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Nondegenerate Curves on S^2 and Orbit Classification of the Zamolodchikov Algebra

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Abstract. The Zamolodchikov algebra is the next case after the Virasoro algebra in the natural hierarchy of the Poisson structures on linear differential equations. We describe here the complete classification of the symplectic leaves of this algebra. It turns out that each symplectic leaf is uniquely defined by the conjugacy class of the monodromy operator and one discrete (2- and 3-valued) invariant arising from the homotopy classes of nondegenerate curves.

1. Introduction

The Zamolodchikov algebra is the algebra generated by the coefficients of the third order linear differential equations on the circle with respect to the quadratic Poisson structure [Z]. There exists a hierarchy of Poisson algebras on linear differential operators of different order on the circle also called the $SL_n(\mathbb{R})$ ($GL_n(\mathbb{R})$)-Gelfand-Dikii algebras or generalized the KdV-structures [GD]. The first Poisson algebra in this series on second order differential equations (more precisely on the Hill's equations) coincides with the Virasoro algebra [Kh]. The classification of the Virasoro coadjoint orbits was obtained in different terms independently by Kuiper [Ku], Lazutkin and Pankratova [LP], Segal [S], Kirillov [Ki].

In the Virasoro case, the Poisson algebra is linear, while for differential operators of any higher order the corresponding structure is quadratic. In the paper [OK] the classification of symplectic leaves (or maximal symplectic submanifolds) of these Poisson brackets for arbitrary order operators was related to the homotopy classification of some special curves on spheres (or in projective spaces). Namely, an n^{th} order linear differential operator on the circle defines a nondegenerate quasiperiodic curve in S^{n-1} (the "projectivization" of its "solution curve," see Sect. 2). It turned out that two differential operators belong to the same symplectic leaf iff the corresponding curves are homotopically equivalent.

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The only continuous (or "local") invariant of a symplectic leaf is the monodromy operator [an element of the group $SL_n(\mathbb{R})$] of the corresponding differential equation [OK]. The discrete (or "global") invariant is the number of connected components of the space of nondegenerate curves with given monodromy. We present the classification of these curves for the case of $SL_3(\mathbb{R})$-bracket. It turns out that the number of connected components is finite and equals two or three for different monodromy matrices in $SL_3(\mathbb{R})$. The different values of this discrete invariant, in fact, split the group $SL_3(\mathbb{R})$ into two parts of nonzero measure. It would be interesting to find the physical sense of this "global" invariant in terms of conformal field theory.

The relation of the SL_3-Gelfand-Dikii structure and the problem of differential geometry was discussed in [O], where the identity monodromy case was considered.

The second section is devoted to a geometric formulation of the main result, and the third section concerns the Poisson aspect of our consideration. Details of the proofs will be published elsewhere.

2. Spaces of Curves

Definition 2.1. A curve $\gamma:[0,1]\to S^2$ is called nondegenerate if at any moment $t\in[0,1]$ its velocity $\dot{\gamma}(t)$ and acceleration $\ddot{\gamma}(t)$ are linearly independent.

This property of a curve depends only on the image $\gamma([0,1])$ in S^2, but not on the particular choice of the parametrization.

For a fixed orientation on S^2 we will consider only "right-oriented" curves, for which the pair $(\dot{\gamma}(t),\ddot{\gamma}(t))$ at an arbitrary moment t defines the given orientation on S^2.

Remark 2.2. The motivation of the definition above is as follows. With any third order linear ordinary differential equation (LDE) $P\phi=0$, one can associate the class Γ_P of GL_3-equivalent curves in $\mathbb{R}^3$. Namely for any curve $\gamma_P\in\Gamma_P$, its coordinates $(\phi_1(t),\phi_2(t),\phi_3(t))=\gamma_P(t)$ (in arbitrary basis) form a fundamental solution of LDE $P\phi=0$. The crucial property of such γ_P is that $\gamma_P(t),\dot{\gamma}_P(t),\ddot{\gamma}_P(t)$ are linearly independent for any t. In particular, this means that the radial projection of the curve $\gamma_P\in\Gamma_P$ along $\gamma_P(t)$ on the standard embedded unit sphere $S^2\subset\mathbb{R}^3$ is a nondegenerate curve.

An analogous description is valid for any dimension and allows to study the topological properties of the space of nondegenerate curves instead of the corresponding spaces of LDE's.

For each LDE on the circle (i.e. with periodic coefficients) we consider its monodromy operator (the transform of its solutions for the period). This operator determines its conjugacy class in GL_n (two monodromy operators can be compared only up to conjugacy, since they act in different spaces of solutions). Now we define the monodromy operator for a nondegenerate curve.

Definition 2.3. A curve $\gamma:[0,1]\to S^2\subset\mathbb{R}^3$ is said to be subordinate to a given monodromy matrix $M\in GL_3^+(\mathbb{R})$ if the image $M(F_0)$ of the flag F_0, spanned by $\langle\gamma(0),\dot{\gamma}(0),\ddot{\gamma}(0)\rangle$ (i.e. "extended initial flag") coincides with the "extended final flag" $F_1=\langle\gamma(1),\dot{\gamma}(1),\ddot{\gamma}(1)\rangle$.

Consider the space of all nondegenerate curves starting with the same initial flag F_0 and subordinated to matrices M from a fixed conjugacy class $\mathcal{M}$ in $GL_3^+(\mathbb{R})$. This space of curves is said to be the space of curves $\Gamma(\mathcal{M})$ subordinated to the given monodromy operator $\mathcal{M}$. [Notice that the spaces $\Gamma(\mathcal{M})$ corresponding to different initial flags F_0 are naturally conjugated by an element of GL_3^+.]

A deformation of a nondegenerate curve within the space $\Gamma(\mathcal{M})$ of nondegenerate curves subordinate to a given monodromy operator $\mathcal{M}$ is called a homotopy of this curve. In particular, we allow homotopies changing the monodromy matrix M and hence the final flag $F_1 = M(F_0)$ (but preserving the flag F_0 and the monodromy operator $\mathcal{M}$, i.e. the conjugacy class of M).

The problem under consideration is to describe the topology of the space of nondegenerate curves with given monodromy operator. This question is closely related to certain problems of infinite-dimensional Lie algebras and conformal field theory (see [OK] or Sect. 2).

In 1970 J. Little found the homotopy classification of all closed nondegenerate curves on S^2. This case corresponds to the identity monodromy $\mathcal{M} = \mathrm{id}$.

Proposition 2.4 [L]. *The space of all right-oriented closed curves on S^2 consists of three connected components with the representatives:*

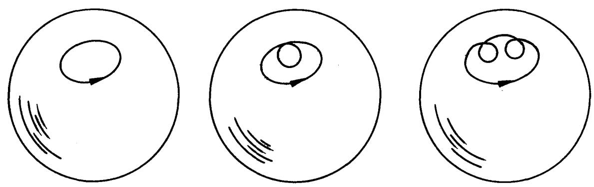

Fig. 1

Our main result is the following classification theorem for nondegenerate curves on S^2 with arbitrary monodromy $\mathcal{M} \subset GL_3^+$.

Theorem 2.5. *The space of all right-oriented nondegenerate curves on S^2 with given monodromy $\mathcal{M}$ consists of two connected components if the Jordan normal form of $\mathcal{M}$ is one of the following:*

$$(*)\quad \begin{pmatrix} -\lambda & & 0 \\ & -\nu & \\ 0 & & \mu \end{pmatrix}, \quad \begin{pmatrix} -\lambda & & 0 \\ & -\lambda & \\ 0 & & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \mu \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix}$$

($\lambda, \mu, \nu > 0$ are distinct real numbers),

and of three components otherwise.

The proof of the theorem is based on a detailed investigation of the deformations, disconjugacy and covering homotopy properties of the corresponding curves. The next two theorems present the main steps of the proof and are of independent interest.

Definition 2.6. The curve $\gamma : [0,1] \to S^2$ is called conjugate if there exists a great circle (an equator) on S^2 having at least three transversal intersections with $\gamma((0,1))$.

The curves violating this property are called disconjugate.

Theorem 2.7. *The space of right-oriented curves on S^2 with given initial and final flags consists of three connected components if for these flags there exists a disconjugate curve connecting them and consists of two components otherwise.*

Denote the space of all right-oriented conjugate curves with a fixed initial flag F_0 by $C(F_0)$ and the map taking each curve to its final flag by $\pi: C(F_0) \to FO_3$. Here FO_3 is the space of all oriented flags on S^2 (coinciding with the space of all oriented flags in the linear space $\mathbb{R}^3$).

Theorem 2.8. *For any flag F_0 the map $\pi: C(F_0) \to FO_3$ satisfies the one-dimensional covering homotopy property.*

Remark 2.9. The analogous classification problem for nondegenerate curves in higher dimensions is still an open question. The number of connected components of nondegenerate curves is known only for closed curves on S^n [SS] and turns out to be equal to three on any S^{2k} and to two on any S^{2k+1} for $k \geqq 1$.

3. Classification of Symplectic Leaves

In this section we recall the general definition of the Gelfand-Dikii quadratic Poisson brackets on the coefficients of LDE and their relation to nondegenerate curves on spheres following [OK]. The Poisson algebra of functions on the space of third order LDE [of the form $\partial^3 + u(t)\partial^2 + v(t)$] is also called the Zamolodchikov- or classical W_3-algebra.

Definition 3.1. Consider the space $\mathscr{L}$ of all differential operators L of the form $\partial^n + \sum_{i=0}^{n-1} u_i(t)\partial^i$, where $\partial = d/dt$, $u_i \in C^\infty(S^1, \mathbb{R})$. The space of all linear functionals on $\mathscr{L}$ is described in terms of "pseudodifferential symbols" $X = \sum_{j=1} a_j(t)\partial^{-j}$, $a_j \in C^\infty(S^1, \mathbb{R})$. Namely, associate with each X the linear functional $l_X(L) = \int_{S^1} \operatorname{res}(XL)dt$, where $\operatorname{res}(XL)$ is a function on S^1 which is defined as follows. Using the Leibnitz rule $\partial^{-1}f = f\partial^{-1} + \sum_{i=1}^{\infty} (-1)^i f^{(i)}\partial^{-1-i}$, we can express the product $X \cdot L$ as a pseudodifferential operator $\sum_{m \in \mathbb{Z}} p_m(t)\partial^m$. Then by definition $\operatorname{res}(XL) = p_{-1}(t)$. The space $\mathscr{L}$ is an affine space (rather than a linear one), but all functionals l_X vanish at the point $L_0 = \partial^n$, so L_0 can be viewed as the origin of $\mathscr{L}$. It is clear that each linear functional on the space $\mathscr{L}$ can be considered of the form l_X where X is a pseudodifferential symbol.

Definition 3.2. The operator $\Omega: l_X \mapsto V_X \in \operatorname{Vect}(\mathscr{L})$ which associates with a linear functional l_X the vector field $V_X(L) = L(XL)_+ - (LX)_+L$ on the space of operators (here the index $+$ denotes the differential part) is called the operator of the second Gelfand-Dikii Poisson structure associated with $GL_n(\mathbb{R})$. This operator defines the quadratic (with respect to L) Poisson bracket on $\mathscr{L}$: $\{l_X, l_Y\}(L) = l_Y(V_X(L))$. The corresponding Poisson algebra of functionals is called the Gelfand-Dikii algebra.

Remark 3.3. The $SL_n(\mathbb{R})$-Gelfand-Dikii bracket is defined on the space $\tilde{\mathscr{L}} = \mathscr{L} \cap \{u_{n-1}(t) \equiv 0\} = \left\{\partial^n + \sum_{i=0}^{n-2} u_i(t)\partial^i\right\}$ by the same formula. The restrictions on $\{X\}$ are determined explicitly from the condition $V_X(L) \in \operatorname{Vect}(\tilde{\mathscr{L}})$ [i.e. $\partial^n + V_X(L) \in \tilde{\mathscr{L}}$].

In the $SL_2(\mathbb{R})$ case, this bracket turns out to be linear and coincides with the Lie-Poisson bracket on the dual space to the Virasoro algebra [Kh].

Remark 3.4. In [OK] the description of symplectic leaves of these Poisson brackets (or maximal nondegenerate submanifolds on which these Poisson structures are invertible) was reduced to the homotopy classification of nondegenerate curves. More precisely, with each n^{th} order operator L on a circle one can associate a nondegenerate curve on $S^{n-1} \subset \mathbb{R}^n$ (see Remark 2.2) with the corresponding monodromy operator.

It should be mentioned that the differential operator L can be uniquely reconstructed if we know the corresponding curve on the sphere and the coefficient $u_{n-1}(t)$. Indeed, the curve on S^1 gives us the homogeneous coordinates of the solution set of L. One complementary condition is provided by the Wronskian $W(t)$ of this set [$W(t)$ satisfies the Liouville equation $\dot{W} = u_{n-1}(t)W$]. In particular, for the $SL_n(\mathbb{R})$ case this condition is $W(t) \equiv \text{const}$.

Theorem 3.5 [OK]. *The complete set of invariants of symplectic leaves of the second Gelfand-Dikii brackets associated with the Lie groups $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$ consists of the monodromy operator (up to conjugacy) and the homotopy class of the corresponding nondegenerate curves subordinate to this monodromy on the sphere S^{n-1}.*

In other words, two differential operators on the circle can be connected by some "Hamiltonian path" in the space $\mathscr{L}$ (i.e., by a path such that its velocity vector at every instant is Hamiltonian with respect to the Gelfand-Dikii bracket) iff they have the same monodromy operator and the same homotopy class of the curves. In some sense, the monodromy is the "continuous" invariant and the homotopy class is the "discrete" one.

For the $SL_2(\mathbb{R})$ case, the classification problem of the Virasoro orbits becomes very transparent from this point of view. In fact, in this case we have to classify nondegenerate curves on S^1 and for every monodromy there exists a countable number of such curves distinguished by the total rotation number [OK].

For the $SL_3(\mathbb{R})$-bracket, this classification is the object of the preceding section and the number of homotopy classes turns out to be finite but depends on the monodromy:

Theorem 3.6 (or 2.5′). *Symplectic leaves of the Zamolodchikov algebra (i.e. the $SL_3(\mathbb{R})$-Gelfand-Dikii bracket) are enumerated by the Jordan normal form of the monodromy operator (belonging to $SL_3(\mathbb{R})$) and a $\mathbb{Z}_2$-invariant for the monodromy of types (*) or a $\mathbb{Z}_3$-invariant otherwise.*

Roughly speaking, the discrete invariant is the parity of the "total rotation number" of the corresponding nondegenerate curves (which are not necessarily closed for a monodromy $\mathscr{M} \neq \text{id}$). Moreover, for some monodromies the disconjugate curves form a separate symplectic leaf.

Remark 3.7. The same classification holds for the $GL_3(\mathbb{R})$-Gelfand-Dikii bracket, where, certainly, the monodromy operator belongs to the bigger group $GL_3(\mathbb{R})$. The identity monodromy case of $SL_3(\mathbb{R})$ was considered in [O].

It would be interesting to find a purely algebraic proof of this result similar to the Virasoro case. The disconjugacy property (Definition 2.6) is closely related to the factorization of a differential operator. In the recent work [W] the Gelfand-Dikii bracket was transferred to the space of solutions of the differential equations through this factorization. Perhaps this approach can lead to a Sturmian-type conjugacy theory for equations of higher order.

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