NORMAL FORMS OF WHITNEY UMBRELLA IN THE PRESENCE OF A CONE

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§1. Results

Let $\mathcal{S}$ be a stratified subvariety of $\mathbf{R}^n$ containing the origin. By $\mathcal{K}$ we denote *the group of contact transformations*, i.e. elements of $\mathcal{K}$ are diffeomorphisms of $(\mathbf{R}^l \times \mathbf{R}^n, 0)$ preserving a) projection on $\mathbf{R}^l$ (inducing diffeomorphisms of $\mathbf{R}^l$) and b) the subspace $(\mathbf{R}^l \times 0)$. Thus two germs f_1 and $f_2 : (\mathbf{R}^l, 0) \to (\mathbf{R}^n, 0)$ are $\mathcal{K}$-equivalent if there exists a diffeomorphism of the source and a germ $M : (\mathbf{R}^l, 0) \to GL(\mathbf{R}^n)$ such that $f_1(x) = M(x)f_2(g(x))$. Let $\mathcal{K}_S$ denote the subgroup of $\mathcal{K}$ consisting of all diffeomorphisms $H \in \mathcal{K}$ such that $H(\mathbf{R}^l \times \mathcal{S}) \subset \mathbf{R}^l \times \mathcal{S}$.

The purpose of this note is to classify the germs of maps $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ under the action of the group $\mathcal{K}_{\mathcal{C}}$ where $\mathcal{C}$ is the usual quadratic cone with the apex at the origin under the assumption that the image of f is a Whitney umbrella. Normal forms for some relative positions of the Whitney umbrella and a germ of a smooth surface in $\mathbf{R}^3$ w.r.t. the left-right and contact equivalence were earlier obtained in [Mo]. Some general properties of the subgroup $\mathcal{K}_S$ where $\mathcal{S}$ is an arbitrary stratified subvariety were studied in details in e.g. [Da].

Recall that any Whitney umbrella $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ determines in the tangent space at the origin of $\mathbf{R}^3$ *the associated complete flag* FL consisting of a distinguished line l and a hyperplane L, see fig.1 and e.g. [AVG]. The line l is the tangent space to the image of f at the origin and the tangent cone to the hypersurface of a Whitney umbrella is the plane L taken with the multiplicity 2. Simply connected components of the complement to the cone $\mathcal{C}$ are called *hyperbolic* while the nonsimply connected component is called *nonhyperbolic.*

Proposition. There exist 6 orbits of the GL_3-action on the set of pairs $(FL, \mathcal{C})$.

1) generic position: the line l lies in the hyperbolic components of the complement to $\mathcal{C}$;

2) generic position: the line l lies in the nonhyperbolic component of the complement to $\mathcal{C}$ and the hyperplane L intersects $\mathcal{C}$ in two lines;

3) generic position: the line l lies in the nonhyperbolic component of the complement to $\mathcal{C}$ and the hyperplane L does not intersect $\mathcal{C}$;

4) codimension= 1: l lies in the nonhyperbolic component of the complement to $\mathcal{C}$ and the hyperplane L is tangent to $\mathcal{C}$;

5) codimension= 1: l lies on $\mathcal{C}$ and the hyperplane L intersects $\mathcal{C}$ in two lines one of which is l;

6) codimension= 2: l lies on $\mathcal{C}$ and the hyperplane L is tangent to $\mathcal{C}$ along l.

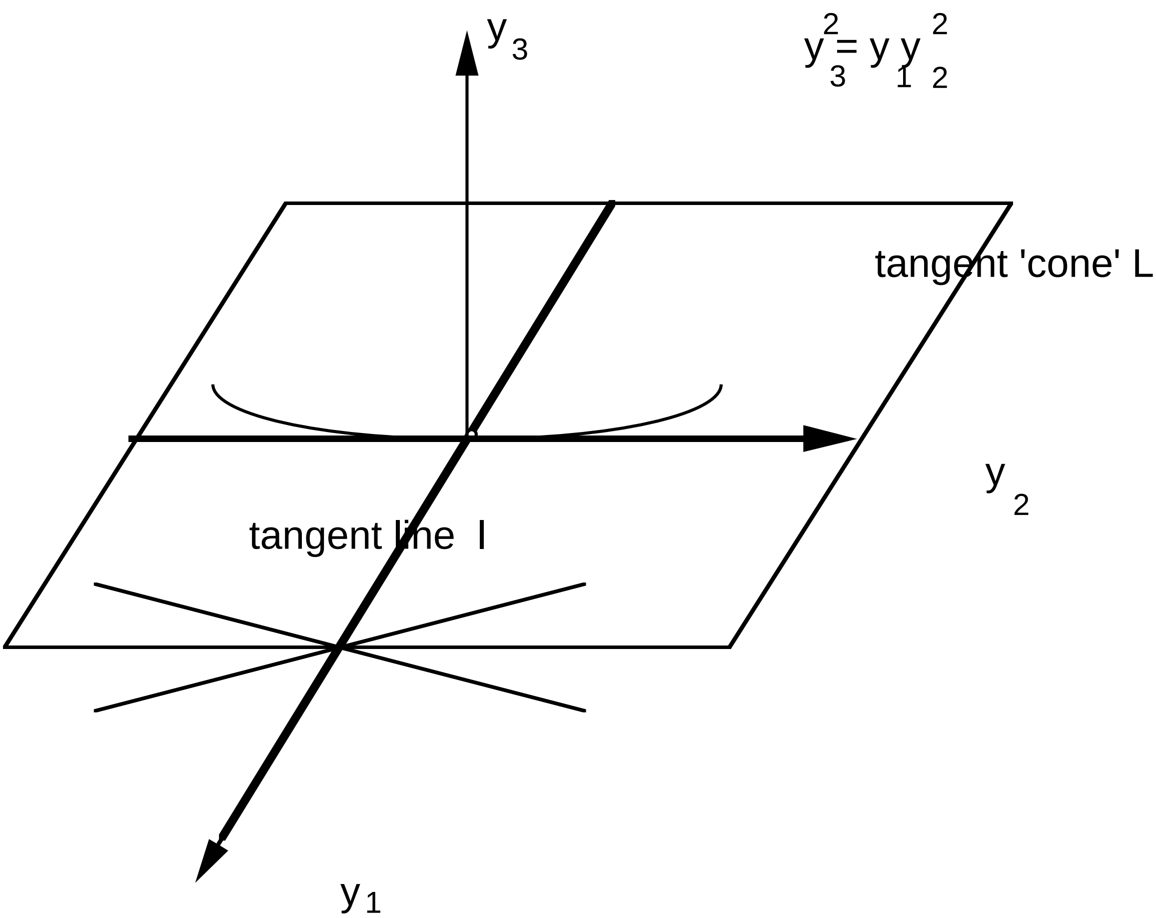

Fig. 1. The standard Whitney umbrella.

Now we are ready to present the list of the normal forms of f.

MAIN THEOREM. A map $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ the image of which is a Whitney umbrella and such that f^*Q has an isolated singularity can be reduced by the $\mathcal{K}_\mathcal{C}$-action to one of the following normal forms according to the relative position of FL w.r.t. $\mathcal{C}$, i.e. to the above orbits. Here $\mathcal{C}$is the usual quadratic cone given by the equation $Q := y_1y_2 - y_3^2 = 0$.

1) $f : y_1 = x_1, y_2 = x_1 + x_2^2, y_3 = 0,$ here $f^*Q \backsimeq A_3^-$;

2) $f : y_1 = x_1, y_2 = -x_1 + x_2^2, y_3 = 0,$ here $f^*Q \backsimeq A_3^-$;

3) $f : y_1 = x_1, y_2 = -x_1, y_3 = x_2^2,$ here $f^*Q \backsimeq A_3^+$;

4) $f_k : y_1 = x_1, y_2 = -x_1 + 2x_2^2 \mp x_2^{k-1}, y_3 = x_2^2,\ k \geq 4,$ here $f_k^*Q \backsimeq A_k^\pm$;

5) $f_k : y_1 = x_1, y_2 = x_2^2 \pm x_1^{k-2}, y_3 = 0,\ k \geq 4,$ here $f^*Q \backsimeq D_k^\pm$;

6) $f_5 : y_1 = x_1, y_2 = x_1x_2, y_3 = x_2^2,$ here $f^*Q \backsimeq D_5$.

Pay attention that in all the cases except case 6 the image of the 2-jet of f is not a Whitney umbrella but a simplier map with nonisolated singularities. This reflects the fact that $\mathcal{K}_\mathcal{C}$-equivalence does not even preserve the stable type of f.

REMARK. In the above formulation we also present the type of singularity of the pullback f^*Q according to the standard classification of singularities of germs of functions (hypersurfaces), see [AVG]. Here $Q = y_1y_2 - y_3^2$ is the quadratic form defining the cone $\mathcal{C}$.

The adjacency diagram of these singularities is given on Fig.2. (Bold numbers on Fig.2 correspond to one of the 6 subcases in the above theorem.)

This note is the continuation of the project started in [Sh] where the classification of the germs of smooth immersions in $\mathbf{R}^n$ under the action of $\mathcal{K}_\mathcal{C}$ was considered and the list of simple singularities was obtained. Here we study the first nontrivial irregular case, i.e. when the image of f is singular. The author is sincerely grateful to V. Goryunov for several discussions of the subject of this paper and especially for the simplification and improvement of the original calculations.

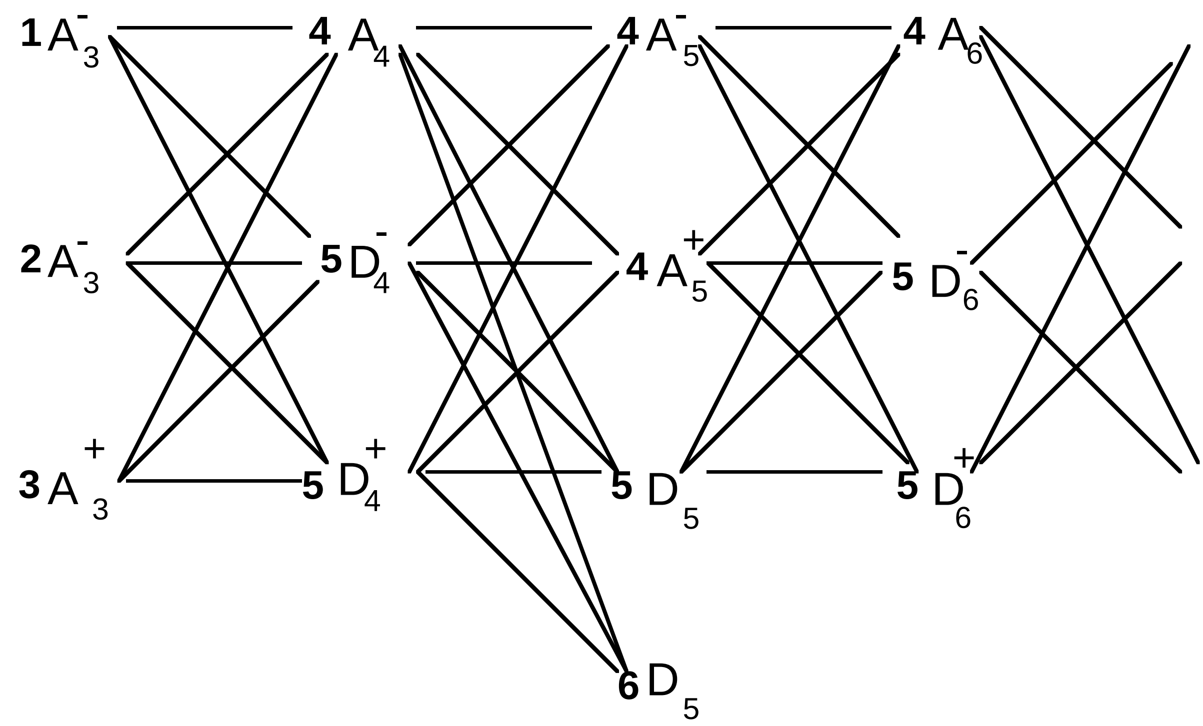

Fig. 2. Adjacency of orbits of K_C-action.

§2. Proofs

We start the proof of the theorem with the following remark.

Remark. The 1-jet of any map $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ of rank 1 can be reduced by the $\mathcal{K}_{\mathcal{C}}$-action to one of the following 3 forms: a) $(x_1, x_1, 0)$, b) $(x_1, -x_1, 0)$ and c) $(x_1, 0, 0)$.

A line through the origin can be reduced to one of the above lines by the action of the linear group preserving $\mathcal{C}$. $\square$

Lemma. A) The 2-jet of any map $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ of rank 1 can be reduced by the $\mathcal{K}_{\mathcal{C}}$-action to one of the following forms: 1) $(x_1, x_1 + x_2^2, 0)$; 2) $(x_1, -x_1 + x_2^2, 0)$; 3) $(x_1, -x_1, x_2^2)$; 4) $(x_1, -x_1 + 2x_2^2, x_2^2)$; 5) $(x_1, x_2^2 \pm x_1^2, 0)$; 5′) $(x_1, x_2^2, 0)$; 6) $(x_1, x_1 x_2, x_2^2)$; 7) (x_1, x_1^2, x_2^2); 8) $(x_1, 0, x_2^2)$; 9) $(x_1, x_1^2, 0)$; 10) $(x_1, x_1, 0)$; 11) $(x_1, -x_1, 0)$; 12) $(x_1, 0, 0)$.

B) The last 6 of the above forms can not occur if the image of f is a Whitney umbrella.

Proofs of this lemma and the main theorem are based on the standard technique of reduction to the normal form described in [AVG] and in our special case in [Sh]. The basic idea is to consider the $\mathcal{O}_x$-module $\mathfrak{M}_{\mathcal{C}}(f)$ of vector fields tangent to the orbit $\mathcal{K}_{\mathcal{C}}(f)$ at f and calculate jets of low order of some basis of $\mathfrak{M}_{\mathcal{C}}(f)$ depending on some low jet of f. According to [Sh] the Lie algebra of $\mathcal{K}_{\mathcal{C}}$ has a basis consisting of 6 vector fields, namely $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and 4 vector fields spanning the basis of the module of vector fields tangent to the nondegenerate cone $\mathcal{C}$. One of these generators is the standard Euler field $(y_1\frac{\partial}{\partial y_1}, y_2\frac{\partial}{\partial y_2}, y_3\frac{\partial}{\partial y_3})$ while the rest are all (2×2)-determinants of the (2×3)-matrix $\begin{pmatrix} \frac{\partial}{\partial y_1}, & \frac{\partial}{\partial y_2}, & \frac{\partial}{\partial y_3} \\ Q_{y_1}, & Q_{y_2}, & Q_{y_3} \end{pmatrix}$, where $Q = y_1y_2 - y_3^2$ is the polynomial defining the cone $\mathcal{C}$ and $Q_{y_i} = \frac{\partial Q}{\partial y_i}$. To get a basis of $\mathfrak{M}_{\mathcal{C}}(f)$ one has to apply the above 6 vector fields to f.

SKETCH OF PROOF OF LEMMA. A) Considering the above vector fields one sees directly that if the 1-jet of f is of type a) or b) (see the above remark) then one can remove all terms of f of order greater than 1 which are divisible by x_1 and the y_1-coordinate can be always kept equal to x_1. Thus for the 1-jets a) and b) we have to consider only the 2-jets: $(x_1, \pm x_1 + \alpha x_2^2, \beta x_2^2)$.

If $\beta = 0$ and $\alpha \neq 0$ then one can normalize $\alpha = 1$ and obtain the normal forms 1),2). (Pay attention that the 2-jets $(x_1, x_1 + x_2^2, 0)$ and $(x_1, x_1 - x_2^2, 0)$ are $\mathcal{K}_{\mathcal{C}}$-equivalent by a linear transformation sending y_1 to y_2 and y_2 to y_1.)

If $\beta \neq 0$ then it can be normalized as $\beta = 1$. In this case after some transformation of the basis of the vector fields one gets for the 1-jet b) the tangent vector field $(0, (\alpha-2)x_2^2, 0)$. This means that for $\alpha \neq 2$ the term αx_2^2 can be removed from y_2-coordinate. This leads to the normal forms 3) and 4). Finally, if $\alpha = \beta = 0$ one gets the normal forms 9) and 10).

We now consider the 1-jet c). In this case one can still remove all terms divisible by x_1 in y_3-coordinate. Thus it suffices to consider to the 2-jets of the form $(x_1, \alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2, \delta x_2^2)$.

Studying the above basis of vector fields one gets that if $\gamma \neq 0$ one can remove δx_2^2 from y_3 and normalize $\gamma = 1$. This reduces consideration to the 2-jets of the form $(x_1, \alpha x_1^2 + \beta x_1x_2 + x_2^2, 0)$. Using complete squares and normalizing α in y_2-coordinate one gets the normal forms 5) and 5$'$).

If $\gamma = 0$ and $\delta \neq 0$ then one normalizes $\delta = 1$ and gets the 2-jet $(x_1, \alpha x_1^2 + \beta x_1x_2, x_2^2)$. The case $\alpha \neq 0$ and $\beta = 0$ leads to the normal form 7). If $\alpha = \beta = 0$ one gets 8). Now if $\beta \neq 0$ one changes $\bar{x}_2 = \beta x_2 + \alpha x_1$ and gets (x_1, x_1x_2, P) where P is some polynomial of degree 2. After that one removes all terms of P divisible by x_1 and obtains the normal form 6). Finally, if all terms of order 2 vanish one gets 11).

B) It remains to explain why the last 6 normal forms do not occur if the image of f is a Whitney umbrella. If a Whitney umbrella is considered as a hypersurface then its tangent cone at the vertex has the equation $h_L^2 = 0$ where h_L is a linear form defining the hyperplane L of its associated flag, see fig. 1. It is easy to show that the tangent cone is the square of a linear form for all ϕ belonging to the orbit $\mathcal{K}_{\mathcal{C}} f$ if the image of f is a Whitney umbrella. This property forbids the normal forms 9)-12).

Let us now assume that f is a map the image of which is a Whitney umbrella and its 2-jet is $\mathcal{K}_{\mathcal{C}}$-reducible to either 7) or 8). Then its associated complete flag (l, L) has the form: l is the y_1-axis and L is the hyperplane $\{y_2 = 0\}$. Now the 2-jet of any f defining a Whitney umbrella with the above complete flag has the form: $y_1 = a_1x_1 + a_{11}x_1^2,\ a_1 \neq 0; y_2 = b_1x_1 + b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2,\ b_{12} \neq 0; y_3 = c_1x_1 + c_{11}x_1^2 + c_{22}x_2^2,\ c_{22} \neq 0$.

In the process of reduction sketched above one can simplify this 2-jet to $y_1 = x_1; y_2 = b_{11}x_1^2 + b_{12}x_1x_2,\ b_{12} \neq 0; y_3 = c_{22}x_2^2,\ c_{22} \neq 0$ and since $b_{12} \neq 0$ this leads us only to the normal form 6) and never to 7) or 8). $\square$

Recall that the k-jet of f is called $\mathcal{K}_{\mathcal{C}}$-*sufficient* if any germ $f + \epsilon$ belongs to the $\mathcal{K}_{\mathcal{C}}$-orbit of f if ϵ contains only terms of order $\geq k + 1$. (See the details and criterion of sufficiency for the group $\mathcal{K}_{\mathcal{C}}$ in [Sh] and [Da].)

Sketch of proof of the main theorem. The 2-jet of f is sufficient in the cases 1), 2), 3) and 6) since the pullbacks of Q defined by these 2-jets give simple singularities. In the case 4) one has to consider only the presented series f_k since one can remove all terms divisible by x_1 in y_2 and y_3-coordinates and all terms $x_2^l,\ l \geq 3$ in y_3. Singularities f_k are pairwise nonequivalent. Finally, for the 2-jets 5) and 5′) we can remove all terms divisible by x_1 in y_3 and all terms of order ≥ 3 divisible by x_2 in y_3. This leads us exactly to the series of f presented in the theorem singularities of which are pairwise nonequivalent. $\square$

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