Computations in the deformed Fomin–Kirillov algebra

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1. Definitions

Consider a (noncommutative) ring R and a positive integer n. For any $i, j \in [n]$ (where $[n] := \{1, 2, \ldots, n\}$), let $X_{i,j} \in R$ be an element. Let $h \in R$ be a further element. Assume that these $n^2 + 1$ elements $X_{i,j}$ and h satisfy the following six axioms:

1. *Centrality:* The element h commutes with all $X_{i,j}$.
2. *Locality:* We have $\left[X_{i,j}, X_{u,v}\right] = 0$ whenever $\{i, j\} \cap \{u, v\} = \varnothing$. Here (and everywhere), $[a, b]$ denotes the commutator $ab - ba$ of two elements a, b of R.
3. *Irreflexivity:* We have $X_{i,i} = 0$ for all i.
4. *Antisymmetry:* We have $X_{i,j} = -X_{j,i}$ for all i, j.
5. *Involutivity:* We have $X_{i,j}^2 = h$ for all $i \neq j$.
6. *Tricyclicity:* We have $X_{i,j}X_{i,k} + X_{j,k}X_{j,i} + X_{k,i}X_{k,j} = h$ for all distinct i, j, k.

Then we say that the family $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$ is a *deformed Fomin–Kirillov family* (short: *dFK family*) in the ring R. Here are some important examples of dFK families:

1. The family with $R = \mathbb{Z}$ and $h = 1$ and $X_{i,j} = \operatorname{sign}(i - j)$, where $\operatorname{sign} x$ denotes the sign of a real number x (that is, 1 if $x > 0$, and 0 if $x = 0$, and -1 if $x < 0$). We call this the *sign family*.

2. The family with $R = \mathbb{Z}[z]$ (a polynomial ring) and $h = z^2$ and $X_{i,j} = \operatorname{sign}(i-j)\, z$. We call this the *blown-up sign family*.

   More generally, for any dFK family $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$ in any ring R and any central element $z \in R$, we can define a new dFK family $\left(zX_{i,j}\right)_{i,j\in[n]} \sqcup \left(z^2 h\right)$ in the same ring R.

3. The *universal dFK family*, in which the $\mathbb{Z}$-algebra R is freely generated by the $X_{i,j}$ and h subject to the six axioms imposed above. We let R_{dFK} denote the ring R in which this family lies. Every dFK family is an image of the universal dFK family under a ring morphism from R_{dFK}.

   Note that [to my knowledge – DG] little is known about the ring R_{dFK}; is it even free as a $\mathbb{Z}$-module?

4. The *universal FK family*, in which $h = 0$ but the $X_{i,j}$ are universal (i.e., the $\mathbb{Z}$-algebra R is freely generated by the $X_{i,j}$ subject to the above six axioms with $h = 0$).

Some remarks about the axioms are in order:

1. The centrality axiom entails that h commutes with all elements of the subring of R generated by the elements of the dFK family $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$. Thus, if the latter elements generate R as a $\mathbb{Z}$-algebra, then h is central in R, and the ring R becomes a $\mathbb{Z}[h]$-algebra.

2. The locality axiom needs to be imposed only for $|\{i,j,u,v\}| = 4$, since all other cases follow from the irreflexivity axiom.

3. The axioms have an S_n-symmetry: The symmetric group S_n of the set $[n]$ (with each permutation $\sigma \in S_n$ fixing h and sending each $X_{i,j}$ to $X_{\sigma(i),\sigma(j)}$) preserves them all. This translates into a genuine S_n-action on the universal ring R_{dFK}, but of course any specific dFK family might not have an S_n-action on it.

   However, there is an S_n-action on the **set** of all dFK families $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$ in a given ring R: Namely, a permutation $\sigma \in S_n$ sends each dFK family $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$ to $\left(X_{\sigma^{-1}(i),\sigma^{-1}(j)}\right)_{i,j\in[n]} \sqcup (h)$.

4. Replacing the ring R by its opposite ring R^{op} also preserves all six axioms and thus transforms dFK families into dFK families. Hence, for the universal ring R_{dFK}, there is an involutive isomorphism $R_{\mathrm{dFK}} \to R_{\mathrm{dFK}}^{\mathrm{op}}$ that fixes all generators h and $X_{i,j}$.

5. Thanks to the irreflexivity and the antisymmetry axioms, we can restrict a dFK family to only the elements h and $X_{i,j}$ with $i < j$ (and even h is unnecessary for $n \geq 2$, since the involutivity axiom rewrites it as $X_{1,2}^2$). However, the

tricyclicity axiom must then be rewritten as two axioms

$$X_{i,j}X_{i,k} - X_{i,j}X_{j,k} + X_{i,k}X_{j,k} = h \qquad \text{for all } i < j < k, \qquad \text{and}$$
$$X_{i,k}X_{i,j} - X_{j,k}X_{i,j} + X_{j,k}X_{i,k} = h \qquad \text{for all } i < j < k.$$

But this viewpoint makes the S_n-action invisible.

6. The universal ring R_{dFK} has a grading, in which all $X_{i,j}$ are homogeneous of degree 1 whereas h is homogeneous of degree 2.

7. If $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$ is a dFK family, then so is $\left(X_{i,j}\right)_{i,j\in[m]} \sqcup (h)$ for each $m \leq n$.

8. The tricyclicity axiom

$$X_{i,j}X_{i,k} + X_{j,k}X_{j,i} + X_{k,i}X_{k,j} = h \tag{1}$$

holds not only for all distinct i, j, k, but also more generally for all $i, j, k \in [n]$ with $|\{i, j, k\}| > 1$. This is because if $i = j$, then $X_{i,j} = 0$ and $X_{k,i}X_{k,j} = X_{k,i}^2 = h$ by involutivity.

The following is an easy consequence of tricyclicity:

Lemma 1.1 (commutator tricyclicity)**.** Let $i, j, k \in [n]$ be arbitrary. Then,

$$\left[X_{i,j},\ X_{i,k}\right] + \left[X_{j,k},\ X_{j,i}\right] + \left[X_{k,i},\ X_{k,j}\right] = 0.$$

This holds even if we don't require involutivity.

Proof. If $i = j$, then (in view of irreflexivity) the claim boils down to

$$[0,\ X_{i,k}] + [X_{i,k},\ 0] + [X_{k,i},\ X_{k,i}] = 0,$$

which is obvious. Thus, we WLOG assume that $i \neq j$. Similarly, we WLOG assume that $j \neq k$ and that $k \neq i$. Hence, i, j, k are distinct.

By the definition of a commutator, we have

$$\begin{aligned}
&\left[X_{i,j},\ X_{i,k}\right] + \left[X_{j,k},\ X_{j,i}\right] + \left[X_{k,i},\ X_{k,j}\right] \\
&= \left(X_{i,j}X_{i,k} - X_{i,k}X_{i,j}\right) + \left(X_{j,k}X_{j,i} - X_{j,i}X_{j,k}\right) + \left(X_{k,i}X_{k,j} - X_{k,j}X_{k,i}\right) \\
&= \underbrace{\left(X_{i,j}X_{i,k} + X_{j,k}X_{j,i} + X_{k,i}X_{k,j}\right)}_{\substack{=h \\ \text{(by tricyclicity)}}} - \underbrace{\left(X_{i,k}X_{i,j} + X_{k,j}X_{k,i} + X_{j,i}X_{j,k}\right)}_{\substack{=h \\ \text{(by tricyclicity, with the roles of } k \text{ and } j \text{ swapped)}}} \\
&= 0.
\end{aligned}$$

$\square$

2. The YJM elements

Now, given any dFK family $\left(X_{i,j}\right)_{i,j\in[n]} \sqcup (h)$, we define the *YJM-like elements* (short *YJM elements*)

$$z_j := \sum_{i\in[n]} X_{i,j} = \sum_{i\neq j} X_{i,j} = \sum_{i<j} X_{i,j} - \sum_{i>j} X_{j,i} \in R$$

for all $j \in [n]$. Note that the S_n-action permutes these elements:

$$\sigma\left(z_j\right) = z_{\sigma(j)} \qquad \text{for all } \sigma \in S_n \text{ and } j \in [n]$$

(this is a literal equality in the universal ring R_{dFK}, and a formal one for arbitrary dFK families).

We claim the following:

Proposition 2.1. The YJM elements $z_1, z_2, \ldots, z_n$ commute.
This holds even if we don't require involutivity.

Proof. It will be slightly more convenient to work with the negatives of these elements. Thus, set

$$\widetilde{z}_j := -z_j = -\sum_{i\in[n]} \underbrace{X_{i,j}}_{\substack{=-X_{j,i} \\ \text{(by antisymmetry)}}} = \sum_{i\in[n]} X_{j,i} = \sum_{i\neq j} X_{j,i}$$

for each $j \in [n]$. It suffices to prove that the elements $\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_n$ commute.

It suffices to show that $\widetilde{z}_i$ commutes with $\widetilde{z}_j$ for all $i \neq j$. Due to the S_n-action, we can WLOG assume that $i = 1$ and $j = 2$ (since $\sigma\left(\widetilde{z}_j\right) = \widetilde{z}_{\sigma(j)}$ for all $\sigma \in S_n$ and $j \in [n]$), so that we must show that $\widetilde{z}_1$ commutes with $\widetilde{z}_2$. (This WLOG assumption will just save us two letters here, but later on it will save us a whole layer of subscripts.)

We now observe that every $j \in [n] \setminus \{1\}$ satisfies

$$\left[\widetilde{z}_1,\ X_{2,j}\right] = \left[X_{2,j},\ X_{2,1}\right] + \left[X_{j,1},\ X_{j,2}\right]. \tag{2}$$

Proof of (2). Let $j \in [n] \setminus \{1\}$. We must prove (2). If $j = 2$, then (2) boils down to $0 = 0 + 0$, since $X_{2,2} = 0$. Thus, we WLOG assume that $j \neq 2$.

Thus,

$$\widetilde{z}_1 = \sum_{i\neq 1} X_{1,i} = X_{1,2} + X_{1,j} + \sum_{i\neq 1,2,j} X_{1,i}.$$

Now take the commutator with $X_{2,j}$. Since all the addends $X_{1,i}$ of the sum $\sum\limits_{i\neq 1,2,j} X_{1,i}$ commute with $X_{2,j}$ (by the locality axiom), we thus conclude that

$$\begin{aligned}\left[\widetilde{z}_1,\ X_{2,j}\right] &= \left[\underbrace{X_{1,2}}_{=-X_{2,1}},\ X_{2,j}\right] + \left[\underbrace{X_{1,j}}_{=-X_{j,1}},\ \underbrace{X_{2,j}}_{=-X_{j,2}}\right] = \underbrace{-\left[X_{2,1},\ X_{2,j}\right]}_{=\left[X_{2,j},\ X_{2,1}\right]} + \left[X_{j,1},\ X_{j,2}\right] \\ &= \left[X_{2,j},\ X_{2,1}\right] + \left[X_{j,1},\ X_{j,2}\right].\end{aligned}$$

This proves (2). □

Furthermore, from $\widetilde{z}_1 = \sum_{i \neq 1} X_{1,i} = \sum_{j \neq 1} X_{1,j}$, we obtain

$$[\widetilde{z}_1,\ X_{2,1}] = \left[\sum_{j \neq 1} X_{1,j},\ X_{2,1}\right] = \sum_{j \neq 1} \left[X_{1,j},\ \underbrace{X_{2,1}}_{=-X_{1,2}}\right] = \sum_{j \neq 1} \underbrace{\left(-\left[X_{1,j},\ X_{1,2}\right]\right)}_{=\left[X_{1,2},\ X_{1,j}\right]}$$
$$= \sum_{j \neq 1} \left[X_{1,2},\ X_{1,j}\right]. \tag{3}$$

Now, recall that $\widetilde{z}_2 = \sum_{i \in [n]} X_{2,i} = \sum_{j \in [n]} X_{2,j}$. Hence,

$$\begin{aligned}
[\widetilde{z}_1,\ \widetilde{z}_2] &= \left[\widetilde{z}_1,\ \sum_{j \in [n]} X_{2,j}\right] = \sum_{j \in [n]} [\widetilde{z}_1,\ X_{2,j}] = [\widetilde{z}_1,\ X_{2,1}] + \sum_{j \neq 1} [\widetilde{z}_1,\ X_{2,j}] \\
&= \sum_{j \neq 1} \left[X_{1,2},\ X_{1,j}\right] + \sum_{j \neq 1} \left(\left[X_{2,j},\ X_{2,1}\right] + \left[X_{j,1},\ X_{j,2}\right]\right) \qquad \text{(by (2) and (3))} \\
&= \sum_{j \neq 1} \underbrace{\left(\left[X_{1,2},\ X_{1,j}\right] + \left[X_{2,j},\ X_{2,1}\right] + \left[X_{j,1},\ X_{j,2}\right]\right)}_{\substack{=0 \\ \text{(by Lemma 1.1)}}} \\
&= 0.
\end{aligned}$$

That is, $\widetilde{z}_1$ and $\widetilde{z}_2$ commute. Hence z_1 and z_2 commute as well. This proves Proposition 2.1. □

3. The shell and cycle lemmas

We shall now generalize the crucial Lemmas 7.2 and 7.3 from [FK]. We call our generalizations the *shell lemma* and the *cycle lemma*, respectively. We state them only for the indices $1, 2, \ldots, m$, since the S_n-action on dFK families allows us to replace these indices by arbitrary distinct elements $i_1, i_2, \ldots, i_m$ of $[n]$.

Given any $i, j_1, j_2, \ldots, j_k \in [n]$, we set

$$X_{i,(j_1,j_2,\ldots,j_k)} := X_{i,j_1} X_{i,j_2} \cdots X_{i,j_k}.$$

For instance, $X_{2,(1,5)} = X_{2,1} X_{2,5}$.

The following generalizes [FK, Lemma 7.2]:

Lemma 3.1 (shell lemma)**.** Let $m \in [n]$. Then,

$$\begin{aligned}
&\sum_{j=2}^{m} X_{1,j}X_{1,j+1}\cdots X_{1,m}X_{1,2}X_{1,3}\cdots X_{1,j} \\
&= h\sum_{j=2}^{m} X_{1,j}X_{1,j+1}\cdots X_{1,m}X_{1,2}X_{1,3}\cdots X_{1,j-2}
\end{aligned}$$

(where the addend $X_{1,j}X_{1,j+1}\cdots X_{1,m}X_{1,2}X_{1,3}\cdots X_{1,j-2}$ on the right hand side has to be understood as $X_{1,j}X_{1,j+1}\cdots X_{1,m-1}$ if $j=2$).

Example 3.2. For $m=1$, this says that $0 = h\cdot 0$, which is obvious.

For $m=2$, this says that $X_{1,2}X_{1,2} = h$, which follows from the involutivity axiom.

For $m=3$, this says that

$$X_{1,2}X_{1,3}X_{1,2} + X_{1,3}X_{1,2}X_{1,3} = h\left(X_{1,2}+X_{1,3}\right).$$

For $m=4$, this says that

$$\begin{aligned}
&X_{1,2}X_{1,3}X_{1,4}X_{1,2} + X_{1,3}X_{1,4}X_{1,2}X_{1,3} + X_{1,4}X_{1,2}X_{1,3}X_{1,4} \\
&= h\left(X_{1,2}X_{1,3} + X_{1,3}X_{1,4} + X_{1,4}X_{1,2}\right).
\end{aligned}$$

Proof of Lemma 3.1. We shall prove the lemma by strong induction on m.

The cases $m=1$ and $m=2$ are immediate (note that the $m=2$ case uses involutivity!). Let us also dispose of the case $m=3$. Put $x_2 = X_{1,2}$, $x_3 = X_{1,3}$ and $y_3 = X_{2,3}$, so the claim for $m=3$ is saying that

$$x_2x_3x_2 + x_3x_2x_3 = h\left(x_2+x_3\right).$$

Tricyclicity gives

$$\begin{aligned}
X_{1,3}X_{1,2} + X_{3,2}X_{3,1} + X_{2,1}X_{2,3} &= h, \qquad \text{that is,} \\
x_3x_2 + y_3x_3 - x_2y_3 &= h
\end{aligned}$$

(since $X_{1,3} = x_3$ and $X_{1,2} = x_2$ and $X_{3,2} = -X_{2,3} = -y_3$ and $X_{3,1} = -X_{1,3} = -x_3$ and $X_{2,1} = -X_{1,2} = -x_2$). Thus,

$$x_3x_2 = h + x_2y_3 - y_3x_3.$$

Thus

$$\begin{aligned}
x_2x_3x_2 + x_3x_2x_3 &= x_2\left(h + x_2y_3 - y_3x_3\right) + \left(h + x_2y_3 - y_3x_3\right)x_3 \\
&= x_2h + \underbrace{x_2^2}_{\substack{=h \\ \text{(by involutivity)}}} y_3 - x_2y_3x_3 + hx_3 + x_2y_3x_3 - y_3 \underbrace{x_3^2}_{\substack{=h \\ \text{(by involutivity)}}} \\
&= x_2h + hy_3 + hx_3 - y_3h \\
&= hx_2 + hy_3 + hx_3 - hy_3 \qquad \text{(since } h \text{ is central)} \\
&= hx_2 + hx_3 = h\left(x_2 + x_3\right).
\end{aligned}$$

This proves the lemma in the case $m = 3$.

Now let $m \geq 4$, and assume that the lemma has already been proved for all smaller values of m. We shall prove it for this m.

Put

$$x_k := X_{1,k} \qquad \text{and} \qquad y_k := X_{2,k} \qquad \text{for all } k \in [n],$$

and set $y := y_m = X_{2,m}$. For every $j \notin \{1,2\}$, tricyclicity applied to the triples $(1,2,j)$ and $(1,j,2)$ gives

$$x_2x_j - y_jx_2 + x_jy_j = h, \tag{4}$$

$$x_jx_2 - x_2y_j + y_jx_j = h \tag{5}$$

(since $X_{1,2} = x_2$ and $X_{1,j} = x_j$ and $X_{2,1} = -X_{1,2} = -x_2$ and $X_{2,j} = y_j$ and $X_{j,1} = -X_{1,j} = -x_j$ and $X_{j,2} = -X_{2,j} = -y_j$). In particular, applying (5) to $j = m$, we find

$$x_mx_2 = h + x_2y - yx_m. \tag{6}$$

Moreover, locality shows that

$$y \text{ commutes with each of } x_3, x_4, \ldots, x_{m-1}. \tag{7}$$

We must prove

$$L_m = hR_m, \tag{8}$$

where

$$\begin{aligned}
L_m &:= \sum_{j=2}^{m} x_jx_{j+1}\cdots x_mx_2x_3\cdots x_j, \\
R_m &:= x_2x_3\cdots x_{m-1} + x_3x_4\cdots x_m + \sum_{j=4}^{m-1} x_jx_{j+1}\cdots x_mx_2x_3\cdots x_{j-2} \\
&\qquad + x_mx_2x_3\cdots x_{m-2}.
\end{aligned}$$

This is just the assertion of the lemma written out in the x-notation.

We shall need three consequences of the induction hypothesis. First, applying the lemma to the indices $1, 2, \ldots, m-1$ gives

$$\begin{aligned}&\sum_{j=2}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 \cdots x_j \\ &= h\left(x_2 x_3 \cdots x_{m-2} + x_3 x_4 \cdots x_{m-1} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 \cdots x_{j-2} \right). \end{aligned} \tag{9}$$

Second, applying the transposition $t_{2,m}$ to this identity gives the same identity for the cyclic list $m, 3, 4, \ldots, m-1$:

$$\begin{aligned}&x_m x_3 x_4 \cdots x_{m-1} x_m + \sum_{j=3}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_j \\ &= h\left(x_m x_3 x_4 \cdots x_{m-2} + x_3 x_4 \cdots x_{m-1} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_{j-2} \right). \end{aligned} \tag{10}$$

Third, applying the lemma to the $m-2$ indices $1, 3, 4, \ldots, m-1$ gives

$$\begin{aligned}&x_3 x_4 \cdots x_{m-1} x_3 + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \\ &= h\left(x_3 x_4 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_{j-2} \right). \end{aligned} \tag{11}$$

For $m = 4$, the sums in (10) and (11) with lower limit exceeding the upper limit are empty; the displayed identities then mean exactly the corresponding small cases of the shell lemma.

We now observe

$$\begin{aligned}
L_m &= \sum_{j=2}^{m} x_j x_{j+1} \cdots \underbrace{x_m x_2}_{\substack{=h+x_2y-yx_m \\ \text{(by (6))}}} x_3 \cdots x_j \\
&= \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} \left(h + x_2 y - y x_m\right) x_3 x_4 \cdots x_j \\
&= h \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \\
&\qquad + \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} x_2 y x_3 x_4 \cdots x_j \\
&\qquad - \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} y x_m x_3 x_4 \cdots x_j \qquad \text{(since } h \text{ is central)} \\
&= h \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \\
&\qquad + \sum_{j=2}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 \underbrace{y x_3 x_4 \cdots x_j}_{\substack{=x_3x_4\cdots x_j y \\ \text{(by (7))}}} \\
&\qquad - \sum_{j=3}^{m} \underbrace{x_j x_{j+1} \cdots x_{m-1} y}_{\substack{=yx_jx_{j+1}\cdots x_{m-1} \\ \text{(by (7))}}} x_m x_3 x_4 \cdots x_j \\
&\qquad \left(\begin{array}{c} \text{here, we have cancelled the last addend } x_2 y x_3 x_4 \cdots x_m \\ \text{of the second sum against the first addend } x_2 x_3 \cdots x_{m-1} y x_m, \\ \text{since they are equal (by (7))} \end{array}\right) \\
&= h \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \\
&\qquad + \sum_{j=2}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 x_4 \cdots x_j y \\
&\qquad - y \sum_{j=3}^{m} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_j \\
&= hA + Uy - yV, \qquad (12)
\end{aligned}$$

where

$$A := \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j,$$
$$U := \sum_{j=2}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 \cdots x_j,$$

and

$$V := \sum_{j=3}^{m} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_j.$$

However,

$$U = \sum_{j=2}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 \cdots x_j = hB,$$

where

$$B := x_2 x_3 \cdots x_{m-2} + x_3 x_4 \cdots x_{m-1} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 \cdots x_{j-2}$$

(by (9)). Moreover,

$$\begin{aligned} V &= \sum_{j=3}^{m} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_j \\ &= x_m x_3 x_4 \cdots x_{m-1} x_m + \sum_{j=3}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_j \\ &= hC, \end{aligned}$$

where

$$C := x_m x_3 x_4 \cdots x_{m-2} + x_3 x_4 \cdots x_{m-1} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_{j-2}$$

(by (10)). Thus, (12) becomes

$$\begin{aligned} L_m &= hA + \underbrace{U}_{=hB} y - y \underbrace{V}_{=hC} = hA + hBy - yhC \\ &= h\left(A + By - yC\right) \qquad (13) \end{aligned}$$

(since h is central). It remains to identify the parenthesized factor $A + By - yC$ here.

Expand $By - yC$ using the definitions of B and C. Since y commutes with $x_3, x_4, \ldots, x_{m-1}$, the middle terms $x_3 x_4 \cdots x_{m-1} y$ and $y x_3 x_4 \cdots x_{m-1}$ cancel in $By - yC$, and we obtain

$$\begin{aligned} By - yC &= \left(x_2 x_3 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_2 x_3 \cdots x_{j-2} \right) y \\ &\qquad - y \left(x_m x_3 x_4 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_3 x_4 \cdots x_{j-2} \right) \\ &= (x_2 y - y x_m) x_3 x_4 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} (x_2 y - y x_m) x_3 x_4 \cdots x_{j-2} \end{aligned}$$

(again using (7)). Using (6) in the form $x_2 y - y x_m = x_m x_2 - h$, we can rewrite this as

$$\begin{aligned} By - yC &= (x_m x_2 - h)\, x_3 x_4 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} \,(x_m x_2 - h)\, x_3 x_4 \cdots x_{j-2} \\ &= x_m x_2 x_3 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_2 x_3 \cdots x_{j-2} \\ &\qquad \underbrace{- h \left(x_3 x_4 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_{j-2} \right)}_{\substack{= x_3 x_4 \cdots x_{m-1} x_3 + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \\ \text{(by (11))}}} \\ &= x_m x_2 x_3 \cdots x_{m-2} + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_2 x_3 \cdots x_{j-2} \\ &\qquad - \left(x_3 x_4 \cdots x_{m-1} x_3 + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \right). \end{aligned}$$

Adding this to

$$\begin{aligned} A &= \sum_{j=2}^{m} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j \\ &= x_2 x_3 \cdots x_{m-1} + x_3 x_4 \cdots x_{m-1} x_3 + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_3 x_4 \cdots x_j + x_3 x_4 \cdots x_m \end{aligned}$$

gives precisely

$$\begin{aligned} A + By - yC &= x_2 x_3 \cdots x_{m-1} + x_3 x_4 \cdots x_m + x_m x_2 x_3 \cdots x_{m-2} \\ &\qquad + \sum_{j=4}^{m-1} x_j x_{j+1} \cdots x_{m-1} x_m x_2 x_3 \cdots x_{j-2} \\ &= R_m \qquad \text{(by the definition of } R_m) . \end{aligned}$$

Hence, (13) rewrites as $L_m = hR_m$. This proves (8), and thus completes the induction step. □

The following generalizes [FK, Lemma 7.3]:

Lemma 3.3 (cycle lemma)**.** Let $m \in [n]$. Then,

$$\sum_{j=1}^{m} X_{j,j+1} X_{j,j+2} \cdots X_{j,m} X_{j,1} X_{j,2} \cdots X_{j,j-1} = \begin{cases} 0, & \text{if } m \text{ is even;} \\ h^{(m-1)/2}, & \text{if } m \text{ is odd.} \end{cases}$$

This holds even if we don't require involutivity.

Example 3.4. For $m = 1$, this says that $1 = h^{(1-1)/2}$, which is obvious.

For $m = 2$, this says that $X_{1,2} + X_{2,1} = 0$, which follows from antisymmetry.

For $m = 3$, this says that $X_{1,2}X_{1,3} + X_{2,3}X_{2,1} + X_{3,1}X_{3,2} = h^{(3-1)/2}$, which follows from tricyclicity.

For $m = 4$, this says that

$$X_{1,2}X_{1,3}X_{1,4} + X_{2,3}X_{2,4}X_{2,1} + X_{3,4}X_{3,1}X_{3,2} + X_{4,1}X_{4,2}X_{4,3} = 0.$$

Proof of Lemma 3.3. Put

$$\varepsilon_r := \begin{cases} 0, & \text{if } r \text{ is even;} \\ h^{(r-1)/2}, & \text{if } r \text{ is odd} \end{cases}$$

for each positive integer r. Thus, we must prove that the left hand side of Lemma 3.3 is ε_m.

We shall use the following notation. If $a_1, a_2, \ldots, a_r$ is a list of r distinct indices, then let

$$C(a_1, a_2, \ldots, a_r) := \sum_{s=1}^{r} \left(\prod_{t=s+1}^{r} X_{a_s,a_t} \right) \left(\prod_{t=1}^{s-1} X_{a_s,a_t} \right),$$

where products are multiplied left-to-right (that is, $\prod_{t=u}^{v} p_t = p_u p_{u+1} \cdots p_v$) and where empty products are understood to be 1. Thus, the assertion to be proved is

$$C(1, 2, \ldots, m) = \varepsilon_m. \tag{14}$$

By the S_n-symmetry of the axioms, once (14) is known for a given m, the analogous identity

$$C\left(a_1, a_2, \ldots, a_m\right) = \varepsilon_m$$

holds for any list $\left(a_1, a_2, \ldots, a_m\right)$ of m distinct indices in place of $1, 2, \ldots, m$.

We prove (14) by strong induction on m. The cases $m = 1$ and $m = 2$ are immediate. Now let $m \geq 3$, and assume that the claim has already been proved for all smaller values of m. Set

$$x := X_{1,m}.$$

Set

$$A_i := X_{i,i+1} X_{i,i+2} \cdots X_{i,m-1} \qquad \text{for all } i \in \{1, 2, \ldots, m-1\} \qquad \text{and}$$
$$B_i := X_{i,2} X_{i,3} \cdots X_{i,i-1} \qquad \text{for all } i \in \{2, 3, \ldots, m\},$$

where empty products are understood to be 1. Thus, for instance, $A_{m-1} = 1$ and $B_2 = 1$.

Let

$$P_1 := X_{1,2} X_{1,3} \cdots X_{1,m-1} \underbrace{X_{1,m}}_{=x} = X_{1,2} X_{1,3} \cdots X_{1,m-1} x$$

be the first addend of $C\left(1, 2, \ldots, m\right)$. For $i \in \{2, 3, \ldots, m-1\}$, the i-th addend of $C\left(1, 2, \ldots, m\right)$ is

$$P_i := A_i X_{i,m} X_{i,1} B_i.$$

Finally, the m-th addend of $C\left(1, 2, \ldots, m\right)$ is

$$\begin{aligned} P_m &:= X_{m,1} X_{m,2} \cdots X_{m,m-1} = -x \underbrace{X_{m,2} X_{m,3} \cdots X_{m,m-1}}_{=B_m} \qquad \left(\text{since } X_{m,1} = -X_{1,m} = -x\right) \\ &= -xB_m. \end{aligned} \tag{15}$$

Hence,

$$C\left(1, 2, \ldots, m\right) = P_1 + \sum_{i=2}^{m-1} P_i + P_m. \tag{16}$$

By the induction hypothesis applied to the list $1, 2, \ldots, m-1$, we have

$$X_{1,2} X_{1,3} \cdots X_{1,m-1} + \sum_{i=2}^{m-1} A_i X_{i,1} B_i = C\left(1, 2, \ldots, m-1\right) = \varepsilon_{m-1}.$$

Multiplying this equality on the right by x, we obtain

$$P_1 + \sum_{i=2}^{m-1} A_i X_{i,1} B_i x = \varepsilon_{m-1} x.$$

In other words,

$$P_1 = \varepsilon_{m-1} x - \sum_{i=2}^{m-1} A_i X_{i,1} B_i x. \tag{17}$$

Now, let $i \in \{2, 3, \ldots, m-1\}$. Then, the element $x = X_{1,m}$ commutes with all factors of B_i by locality, since these factors have the form $X_{i,k}$ with $k \in \{2, 3, \ldots, i-1\}$. Hence,

$$B_i x = x B_i. \tag{18}$$

On the other hand, tricyclicity applied to the triple $(i, m, 1)$ gives

$$X_{i,m} X_{i,1} + X_{m,1} X_{m,i} + X_{1,i} X_{1,m} = h.$$

Since $X_{1,m} = x$, $X_{m,1} = -x$, $X_{m,i} = -X_{i,m}$ and $X_{1,i} = -X_{i,1}$, this becomes

$$X_{i,m} X_{i,1} + x X_{i,m} - X_{i,1} x = h.$$

In other words,

$$X_{i,m} X_{i,1} - X_{i,1} x = h - x X_{i,m}. \tag{19}$$

Now, the definition of P_i yields

$$\begin{aligned} P_i - A_i X_{i,1} B_i x &= A_i X_{i,m} X_{i,1} B_i - A_i X_{i,1} \underbrace{B_i x}_{\substack{= x B_i \\ \text{(by (18))}}} \\ &= A_i \underbrace{\left(X_{i,m} X_{i,1} - X_{i,1} x \right)}_{\substack{= h - x X_{i,m} \\ \text{(by (19))}}} B_i = A_i \left(h - x X_{i,m} \right) B_i \\ &= h A_i B_i - A_i x X_{i,m} B_i \qquad \text{(by centrality)} . \end{aligned} \tag{20}$$

But $x = X_{1,m}$ commutes with all factors of $A_i = X_{i,i+1} X_{i,i+2} \cdots X_{i,m-1}$ by locality, so we have $A_i x = x A_i$. Thus, (20) rewrites as

$$P_i - A_i X_{i,1} B_i x = h A_i B_i - x A_i X_{i,m} B_i. \tag{21}$$

Now (16) becomes

$$\begin{aligned}
C(1,2,\ldots,m) &= P_1 + \sum_{i=2}^{m-1} P_i + P_m \\
&= \varepsilon_{m-1}x - \sum_{i=2}^{m-1} A_i X_{i,1} B_i x + \sum_{i=2}^{m-1} P_i + P_m \qquad \text{(by (17))} \\
&= \varepsilon_{m-1}x + \sum_{i=2}^{m-1} \underbrace{\left(P_i - A_i X_{i,1} B_i x\right)}_{\substack{= hA_iB_i - xA_iX_{i,m}B_i \\ \text{(by (21))}}} + \underbrace{P_m}_{\substack{=-xB_m \\ \text{(by (15))}}} \\
&= \varepsilon_{m-1}x + h\sum_{i=2}^{m-1} A_iB_i - x\sum_{i=2}^{m-1} A_iX_{i,m}B_i - xB_m \\
&= \varepsilon_{m-1}x + h \underbrace{\sum_{i=2}^{m-1} A_iB_i}_{\substack{=C(2,3,\ldots,m-1) \\ \text{(by the definition of } C(2,3,\ldots,m-1))}} - x \underbrace{\left(\sum_{i=2}^{m-1} A_iX_{i,m}B_i + B_m\right)}_{\substack{=C(2,3,\ldots,m) \\ \text{(by the definition of } C(2,3,\ldots,m))}} \\
&= \varepsilon_{m-1}x + h \underbrace{C(2,3,\ldots,m-1)}_{\substack{=\varepsilon_{m-2} \\ \text{(by the induction hypothesis)}}} - x \underbrace{C(2,3,\ldots,m)}_{\substack{=\varepsilon_{m-1} \\ \text{(by the induction hypothesis)}}} \\
&= \varepsilon_{m-1}x + h\varepsilon_{m-2} - x\varepsilon_{m-1} = h\varepsilon_{m-2} + \underbrace{\varepsilon_{m-1}x - x\varepsilon_{m-1}}_{\substack{=0 \\ \text{(since } \varepsilon_{m-1} \text{ is central} \\ \text{(because } \varepsilon_{m-1} \text{ is 0 or a power of } h))}} \\
&= h\varepsilon_{m-2} = \varepsilon_m \qquad \text{(this follows from the definition of } \varepsilon_r\text{)}.
\end{aligned}$$

This proves (14), and thus completes the induction step. □

Remark 3.5. I think the above proof of Lemma 3.3 does **not** use the involutivity axiom! But this needs to be properly checked.

The following proposition is a partial generalization of Lemma 3.3; we will not use it in what follows:

Proposition 3.6 (endpoint recurrence for cycle sums)**.** Extend the notation $C(a_1, a_2, \ldots, a_r)$ from the proof of Lemma 3.3 to arbitrary (not necessarily repetition-free) lists by setting

$$C(a_1, a_2, \ldots, a_r) := \sum_{s=1}^{r} \left(\prod_{t=s+1}^{r} X_{a_s,a_t} \right) \left(\prod_{t=1}^{s-1} X_{a_s,a_t} \right),$$

where empty products are 1. Let $m \geq 3$, and let $a_1, a_2, \ldots, a_m$ be a list of elements of $[n]$ such that a_1 and a_m are distinct and neither a_1 nor a_m occurs among

$a_2, a_3, \ldots, a_{m-1}$. Then

$$\begin{aligned} & C\left(a_1, a_2, \ldots, a_m\right) \\ &= C\left(a_1, a_2, \ldots, a_{m-1}\right) X_{a_1,a_m} + hC\left(a_2, a_3, \ldots, a_{m-1}\right) - X_{a_1,a_m} C\left(a_2, a_3, \ldots, a_m\right) . \end{aligned}$$

Proof. Set

$$x := X_{a_1,a_m}$$

(so that $X_{a_m,a_1} = -x$). For $i \in \{1, 2, \ldots, m-1\}$, set

$$A_i := X_{a_i,a_{i+1}} X_{a_i,a_{i+2}} \cdots X_{a_i,a_{m-1}},$$

and for $i \in \{2, 3, \ldots, m\}$, set

$$B_i := X_{a_i,a_2} X_{a_i,a_3} \cdots X_{a_i,a_{i-1}},$$

with empty products understood to be 1. Thus the i-th summand of $C\left(a_1, a_2, \ldots, a_m\right)$ is

$$\begin{aligned} P_1 &:= A_1 x && \text{for } i = 1, \\ P_i &:= A_i X_{a_i,a_m} X_{a_i,a_1} B_i && \text{for } i \in \{2, 3, \ldots, m-1\}, && \text{and} \\ P_m &:= X_{a_m,a_1} B_m = -x B_m && \text{for } i = m. \end{aligned}$$

Hence

$$C\left(a_1, a_2, \ldots, a_m\right) = P_1 + \sum_{i=2}^{m-1} P_i + P_m. \tag{22}$$

On the other hand,

$$C\left(a_1, a_2, \ldots, a_{m-1}\right) = A_1 + \sum_{i=2}^{m-1} A_i X_{a_i,a_1} B_i.$$

Multiplying this equality on the right by x, we get

$$C\left(a_1, a_2, \ldots, a_{m-1}\right) x = P_1 + \sum_{i=2}^{m-1} A_i X_{a_i,a_1} B_i x. \tag{23}$$

Fix $i \in \{2, 3, \ldots, m-1\}$. We claim that $B_i x = x B_i$. Indeed, if some factor in B_i is 0, then both sides are 0. Otherwise, each factor of B_i has the form X_{a_i,a_t} with $t \in \{2, 3, \ldots, i-1\}$ and with a_i, a_t, a_1, a_m all distinct; so locality gives its commutation with x, and therefore $B_i x = x B_i$. The same argument gives $A_i x = x A_i$.

Tricyclicity applied to the triple (a_i, a_m, a_1) gives

$$X_{a_i,a_m} X_{a_i,a_1} + X_{a_m,a_1} X_{a_m,a_i} + X_{a_1,a_i} X_{a_1,a_m} = h.$$

Using $X_{a_m,a_1} = -x$, $X_{a_m,a_i} = -X_{a_i,a_m}$ and $X_{a_1,a_i} = -X_{a_i,a_1}$, this becomes

$$X_{a_i,a_m} X_{a_i,a_1} + x X_{a_i,a_m} - X_{a_i,a_1} x = h,$$

or equivalently

$$X_{a_i,a_m} X_{a_i,a_1} - X_{a_i,a_1} x = h - x X_{a_i,a_m}. \tag{24}$$

Therefore,

$$\begin{aligned} P_i - A_i X_{a_i,a_1} B_i x &= A_i X_{a_i,a_m} X_{a_i,a_1} B_i - A_i X_{a_i,a_1} B_i x \\ &= A_i \left(X_{a_i,a_m} X_{a_i,a_1} - X_{a_i,a_1} x \right) B_i \\ &= A_i \left(h - x X_{a_i,a_m} \right) B_i \qquad \text{(by (24))} \\ &= h A_i B_i - x A_i X_{a_i,a_m} B_i, \end{aligned} \tag{25}$$

where we have also used $A_i x = x A_i$.

Substituting (23) and (25) into (22), and using $P_m = -x B_m$, we obtain

$$\begin{aligned} C(a_1, a_2, \ldots, a_m) = C(a_1, a_2, \ldots, a_{m-1})\, x + h \sum_{i=2}^{m-1} A_i B_i \\ - x \left(\sum_{i=2}^{m-1} A_i X_{a_i,a_m} B_i + B_m \right). \end{aligned}$$

By the definition of C, we have

$$\sum_{i=2}^{m-1} A_i B_i = C(a_2, a_3, \ldots, a_{m-1})$$

and

$$\sum_{i=2}^{m-1} A_i X_{a_i,a_m} B_i + B_m = C(a_2, a_3, \ldots, a_m).$$

Since $x = X_{a_1,a_m}$, the claimed recurrence follows. □

4. Symmetric functions of the YJM elements

We now aim to prove the following generalization of [FK, Theorem 7.1 (first part)]. The second formula below is best written without choosing a square root of h; equivalently, it says that the roots are

$$(n-1)\sqrt{h},\ (n-3)\sqrt{h}, \ldots,\ -(n-3)\sqrt{h},\ -(n-1)\sqrt{h}$$

if a formal square root of h is adjoined.

Theorem 4.1. **(a)** For every $m \geq 1$, the power sum

$$p_m(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^m$$

belongs to $\mathbb{Z}[h]$ in the universal dFK ring. Hence, after tensoring with $\mathbb{Q}$, every symmetric polynomial in $z_1, \ldots, z_n$ with rational coefficients belongs to $\mathbb{Q}[h]$.

(b) In the univariate polynomial ring $(R \otimes_{\mathbb{Z}} \mathbb{Q})[Y]$ (where Y is central), we have

$$\prod_{i=1}^{n}(Y - z_i) = \begin{cases} \prod\limits_{\substack{1 \leq r \leq n-1; \\ r \text{ odd}}} (Y^2 - r^2 h), & \text{if } n \text{ is even,} \\ Y \prod\limits_{\substack{2 \leq r \leq n-1; \\ r \text{ even}}} (Y^2 - r^2 h), & \text{if } n \text{ is odd.} \end{cases}$$

(c) If the subring of R generated by the dFK family is $\mathbb{Z}$-torsion-free, then the equality in **(b)** already holds in $R[Y]$, and every symmetric polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$ satisfies $f(z_1, \ldots, z_n) \in \mathbb{Z}[h]$.

Proof. We work first in the universal dFK ring. The power-sum assertion in **(a)** will be proved integrally. Only after that step will we tensor with $\mathbb{Q}$ in order to pass from power sums to arbitrary symmetric polynomials. Since every dFK family is an image of the universal one, all the resulting identities specialize to arbitrary dFK families.

Recall that the power-sum symmetric polynomials p_m for $m = 1, 2, \ldots, n$ generate all symmetric polynomials in n indeterminates over $\mathbb{Q}$. Applying these polynomials to the commuting elements $z_1, z_2, \ldots, z_n$ of R, we find

$$p_m(z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} z_i^m = \sum_{i=1}^{n} \sum_{\substack{j_1, j_2, \ldots, j_m \in [n]; \\ j_k \neq i \text{ for all } k}} X_{j_1,i} X_{j_2,i} \cdots X_{j_m,i}$$

(by multiplying out z_i^m using the definition of z_i). Due to antisymmetry, we can flip each $X_{j_k,i}$ to transform this into

$$\begin{aligned} &(-1)^m \, p_m(z_1, z_2, \ldots, z_n) \\ &= \sum_{i=1}^{n} \sum_{\substack{j_1, j_2, \ldots, j_m \in [n]; \\ j_k \neq i \text{ for all } k}} X_{i,j_1} X_{i,j_2} \cdots X_{i,j_m} \\ &= \sum_{\substack{i, j_1, j_2, \ldots, j_m \in [n]; \\ j_k \neq i \text{ for all } k}} X_{i,j_1} X_{i,j_2} \cdots X_{i,j_m}. \end{aligned}$$

We shall now prove a family of auxiliary assertions integrally in the universal dFK ring R_{dFK}. For $m \in \mathbb{N}$ and $r \in \{0, 1, \ldots, m\}$, define

$$A_{m,r} := \sum_{\substack{i, j_1, j_2, \ldots, j_m \in [n]; \\ j_k \neq i \text{ for all } k; \\ j_1, j_2, \ldots, j_r \text{ are distinct}}} X_{i,j_1} X_{i,j_2} \cdots X_{i,j_m}$$

(where the distinctness condition is vacuous when $r = 0$). We claim that

$$A_{m,r} \in \mathbb{Z}[h] \qquad \text{for all } m \in \mathbb{N} \text{ and all } r \in \{0, 1, \ldots, m\}. \tag{26}$$

We prove this claim by induction on m. The case $m = 0$ is clear, since $A_{0,0} = n$. Now let $m > 0$, and assume that the claim has already been proved with m replaced by all smaller nonnegative integers.

First consider $A_{m,m}$. This is the fully distinct part:

$$A_{m,m} = \sum_{\substack{j_0, j_1, \ldots, j_m \in [n] \\ \text{are distinct}}} X_{j_0,j_1} X_{j_0,j_2} \cdots X_{j_0,j_m}.$$

Each ordered $(m+1)$-tuple of distinct indices can be uniquely written as a cyclic rotation of an ordered tuple $(u_0, u_1, \ldots, u_m)$ satisfying $u_0 < u_k$ for all $k \in [m]$. Grouping the above sum into these cyclic orbits gives

$$A_{m,m} = \sum_{\substack{u_0, u_1, \ldots, u_m \in [n] \\ \text{are distinct;} \\ u_0 < u_k \text{ for each } k \in [m]}} \sum_{s=0}^{m} X_{u_s,u_{s+1}} X_{u_s,u_{s+2}} \cdots X_{u_s,u_m} X_{u_s,u_0} X_{u_s,u_1} \cdots X_{u_s,u_{s-1}},$$

where the subscripts of the u's are read cyclically. The inner sum belongs to $\mathbb{Z}[h]$ by Lemma 3.3 (applied to $m+1$ and the list $u_0, u_1, \ldots, u_m$ instead of m and the list $1, 2, \ldots, m$; as we said, the S_n-action on R_{dFK} allows us to arbitrarily permute the indices); in fact it is 0 if $m+1$ is even and $h^{m/2}$ if $m+1$ is odd. Hence $A_{m,m} \in \mathbb{Z}[h]$.

It remains to descend from $r = m$ to $r = 0$. We shall show that, for each $r \in \{1, 2, \ldots, m-1\}$, we have

$$A_{m,r} = A_{m,r+1} + hr(n-r) A_{m-2,r-1}. \tag{27}$$

Assume this for the moment. Since $A_{m,m} \in \mathbb{Z}[h]$ and $A_{m-2,r-1} \in \mathbb{Z}[h]$ by the induction hypothesis, formula (27) yields, by downward induction on r, that $A_{m,r} \in \mathbb{Z}[h]$ for every $r \in \{1, 2, \ldots, m\}$. Finally, $A_{m,0} = A_{m,1}$, since no condition is added when passing from $r = 0$ to $r = 1$. Thus (26) follows.

It remains to prove the recurrence (27). Fix $r \in \{1, 2, \ldots, m-1\}$. In the sum defining $A_{m,r}$, split according to whether j_{r+1} is distinct from $j_1, j_2, \ldots, j_r$ or is equal to one of them. The part where it is distinct is precisely $A_{m,r+1}$. Thus,

$$A_{m,r} = A_{m,r+1} + \sum_{u=1}^{r} B_{m,r,u}, \tag{28}$$

where $B_{m,r,u}$ is the part of the sum in which $j_{r+1} = j_u$, namely

$$B_{m,r,u} := \sum_{\substack{i,j_1,j_2,\ldots,j_m \in [n]; \\ j_k \neq i \text{ for all } k; \\ j_1,j_2,\ldots,j_r \text{ are distinct;} \\ j_{r+1}=j_u}} X_{i,j_1} X_{i,j_2} \cdots X_{i,j_m}.$$

We compute $B_{m,r,u}$. Put $w := r - u + 1$. Rename $j_u, j_{u+1}, \ldots, j_r$ as $p_1, p_2, \ldots, p_w$, and rename $j_{r+2}, j_{r+3}, \ldots, j_m$ as $q_1, q_2, \ldots, q_{m-r-1}$. Then[1]

$$B_{m,r,u} = \sum_{\substack{i,j_1,\ldots,j_{u-1} \in [n] \\ \text{are distinct}}} \sum_{\substack{p_1,\ldots,p_w \in [n] \setminus \{i,j_1,\ldots,j_{u-1}\} \\ \text{are distinct}}} \sum_{q_1,\ldots,q_{m-r-1} \in [n] \setminus \{i\}} \left(X_{i,j_1} \cdots X_{i,j_{u-1}}\right) \left(X_{i,p_1} X_{i,p_2} \cdots X_{i,p_w} X_{i,p_1}\right) \left(X_{i,q_1} \cdots X_{i,q_{m-r-1}}\right).$$

For fixed $i, j_1, \ldots, j_{u-1}, q_1, \ldots, q_{m-r-1}$, the sum over $p_1, \ldots, p_w$ of the middle factor is

$$\sum_{\substack{p_1,\ldots,p_w \in [n] \setminus \{i,j_1,\ldots,j_{u-1}\} \\ \text{are distinct}}} X_{i,p_1} X_{i,p_2} \cdots X_{i,p_w} X_{i,p_1} = h \sum_{\substack{p_1,\ldots,p_w \in [n] \setminus \{i,j_1,\ldots,j_{u-1}\} \\ \text{are distinct}}} X_{i,p_1} X_{i,p_2} \cdots X_{i,p_{w-1}}$$

(by grouping ordered w-tuples into cyclic orbits and applying Lemma 3.1).

In the right hand side, the factor $X_{i,p_1} \cdots X_{i,p_{w-1}}$ no longer depends on p_w. Once $i, j_1, \ldots, j_{u-1}, p_1, \ldots, p_{w-1}$ have been chosen, there are exactly

$$n - 1 - (u - 1) - (w - 1) = n - r$$

choices for p_w. Hence

$$B_{m,r,u} = h\,(n-r) \sum_{\substack{i,j_1,\ldots,j_{u-1} \in [n] \\ \text{are distinct}}} \sum_{\substack{p_1,\ldots,p_{w-1} \in [n] \setminus \{i,j_1,\ldots,j_{u-1}\} \\ \text{are distinct}}} \sum_{q_1,\ldots,q_{m-r-1} \in [n] \setminus \{i\}} \left(X_{i,j_1} \cdots X_{i,j_{u-1}}\right) \left(X_{i,p_1} X_{i,p_2} \cdots X_{i,p_{w-1}}\right) \left(X_{i,q_1} \cdots X_{i,q_{m-r-1}}\right).$$

The first $u - 1$ letters followed by the $w - 1 = r - u$ letters $p_1, \ldots, p_{w-1}$ form a list of $r - 1$ distinct letters, each different from i, while the q's are arbitrary letters different from i. Therefore the last displayed sum is exactly $A_{m-2,r-1}$. Thus

$$B_{m,r,u} = h\,(n-r)\,A_{m-2,r-1}.$$

Summing this over $u = 1, 2, \ldots, r$ in (28) proves (27).

[1]Here and below, an empty product is understood as 1.

We have now shown (26). In particular,

$$\begin{aligned}(-1)^m p_m(z_1, z_2, \ldots, z_n) &= \sum_{\substack{i,j_1,j_2,\ldots,j_m \in [n];\\ j_k \neq i \text{ for all } k}} X_{i,j_1} X_{i,j_2} \cdots X_{i,j_m} \\ &= A_{m,0} \in \mathbb{Z}[h].\end{aligned}$$

Hence all power-sum symmetric polynomials in $z_1, \ldots, z_n$ belong to $\mathbb{Z}[h]$. Consequently, after rationalization, every symmetric polynomial in $z_1, \ldots, z_n$ with rational coefficients belongs to $\mathbb{Q}[h]$. This proves part **(a)**.

Apply this in particular to the elementary symmetric polynomials. Thus, for each k, there is a polynomial $g_k \in \mathbb{Q}[h]$ such that

$$e_k(z_1, z_2, \ldots, z_n) = g_k(h)$$

in the universal dFK ring. We determine g_k by evaluating in the blown-up sign family. In that family, $h = z^2$ and

$$z_i = \sum_{a \in [n]} \underbrace{X_{a,i}}_{= \operatorname{sign}(a-i) z} = \underbrace{\left(\sum_{a \in [n]} \operatorname{sign}(a-i) \right)}_{= n+1-2i} z = (n+1-2i) z.$$

Therefore

$$\prod_{i=1}^{n} (Y - z_i) = \prod_{i=1}^{n} (Y - (n+1-2i) z).$$

Pairing the opposite roots on the right hand side (i.e., pairing the factor for i with the factor for $n+1-i$) gives

$$\prod_{i=1}^{n} (Y - (n+1-2i) z) = \begin{cases} \prod\limits_{\substack{1 \le r \le n-1;\\ r \text{ odd}}} \left(Y^2 - r^2 z^2\right), & \text{if } n \text{ is even;} \\ Y \prod\limits_{\substack{2 \le r \le n-1;\\ r \text{ even}}} \left(Y^2 - r^2 z^2\right), & \text{if } n \text{ is odd.} \end{cases}$$

Since the substitution map $\mathbb{Q}[h] \to \mathbb{Q}[z]$ sending h to z^2 is injective, this identifies all coefficients of $\prod\limits_{i=1}^{n} (Y - z_i)$ in the rationalized universal dFK ring and proves part **(b)**.

It remains only to justify part **(c)**. The two assertions in parts **(a)** and **(b)**, when applied to a symmetric polynomial with integer coefficients, say that the desired integral identity becomes true after tensoring with $\mathbb{Q}$. Hence the difference between the two sides is a $\mathbb{Z}$-torsion element of the subring generated by the dFK family. If this subring is $\mathbb{Z}$-torsion-free, this difference is 0. Since the right hand side in part **(b)** has coefficients in $\mathbb{Z}[h]$, the elementary symmetric polynomials

$e_1(z_1,\ldots,z_n),\ldots,e_n(z_1,\ldots,z_n)$ then belong to $\mathbb{Z}[h]$. Finally, $\mathbb{Z}[X_1,\ldots,X_n]^{S_n} = \mathbb{Z}[e_1,\ldots,e_n]$. Therefore every symmetric polynomial with integer coefficients, evaluated at $z_1,\ldots,z_n$, belongs to $\mathbb{Z}[h]$ as well. This proves part **(c)**. □

Proposition 4.2 (the second elementary symmetric function)**.** In the universal dFK ring one has, integrally,

$$e_1(z_1,\ldots,z_n) = 0, \qquad e_2(z_1,\ldots,z_n) = -\binom{n+1}{3}h.$$

Thus the first two elementary-symmetric coefficients of the characteristic polynomial have no integral torsion ambiguity.

Proof. The first identity follows at once from antisymmetry. Put

$$y_i := -z_i = \sum_{a\neq i} X_{ia}.$$

Since $e_2(y_1,\ldots,y_n) = e_2(z_1,\ldots,z_n)$, we expand $\sum_{i<j} y_i y_j$ and group its monomials according to the number of distinct indices occurring.

Terms involving only two indices are $X_{ij}X_{ji} = -h$, one for each unordered pair, and hence contribute $-\binom{n}{2}h$.

Fix next three indices $a < b < c$ and write

$$A = X_{ab}, \qquad B = X_{ac}, \qquad C = X_{bc}.$$

The terms whose set of indices is exactly $\{a,b,c\}$, coming respectively from $y_a y_b$, $y_a y_c$, and $y_b y_c$, have total

$$\begin{aligned}(AC - BA + BC) + (-BC - AB - AC) + (-CB + AC + AB)\\ = AC - BA - CB = -h.\end{aligned}$$

The last equality is the tricyclic relation written as $-AC + BA + CB = h$. Thus all three-index terms contribute $-\binom{n}{3}h$.

Finally, fix four distinct indices. A term involving all four is a product of two disjoint edges. For each of the three perfect matchings of the four indices, the four choices of orientations contribute with signs $+,-,-,+$. Since disjoint edges commute, these four contributions cancel. Hence the four-index part is zero.

Adding the three cases gives

$$e_2(z_1,\ldots,z_n) = -\left(\binom{n}{2} + \binom{n}{3}\right)h = -\binom{n+1}{3}h,$$

as claimed. □

4.1. A precise reduction of the integral problem

The rationalization in Theorem 4.1 enters only when we pass from power sums to arbitrary symmetric functions. We now record a different route to the integral statement. It is useful for two reasons. First, it isolates a single multiplicative identity which would imply the full integral theorem. Second, it clarifies the relation with Kirillov's three-term algebra without assuming any torsion-freeness of R_{dFK}.

Let

$$A := R_{\mathrm{dFK}}$$

and make the base change

$$\widetilde{A} := A \otimes_{\mathbb{Z}[h]} \mathbb{Z}[s], \qquad h \longmapsto s^2.$$

Since

$$\mathbb{Z}[s] = \mathbb{Z}[h] \oplus s\mathbb{Z}[h]$$

as a $\mathbb{Z}[h]$-module, the natural map $A \to \widetilde{A}$ is injective. This observation does not require A to be torsion-free.

For $i \neq j$ put

$$u_{ij} := X_{ij} + s, \qquad \beta := 2s.$$

Thus $u_{ij} + u_{ji} = \beta$.

Proposition 4.3. The elements u_{ij} in $\widetilde{A}$ satisfy Kirillov's defining relations for the algebra $3T_n(\beta)$, and in addition

$$q_{ij} := u_{ij}^2 - \beta u_{ij} = 0 \qquad (i \neq j).$$

Proof. The square relation is immediate:

$$u_{ij}^2 = (X_{ij} + s)^2 = X_{ij}^2 + 2sX_{ij} + s^2 = 2s(X_{ij} + s) = \beta u_{ij}.$$

Locality is inherited from the X_{ij}. For $i < j < k$, set $A = X_{ij}$, $B = X_{jk}$ and $C = X_{ik}$. The two tricyclic relations are equivalently

$$AB = CA + BC - h, \qquad BA = AC + CB - h.$$

Using $h = s^2$ gives

$$\begin{aligned} u_{ij}u_{jk} &= (A+s)(B+s) \\ &= CA + BC + sA + sB \\ &= u_{ik}u_{ij} + u_{jk}u_{ik} - \beta u_{ik}, \end{aligned}$$

and similarly

$$u_{jk}u_{ij} = u_{ij}u_{ik} + u_{ik}u_{jk} - \beta u_{ik}.$$

These are precisely the two three-term relations. □

We next formulate the one multiplicative identity that is needed. Introduce a central formal variable t, put

$$H_{ij} = 1 + tu_{ij}, \qquad Q = 1 + \beta t,$$

and define

$$\Theta_j = H_{j-1,j}^{-1} H_{j-2,j}^{-1} \cdots H_{1j}^{-1} H_{jn} H_{j,n-1} \cdots H_{j,j+1},$$

with the evident convention when one of the strings is empty. Since $q_{ij} = 0$, we have

$$H_{ij} H_{ji} = Q,$$

so all inverses above exist in $\widetilde{A}[[t]]$. Finally set

$$\widehat{\Theta}_j := Q^{j-1} \Theta_j.$$

Lemma 4.4 (Yang–Baxter relations and commutativity)**.** In the above specialization the elements H_{ij} satisfy locality, unitarity

$$H_{ij} H_{ji} = Q,$$

and the multiplicative Yang–Baxter relation

$$H_{ij} H_{ik} H_{jk} = H_{jk} H_{ik} H_{ij} \qquad (i < j < k).$$

Consequently the multiplicative Dunkl elements $\Theta_1, \ldots, \Theta_n$ (and hence $\widehat{\Theta}_1, \ldots, \widehat{\Theta}_n$) pairwise commute. Moreover,

$$\prod_{j=1}^{n} \Theta_j = 1.$$

These assertions use only the defining relations, not the general multiplicative identity discussed below.

Proof. Only the Yang–Baxter relation needs a check. Put $a = u_{ij}$, $b = u_{ik}$ and $c = u_{jk}$. The coefficient of t^2 in the difference of its two sides vanishes by the two three-term relations. For the coefficient of t^3, these relations give

$$ab = ca - bc + \beta b, \qquad ba = ac - cb + \beta b.$$

Using $b^2 = \beta b$ and $c^2 = \beta c$, we obtain

$$abc = cac = cba.$$

Thus the Yang–Baxter relation follows. All oriented versions needed below follow from this one by permuting the three indices and using $H_{ij}^{-1} = Q^{-1} H_{ji}$.

Pairwise commutativity of the Θ_j is then the standard Yang–Baxter "train" argument: move the factors of one Dunkl word through the other; disjoint moves use locality and each move involving three indices is exactly the displayed Yang–Baxter relation. Inverses cause no difficulty because $H_{ij}^{-1} = Q^{-1}H_{ji}$. This argument is the usual one for multiplicative Dunkl elements (cf. [FKYB]) and is independent of Kirillov's Gaussian identity. Finally, in the ordered product $\Theta_1 \cdots \Theta_n$ the factors H_{ij} and H_{ij}^{-1} cancel successively after locality and the same Yang–Baxter train moves; hence the product is 1. □

Question 4.5 (special multiplicative identity)**.** Does the following ordered characteristic-polynomial identity hold in $\widetilde{A}[[t]][U]$?

$$\prod_{j=1}^{n}(U - \widehat{\Theta}_j) = \prod_{r=0}^{n-1}(U - Q^r), \tag{29}$$

where the product on the left is taken in increasing order of j and U is central.

Proposition 4.6. If (29) holds in the above $q_{ij} = 0$ specialization, then the characteristic-polynomial identity in Theorem 4.1**(b)** holds already in $R_{\mathrm{dFK}}[Y]$. Consequently every symmetric polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$ satisfies

$$f(z_1, \ldots, z_n) \in \mathbb{Z}[h]$$

without any torsion-freeness hypothesis.

Proof. Let

$$\vartheta_j := \sum_{a \neq j} u_{ja}.$$

Expanding the definition of Θ_j to first order in t gives

$$\Theta_j = 1 + t\big(\vartheta_j - (j-1)\beta\big) + O(t^2).$$

Indeed, every factor H_{aj}^{-1} with $a < j$ contributes $-u_{aj} = u_{ja} - \beta$, whereas every factor H_{ja} with $a > j$ contributes u_{ja}. Since

$$Q^{j-1} = 1 + (j-1)\beta t + O(t^2),$$

we obtain the cancellation

$$\widehat{\Theta}_j = 1 + t\vartheta_j + O(t^2). \tag{30}$$

Assume (29) and put $U = 1 + tW$ in (29). By (30), the j-th factor on the left is

$$1 + tW - \widehat{\Theta}_j = t(W - \vartheta_j) + O(t^2),$$

whereas the r-th factor on the right is

$$1 + tW - Q^r = t(W - r\beta) + O(t^2).$$

Comparing the coefficient of t^n yields

$$\prod_{j=1}^{n}(W-\vartheta_j)=\prod_{r=0}^{n-1}(W-r\beta). \tag{31}$$

No division by an integer has occurred.

Now

$$\vartheta_j=\sum_{a\neq j}(X_{ja}+s)=(n-1)s-z_j,$$

because $z_j=\sum_{a\neq j}X_{aj}=-\sum_{a\neq j}X_{ja}$, and $\beta=2s$. Substituting $W=(n-1)s-Y$ into (31) gives

$$\prod_{j=1}^{n}(Y-z_j)=\prod_{r=0}^{n-1}\big(Y-(n-1-2r)s\big).$$

Pairing opposite roots and using $s^2=h$, the right hand side becomes

$$\begin{cases}\displaystyle\prod_{\substack{1\le r\le n-1;\\ r\text{ odd}}}(Y^2-r^2h), & n\text{ even},\\[2ex] \displaystyle Y\prod_{\substack{2\le r\le n-1;\\ r\text{ even}}}(Y^2-r^2h), & n\text{ odd}.\end{cases}$$

This polynomial belongs to $\mathbb{Z}[h,Y]$. Since $A\to\widetilde{A}$ is injective, the identity descends from $\widetilde{A}[Y]$ to $A[Y]$. Its coefficients show that all elementary symmetric polynomials in the z_j belong to $\mathbb{Z}[h]$. The final assertion follows from $\mathbb{Z}[X_1,\dots,X_n]^{S_n}=\mathbb{Z}[e_1,\dots,e_n]$. □

4.2. The first multiplicative coefficient: a direct proof

We can prove the first nontrivial coefficient of (29) directly, without using Kirillov's general theorem. The argument is a multiplicative version of the cycle lemma and will also be useful for attempts at the remaining coefficients.

Since $H_{ij}H_{ji}=Q$, we have

$$QH_{ij}^{-1}=H_{ji}.$$

Consequently the renormalized multiplicative Dunkl element has the particularly simple star form

$$\widehat{\Theta}_j=H_{j,j-1}H_{j,j-2}\cdots H_{j,1}H_{j,n}H_{j,n-1}\cdots H_{j,j+1}. \tag{32}$$

Thus every factor in $\widehat{\Theta}_j$ is oriented away from the centre j.

There is also a useful reduced form for products of several stars. For $I \subseteq [n]$, write $I = \{i_1 < \cdots < i_k\}$ and set

$$\nu(I) = \#\{(a,b) : a \notin I,\ b \in I,\ a < b\}.$$

Define the ordered cut product

$$B_I := \left[\prod_{b\in I}^{\nearrow}\left(\prod_{\substack{a\notin I\\ a<b}}^{\searrow} H_{ba}\right)\right]\left[\prod_{a\in I}^{\nearrow}\left(\prod_{\substack{b\notin I\\ a<b}}^{\searrow} H_{ab}\right)\right]. \tag{33}$$

Thus every edge crossing the cut $I \mid I^c$ occurs exactly once, oriented from I to I^c; the arrows specify the order.

Lemma 4.7 (cut reduction)**.** For every $I \subseteq [n]$,

$$B_I = Q^{\nu(I)}\Theta_I = Q^{-k(k-1)/2}\prod_{i\in I}\widehat{\Theta}_i, \qquad \Theta_I := \prod_{i\in I}\Theta_i.$$

Consequently the k-th coefficient of (29) is equivalent to the square-free cut identity

$$\sum_{\substack{I\subseteq[n]\\ |I|=k}} B_I = \begin{bmatrix} n \\ k \end{bmatrix}_Q. \tag{34}$$

Proof. Multiply the Dunkl words Θ_i for $i \in I$ in increasing order. For a pair $a < b$ both belonging to I, the factor H_{ab} in the a-word and the factor H_{ab}^{-1} in the b-word can be brought together and cancelled. To move them through the intervening words one uses locality for disjoint indices and an oriented Yang–Baxter relation from Lemma 4.4 whenever a third index is met. Performing these train moves successively in increasing order of the centres (and, inside each train, from the nearest internal edge outwards) cancels every internal edge exactly once. The Yang–Baxter move preserves the relative order of the uncancelled boundary edges, so the word which remains is the one displayed in (33), apart from the inverse orientation described next.

If $a < b$, $b \in I$ and $a \notin I$, the surviving factor coming from the b-word is $H_{ab}^{-1} = Q^{-1}H_{ba}$. There are precisely $\nu(I)$ such factors, so multiplication by $Q^{\nu(I)}$ converts all of them to the outward orientation and gives exactly the ordered word (33). This proves the first equality. Since

$$\sum_{r=1}^{k}(i_r - 1) = \nu(I) + \binom{k}{2},$$

the second equality follows from $\widehat{\Theta}_i = Q^{i-1}\Theta_i$. Summing over I shows that (34) is exactly the k-th elementary-symmetric coefficient of (29). $\square$

For a cyclically ordered list $a_1, \dots, a_r$ of distinct indices, set

$$C_u(a_1, \dots, a_r) := \sum_{q=1}^{r} \left(\prod_{\ell=q+1}^{r} u_{a_q, a_\ell} \right) \left(\prod_{\ell=1}^{q-1} u_{a_q, a_\ell} \right).$$

The order of the factors is the displayed cyclic order.

Lemma 4.8 (shifted cycle lemma)**.** For every cyclically ordered list of $r \geq 1$ distinct indices,

$$C_u(a_1, \dots, a_r) = \beta^{r-1}.$$

Proof. Recall that $u_{ij} = X_{ij} + s$, $h = s^2$ and $\beta = 2s$. Expand every summand in $C_u(a_1, \dots, a_r)$ by choosing, in each factor, either the X-part or the scalar part s. Fix a nonempty subset $T \subseteq \{a_1, \dots, a_r\}$. Collect all terms for which the centre belongs to T, the X-part is chosen precisely on the other vertices of T, and the scalar part s is chosen on the vertices outside T. Their sum is

$$s^{r-|T|} C_X(T),$$

where $C_X(T)$ is exactly the cyclic sum occurring in Lemma 3.3, with the cyclic order induced from $a_1, \dots, a_r$. Therefore

$$C_X(T) = \begin{cases} 0, & |T| \text{ even}, \\ s^{|T|-1}, & |T| \text{ odd}. \end{cases}$$

Hence every odd subset T contributes the same scalar s^{r-1}, while every even subset contributes 0. There are 2^{r-1} odd subsets of an r-set. Thus

$$C_u(a_1, \dots, a_r) = 2^{r-1} s^{r-1} = (2s)^{r-1} = \beta^{r-1}.$$

□

Theorem 4.9 (the $k = 1$ case of the special multiplicative identity)**.** In $\widetilde{A}[[t]]$ one has

$$\sum_{j=1}^{n} \widehat{\Theta}_j = 1 + Q + Q^2 + \cdots + Q^{n-1}. \tag{35}$$

Equivalently, the coefficient of U^{n-1} in (29) is correct.

Proof. Use the star expression (32) and expand in powers of t. Fix $m \in \{0, 1, \dots, n-1\}$. To obtain a term of degree m in t from a fixed star, one chooses m of its $n-1$ edges. If we group all such terms by the $(m+1)$-element set S consisting of the centre together with the chosen neighbours, then the sum over all possible centres in S is precisely $C_u(S)$, with the cyclic order induced by (32). By Lemma 4.8, this sum is β^m. Hence

$$[t^m] \sum_{j=1}^{n} \widehat{\Theta}_j = \binom{n}{m+1} \beta^m.$$

On the other hand,

$$\sum_{r=0}^{n-1} Q^r = \sum_{r=0}^{n-1} (1+\beta t)^r,$$

and the hockey-stick identity gives

$$[t^m] \sum_{r=0}^{n-1} (1+\beta t)^r = \beta^m \sum_{r=m}^{n-1} \binom{r}{m} = \binom{n}{m+1} \beta^m.$$

The coefficients agree for every m, proving (35). $\square$

Proposition 4.10 (reciprocity)**.** The special multiplicative identity is self-dual under $k \leftrightarrow n-k$. More precisely, the $k=1$ identity of Theorem 4.9 implies the $k=n-1$ coefficient of (29); the coefficients $k=0$ and $k=n$ are automatic. In general, a proof of the k-th coefficient for all n and all values of the formal parameter implies the $(n-k)$-th coefficient.

Proof. Write $S_j(t) := \widehat{\Theta}_j(t)$ in the star form (32). Since

$$H_{ij}(t)^{-1} = Q^{-1} H_{ji}(t), \qquad Q = 1+\beta t,$$

and $u_{ji} = \beta - u_{ij}$, we obtain, after reversing the order of the factors,

$$S_j(t)^{-1} = H_{j,j+1}(t') \cdots H_{j,n}(t') H_{j,1}(t') \cdots H_{j,j-1}(t'), \qquad t' = -\frac{t}{Q}.$$

Indeed,

$$Q^{-1} H_{aj}(t) = Q^{-1}\big(Q - t u_{ja}\big) = 1 - \frac{t}{Q} u_{ja} = H_{ja}(t').$$

The displayed inverse is therefore the same kind of outward star as before, but with the opposite cyclic order. Lemma 4.8 and the proof of Theorem 4.9 apply verbatim to either cyclic order. Since

$$1 + \beta t' = 1 - \frac{\beta t}{Q} = Q^{-1},$$

we get

$$\sum_{j=1}^{n} S_j(t)^{-1} = 1 + Q^{-1} + \cdots + Q^{-(n-1)}. \tag{36}$$

The unrenormalized multiplicative Dunkl elements satisfy $\prod_{j=1}^{n} \Theta_j = 1$; this also follows directly from their ordered factorizations, using locality to cancel the inverse pairs. Consequently

$$\prod_{j=1}^{n} S_j(t) = Q^{0+1+\cdots+(n-1)} = Q^{\binom{n}{2}}.$$

Because the S_j commute, (36) gives

$$e_{n-1}(S_1,\dots,S_n) = Q^{\binom{n}{2}} \sum_j S_j^{-1} = Q^{\binom{n}{2}} \sum_{r=0}^{n-1} Q^{-r},$$

which is exactly $e_{n-1}(1,Q,\dots,Q^{n-1})$. The case $k=n$ is the product identity just displayed.

The same argument with $e_k(S_1^{-1},\dots,S_n^{-1})$ shows the general $k \leftrightarrow n-k$ reduction. Reversal of the cyclic order causes no problem: it is carried to the standard order by the index-reversing permutation $i \mapsto n+1-i$, which is an automorphism of the defining relations. □

Remark 4.11 (what remains)**.** Theorem 4.9 settles the first elementary-symmetric coefficient of Question 4.5 by a self-contained argument, and Proposition 4.10 gives the reciprocal coefficient $k = n-1$ as well. The same calculation also explains why the q-integer $[n]_Q$ appears: its coefficient of t^m counts the $(m+1)$-subsets on which the shifted cycle lemma is applied. By reciprocity it is enough, in principle, to treat $2 \le k \le \lfloor n/2 \rfloor$.

For these middle values, Lemma 4.7 shows that the problem is exactly the square-free cut identity (34). Thus the remaining difficulty is no longer the reduction of products of Dunkl elements, but the cancellation in the sum of the ordered boundary products B_I. This is the multiplicative analogue of the Pieri identities used in the Fomin–Kirillov and Postnikov approaches [FK, Pos]. We have not found a complete direct proof of this multi-centre cancellation. Hence the genuinely unresolved part of Question 4.5 is concentrated in (34) for $2 \le k \le \lfloor n/2 \rfloor$.

Remark 4.12 (status of Kirillov's general multiplicative theorem)**.** A substantially more general formula is stated by Kirillov in [Kir12, Theorem 2.1] and again in [Kir16, Theorem 3.12]. After the specialization $q_{ij} = 0$, it reads

$$\sum_{\substack{I\subseteq[n]\\|I|=k}} \Theta_I Q^{\nu(I)} = \begin{bmatrix} n \\ k \end{bmatrix}_Q, \qquad \nu(I) = \#\{(a,b) : a \notin I,\ b \in I,\ a<b\}.$$

Together with the elementary renormalization $\widehat{\Theta}_j = Q^{j-1}\Theta_j$, this is equivalent to the full characteristic identity (29).

The bibliographical status deserves some care. The 2012 and 2016 texts do more than merely announce the formula: they give the Yang–Baxter setup, reduced expressions, consequences of the formula, and refer explicitly to a proof. In particular, the 2016 paper says that the corresponding shifted identity is proved by induction using local three-term relations and is reduced to Theorem 3.12; it then says that the $\beta = 0$ proof is contained in [KMunpub, Theorem 1] and "can be immediately extended" to $\beta \neq 0$. However, the cited Kirillov–Maeno item is still listed in the same bibliography as "in preparation." We have not found the missing complete induction in the published versions themselves.

Thus the evidence strongly suggests that Kirillov and Maeno had a proof, but the presently accessible references do not provide a self-contained published proof of the precise formula needed here. Closely related Yang–Baxter identities are proved in Fomin–Kirillov [FKYB], and the additive/quantum Pieri mechanism is proved by Postnikov [Pos]; these do not by themselves supply the present $\beta \neq 0$, $q_{ij} = 0$ Gaussian identity. Accordingly, we use only the coefficients proved directly above and keep (34) for the middle coefficients as an explicit problem.

5. Open questions

The integral problem has now been reduced to a particularly concrete form.

Question 5.1. Can one prove the characteristic-polynomial identity of Theorem 4.1**(b)** directly over $\mathbb{Z}[h]$? In view of Proposition 4.6, one possible route is to complete the special multiplicative identity (29). Theorem 4.9 proves its $k = 1$ coefficient directly and Proposition 4.10 gives $k = n - 1$ and the complementary reduction. Thus it suffices to establish the multi-centre cancellation for $2 \leq k \leq \lfloor n/2 \rfloor$, without appealing to the general statement discussed in Remark 4.12.

Question 5.2. Is h a regular element of R_{dFK}? Equivalently, is the map

$$R_{\mathrm{dFK}} \longrightarrow R_{\mathrm{dFK}}, \qquad u \longmapsto hu,$$

injective?

Question 5.3. Is R_{dFK} a torsion-free $\mathbb{Z}$-module? More ambitiously, can one construct a useful $\mathbb{Z}[h]$-module basis or a PBW-type normal form for R_{dFK}?

The specialization $h = 0$ is the universal Fomin–Kirillov algebra. Its size and structure are notoriously subtle. Bärligea [Bar] gives a proof claim for infinite-dimensionality in ranks $n \geq 6$; independently of the status of that result, the deformation questions above remain meaningful and are logically separate from the dimension problem at $h = 0$.

A potentially useful auxiliary direction is to study variants in which the square relation $X_{ij}^2 = h$ is weakened or removed, where noncommutative Gröbner methods may become more accessible. Mészáros [Mes, § 10] constructs such methods for a related subdivision algebra. At present we do not identify that algebra with a quotient or extension of R_{dFK}.

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