclusion $D_n[I] \subseteq L_n[I]$ is a weak homotopy equivalence.
Proof. If $I$ is finite this follows directly from the lemma, so let us assume $I$ infinite. Any infinite lower set is a union of nested finite lower sets:

$$I = \bigcup_{k \in \mathbb{N}} I_k, \quad \forall k_0, k_1 \in \mathbb{N}, \ k_0 < k_1 \implies I_{k_0} \subseteq I_{k_1}.$$ 

Use the lemma to construct a valid complex $D_n[I_0]$, then use it again to extend it to $D_n[I_1]$, and so on: the desired complex $D_n[I]$ is the union of these complexes, and is clearly valid.

By compactness, any continuous map $g_0 : S^k \to \mathcal{L}_n$ has image contained in $\mathcal{L}_n[I]$ for some finite lower set $I_k$. Since the inclusion $D_n[I_k] \subseteq \mathcal{L}_n[I_k]$ is a homotopy equivalence (from the lemma), $g_0$ is homotopic to a map $g_1 : S^k \to D_n[I_k] \subseteq D_n[I]$. Conversely, assume that a map $h : S^k \to D_n[I]$ can be extended to $H_0 : D^{k+1} \to D_n[I]$. Again by compactness, there exists a finite lower set $I_k$ such that the image of $H_0$ is contained in $\mathcal{L}_n[I_k]$ and the image of $h$ is contained in $D_n[I_k]$. Again from the lemma, the inclusion $D_n[I_k] \subseteq \mathcal{L}_n[I_k]$ is a homotopy equivalence and there exists therefore another extension of $h$ of the form $H_1 : D^{k+1} \to D_n[I_k] \subseteq D_n[I]$. This completes the proof that the inclusion $D_n[I] \subseteq \mathcal{L}_n[I]$ is a weak homotopy equivalence.

\[\square\]

10 The complexes $D_n^{(0)} \subset D_n^{(1)}$

In this section we consider a few simple examples of valid complexes. Recall that $I^{(0)} \subset \mathcal{W}_n$ is the set of words of dimension 0. For each $w \in I^{(0)}$, let $c_w \in \mathcal{P}_n(w)$ be an arbitrary curve: we think of $c_w$ as a vertex of the complex $D_n$. In other words, the 0-skeleton of $D_n$ is the infinite countable set of vertices $D_n^{(0)} = \{c_w, w \in \mathcal{W}^{(0)}\}$; the inclusion $D_n^{(0)} \subseteq \mathcal{L}_n^{(0)}$ is a homotopy equivalence.

The subset of $I^{(1)} \subseteq \mathcal{W}_n$ of words of dimension at most 1 is also a lower set. A word of dimension 1 is of the form $w = w_0[a_ka_l]w_1$ where $w_0, w_1 \in I^{(0)} \subset \mathcal{W}_n$ are (possibly empty) words of dimension 0 and $k \neq l$ so that $[a_ka_l] \in \mathcal{A}_n$ is an element of dimension 1. There are three cases: case (i) is $l = k - 1$, case (ii) is $l = k + 1$ and case (iii) is $l > k + 1$ (if $l < k - 1$ we write $a_la_k$ instead; notice that in this case the permutations $a_k$ and $a_l$ commute). In each case the stratum $\mathcal{P}_n(w)$ is a hypersurface with an open stratum on either side: let us call them $\mathcal{P}_n(w^+)$ and $\mathcal{P}_n(w^-)$. The words $w^+, w^- \in \mathcal{W}_n$ have dimension 0; their values according to case are:

(i) $w^- = w_0a_lw_1, \ w^+ = w_0a_ka_la_kw_1$;

(ii) $w^- = w_0a_ka_kw_1, \ w^+ = w_0a_lw_1$;

(iii) $w^- = w_0a_ka_lw_1, \ w^+ = w_0a_la_kw_1$. 

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From Lemma 6.5 it follows easily that the only two words \( \tilde{w} \in W_n \) such that \( \tilde{w} \preceq w \) are \( \tilde{w} = w \). In order to construct the 1-skeleton \( D_n(1) \) of \( D_n \), we add an oriented edge \( c_{w} \) from \( c_{w^{-}} \) to \( c_{w^{+}} \). The edge may be assumed to be contained in \( P_n(w^{-}) \cup P_n(w) \cup P_n(w^{+}) \) and to cross \( P_n(w) \) transversally and precisely once; edges are also assumed to be simple, and disjoint except at endpoints. Again, the inclusion \( D_n(1) \subset L_n(1) \) is a homotopy equivalence.

The four sides of Figure 1 above provide examples of edges of cases (i) and (ii). Figure 4 shows the edges of \( D_n \). Here we omit the initial and final words \( w_0 \) and \( w_1 \) (since they are not involved anyway). We also prefer to present examples (instead of spelling out conditions as above; a more formal discussion is given in Sections 8 and ??). Thus, cases (i), (ii) and (iii) are represented by \( [ba] \), \( [ab] \) and \( [ac] \), respectively.

![Figure 4: Examples of edges of \( D_n \).](image)

In a homological language, we write

\[
\partial[b] = ba - a, \quad \partial[a] = b - ab, \quad \partial[c] = ca - ac.
\]

Notice that in all cases \( \tilde{w} = \tilde{w}^{+} = \tilde{w}^{-} \in \text{Quat}_{n+1} \), consistently with the fact that we are constructing disjoint complexes \( D_n(z) \), \( z \in \text{Quat}_{n+1} \).

We take the occasion to compute the connected components of \( D_n(z) \) (i.e., we are reproving in our notation the main result of [6]). We need only consider words of dimension 0 (the vertices of \( D_n \)), i.e., words in the generators \( a_k \), \( 1 \leq k \leq n \). Write \( w_0 \sim w_1 \) if \( c_{w_0} \) and \( c_{w_1} \) are in the same connected component; thus, if \( w_0 \preceq w_1 \) then \( w_0 \sim w_1 \) and if \( w_0 \sim w_1 \) then \( \hat{w}_0 = \hat{w}_1 \) (in \( \text{Quat}_{n+1} \)).

For \( w \) the empty word, the vertex \( c_w \) is not attached to any edge and thus forms a contractible connected component. This of course corresponds to \( P_n(()) \), the component of convex curves; notice that \( \hat{a}a\hat{a}a = \hat{w} = 1 \) but \( aaaa \not\sim w \).

**Proposition 10.1.** Consider two non-empty words \( w_0, w_1 \in D_n \) of dimension 0. Then \( w_0 \sim w_1 \) if and only if \( \hat{w}_0 = \hat{w}_1 \in \text{Quat}_{n+1} \).

The proof we present now is rather direct.

\[ \text{Proof.} \] We have already seen that \( w_0 \sim w_1 \) implies \( \hat{w}_0 = \hat{w}_1 \). We prove the other implication. A basic word is either:

(i) a non-empty word \( a_{k_1}a_{k_2}\cdots a_{k_l} \) with \( k_1 < k_2 < \cdots k_l \);

(ii) of the form \( aaa a_{k_2}\cdots a_{k_l} \) with \( k_1 < k_2 < \cdots k_l \) (here the word \( aa \) corresponding to \( l = 0 \) is allowed);
In particular, words of length 1 are basic. Clearly, for each \( z \in \text{Quat}_{n+1} \) there exists a unique basic word \( w \) with \( z = \hat{w} \).

Using the edges above, first notice that
\[
\begin{align*}
\text{aaa} & \sim \text{abab} \sim \text{bb} \sim \text{bcbc} \sim \text{cc} \sim \cdots \sim a_k a_k; \\
\text{a}_{k+1}a_k & \sim a_{k+1}a_k a_k a_{k+1} \sim a_{k+1}a_k a_{k+1};
\end{align*}
\]
furthermore, \( \text{baa} \sim \text{babab} \sim \text{aab} \) and, for \( k > 2 \), \( a_k a a \sim a_{k+1} a a \). Also,
\[
a \sim \text{bab} \sim \text{abababa} \sim a a a a.
\]
Thus \( aa \) commutes with all generators \( a_k \) and can be brought to the beginning of the word; other generators either commute (\( a_k a_l \sim a_l a_k \) if \( |k - l| \neq 1 \)) or anticommute (\( a_{k+1} a_k \sim a a a_k a_{k+1} \)). Thus, for an arbitrary non-empty word \( w \), generators can be arranged in increasing order of index, at the price of creating copies of \( aa \) which are taken to the beginning of the word. Duplicate generators can also be transformed into further copies of \( aa \). Finally, if there are more than 4 copies of \( a \), they can be removed 4 by 4 thus arriving at a basic word.

**11 Product cells and other simplifications**

This definition of valid complexes is perhaps too abstract. In this and the next section we provide a more explicit description. In this section, we show that in many cases the cell \( c_w \) need not be constructed from scratch, but may be produced from previous cells. In the next section we shall consider the cases for which no such reduction is possible.

As a first simple example, consider \( w \) of the form \( w = w_0 \alpha_1 w_1 \cdots \alpha_m w_m \) where \( \dim(w_j) = 0 \) and \( \dim(\alpha_j) = 1 \) (some of the \( w_j \) may equal the empty word). Set \( c_w \) to be a product cell of dimension \( m \), i.e., the \( m \)-th dimensional cube
\[ c_w = c_{w_0} \times c_{\alpha_1} \times c_{w_1} \times \cdots \times c_{\alpha_m} \times c_{w_m}. \]
Notice that its boundary is contained in \( D_n[I_w] \), so the gluing instructions are legitimate. See Figure 5 for the following examples:
\[
\begin{align*}
\partial([ba][ab]) &= [ba]a b a + bab[ab] - [ba]b - a[ab], \\
\partial([ac][b][ac]) &= [ac]b a c + cab[ac] - [ac]b c a - a c b[ac].
\end{align*}
\]
More generally, if \( w \in \mathcal{W}_n \) is a word of length greater than 1, possibly containing more than one letter of positive dimension, define \( c_w \) as a product cell. As
before, write \( w = w_0 \alpha_1 w_1 \cdots \alpha_n w_m \) where \( \dim(w_j) = 0 \) and \( \dim(\alpha_j) > 0 \) (some of the \( w_j \) may equal the empty word). Set
\[
c_w = c_{w_0} \times c_{\alpha_1} \times c_{w_1} \times \cdots \times c_{\alpha_m} \times c_{w_m}.
\]
Here we assume the cells \( c_{\alpha_j} \) to have been previously constructed.

Finally, let \( \alpha \in A_n \) be a letter of dimension \( k \) (i.e., a permutation) such that \( 1 \alpha = 1 \), write \( \alpha = [a_{n_1} \cdots a_{n_{k+1}}] \) and \( s = -1 + \min n_j > 0 \). Set \( \hat{\alpha} = [a_{n-s} \cdots a_{n_{k+1}-s}] \) the cell \( c_{\hat{\alpha}} \) is assumed to be already constructed. Define \( c_\alpha \) from \( c_{\hat{\alpha}} \) by adding \( s \) to the index of every generator of every letter. Notice that this fits with our construction of the 1-skeleton; see also Figure 3 for \( c_{[bc\hat{b}]} \).

**12 Transversal sections**

In this section we show how to construct an explicit transversal section to \( \mathcal{P}_n(\alpha) \subset \mathcal{L}_n \) for \( \alpha \in A_n \), \( \dim(\alpha) = k \). We first present this as an algorithm, then provide examples.

Let \( P_\alpha, H \in B_{n+1} \) be permutation matrices as in Section 4; let
\[
E = \text{diag}(\det(H P_\alpha), 1, \cdots, 1) \in \text{Diag}_{n+1}, \quad Q_0 = E H P_\alpha \in B_{n+1}^+.
\]

We first construct an explicit transversal section \( \psi : \mathbb{R}^{k+1} \to SO_{n+1} \) to the Bruhat cell \( \text{Bru}_{Q_0} \subset SO_{n+1} \) passing through \( Q_0 = \psi(0) \). There are \( k + 1 \) zero entries in \( Q_0 \) which are simultaneously below a nonzero entry and to the left of a nonzero entry: these are the pairs \( (i, j) \) for which \( j < (n + 2 - i) \alpha \) and \( n + 2 - (j \alpha^{-1}) < i \). First put a new variable \( x_l \), \( 1 \leq l \leq k + 1 \), in each such position; number the positions in the same order you would read or write them on a page (top to bottom and left to right). This defines a matrix \( \tilde{M} \in (\mathbb{R}[x_1, \ldots, x_{k+1}])^{(n+1) \times (n+1)} \) or, equivalently, a function \( \tilde{\psi} : \mathbb{R}^{k+1} \to \text{GL}_{n+1}^+ \) where \( \tilde{\psi}(x) \) is obtained by evaluating \( \tilde{M} \) at \( x \in \mathbb{R}^{k+1} \). Finally, do a QR factorization: in order to compute \( \psi(x) \) compute \( M = \tilde{\psi}(x) \), factor \( M = QR \) where \( Q \in SO_{n+1} \), \( R \in U_{n+1}^+ \) and set \( \psi(x) = Q; \psi : \mathbb{R}^{k+1} \to SO_{n+1} \) is the desired transversal section to the Bruhat cell \( \text{Bru}_{Q_0} \).

We now construct \( \phi : \mathbb{R}^k \to \mathcal{L}, \) the transversal section to \( \mathcal{P}_n(\alpha) \subset \mathcal{L} \). Consider \( \mathbb{R}^k \subset \mathbb{R}^{k+1} \) defined by \( x_{k+1} = 0 \). Let \( N \) be the lower triangular nilpotent matrix whose only nonzero entries are \( N_{j+1,j+1} = 1 \). For each \( x \in \mathbb{R}^k \) define a curve \( \tilde{\phi}(x; \cdot) : \mathbb{R} \to GL_{n+1}^+ \) by the IVP
\[
\frac{d}{dt} \tilde{\phi}(x; t) = \tilde{\phi}(x; t) N, \quad \tilde{\phi}(x; 0) = \tilde{\psi}(x),
\]

Figure 5: The cells \([ba][ab]\) and \([ac]b[ac]\).
so that \( \tilde{\phi}(x; t) = \tilde{\psi}(x) \exp(tN) \). Since entries of \( \tilde{\phi}(x; t) \) are polynomials in \( x \) and \( t \), we may equivalently consider the matrix \( M \in (\mathbb{R}[x; t])^{(n+1) \times (n+1)} \), \( M(x, t) = \tilde{\phi}(x; t) \), whose entries are polynomials in \( x \) and \( t \), of degree at most \( n \) in the variable \( t \) and satisfying

\[
(M)_{i,j+1} = \frac{d}{dt}(M)_{i,j}.
\]

Let us see a few simple examples. Take \( n = 2 \) and \( \alpha = [aba] = [321] \), so that \( k = 2 \). We have \( E = I \) and

\[
\tilde{M} = \begin{pmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
y & z & 1
\end{pmatrix}, \quad M = \begin{pmatrix}
1 & 0 & 0 \\
t + x & 1 & 0 \\
\frac{t^2}{2} + y & t & 1
\end{pmatrix};
\]

here \( x = x_1 \), \( y = x_2 \) and \( z = x_3 \); recall that the variable \( z = x_3 \) is dropped in \( M \).

Take \( n = 3 \) and \( \alpha = [acb] = [3142] \), so that \( k = 2 \). We have \( E = \text{diag}(-1, 1, 1, 1) \) and

\[
\tilde{M} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & x & 0 & 1 \\
1 & 0 & 0 & 0 \\
y & z & 1 & 0
\end{pmatrix}, \quad M = \begin{pmatrix}
-t & -1 & 0 & 0 \\
\frac{t^3}{6} + xt & \frac{t^2}{2} + x & t & 1 \\
1 & 0 & 0 & 0 \\
\frac{t^2}{2} + y & t & 1 & 0
\end{pmatrix};
\]

here \( x = x_1 \), \( y = x_2 \) and \( z = x_3 \).

Recall from Section 4 that \( m_j(t) \) is the determinant of the south-west \( j \times j \) minor of \( \Gamma_x(t) \). Similarly, let \( p_{a_j}(t) = p_j(t) \) be the determinant of the corresponding minor of \( M \), so that \( p_j \) is an explicit polynomial in rational coefficients in \( t \) and \( x \) (or \( x_i \) for \( 1 \leq i \leq k \)). Following the examples above, for \( \alpha = [aba] \) we have

\[
p_a(t) = \frac{t^2}{2} + y, \quad p_b(t) = \frac{t^2}{2} + xt - y
\]

and for \( \alpha = [acb] \) we have

\[
p_a(t) = \frac{t^2}{2} + y, \quad p_b(t) = t, \quad p_c(t) = \frac{t^2}{2} - x.
\]

Notice that the leading coefficient of \( p_j \) is always a constant multiple of \( t^{\text{mult}_j(\alpha)} \). In particular, there exists \( r > 0 \) such that if \( |x| \leq 1 \) then all roots (in \( t \)) of all polynomials \( p_j \) have absolute value smaller than \( r \). In particular, there exist \( E_0, E_1 \in \text{Diag}_{n+1}^{+} \) such that:

\[
\forall x \in \mathbb{D}^k, \forall t \leq -r, \exists U_0, U_1 \in \text{Up}^+, \tilde{\phi}(x; t) = U_0 E_0 A^\top U_1;
\]

\[
\forall x \in \mathbb{D}^k, \forall t \geq r, \exists U_0, U_1 \in \text{Up}^+, \tilde{\phi}(x; t) = U_0 E_1 A U_1;
\]

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here $A$ is the Arnold matrix. In our example, $E_0 = \text{diag}(+1, -1, -1, +1)$ and $E_1 = -I$.

We now construct the desired transversal surface $\phi : \mathbb{D}^k \to \mathcal{L}_n(E_0E_1) \subset \mathcal{L}_n$. Given $x \in \mathbb{D}^k$ and $t \in [\frac{1}{4}, \frac{3}{4}]$, take $\tilde{t} = 4rt - 2r$ (so that $\tilde{t} \in [-r, r]$) and compute $M = E_0\tilde{\phi}(x; \tilde{t})$ and perform a QR factorization $M = QR$ (with $Q \in \text{SO}_{n+1}$ and $R \in \text{Up}^+$): set $\phi(x)(t) = Qe_1$ so that $\tilde{\mathcal{F}}(\phi(x))(t) = Q$. For each $x$, the arcs $\phi(x)|_{[0, \frac{1}{4}]}$ and $\phi(x)|_{[\frac{3}{4}, 1]}$ are chosen arbitrarily as a convex arcs from $\tilde{\mathcal{F}}(\phi(x))(0) = I$ to $\tilde{\mathcal{F}}(\phi(x))(\frac{1}{4})$ and from $\tilde{\mathcal{F}}(\phi(x))(\frac{3}{4})$ to $\tilde{\mathcal{F}}(\phi(x))(1) = E_0E_1$, respectively. Notice that $\tilde{\mathcal{F}}(\phi(x))(\frac{1}{4})$ and $\tilde{\mathcal{F}}(\phi(x))(\frac{3}{4})$ belong to the correct open (signed) Bruhat cells for the desired convex arcs to exist. Notice also that the set of such arcs is (nonempty and) contractible, showing that the choice is essentially unique. The convex arcs at the extremes are therefore not interesting: we shall not talk about them. Also, given $x$, the functions $p_j(\tilde{t})$ and $m_j(t)$ differ by a positive (algebraic) multiplicative factor so that the multiplicity vector and the itinerary can be defined from $p$ exactly as from $m$. We shall therefore restrict out attention to the matrix $M$.

**Lemma 12.1.** Consider $\alpha \in A_n$, $\dim(\alpha) = k$ and construct the map $\phi : \mathbb{D}^k \to \mathcal{L}_n$ as above. This map is transversal to $\mathcal{P}_n(\alpha)$, with a unique intersection at $x = 0$.

**Proof.** Easy. [TO BE WRITTEN, BUT HERE??] \hfill $\square$

Notice that the map $\phi$ provides us with a transversal orientation to $\mathcal{P}_n(\alpha)$ and therefore to the cells of $\mathcal{D}_n$.

In our first example ($\alpha = [aba]$), $p_a$ has two real roots $t = \pm \sqrt{-2y}$ if $y < 0$ and $p_b$ has two real roots $t = -x \pm \sqrt{x^2 + 2y}$ if $y > -\frac{x^2}{2}$. Thus, if $y > 0$ the itinerary is $bb$ and if $y < -\frac{x^2}{2}$ the itinerary is $aa$. If $x < 0$ (resp. $x > 0$) and $-\frac{x^2}{2} < y < 0$ the itinerary is $abab$ (resp. $baba$); the reader should compare these results with Figures 1 and 3.

In our second example ($\alpha = [acb]$), $p_b$ has a simple root at $t = 0$. If $y > 0$, $p_a$ has no real roots; if $y < 0$, $p_a$ has roots $t = \pm \sqrt{-y}$. Similarly, for $x < 0$, $p_c$ has no real roots and for $x > 0$, $p_c$ has roots $t = \pm \sqrt{x}$. It is now easy to verify the itineraries in Figure 10. Notice that this section is transversal to $\mathcal{P}_3([acb])$ (as promised) but is not transversal to $\mathcal{P}_3([acb][ac])$.

**Figure 6:** A transversal section to $\mathcal{P}_3([acb])$.

Notice that at a point $(x, y)$ where the itinerary includes the letter $[ab]$ (or any individual letter of larger multiplicity vector), both the resultant $\text{Res}_t(p_a, p_b)$ and the discriminant $\text{Disc}_t(p_a)$ must be equal to zero. Define

$$q_{[ab]} = \gcd(\text{Res}_t(p_a, p_b), \text{Disc}_t(p_a)).$$
More generally, the polynomial

$$q_{[a, a_j]} = \begin{cases} \gcd(\text{Res}_t(p_i, p_j), \text{Disc}_t(p_i)), & |i - j| = 1, \\ \text{Res}_t(p_i, p_j), & j > i + 1 \end{cases}$$

equals zero whenever the itinerary includes the letter $[a_i a_j]$. In each connected component of the union of the zero sets of the polynomials $q_*$, the itinerary is constant equal to a word of dimension 0, a vertex of $\mathcal{D}_n$.

In the example $\alpha = [acb]$, $q_{[ba]} = q_{[ac]} = 1$, $q_{[ab]} = y$, $q_{[bc]} = x$ and $q_{[ac]} = (x + y)^2$. This is consistent with Figure 10, but notice that $q_{[ac]} = 0$ on the dotted diagonal half-line in the second quadrant even though neither an $[ac]$ letter nor any change of itinerary occurs on that line (but both complex roots of $p_a$ and $p_c$ coincide at these points).

For each $l$, $1 \leq l \leq k$, let $(i_l, j_l)$ be the position of $x_l$ in $\tilde{M}$: set $d_l = (n + 2 - i_l)\alpha - j_l > 0$ (for $\alpha = [acb]$ we have $d_1 = d_2 = 2$). The following lemma shows that it is enough to study the map $\phi$ on the sphere $S^{k-1}$ and that this restriction has symmetries.

**Lemma 12.2.** (i) For all $\lambda \in \mathbb{R}$,

$$p_j(\lambda^{d_1} x_1, \ldots, \lambda^{d_i} x_l, \ldots, \lambda^{d_k} x_k; \lambda t) = \lambda^{\text{mult}_j(\alpha)} p_j(x_1, \ldots, x_k; t).$$

(ii) For $\lambda > 0$, the itineraries of the curves $\phi(x_1, \ldots, x_k)$ and $\phi(\lambda^{d_i} x_1, \ldots, \lambda^{d_k} x_k)$ are equal.

(iii) The itinerary of the curve $\phi((-1)^{d_1} x_1, \ldots, (-1)^{d_k} x_k)$ is obtained from that of $\phi(x_1, \ldots, x_k)$ by time reversal.

**Proof.** Easy. [TO BE WRITTEN]

### 13 From sections to cells

Our construction of a transversal section in the previous section comes with an orientation. For many cells, our section provides a natural definition of the cell: just follow the construction in Sections 8, 10 and ???. The reader is invited to verify that, for cells of dimension 1, this construction obtains the cells described in Section ??; even the orientation, which then appeared to be rather arbitrary, is consistent. In the next few sections we will follow this procedure to obtain several higher dimensional cells.

In the case $\alpha = [acb]$, however, if we want to construct a CW complex we are not allowed to glue part of the boundary of $c_{[acb]}$ on the 2-dimensional cell
We need therefore to perturb our section so that it becomes transversal to both $\mathcal{P}_3([acb])$ and $\mathcal{P}_3([ac]b[ac])$.

Take

$$M = \begin{pmatrix}
-t & -1 & 0 & 0 \\
\frac{t^3}{6} + xt & \frac{t^2}{2} + x & t & 0 \\
ut + 1 & u & 0 & 0 \\
\frac{t^2}{2} + y & t & 1 & 0
\end{pmatrix}$$

where $u$ is to be thought of as a fixed real number of small absolute value: say, $|u| < 1/4$. The construction in the previous paragraphs corresponds of course to $u = 0$. Figure ?? shows the resulting sections: the green cusp is given by the polynomial $q_{[ac]}$ which is now no longer a perfect square.

Figure 7: Two other transversal sections to $\mathcal{P}_3([acb])$ ($u < 0$ and $u > 0$); two different valid cells.

Notice that the two diagrams differ not just geometrically but combinatorially. For $u < 0$, the itinerary $acbac$ appears and $cabca$ does not; for $u > 0$ it is the other way round. Two different valid cells $c_{[acb]}$ are given by these two diagrams. Fortunately, the presence of the product cell $[ac]b[ac]$ in $\mathcal{D}_3^{[1]}$, shown in Figure 5, tells us that this choice does not affect the homotopy type of $\mathcal{D}_3$. In homological language we write

$$\partial [acb] = c[ab]c - [cb] - [ab] + a[cb]a + \begin{cases}
-acb[ac] + [ac]bac & \text{or} \\
[ac]bca - cab[ac].
\end{cases}$$

On a related note, the five immediate predecessors of $[acb]$ are: $[cb]$, $[ab]$, $c[ab]c$, $a[cb]a$ and $[ac]b[ac]$. This may surprise us, since, after all, $\dim([ac]b[ac]) = \dim([acb]) = 2$ (and $\mult([ac]b[ac]) = \mult([acb])$). This shows that $\mathcal{W}_n$ is not a graded poset for $n \geq 3$ (and is of course consistent with previous results). This difficulty is also the reason introducing the poset structure was necessary for our definition of valid complexes and for our proof of Lemma 8.1.

14 The complex $\mathcal{D}_n^{[2]}$

Given $n$ and the simplifications in the previous section, there is a finite (and short) list of possibilities of letters of dimension 2. In $\mathcal{A}_2$, the only letter of dimension 2 is $[aba]$: Figure 2 shows the elements of $\mathcal{W}_2$ (or $\mathcal{W}_n$) below $[aba]$, i.e., the smallest lower set containing $[aba]$. See also Figures 1 and 3 for a transversal surface to the submanifold $\mathcal{P}_2([aba])$ and for the cell $c_{[aba]}$. 
In \( \mathcal{A}_3 \) we also have \([bcb]\) (which is of course similar to \([aba]\), as in Figure 3) and \([acb]\), which we already discussed and for which valid cells are shown in Figure 10. The three remaining cells are and \([abc]\), \([bac]\) and \([cba]\), for which transversal sections and valid cells are shown in Figure 6; in a homological notation:

\[
\partial[abc] = abc[ab] + a[bc] + [ac] - [bc]a - [ab]cba + ab[ac]ba,
\partial[acb] = [ac]b - [bc]ab - bc[ba] - b[ac] - ba[bc] - [ba]cb,
\]

Figure 8: The cells \(c_{[abc]}\), \(c_{[bac]}\) and \(c_{[cba]}\).

In \( \mathcal{D}_4 \) there exist faces similar to those above (such as \([bcd]\), which is similar to \([abc]\)) but also a few genuinely new ones: \([abd]\), \([acd]\), \([adc]\) and \([bad]\), all shown in Figure 8; more generally, we have

\[
\partial[aba_k] = ab[aa_k] + a[ba_k]a + [aa_k]ba + a_k[ab] - [ba_k] - [ab]a_k, \quad k \geq 4;
\partial[aa_k a_{k+1}] = a[a_k a_{k+1}] + [a a_{k+1}] - [a_k a_{k+1}]a
- a_k a_{k+1} [aa_k] - a_k [aa_{k+1}] a_k - [aa_k] a_{k+1} a_k, \quad k \geq 3;
\partial[aa_{k+1} a_k] = a[a_{k+1} a_k] + [a a_{k+1}] a_{k+1} + a_{k+1} [aa_k] a_{k+1}
+ a_{k+1} a_{k+1} [aa_k] - [a_{k+1} a_k] a - [aa_k], \quad k \geq 3;
\partial[baa_k] = [aa_k] + a_k [ba] - [ba_k] ab - b [aa_k] b - ba [ba_k] - [ba] a_k, \quad k \geq 4.
\]

The only genuinely new cell in \( \mathcal{D}_5 \) is \([a_1 a_3 a_5]\), shown in Figure 9; we have

\[
\partial[aa_k a_l] = a[a_k a_l] + [a a_l] a_k + a_l [aa_k] - [a_k a_l] a - a_k [aa_l] - [aa_k] a_l, \quad 3 \leq k < l - 1.
\]

Figure 9: The cells \(c_{[abd]}\), \(c_{[acd]}\), \(c_{[adc]}\) and \(c_{[bad]}\)

Figure 10: The cell \(c_{[a_1 a_3 a_5]}\).

15 The 3-skeleton of \( \mathcal{D}_n \)

We now present the 3-skeleton of \( \mathcal{D}_n \). For each word \( w \in \mathcal{W}_n \) of dimension 3 we must glue a 3-cell \( c_w \) to the already constructed 2-skeleton. Thus, for each such \( w \) we must describe the boundary \( \partial c_w \). If \( w \) has three letters of dimension 1 or
one letter of dimension 2 and one letter of dimension 1 then $c_w$ is a product cell and its description follows from what we already saw. We must therefore describe $\partial c_\alpha$ for each letter of dimension 3. This is essentially equivalent to describing the map $\phi : S^2 \to L_n$ described in the previous section.

Let us consider a first example in some detail. Take $\alpha = [abac] = [4213]$. Then we have

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & z & x & 1 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} \frac{t^2}{2} & t & 1 & 0 \\ 1 & 0 & 0 & 0 \\ t + x & 1 & 0 & 0 \\ \frac{t^3}{6} + zt + y & \frac{t^2}{2} + z & t & 1 \end{pmatrix},$$

$$p_a = \frac{1}{6}t^3 + zt + y, \quad p_b = \frac{1}{3}t^3 + \frac{1}{2}xt^2 + (xz - y), \quad p_c = t,$$

$$q_{[ab]} = \frac{8}{9}z^3 + y^2, \quad q_{[ba]} = x^3 + 6xz - 6y, \quad q_{[ac]} = -y, \quad q_{[bc]} = xz - y, \quad q_{[cb]} = 1.$$ 

Plots of the intersection of these surfaces with the unit sphere reveal that space is decomposed into 18 open regions with distinct words of dimension 0.

Figure 11: A transversal section to $\mathcal{P}_3([abac])$. The north pole is $e_2$ and $-e_3$ is in the center.

In Figure 11 we open the unit sphere as a world map, with poles $\pm e_2$; north (up) is $+e_2$ and south is $-e_2$. The horizontal line in the middle is the equator $x_2 = 0$, with the points $e_3, e_1, -e_3, -e_1$ and again $e_3$ from left to right. Zero sets of $q_{[ab]}, q_{[ba]}, q_{[ac]}$ and $q_{[bc]}$ are shown in black, red, green and blue, respectively.

Words for some open regions are also shown. For instance, $(x, y, z) = (0, 1, 0)$ is in the interior of one of these regions and the corresponding word is $acb$. These 18 words are the vertices in $\mathcal{D}_3$ of the desired cell $c_{[abac]}$. There are 12 half-curves originating from the origin, which can be obtained by taking intersections of the above surfaces. For instance, $(0, 0, z)$ for $z > 0$ has the word $[bac]$ but for $z < 0$ has the word $a[bac]a$. In $\mathcal{D}_3$, these are the faces of the desired cell, whose combinatorics is shown in Figure 12. We thus obtain the list of all 12 immediate predecessors in the poset $W_n$ of the word $[abac]$:
Figure 12: A valid boundary for \([abac]\).

Figure 13: A transversal section to \(\mathcal{P}_3([abcb])\). Poles \(\pm e_3\); the horizontal line in the middle is the equator \(x_3 = 0\), with the point \(e_1\) in the center. Zero sets of \(q_{[ab]}, q_{[ac]}, q_{[bc]}\) and \(q_{[cb]}\) are shown in black, green, blue and magenta, respectively.

There are twenty letters of dimension 3 in \(A_4\): the five in the previous list, another five which are obtained by translating indices ([bcbd], [bcde], \ldots), two which break into blocks ([abad] and [acde]) and eight which are really new: [abcd], [abdc], [acbd], [adcb], [badc], [bcad] and [dcba]. Valid gluing maps \(\partial c_\alpha : S^2 \to \mathcal{D}_3\) for their cells are shown in Figures 21, \ldots
Figure 14: A valid boundary for $[abcb]$.

Figure 15: A transversal section to $P_3([bacb])$. Poles $\pm e_1$; the horizontal line in the middle is the equator $x_1 = 0$, with the points $-e_3$ and $e_2$ to the left and right of the center, respectively. Zero sets of $q_{[ab]}$, $q_{[ba]}$, $q_{[ac]}$, $q_{[bc]}$ and $q_{[cb]}$ are shown in black, red, green, blue and magenta, respectively. There are points $[acb]b$ in $e_1$ and $b[acb]$ in $-e_1$, not shown. As in Figure 10, the polynomial $q_{[ac]}$ is a square and the green lines are actually double. The one at the center has the effect of swapping two pairs $ac \leftrightarrow ca$; the one appearing twice at both ends of the figure detects the coincidence of complex roots and therefore swaps nothing.

Figure 16: A valid boundary for $[bacb]$. 
Figure 17: A valid boundary for \([acba]\).

Figure 18: A valid boundary for \([bcba]\).
Figure 19: A valid boundary for $[abcb]$. 

Figure 20: A transversal section to $\mathcal{P}_3([abcd])$. Poles $\pm e_2$; the horizontal line in the middle is the equator $x_2 = 0$, with the points $e_1$ and $-e_3$ to the left and right of the center, respectively.
Figure 21: A valid boundary for $[abcd]$. 

Figure 22: A transversal section to $P_3([abdc])$. Poles $\pm e_2$; the horizontal line in the middle is the equator $x_2 = 0$, with the points $e_1$ and $-e_3$ to the left and right of the center, respectively. The line $q_{[bd]} = 0$ at the center has the effect of swapping two pairs $bd \leftrightarrow db$; the one appearing twice at both ends of the figure swaps nothing.

Figure 23: A valid boundary for $[abcd]$. 

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Figure 24: A transversal section to $\mathcal{P}_3([abcd])$. Poles $\pm e_2$; the horizontal line in the middle is the equator $x_2 = 0$, with the points $e_1$ and $-e_3$ to the left and right of the center, respectively. The line $q_{[bd]} = 0$ at the center has the effect of swapping two pairs $bd \leftrightarrow db$; the one appearing twice at both ends of the figure swaps nothing.

Figure 25: A valid boundary for $[abcd]$.

Figure 26: A valid boundary for $[acbd]$.

Figure 27: A transversal section to $\mathcal{P}_3([acbd])$. Poles are $\pm e_1$.

Figure 28: A transversal section to $\mathcal{P}_3([bacd])$. Poles are $\pm e_1$. 
16 Loose and tight maps

In this section $\ast$ denotes the concatenation of curves: if $\gamma_0 \in \mathcal{L}_n(1)$ and $\gamma_1 \in \mathcal{L}_n(z)$ then $\gamma_0 \ast \gamma_1 \in \mathcal{L}_n(z)$ is given by

$$(\gamma_0 \ast \gamma_1)(t) = \begin{cases} 
\gamma_0(2t), & t \leq 1/2; \\
\gamma_1(2t - 1), & t \geq 1/2.
\end{cases}$$

We have $w_{\gamma_0 \ast \gamma_1} = w_{\gamma_0} \eta w_{\gamma_1}$ (where $\eta \in \mathcal{A}_n$ is the Coxeter element).

Let $\gamma_0 \in \mathcal{L}_n(1)$ be a fixed non-convex closed curve. Let $M$ be a compact manifold and consider a map $\phi : M \to \mathcal{L}_n(z)$: $\phi$ is loose if $\phi$ is homotopic to

$$\gamma_0 \ast \phi : M \to \mathcal{L}_n(z)
\quad s \mapsto \gamma_0 \ast \phi(s);$$

otherwise, $\phi$ is tight. These definitions generalize those in [4]; there, $n = 2$ and $\gamma_0$ is a fixed curve. It is easy to see, however, that the definition does not depend on the choice of $\gamma_0$. Indeed, if $\gamma_1 \in \mathcal{L}_n(1)$ is another non-convex closed curve then $\gamma_0$ and $\gamma_1$ are homotopic. Also, $\gamma_0$ is homotopic to $\gamma_0 \ast \gamma_0$.

The following result partially justifies the interest in considering loose maps.

**Lemma 16.1.** Let $M$ be a compact manifold and consider maps $\phi_0, \phi_1 : M \to \mathcal{L}_n(z)$. If $\phi_0$ and $\phi_1$ are both loose and $\mathfrak{F}_{\phi_0}, \mathfrak{F}_{\phi_1} : M \to \Omega \Spin_{n+1}$ are homotopic then $\phi_0$ and $\phi_1$ are homotopic.

**Proof.** This result is proved in [5] and, for $n = 2$, in [4]. Since this is such a crucial result we present here a brief sketch of proof.

![Figure 29: Any curve becomes locally convex if we add loops.](image)

The idea is the following: if $\phi_0$ is loose then it is homotopic to $\gamma_0 \ast \phi_0$ and therefore to $\gamma_0 \ast \cdots \ast \gamma_0 \ast \phi_0$ (for a very large number of copies of $\gamma_0$). The copies of $\gamma_0$ are then spread along $\phi_0$. At this stage our curve looks like a phone wire (see Figure 25); the many loops allow us to follow the homotopy in $\Omega \Spin_{n+1}$ and keep our curves locally convex. \qed
We now restate Lemma 6.6 in [4], with minor changes. First of all, we are now in arbitrary dimension. Second, the concept of “adding two loops” has to be adapted. The proof, however, is very much the same.

**Lemma 16.2.** Let $K$ be a compact manifold and $f : K \rightarrow \mathcal{L}_n$ a continuous map. Assume that:

- $t_0 \in (0, 1)$ and $t_{1,-}, t_{1,+}, t_{2,-}, \ldots, t_{J,+} : K \rightarrow (0, 1)$ are continuous functions with $t_0 < t_{1,-} < t_{1,+} < t_{2,-} < \cdots < t_{J,+}$;
- $K = \bigcup_{1 \leq j \leq J} U_j$, where $U_j \subset K$ are open sets;
- for $p \in U_j$, $\mathcal{S}_{f(p)}(t_{j,-}(p)) = \mathcal{S}_{f(p)}(t_{j,+}(p))$ and $f(p)|_{[t_{j,-}(p), t_{j,+}(p)]}$ is non-convex.

Then $f$ is loose.

**Proof.** [WHAT SHOULD WE WRITE HERE?] \hfill \Box

We want to adapt these definitions to our combinatorial notation. If $\phi : M \rightarrow \mathcal{D}_n(z_1)$ is a map and $w_0 \in \mathcal{W}_n$ is a word with $\hat{w}_0 = z_0$ then $w_0\phi : M \rightarrow \mathcal{D}_n(z_0 z_1)$ is defined by $(w_0\phi)(s) = w_0\phi(s)$ (for all $s \in M$). The smooth case corresponds to the special case $w_0 = w\eta$, $\hat{w}_0 = 1$. The special case in [4] corresponds to $w_0 = \eta\eta$.

**Lemma 16.3.** Let $M$ be a compact manifold and consider a map $\phi : M \rightarrow \mathcal{D}_n(z)$: $\phi$ is loose if and only if $\phi$ is homotopic to $aaaa\phi$.

We now provide a combinatorial version of Lemma 14.2.

**Lemma 16.4.** Let $M$ be a compact manifold and consider a map $\phi : M \rightarrow \mathcal{D}_n(z)$. Assume that the word of every top dimensional face includes the substring $aaaa$ (at some point). Then $\phi$ is loose.

**Proof.** Take a smooth generic perturbation $f : M \rightarrow \mathcal{L}_n$ of $\phi$ so that top dimensional faces become vertices $v_1, \ldots, v_J$ of a decomposition of $M$. The substrings $aaaa$ exist in the open neighborhoods of these vertices. By slightly shrinking the neighborhoods, we obtain the open sets $U_j$. The curves can be deformed so that the $aaaa$ substring contains a closed non-convex arc, as desired. \hfill \Box
17 The map $H_*$

Let $I_\ast \subset W_n$ be the set of non-empty words whose first letter has dimension at most 1; $I_\ast$ is a lower set and $L_n[I_\ast] \subset L_n$ is an open set; the inclusion $D[I_\ast] \subset L[I_\ast]$ is a weak homotopy equivalence.

Lemma 17.1. The inclusion $i : D[I_\ast] \to L_n$ is a loose map.

Proof. We construct an explicit homotopy map $H_\ast$. The $H_\ast$ map takes any cell $c_w : D^k \to D_n, w \in I_\ast$, to a homotopy $c_w^{H_\ast} : [0, 1] \times D^k \to D_n$ from $c_w$ to $c_{aaaaw}$ in such a way that $\partial c_w^{H_\ast} = c_{aaaaw} - c_w + c_w^{H_\ast}$: this condition guarantees that these cells can be glued to define the desired homotopy $H_\ast$. We omit the $c$'s for conciseness. We also have $(\alpha w)^{H_\ast} = \alpha w^{H_\ast}$; it is therefore sufficient to define $\alpha w^{H_\ast}$ for each letter $\alpha \in A_n, \dim(\alpha) \leq 1$.

For a non-empty word $w \in W_n$, we define $w^{H_\ast}$, a path in $D_n$ from the vertex $c_w$ to the vertex $c_{aaaaw}$ (notice that for the empty word no such path exists). The simplest example is:

$$a^{H_\ast} = (a \leftarrow bab \leftarrow ababababa \leftarrow \cdots \leftarrow ababababa \leftarrow \cdots \leftarrow aaaaaw),$$

which already appeared implicitly when we discussed connectivity. If the word starts with $a$, we fall back on this example:

$$(aw)^{H_\ast} = a^{H_\ast}w = (aw \to babw \leftarrow \cdots \leftarrow abababaw \leftarrow \cdots \leftarrow aaaaaw).$$

Next, define

$$b^{H_\ast} = (b \leftarrow aba \rightarrow aaaaaba \rightarrow aaaaab) = (b \leftarrow aba \to babba \leftarrow \cdots \leftarrow ababababa \leftarrow \cdots \leftarrow aaaaaba \to aaaab)$$

and $(bw)^{H_\ast} = b^{H_\ast}w$. Define recursively

$$a_k^{H_\ast} = (a_k \leftarrow a_k a_{k+1} \leftarrow a_k a_{k+1} a_k \leftarrow \cdots \leftarrow a_k a_{k+1} a_k a_k a_{k+1} \leftarrow \cdots \leftarrow a_{k}a_{k+1}a_k)$$

and $(a_kw)^{H_\ast} = a_k^{H_\ast}w$.

We are left therefore with the problem of consistently defining $a^{H_\ast}$ for $\alpha$ a letter of dimension 1. This is done on a case by case basis: $[a_k a_{k+1}], [a_{k+1} a_k]$ and $[a_k a_l]$ for $l > k + 1$.

We first observe that the case $[a_k a_{k+1}]$ is rather trivial: the square collapses to the segment in the definition of $a_k^{H_\ast}$. We define $[ac]^{H_\ast}$ in Figure 26.

The top and bottom hexagons (they look like triangles, but they have six vertices each) are $[abc]$ and $aaa[aabc]$, as indicated. Each of the three central trapezoidal regions is actually composed of six product cells (since $a^{H_\ast}$ is not one
edge, but six). The definition of \([ak a_{k+2}]^H_\star\) is similar: just substitute \(a_k\), \(a_{k+1}\) and \(a_{k+2}\) for \(a\), \(b\) and \(c\) in Figure 26 (a minor difference is that \(a_k^H_\star\) is actually \(4 + 2k\) edges). We define \([ad]^H_\star\) in Figure 27.

Notice that the second of the four central trapezoidal regions uses the previous construction of \([ac]^H_\star\). The other three consist of several product cells. Again, the definition of \([ak a_{k+3}]^H_\star\) is similar. The definition of \([ak a_{k+4}]^H_\star\) uses that of \([ak a_{k+3}]^H_\star\) and so on, recursively.

This takes care of the cases \([ak a_l]\) for \(l > k\); we are left with the case \([a_{k+1} a_k]\).

We now describe \([cb]^H_\star\) in Figure 29.

This definition uses a specific cell for \([acb]\); a similar diagram works for the other cell. The definition of \([a_{k+1} a_k]^H_\star\) for \(k > 2\) is similar, just substitute \(a_{k-1}\), \(a_k\) and \(a_{k+1}\) for \(a\), \(b\) and \(c\) (except for the initial \(aaaa\) in the lower part of the diagram).

Finally, in order to define \([ba]^H_\star\) we have to fill in the trapezoidal region, indicated by (?) in Figure 28; notice that there is a partial collapse at the top.

This can be done explicitly but is rather messy; we prefer to use Lemmas 14.1 and 14.4. We need to show that the map from the circle to the trapezoid is loose. Draw the circle as the interior curve in the right part of Figure 28. First apply \(H_\star\) to the initial \(a\) on the left arc; this defines our map in the region with vertices \(abaab\), \(aaaaabaab\), \(aaaaaabaab\), \(aaaaabab\). Next apply \(H_\star\) to the final \(b\) on the right arc, including the segments \(abaab\), \(aaaaabaab\) and \(aaaaabab\), \(aaaaaabaab\); this defines our map in the rest of the region. Notice that the two operations above commute, in the sense that we could have first applied \(H_\star\) to the final \(b\) and only then to the initial \(a\) with the same end result. Lemma 14.4 implies that the map on the outer curve is loose, and therefore so is the map in the inner curve. Since \(\Omega\ Spin_{n+1}\) is simply connected, Lemma 14.1 implies that \([ba]^H_\star\) can be defined. This completes the construction of \(w^H_\star\).

\[\square\]
18 Easy letters and the map $H_*$

A letter $\alpha \in \mathcal{A}_n$ is easy if $\dim(\alpha) \leq 1$ or $\alpha = [a_k a_l a_{l+1}]$ (where $k < l$) or $\alpha = [a_k a_{k+2} a_{k+1}]$; otherwise, $\alpha$ is hard. Thus, for instance, the only hard letter in $\mathcal{A}_2$ is $[aba]$. Notice that the easy 2-cells are the ones explicitly used in the construction of $H_*$; the relevance of this remark will become evident below.

Let $\mathcal{I}_* \subset \mathcal{W}_n$ be the set of non-empty words including at least one easy letter: $\mathcal{I}_*$ is a lower set and $\mathcal{L}_n[\mathcal{I}_*] \subset \mathcal{L}_n$ is an open set. For $n = 2$, $$\mathcal{I}_* = \mathcal{W}_2 \setminus \{(), [aba], [aba][aba], \ldots, [aba][aba] \cdots [aba], \ldots\}.$$ 

**Lemma 18.1.** The inclusion $i : \mathcal{D}_n[\mathcal{I}_*] \to \mathcal{L}_n$ is a loose map. Moreover, the required homotopy between $i$ and $\text{aaaai}$ can be constructed with image in $\mathcal{L}_n[\mathcal{I}_*]$.

**Proof.** Let $\mathcal{I}_{*2} \subset \mathcal{W}_n$ be the set of non-empty words whose first letter is easy. Clearly, $\mathcal{I}_{*2}$ is a lower set with $\mathcal{I}_* \subset \mathcal{I}_{*2} \subset \mathcal{I}_*$. We show how to extend $H_*$ to $\mathcal{D}[\mathcal{I}_{*2}]$ (for simplicity we also call this extension $H_*$).

First notice that $H_*$ can be extended rather trivially to cells of the form $\alpha = [a_k a_{k+1} a_{k+2}]$. Indeed, the definition of $H_*$ at the side $[a_k a_{k+2}]$ (see Figure 26) causes the map $\alpha^B = aaaa - \alpha + (\partial a)^{H_*}$ to collapse. Figure 26 thus also gives us a construction for $\alpha^{H_*}$. The reader may find it helpful to compare this situation with the construction of $[ab]^{H_*}$.

A similar construction obtains $\alpha^{H_*}$ for $\alpha = [a_k a_l a_{l+1}]$ (where $k < l$): use $[a_k a_{l+1}]^{H_*}$ and Figure 27. Similarly, for $\alpha = [a_k a_{k+2} a_{k+1}]$ we define $\alpha^{H_*}$ using $[a_{k+2} a_{k+1}]^{H_*}$ and Figure 29.

We now construct an (almost) explicit homotopy $H_*$ cell by cell. Consider a word $w_0 \in \mathcal{W}_n[\mathcal{I}_*]$; we define $w_0^{H_*}$ by applying $H_{*2}$ to each easy letter of $w_0$; hard letters are left alone. Strictly speaking, this is not a consistent definition: if $w_0 \preceq w_1$ both belong to $\mathcal{W}_n[\mathcal{I}_*]$ and, furthermore, the gluing instructions for $\partial w_1$ involve $w_0$ then the homotopy $H_*$ should agree on these two cells, but does
not quite: the homotopy $H_{*2}$ may have been applied to more letters in $w_0$ then $w_1$ knows about. We present a few examples.

[EXAMPLES HERE]

Moreover, going through the construction of $H_*$ in the proof of Lemma 15.1 and of $H_*$ above, we see that all cells created belong to $I_{*2}$. This requires some clarification for $[ba]^H_*$ since in these cases the construction relies on Lemma 14.1 and is therefore not explicit.

[EXPLAIN MORE]

**Corollary 18.2.** Consider $z \in \text{Quat}_n$. The inclusion $\mathcal{D}_n[I_\ast](z) \subset \Omega \text{Spin}_{n+1}$ is a weak homotopy equivalence.

For $n = 2$ and $z = \pm 1$, the set $\mathcal{L}_n[I_\ast](z)$ is called $Y_z$ in [4]. A major intermediate result in that paper is that the inclusion $\mathcal{L}_n[I_\ast](z) = Y_z \subset \Omega S^3$ is a weak homotopy equivalence, a special case therefore of our last corollary.

Item (iii) of Theorem 1 now follows directly from the next result.

**Corollary 18.3.** Every connected component of $\mathcal{D}_n$ is simply connected.

*Proof.* The connected component of the convex curves is contractible. Otherwise, it follows from Corollary 16.3 that $\mathcal{D}_n[I_\ast](z)$ is simply connected. But $\mathcal{D}_n(z)$ is obtained from this by attaching cells of dimension at least 2. □

19 Good upper sets

A subset $Y \subset A_n$ is a $(\preceq)$-upper set if it is an upper set for the order $\preceq$, i.e., if

$$\forall \alpha \in A_n \forall \beta \in A_n ((\alpha \preceq \beta) \land (\alpha \in Y)) \rightarrow (\beta \in Y).$$

Notice that here we use the order $\preceq$ of the poset $W_n$; this should not be confused with a $(\leq)$-upper set (for the Bruhat order in $S_{n+1}$). Notice that a $(\preceq)$-upper set $Y \subset A_n \subset W_n$ is also an upper subset of $W_n$ (with $\preceq$). Recall that

$$\alpha \trianglerighteq \beta \rightarrow \alpha \preceq \beta, \quad \alpha \preceq \beta \rightarrow ((\alpha \preceq \beta) \land (\hat{\alpha} = \hat{\beta}))$$

where $\preceq$ denotes the Bruhat order in $A_n \subset S_{n+1}$ and $\hat{\alpha} \in \text{Quat}_{n+1}$.

For a $(\preceq)$-upper set $Y \subset A_n$, partition $W_n = \mathcal{I}_Y \sqcup \mathcal{U}_Y$, where $\mathcal{U}_Y$ is the upper set of words *all* of whose letters belong to $Y$ (this includes the empty word). Clearly, $\mathcal{I}_Y \subset W_n$ is the lower set of non-empty words including *at least one* letter not in $Y$. Thus, $Y_n(Y) = \mathcal{L}_n[\mathcal{I}_Y] \subset \mathcal{L}_n$ is an open subset, $\mathcal{M}_n(Y) = \mathcal{L}_n \setminus Y_n(Y) \subset \mathcal{L}_n$ is a closed subset and $\mathcal{D}_n[\mathcal{I}_Y] \subset \mathcal{D}_n$ is a subcomplex.
A \((\leq)\)-upper set \(Y \subset A_n\) is **good** if there exists homotopies

\[
H : [0, 1] \times L_n[I_Y] \to L_n[I_Y], \quad H : [0, 1] \times D_n[I_Y] \to D_n[I_Y]
\]
such that \(H(0, \cdot)\) is the identity and \(H(1, w) = aaaaw\) (one homotopy exists if and only if the other one does). In Lemma 16.2, we saw an example of a good set: 
\(Y_{\text{hard}}\) consists of all hard letters, thus including all cells of dimension at least 3 and some letters of dimension 2. The following result generalizes Corollary 16.3.

**Lemma 19.1.** Let \(Y \subset A_n\) be a good set; let \(z \in \text{Quat}_{n+1}\). The inclusions 

\[
D_n[I_Y](z) \subset \Omega \text{Spin}_{n+1} \quad \text{and} \quad Y_n(Y; z) \subset \Omega \text{Spin}_{n+1}
\]

are weak homotopy equivalences.

The point of introducing these definitions is to formulate the next lemma, which will allow us to prove that smaller sets are also good.

**Lemma 19.2** (Reduction Lemma). Let \(Y_0 \subset A_n\) be a good set. Let \(\alpha, \beta \in Y_0\) be such that \(\dim(\alpha) \geq 1\), \(\alpha \lhd \beta\) and \(\partial\beta \smallsetminus \alpha\) is contained in \(D_n[I_{Y_0}]\). Then 

\(Y_1 = Y_0 \setminus \{\alpha, \beta\}\) is also a good set.

**Proof.** Let \(H\) be the homotopy from \([0, 1] \times D_n[I_{Y_0}]\) to \(D_n[I_{Y_0}]\), as in the definition. We need to extend the homotopy \(H\) to \([0, 1] \times D_n[I_{Y_1}]\) by defining \(\alpha^H\) and \(\beta^H\) consistently with previous definitions. We perform the same construction done in Figure 26 for \(\alpha = [ac] \lhd [abc]\), Figure 27 for \(\alpha = [ad] \lhd [acd]\) and Figure 29 for \(\alpha = [cb] \lhd [acb]\) (see also the proof of Lemma 16.2). More generally, consider the boundary conditions \(\alpha^B\) for \(\alpha^H\): this is a cylinder with \(\alpha\) in one end, \(aaaaa\alpha\) on the other and sides \((\partial\alpha)^H\). Start by attaching copies of \(\beta\) and \(aaaa\beta\) to the ends, thus covering the faces \(\alpha\) and \(aaaaa\alpha\). The new boundary is now by hypothesis in \(D_n[I_{Y_0}]\) and can therefore be filled in face by face (as in the examples). This completes the construction of \(\alpha^H\).

In order to construct \(\beta^H\), proceed as in the proof of Lemma 16.2: the construction of \(\alpha^H\) made \(\beta^H\) somewhat trivial. (Notice that the constructions of the pairs \(b^H \lhd [ab]^H\), \(c^H \lhd [bc]^H\), ... can be considered as very low dimensional and somewhat degenerate examples of the same construction. The pair \(a^H \lhd [ba]^H\) does not fit, it had to be done otherwise.)

\[\square\]

**Proposition 19.3.** For \(n = 3\), the following set is good:

\[
Y_3 = \{[aba], [bcb], [cba], [bacb], [abacba]\}.
\]

**Proof.** From Lemma 16.2, the following set is good:

\[
Y_{\text{hard}} = \{[aba], [bac], [bcb], [cba], [abac], [abcb], [acba], [bacb], [bcba], [abacb], [abcba], [bacba], [abacba]\}.
\]
We apply the Reduction Lemma (Lemma ??) in this order to the following pairs: 
\[ bac \triangleleft [bac], \ [abcb] \triangleleft [abcb], \ [acba] \triangleleft [acba]. \] 
We must now verify the hypothesis, namely, that at each step \( \partial \beta \setminus \alpha \) is contained in the current \( D_n[I_Y] \). Indeed, \( \ldots \) [EXPLAIN]

A permutation \( \pi \in S_{n+1} \) is parity alternating (or PA) if and only if \( i^\alpha \equiv (i+1)^\alpha \mod 2 \) for all \( i \in \{1, \ldots, n\} \). The set of parity alternating permutations is a subgroup of \( S_{n+1} \). We use the same notation for a letter \( \alpha \in A_n \). Thus, for instance, for \( n = 2 \) the only parity alternating letter is \([321] = [aba] \); the identity \([123]\) is also a PA permutation but is not a letter. For \( n = 3 \), the parity alternating letters are

\[
\begin{align*}
[2143] &= [ac],
[3214] &= [aba],
[4123] &= [abc],
[1432] &= [bc],
[2341] &= [cba],
[3412] &= [bacb],
[4321] &= [abac].
\end{align*}
\]

For \( n \geq 4 \), all parity alternating letters have dimension at least 2. The number of parity alternating letters is \( (\frac{n}{2})!(\frac{n}{2} + 1)! - 1 \) for \( n \) even and \( 2((\frac{n+1}{2})!)^2 - 1 \) for \( n \) odd. Recall that \( Z(\text{Quat}_{n+1}) \) (the center of the group) equals \( \{\pm1\} \) for \( n \) even and \( \{\pm1, \pm\hat{a}_1\hat{a}_3\cdots\hat{a}_n\} \) for \( n \) odd. For instance, \( Z(\text{Quat}_4) = \{\pm1, \pm\hat{a}_1\hat{a}_3\hat{a}_5\} \). A letter \( \alpha \in A_n \) is parity alternating if and only if \( \hat{\alpha} \in Z(\text{Quat}_{n+1}) \). Let \( Y_{PA} \subset A_n \) be the set of parity alternating letters: \( Y_{PA} \) is clearly a \( (\leq) \)-upper set. Let \( Y_n = Y_n(Y_{PA}) \subset L_n \) be the set of complicated curves; let \( M_n = L_n \setminus Y_n \subset L_n \) be the set of multiconvex curves. The following proposition completes the proof of Theorem ?? (and therefore of item (i) of Theorem 1).

**Proposition 19.4.** For \( n \geq 4 \), the set \( Y_{PA} \subset A_n \) is good.

**Proof.** Again, the idea is to use the Reduction Lemma several times. We first describe a matching among letters \( \alpha \in A_n \setminus Y_{PA} \). Given such \( \alpha \), let \( i \) be the smallest integer for which \( i^\alpha \equiv (i+1)^\alpha \mod 2 \). We say that \( \alpha \) is small if \( i^\alpha < (i+1)^\alpha \) and large otherwise. For small \( \alpha \), set \( \beta = [a_1\alpha] \) so that \( \alpha \triangleleft \beta \). Some of the first pairs are

\[
\begin{align*}
a &\triangleleft [ba], \quad b &\triangleleft [ab], \quad c &\triangleleft [bc], \quad d &\triangleleft [cd], \ldots \quad [cb] &\triangleleft [acb], \quad [dc] &\triangleleft [bdc], \ldots
\end{align*}
\]

which are consistent with the contraction above (except that we can not use the Reduction Lemma for \( a \triangleleft [ba] \)). We also have

\[
\begin{align*}
[ac] &\triangleleft [adc], \quad [ad] &\triangleleft [bad], \quad [bd] &\triangleleft [abd]
\end{align*}
\]

which are different from our previous construction but which also work well. We proceed in increasing order of dimension. We need to prove that the hypothesis of the Reduction Lemma holds.

Let \( \alpha \triangleleft \beta \) be a pair with \( \dim(\alpha) = k \), \( \dim(\beta) = k + 1 \). We need to prove that \( \partial \beta \setminus \alpha \) is contained in \( D_n[I_Y] \) at that stage. We prove that if \( w \leq \beta \), \( w \neq \beta \) and
$w \neq \alpha$ then $w$ includes a non parity alternating letter $\gamma$ with either $\dim(\gamma) < k$ or $\dim(\gamma) = k$ and $\gamma$ large (recall that $w$ is nonempty). We consider three cases: $w$ consists of letters of dimension less than $k$ (only); $w = \gamma \triangleright \beta$; $\ell(w) > 1$ and includes a letter $\gamma \triangleleft \beta$.

For the first case, $\hat{w} = \hat{\beta} \notin Z(\text{Quat}_{n+1})$ implies that at least one letter $\gamma$ of $w$ satisfies $\hat{\gamma} \notin Z(\text{Quat}_{n+1})$ and therefore $\gamma \notin Y_{PA}$. Since $\dim(\gamma) < k$ we are done.

For the second case, we must prove that $\gamma$ is large. Let $\beta = (i,i+1)\alpha = (i_0i_1)\gamma$, $i_0 < i_1$ and $i_0^\beta \equiv i_1^\beta \pmod{2}$. We must prove that $i^\gamma > (i+1)^\gamma$. If $\{i,i+1\}$ and $\{i_0,i_1\}$ are disjoint then $i^\gamma = i^\beta > (i+1)^\beta = (i+1)^\gamma$ (and we are done). If $i_0 = i$ and $i_1 > i+1$ we must have $(i+1)^\beta < i_1^\beta < i^\beta$ and therefore $i^\gamma = i_1^\beta > (i+1)^\beta = (i+1)^\gamma$. If $i_0 = i+1$ we have $i_1^\beta > i_0^\beta$ and $i^\gamma = i^\beta > i_0^\beta > i_1^\beta = (i+1)^\gamma$. The cases $i_1 = i$ and $i_1 = i+1$ are similar.

For the third case, if $\gamma \triangleright \beta$ we may argue as in the second case; we may therefore assume $\gamma \triangleleft \beta$, $\gamma \triangleright \beta$. Let $\beta = (i,i+1)\alpha = (i_0i_1)\gamma$, $i_0 < i_1$ and $d = i_0^\beta - i_1^\beta$ so that $d > 0$ is odd and

$$\mult_k(\gamma) = \mult_k(\beta) - d \left[ i_0 \leq k < i_1 \right].$$

In particular,

$$\sum_\delta \mult_k(\delta) = \begin{cases} \text{odd}, & i_0 \leq k < i_1, \\ 0, & \text{otherwise} \end{cases}$$

where $\delta$ runs over the letters of $w$ distinct from $\gamma$. There exists therefore such $\delta$ with $\hat{\delta} \notin Z(\text{Quat}_{n+1})$ and therefore $\delta \in A_n \setminus Y_{PA}$. If $\dim(\delta) < k$ we are done. Otherwise we have the situation of Lemma ?? with $\alpha_0 = \alpha$ and $\alpha_1 = \delta$. Going through the possibilities in the lemma, however, we see that none of them apply (they all have small dimension). This completes the proof.

\section{Tight maps}

A few examples of tight maps are in order.

**Proposition 20.1.** Consider the map $[aba]^B : \mathbb{S}^2 \to \mathcal{D}_2$ sketched in Figure ??: the bottom is $[aba]$, the top is $aaaa[aba]$ and the sides are $(\partial[aba])^{H^*}$: this map is tight.

Notice that extending $[aba]^B$ to a closed ball would provide a good candidate for the (non-existent!) $[aba]^{H^*}$. As we shall see, no such extension exists.

This proposition is already proved in [4] (in a different language).
Proof. The map $aaaa[aba]^B$ (obtained from $[aba]^B$ by attaching $aaaa$ at the start) is homotopic to a constant.

From Proposition 7.1, the set $\{[aba]\}$ is an upper set, implying that $P_2([aba])$ is a closed contractible submanifold of codimension 2. The transversal bundle is trivial, being described by Figure 1. The map $[aba]^B$ has exactly one transversal intersection with $P_2([aba])$, proving that $[aba]^B$ is not homotopic to a constant. 

\[\square\]

**Proposition 20.2.** Consider the map $[cba]^B : S^2 \to D_3$ sketched in Figure ??: the bottom is $[cba]$, the top is $aaaa[aba]$ and the sides are $(\partial[cba])^H$: this map is tight.

Proof. This proof has to be postponed. \[\square\]

Towards proving Theorem ??, we now describe the upper set $U_{Y_3}$ in more detail; here

$$Y_3 = \{[aba], [bcb], [cba], [bacb], [abacba]\}$$

is the upper set described in Proposition ??, Define $h = (h_0, h_1) : Y_3 \to \mathbb{N}^2$ by

$$h([aba]) = h([bacb]) = h([bcb]) = (1, 0), \quad h([cba]) = (0, 1), \quad h([abacba]) = (1, 1)$$

so that $\hat{\alpha} = (-1)^{h_0(\alpha)}(\hat{a}\hat{c})^{h_1(\alpha)}$ for all $\alpha \in Y_3$. Extend $h$ to $U_{Y_3}$: if $w = \alpha_1 \cdots \alpha_k$ then $h(w) = h(\alpha_1) + \cdots + h(\alpha_k)$. Clearly $\hat{\alpha} = (-1)^{h_0(w)}(\hat{a}\hat{c})^{h_1(w)}$; also, a case by case analysis shows that if $w_0, w_1 \in U_{Y_3}$ and $w_0 \preceq w_1$ then $h(w_0) = h(w_1)$. The sets $U_3^{(i,j)} = \mathcal{L}_3[U_3]$, $\Sigma_3^{(i,j)} \subset \mathcal{L}_3 \subset \mathcal{L}_3[\mathcal{L}_3^{(i,j)}]$ are therefore also upper sets and $\mathcal{M}_3^{(i,j)} = \mathcal{L}_3[U_3^{(i,j)}]$ are closed subsets: we claim they are contractible topological submanifolds of codimension $2(i+j)$.

We describe the upper set $U_3^{(i,j)}$. The words of minimal dimension in this set consist of $i$ letters in $\{[aba], [bcb]\}$ and $j$ letters $[cba]$, in any order. There are $2^i(i+j)$ such words and each such $w$ corresponds to a cell $c_w$ of dimension $2(i+j)$ in $D_3$ or to a contractible smooth submanifold $P_3(w) \subset L_3$. When two such words $w_+$ and $w_-$ differ by a single letter, they are of the form $w_+ = w_0[aba]w_1$ and $w_- = w_0[bcb]w_1$: in this case, their cells are contained in the boundary of $c_{w_0}, w = w_0[bcb]w_1$. Alternatively, the smooth sub manifold $P_3(w)$ (of codimension $2(i+j)+1$) glues the two smooth sub manifolds $P_3(w_+)$ and $P_3(w_-)$ (of codimension $2(i+j)$), so that the set $P_3(w_+) \cup P_3(w) \cup P_3(w_-)$ is a contractible topological submanifold, but is not smooth along $P_3(w)$. 

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References


