Semiclassical expansion for exactly solvable differential operators

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Abstract

Below we study a linear differential equation $\mathcal{M}(v(z,\eta)) = \eta^M v(z,\eta)$, where $\eta > 0$ is a large spectral parameter and $\mathcal{M} = \sum_{k=1}^{M} \rho_k(z)\frac{d^k}{dz^k}$, $M \geq 2$ is a differential operator with polynomial coefficients such that the leading coefficient $\rho_M(z)$ is a monic complex-valued polynomial with $\deg[\rho_M] = M$ and other $\rho_k(z)$'s are complex-valued polynomials with $\deg[\rho_k] \leq k$. We prove the Borel summability of its WKB-solutions in the Stokes regions. For $M = 3$ under the assumption that ρ_M has simple zeros, we give the full description of the Stokes complex (i.e. the union of all Stokes curves) of this equation. Finally, we show that for the Euler-Cauchy equations, their WKB-solutions converge in the usual sense.

1 Introduction

1.1 Set-up of the problem

In mathematical physics, a linear differential operator

$$\mathcal{M} = \sum_{k=1}^{M} \rho_k(z)\frac{d^k}{dz^k} \tag{1}$$

with polynomial complex-valued coefficients is called *exactly solvable* if:
(i) $\deg[\rho_k] \leq k$, $1 \leq k \leq M$;
(ii) there exists (at least one) $1 \leq \ell \leq M$ such that $\deg[\rho_\ell] = \ell$.

This terminology is motivated by the fact that any exactly solvable operator $\mathcal{M}$ preserves the infinite flag of linear subspaces of polynomials whose degree does not exceed a given non-negative integer n. Therefore one can explicitly find the sequence of its eigenvalues as well as the sequence of corresponding eigenfunctions in the form of polynomials of consecutive degrees $n = 0, 1, \dots$ by using linear algebra methods. In other words, one can solve a certain spectral problem for $\mathcal{M}$ exactly and explicitly which explains the terminology. We will call these polynomial eigenfunctions *eigenpolynomials* of $\mathcal{M}$ and denote by $\{Q_n^{\mathcal{M}}(z)\}_{n=0}^{\infty}$ the sequence of monic eigenpolynomials of $\mathcal{M}$. (For any exactly solvable $\mathcal{M}$, its monic eigenpolynomials $Q_n^{\mathcal{M}}(z)$ are unique for all sufficiently large n). We say that an exactly solvable operator $\mathcal{M}$ is *non-degenerate* if $\deg[\rho_M] = M$.

Exactly solvable operators appeared already in the 1930's in connection with the so-called Bochner-Krall problem asking which exactly solvable operators have sequences of eigenpolynomials which are orthogonal with respect to an appropriate linear functional. Different results on the asymptotic behavior of sequence of eigenpolynomials can be found in e.g. [1, 2, 3, 4] and on the Bochner-Krall problem in e.g. [5, 6, 7, 8, 9, 10]. One should mention that the Bochner-Krall problem still remains widely open.

Notation 1 Given a polynomial $P(z)$ of degree n, denote by $\mu_P = \frac{1}{n}\sum_{i=1}^{n}\delta(z-u_i)$ its root-counting measure where $u_1, \dots, u_n$ are the roots of $P(z)$ (with repetitions) and $\delta(z-u)$ is Dirac's delta measure supported at u.

It has been conjectured in [4] and shown in [3] that for any non-degenerate exactly solvable operator $\mathcal{M}$, the sequence $\{\mu_n^{\mathcal{M}}\}$ of the root-counting measures of its sequence $\{Q_n^{\mathcal{M}}(z)\}$ of eigenpolynomials converges in the weak sense to a probability measure $\mu^{\mathcal{M}}$ depending only on the leading coefficient $\rho_M(z)$. Moreover $\mu^{\mathcal{M}}$ is supported on an embedded graph in $\mathbb{C}$ which is topologically a tree whose leaves (i.e. vertices of valency 1) are exactly all roots of $\rho_M(z)$, see [3, Th.3]. Further, the support of $\mu^{\mathcal{M}}$ lies inside the convex hull of these roots and can be straightened out in a certain local canonical coordinate which is very natural from the point of view of semiclassical asymptotic for solutions of a linear ODE. More information about $\mu^{\mathcal{M}}$ can be found in [4, 3].

In particular, for $M = 3$, $\operatorname{supp}[\mu^{\mathcal{M}}]$ is a tree with leaves given by the zeros of $\rho_3(z)$. Hence, if all these zeros are simple, $\operatorname{supp}[\mu^{\mathcal{M}}]$ is the union of three smooth Jordan arcs $\{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\}$ connecting each zero of $\rho_3(z)$ to a common point v contained in the convex hull of the zeros of $\rho_3(z)$, see Figure 1. (The angle between any pair of these arcs at v is 120^o, see [3] and Lemmas 3 and 4).

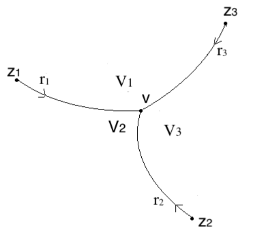

Figure 1: Typical structure of $\operatorname{supp}[\mu^{\mathcal{M}}]$ for $M = 3$

Notation 2 Given a non-degenerate exactly solvable operator $\mathcal{M}$, set $\Omega := \mathbb{C} \setminus \operatorname{supp}[\mu^{\mathcal{M}}]$. For a given Jordan arc τ connecting ∞ and an arbitrary point $z_0 \in \operatorname{supp}[\mu^{\mathcal{M}}]$, we will denote by Ω_τ the open connected set $\Omega \setminus \tau$ and by $\mathcal{Z}$ the set of zeros of ρ_M. Given an open set $U \subset \mathbb{C}$, let $\mathcal{H}(U)$ stand for the space of analytic functions in U. For a set A, let $\mathring{A}$ denote the interior of A. If γ is a closed Jordan curve in $\mathbb{C}$, we denote by $int(\gamma)$ and $ext(\gamma)$ the bounded and unbounded connected components of $\mathbb{C} \setminus \gamma$ respectively. For any oriented Jordan arc $\tau \subset \mathbb{C}$, we denote by τ^+ the (local) side to the left of τ.

Denote by w_1 the branch of $\frac{1}{\sqrt[M]{\rho_M(z)}}$ in Ω which has asymptotic $\dfrac{1}{z}$ near ∞. Introduce the other branches of $\frac{1}{\sqrt[M]{\rho_M(z)}}$ as

$$w_j(z) := e^{\frac{2\pi i(j-1)}{M}} w_1(z), \ z \in \Omega, \ j = 2, \dots, M. \tag{2}$$

Further, define Φ_0 as the primitive of $w_1(z)$ in Ω_τ such that

$$\lim_{z\to\infty} \Phi_0(z) - \ln z = 0$$

and define Φ_1 as the primitive of the function $\frac{(M-1)\rho_M'(z)}{2M\rho_M(z)} - \frac{\rho_{M-1}(z)}{M\rho_M(z)}$ in Ω_τ such that

$$\lim_{z\to\infty} \Phi_1(z) - \left(\frac{M-1}{2} - \frac{\rho_{M-1,M-1}}{M}\right) \ln z = 0,$$

where $\rho_{M-1,M-1}$ is the coefficient of z^{M-1} in $\rho_{M-1}(z)$.

In a recent publication [11] the first author has established the following WKB-expansion for the sequence $\{p_n^{\mathcal{M}}(z)\}$ of monic eigenpolynomials of a given non-degenerate exactly solvable operator $\mathcal{M}$ originally conjectured in [4].

Theorem A (see Theorem 1 of [11]) *For a non–degenerate exactly solvable operator $\mathcal{M}$ of order $M \geq 2$ and the sequence $\{p_n^{\mathcal{M}}(z)\}$ of its monic eigenpolynomials, when $n \to \infty$ one has the asymptotic expansion in the sense of Poincaré*

$$Q_n^{\mathcal{M}}(z) \sim \exp\left(n\Phi_0(z) - \left(\frac{M-1}{2} - \frac{\rho_{M-1,M-1}}{M}\right)\Phi_0(z) + \Phi_1(z)\right)\left(1 + \frac{C_1(z)}{n} + \frac{C_2(z)}{n^2} + \dots\right)$$

uniformly on compacts subsets $K \subset \Omega$, where for $j \geq 1$, C_j are analytic in Ω.

(The definition of the above asymptotic expansion can be found in e.g. (7.03), p.16 of [12]).

In the present article we attempt to extend the existing results of the exact WKB-analysis to the case of exactly solvable operators. In particular, we establish the Borel summability of the WKB-solutions of the operator $\mathcal{M}$ in the regions bounded by the Stokes curves (see exact statements below). Notice that the description of the global geometry of Stokes curves is a challenging open problem for very many types of operators. In this direction, we give the full description of the Stokes curves in case of ρ_3 having simple roots. Typically the study of the WKB-solutions of higher order differential operators is carried out by using factorization in lower order operators [13, 14, 15, 16, 17]. In what follows, we provide an example of an exactly solvable operator not admitting such a factorization. Finally, it is well-known that, in general, the WKB-series diverges in the usual sense. However, as we show below, in the case of the Euler-Cauchy differential operators, their WKB-series converges.

1.2 Short historical account

Analysis of the Borel resummed WKB-solutions has been a topic of interest for at least half a century. Bender and Wu [18] were the first to notice the relevance of the Borel summability to the analysis of the WKB-solutions. In [19] Voros studied the special case of the second order Schrödinger equation with a quartic potential establishing the connection formulae for its WKB-solutions, and in [20] Silverstone discussed the connection problem further. At the same time, Ecalle was developing the theory of resurgent functions [21, 22], which is also based on the Borel sums. Extending these contributions, several researchers [23, 24, 25, 26] introduced what is now known as the *exact* or the *complex WKB-analysis*, see [27] for some historical remarks.

The study of the regions in which the WKB-solutions are Borel summable is a fundamental problem of the exact WKB-analysis. For the second order Schrödinger type linear ordinary differential equations

$$\frac{d^2y}{dz^2} - \eta^2 Q(z) y = 0,$$

when Q is a rational function and η is a large positive parameter, this problem has been solved in [28].

For the ordinary differential equations of the form

$$\frac{d^2y}{dz^2} - \eta^2 \left(f_0(z) + \frac{f_1(z)}{\eta} + \frac{f_2(z)}{\eta^2} \right) y = 0,$$

where f_0, f_1, f_2 are analytic in some domain and large $\Re[\eta] > 0$, the Borel summability was established in [29], see also references in [30].

In [31] the authors studied the Borel summability of the WKB-solutions for higher order linear differential equations of the form

$$\sum_{j=0}^{n} a_j(z) \left(\eta^{-1} \frac{\partial}{\partial z} \right)^j y(z,\eta),$$

with polynomial coefficients and large positive parameter η by using reduction to the linear second order differential equations via middle convolutions. In a recent publication [30], the author considers differential operators of the form

$$\left(-\frac{\partial^n}{\partial z^n} + \sum_{k=0}^{n-2} \eta^{n-k} f_k(z,\eta) \frac{\partial^n}{\partial z^n} \right) y(z,\eta),$$

where $n \geq 2$ and f_k are analytic functions of z varying in some bounded domain or on a Riemann surface $\mathbb{D}$, possessing as $\eta \to \infty$ and z varying uniformly in $\mathbb{D}$ asymptotic expansions of the form

$$f_0(z,\eta) \sim \sum_{k=0}^{\infty} \frac{f_{0,k}(z)}{\eta^k}, \quad f_k(z,\eta) \sim \sum_{k=0}^{\infty} \frac{f_{k,m}(z)}{\eta^k}.$$

They proved the Borel summability of the WKB-solutions when z varies in some subdomain of $\mathbb{D}$.

The present manuscript is organized as follows. In Section 2 we discuss some basic aspects of the general theory of the Borel summability and formulate our main results. Section 3 is devoted to a number of technical results required for the main proofs which we carry out in Section 4. In Section 5 we give an example of an exactly solvable differential operator which can not be factorized into the WKB-type linear differential operators of lower order. Finally, in Section 6 we show that the Euler-Cauchy operators admit convergent WKB-solutions.

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2 Preliminaries, basic notions, and formulation of main results

2.1 Preliminaries

Below we will often use of the general notion of an operator of the WKB-type, introduced in [15], see also [14]. Let $U \subset \mathbb{C}$ be an open subset, $(z, y) \in U \times \mathbb{C}$ and $(z, y; \zeta, \eta) \in T^*(U \times \mathbb{C})$, where T^* denotes the cotangent bundle.

Definition 1 *A differential operator P of the WKB-type on an open set $U \subset \mathbb{C}_z$ is a microdifferential operator of order 0 defined on $(z, y; \zeta, \eta) \in T^*(U \times \mathbb{C}), \eta \neq 0$ commuting with the differentiation with respect to y, i.e. $[P, \partial_y] := P\partial_y - \partial_y P = 0$. Thus, its total symbol $\sigma_0(P)$ is a formal power series of the form:*

$$\sigma_0(P) = \sum_{j \geq 0} \eta^{-j} P_j \left(z, \frac{\zeta}{\eta}\right),$$

where $(P_j(z, \zeta))_{j \geq 0}$ are holomorphic functions in $z \in U$ and entire functions in ζ (in our current context they are actually polynomials in ζ), and they satisfy the following growth condition:

– there exists a constant $c_0 > 0$ such that for each compact set K in $U \times \mathbb{C}_\zeta$, we can find another constant c_K for which

$$\sup_K |P_j(z, \zeta)| \leq c_K j! c_0^j.$$

Following the traditional terminology of the microlocal analysis, we call $\sigma(P) := P_0(z, \zeta)$ the principal symbol of the operator P.

Example 1 In [31] the authors studied the Stokes geometry of the WKB-solutions of linear n-th order differential operators of the form

$$\mathcal{S}\phi(z, \eta) = \sum_{j=0}^{n} a_j(z) \left(\eta^{-1} \frac{\partial}{\partial z}\right)^j \phi(z, \eta), \tag{3}$$

where $a_j(z)$ are polynomials, a_n is a non–zero complex constant, and $\eta > 0$ is a large parameter. By definition, $\sigma_0(\mathcal{S})$ can be expressed as $\sigma_0(\mathcal{S}) = P_0\left(z, \frac{\zeta}{\eta}\right) = \sum_{j=0}^{n} a_j(z) \left(\frac{\zeta}{\eta}\right)^j$. Hence, $\mathcal{S}$ is of the WKB-type.

Consider the linear differential operator

$$\mathcal{L} = \mathcal{M} - \eta^M \tag{4}$$

where $\mathcal{M}$ given by (1) is an exactly solvable operator. Let $v(z, \eta)$ be a family of solutions to the equation

$$\mathcal{L}(v(z, \eta)) = 0,$$

where (unless otherwise specified) η is assumed to be a large positive parameter. In other words, $v(z, \eta)$ satisfies the relation

$$v^{(M)}(z, \eta) + \sum_{k=1}^{M-1} \frac{\rho_k(z)}{\rho_M(z)} v^{(k)}(z, \eta) - \eta^M \frac{v(z, \eta)}{\rho_M(z)} = 0. \tag{5}$$

Example 2 Let $\mathcal{L}$ be defined as in (4). Then $\eta^{-M}\mathcal{L}$ is of the WKB-type. Indeed,

$$\sigma_0(\eta^{-M}\mathcal{L}) = \sum_{k=1}^{M} \eta^{-M+k} \rho_k(z) \frac{\zeta^k}{\eta^k} - 1 \tag{6}$$

$$= -1 + \rho_M(z) \left(\frac{\zeta}{\eta}\right)^M + \sum_{j=1}^{M-1} \eta^{-j} \rho_{M-j}(z) \left(\frac{\zeta}{\eta}\right)^{M-j}. \tag{7}$$

In case of a linear differential operator P given by (4) or (3), the fundamental role in its Stokes geometry is played by the branch points of its *symbol curve* $\Gamma_P \subset \mathbb{C}^2 \simeq \mathbb{C}_z \times \mathbb{C}_\zeta$ given by the *symbol equation*

$$\sigma(P)(z, \zeta) = 0. \tag{8}$$

Observe that for (4), its symbol equation reduces to

$$\sigma(\mathcal{L})(z, \zeta) = \rho_M(z)\zeta^M - 1 = 0. \tag{9}$$

This equation plays a fundamental role in the asymptotic analysis of the operator $\mathcal{L}$ and will be referred to as the *characteristic equation* of (4).

Projecting Γ_P on the first coordinate $\mathbb{C}_z$ we obtain the finite subset of Γ_P consisting of the *branching points* of this projection, i.e. points near which the projection is not a local diffeomorphism. To globalize the situation, one usually considers the compactification $\widehat{\Gamma}_P \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_\zeta$ of $\Gamma_P \subset \mathbb{C}_z \times \mathbb{C}_\zeta$, its projection on $\mathbb{C}P^1_z$, and its branching points.

When (8) reduces to an algebraic equation $b_n(z)\zeta^n + b_{n-1}(z)\zeta^{n-1} + \ldots + b_0(z) = 0$ (see [32, p.185]) the *set of critical values* of $\widehat{\Gamma}_P$ is a finite subset of $\mathbb{C}P^1_z$ whose points satisfy at least one of the following 3 conditions:

- $z = \infty$;
- $b_n(z) = 0$;
- $z \in \mathbb{C}_z$ is a point at which the characteristic equation (8) has a multiple root in the variable w.

It is known that the projection of any branch point of $\widehat{\Gamma}_P$ to $\mathbb{C}P^1_z$ lies among its critical values, cf. [32, Th 4.14.3 p.186].

Definition 2 *A WKB-solution of a linear differential operator P of the WKB-type is a formal solution of the form:*

$$\begin{aligned}\psi(z,\eta,z^*) &= \eta^{-\frac{1}{2}} \exp\left(\int_{z^*}^{z} S(\zeta,\eta)d\zeta\right) \\ &= \eta^{-\frac{1}{2}} \exp\left(\eta \int_{z^*}^{z} \sum_{k=0}^{\infty} \frac{S_k(\zeta)d\zeta}{\eta^k}\right) \\ &= \exp\left(\eta \int_{z^*}^{z} S_0(\zeta)d\zeta\right) \sum_{n=0}^{\infty} \frac{\phi_n(z)}{\eta^{n-\frac{1}{2}}},\end{aligned}$$

where S_k are (locally) holomorphic functions, and z^* is some reference point.

Remark 1 Some authors instead of $\eta^{-\frac{1}{2}}$ in the above definition consider the more general normalizing factor $\eta^{-\alpha}, \alpha \geq 0$, cf. [28].

A WKB-solution for (4) can be constructed by substituting the expression $e^{\int^z S(\xi,\eta)d\xi}$ in (4) and solving it for $S(\xi,\eta)$. In Lemma 5 we show that S should satisfy a generalized Riccati equation of order $M-1$. Expanding $S(z,\eta) = \sum_{k=0}^{\infty} S_k(z)\eta^{1-k}$, one can easily check that $S_0(z)$ is a solution of the simple algebraic equation

$$\rho_M(z) S_0^M(z) - 1 = 0 \quad \leftrightarrow \quad S_0(z) = \frac{1}{\sqrt[M]{\rho_M(z)}},$$

and the remaining $S_n, n > 0$ can be obtained recursively from $S_{n-1}, \dots, S_0$, cf. [33]. To obtain a WKB-solution, (considering z^* as the reference point) the indefinite integral in the expression is substituted by a definite integral $\int_{z^*}^{z} S(\xi,\eta)d\xi$.

In our case, the functions S_n may have singularities at each zero z_k of $\rho_M(z)$ so that the above integral of $S(\xi,\eta)$ can not be defined in the usual sense when $z^* = z_k$. Indeed, from [11, Prop.1 b)], the function S_1 is given by

$$S_1(z) = \frac{(M-1)\rho_M'(z)}{2M\rho_M(z)} - \frac{\rho_{M-1}(z)}{M\rho_M(z)}.$$

In particular, if $\rho_{M-1} \neq \frac{(M-1)}{2}\rho_M'$, we have $S_1(z) \sim a_k(z-z_k)^{-m_k}$ in a neighborhood of z_k, where m_k is the multiplicity of the root z_k and $a_k \in \mathbb{C}$. Notice also that for $\rho_M(z) = (z-z_k)^M$, the function S_0 reduces to $\dfrac{\sqrt[M]{1}}{z-z_k}$.

For this reason, when the reference point is taken at z_k we will interpret the integral $\int_{z_k}^{z}$ in the sense of the *Hadamard finite part*. We recall its definition introduced by Hadamard in order to deal with some divergent integrals, cf. [34, Ch.1]. For simplicity, let $f(x), x \in \mathbb{R}$ be given by

$$f(x) = a(x-c)^{-\nu} + b(x-c)^{-1} + s(x), \tag{10}$$

where $c \in \mathbb{R}, \Re[\nu] > 1, \nu \neq 1$, and $s(x)$ is integrable on $[c,C]$. Choosing any δ such that $c < c+\delta < C$, set $J(\delta) = \int_{c+\delta}^{C} f(x)dx$. Then term-by-term integration yields

$$J(\delta) = -\frac{a}{\nu-1}(C-c)^{-\nu+1} + b\ln(C-c) + \frac{a}{\nu-1}\delta^{-\nu+1} - b\ln\delta + \int_{c+\delta}^{C} s(x)dx.$$

When $\delta \to 0$, the function $J(\delta)$ has no finite limit because of the terms $\frac{a}{\nu-1}\delta^{-\nu+1} - b\ln\delta$, but the remaining terms in the right-hand side have a limit which is called the *finite part of the integral* $\int_{c+\delta}^{C} f(x)dx$ when $\delta \to 0$. We will use the notation $\mathtt{Fp}\int_{c}^{C} f(x)dx$ to represent this finite part. Notice that from (10) we have

$$\mathtt{Fp}\int_{c}^{C} f(x)dx = -\frac{a}{\nu-1}(C-c)^{-\nu+1} + b\ln(C-c) + \int_{c}^{C} s(x)dx. \tag{11}$$

The definition (11) is easily extended to the case when the integration is taken along a smooth arc γ in the complex plane.

Observe that in the domain Ω, the algebraic function $S_0(z)$ splits into M single-valued branches given by $w_1(z), \ldots, w_M(z)$, see above. We denote $S_0(z) = w_j(z),\ j = 1, \ldots, M$; the choice of the branch will be made according to the index j of the function w_j.

Once we have chosen $S_0(z)$, we can recursively define the consecutive terms $S_k(z),\ k \geq 1$ of the corresponding WKB-solution which will also be single-valued functions in Ω_τ. In this way we obtain M (formal) WKB-solutions $\psi_j,\ j = 1, \ldots, M$ of (4) in Ω_τ. The following definition plays an important role in the proof of the Borel summability of ψ_j formulated in Theorem 1,

Notation 3 *Take $z_k \in \mathcal{Z}$ and let ψ_j be a WKB-solution. Denote by $\mathcal{D}(z_k)$ the set of indices n for which the integral $\int_{z_k}^{z} S_n(\zeta)d\zeta$ diverges, cf. Lemma 7.*

It might happen that the WKB-solutions converge in some subdomain of Ω_τ, as in e.g. Theorem 6. But, generally, $\psi_j,\ j = 1, \ldots, M$ are diverging in the whole Ω. Due to the important discovery of Voros [19] and Ecalle [22], the use of the Borel resummation technique (or the Borel-Laplace method) with respect to a large parameter η rescues the situation; we recall the definitions of Voros [19] below.

Definition 3 *Let $\eta > 0$ be a large parameter and $y_0, f_n,\ \alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ be constants. For an infinite series*

$$f(\eta) = \exp(\eta y_0) \sum_{n=0}^{\infty} f_n \eta^{-(n+\alpha)},$$

the Borel transform $f_B(y)$ and the Borel sum $F(\eta)$ of f are defined as

$$f_B(y) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\alpha)} (y + y_0)^{n+\alpha-1} \quad \text{and} \quad F(\eta) = \int_{-y_0}^{\infty} e^{-y\eta} f_B(y) dy,$$

respectively provided that the right-hand sides exist. Here $\Gamma(s)$ is Euler's Γ-function and the integration path is taken parallel to the positive real axis.

Remark 2 Following Definition 2 of the WKB-series, we will use $\alpha = \frac{1}{2}$ throughout the manuscript. The definition of the Borel transform might vary in the literature. While most of the authors use the above definition, some use

$$f_B(y) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n+\alpha)} (y + y_0)^{n+\alpha},$$

see [35, 19].

We will denote by $\psi_{j,B}(z, y, z^*)$ the Borel transform of the WKB-solution $\psi_j(z, \eta, z^*)$. It is defined as

$$\psi_{j,B}(z, y, z^*) = \sum_{n \geq 0} \frac{f_n(z, z^*)}{\Gamma(n + \frac{1}{2})} (y + y_0(z, z^*))^{n - \frac{1}{2}}, \tag{12}$$

where $y_0(z, z^*) = \int_{z^*}^{z} S_0(\zeta)d\zeta$ and f_n are determined recursively once S_0 is fixed, cf. Lemma 7. Denote by $\Psi_j(z, \eta, z^*)$ the Borel sum of $\psi_j(z, \eta, z^*)$.

For $\psi(z, \eta) = \exp(\eta y_0(z)) \sum_{n \geq 0} \phi_n(z) \eta^{-(n+\alpha)}, \alpha > 0, \alpha \notin \mathbb{Z}$, it is immediate from the definition that

$$\left[\frac{\partial \psi}{\partial z}\right]_B = \frac{\partial}{\partial z} \psi_B \quad \text{and} \quad [\eta^m \psi]_B = \left(\frac{\partial}{\partial y}\right)^m \psi_B, m = 1, 2, \ldots$$

In particular, if $\psi(z, \eta)$ is a formal solution of the linear differential equation (5), then its Borel transform $\psi_B(z, y)$ satisfies the linear partial differential equation

$$\mathcal{L}_B \left(z, y, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y}\right) = \sum_{k=1}^{M} \rho_k(z) \frac{\partial^k u}{\partial z^k} - \frac{\partial^M u}{\partial y^M} = 0, \tag{13}$$

which coincides with the Borel transform of the operator $\mathcal{L}$.

The behavior of the WKB-solutions crucially depends on the critical points of the characteristic equation since these solutions do not provide a single-valued fundamental system in full neighborhoods of the critical points, cf. [36, Def.3.1-2 p.39]. This difficulty leads to the notion of turning points.

Definition 4 (see Def. 1.2.1 p.21 [37], [38, 39, 17]) *Let P be a differential operator of the WKB-type in an open set $U \subset \mathbb{C}_z$. A critical point a of (8) is called a turning point of P. When (8) reduces to an algebraic equation with coefficients in $\mathcal{H}(U)$ and $b_n(a) = 0$, we will additionally say that a is of pole-type, and if $b_n(a) \neq 0$, we refer to a as an ordinary turning point. When two roots $\zeta_j(z)$ and $\zeta_{j'}(z), j \neq j'$ of the symbol equation merge at a turning point a, we say that a has type (j, j'). (Notice that if more than two roots collide at a then several types are assigned to a.)*

If a is a turning point of type (j, j'), then a curve emanating from the point a and defined by the equation

$$\Im\left[\int_a^z (\zeta_j(\xi) - \zeta_{j'}(\xi))d\xi\right] = 0,$$

is called a Stokes curve of type (j, j') emanating from a.

We denote by $\mathcal{S}_{a,j}$ the set of all Stokes curves of type $(j, j'), 1 \leq j' \leq M, j \neq j'$ emanating from a.

An ordinary turning point of a linear ODE at which exactly two roots ζ_j and $\zeta_{j'}$ of its symbol equation collide is called simple, that is

$$\left.\frac{\partial\sigma(P)}{\partial z}\right|_{(z,\zeta)=(a,\zeta_j(a))} \neq 0.$$

The multiplicity of a pole-type turning point $z = a$ is defined as the multiplicity of the pole of $b_n(z)$ at $z = a$.

Remark 3 In Section 6 we study the Euler-Cauchy operator. In this case we can not use the preceding definition for the Stokes curves since the integral is not convergent in the usual sense. One can be tempted to define

$$\Im\left[\mathtt{Fp}\int_0^z (\zeta_j(\xi) - \zeta_{j'}(\xi))d\xi\right] = \Im\left[(e^{\frac{\pi(j-1)\imath}{M}} - e^{\frac{\pi(j'-1)\imath}{M}})\ln z\right] = 0. \tag{14}$$

Later we will show that there is no Stokes phenomenon present.

Remark 4 Although for a generic linear ODE depending on a parameter, all its ordinary turning points are typically simple, equation (4) we consider below is highly non-generic. Namely, one can easily observe from the characteristic equation (9) that the set of ordinary turning points of (4) coincides with the zero locus of $\rho_M(z)$. Moreover each of these zeros is a pole-type turning point of every type (j, j'), for all $1 \leq j < j' \leq M$. Notice also that in our case, the definition of the Stokes curves emanating from the turning point a reduces to

$$\Im\left[\int_a^z (w_j(\xi) - w_{j'}(\xi))d\xi\right] = 0.$$

Remark 5 Let a be an ordinary turning point of type (j, j'). Some authors define a Stokes curve as given by

$$\Re\left[\int_a^z (\zeta_j(\xi) - \zeta_{j'}(\xi))d\xi\right] = 0, \tag{15}$$

emanating from a, see e.g. [39, p.292].

It is well-known that in the second order case the Stokes regions (the regions where the Borel sum of the WKB-solutions is well-defined) are domains in the z-plane bounded by the Stokes curves, cf. [27, p.26]). For linear ODE of order greater than 2, Stokes regions are much more difficult to describe since the totality of the Stokes curves emanating from the (original) turning points are not enough to describe the boundaries of the Stokes regions. As was first noticed in [40], see also [13], the Borel summability of the WKB-solutions may fail on *new Stokes curves* obtained from ordered crossing points of the original Stokes curves, in the terminology of [40]. Thus, new Stokes curves emanating from new turning points are a natural generalization of the original Stokes curves emanating from the original turning points.

Due to results of Voros (see [19]) who first recognized that the Borel transform is a solution of a linear partial differential operator and to microlocal analysis [13, 41], new Stokes curves can be defined as the Stokes curves emanating from "new" singularities of the bicharacteristic strip. These are baptised in [13], where this concept was introduced as "new turning points" or "virtual turning points".

Definition 5 (see p.29, [37], [42, 41, 43]) *A bicharacteristic strip $\mathcal{BS}(t)$ associated with a linear partial differential operator is a complex-analytic curve $\mathcal{BS}(t) = (z(t), y(t); \zeta(t), \epsilon(t))_{t\in\mathbb{C}}$ in the cotangent bundle $T^*\mathbb{C}^2_{(z,y)}$ with coordinates*

$(z, y; \zeta, \epsilon)$ where ζ is dual to z and ϵ is dual to y defined by the following system of Hamilton–Jacobi equations:

$$\frac{dz}{dt} = \frac{\partial \sigma}{\partial \zeta} \tag{16}$$

$$\frac{dy}{dt} = \frac{\partial \sigma}{\partial \epsilon} \tag{17}$$

$$\frac{d\zeta}{dt} = -\frac{\partial \sigma}{\partial z} \tag{18}$$

$$\frac{d\epsilon}{dt} = -\frac{\partial \sigma}{\partial y}, \tag{19}$$

$$\sigma(z, y, \zeta, \epsilon) = 0, \tag{20}$$

where σ denotes the principal symbol of the operator. The image of the projection of a bicharacteristic strip $\mathcal{BS}(t)$ to the base $\mathbb{C}^2_{(z,y)}$ is called a bicharacteristic curve and is denoted by $\mathcal{BC}(t) := \{(z(t), y(t))\}_{t \in \mathbb{C}}$.

Remark 6 One can check that since the initial condition $\mathcal{BS}(t_0)$ of a bicharacteristic strip lies on the hypersurface $\sigma(z, y, \zeta, \epsilon) = 0$ then the whole bicharacteristic strip $\mathcal{BS}(t)$ lies on it as well.

A fundamental result of the microlocal analysis claims that the singularities of solutions of a linear partial differential equation with simple (in the sense of microlocal analysis [41, Ch.II]) characteristics, propagate along the bicharacteristic strips, see also [43, Cor.7.2.2]. Notice that by (18), for a WKB-type differential operator with ordinary simple turning points one has by definition $\frac{\partial \sigma}{\partial z}(a, 0, \zeta_0, 1) \neq 0$ which implies that the bicharacteristic strip emanating from $(a, 0, \zeta_0, 1)$ is locally non-singular in $T^*\mathbb{C}^2_{(z,y)}$. The singularities of the Borel transform belong to the same non-singular bicharacteristic strip and coalesce at a turning point. Such singularities are then called "cognate", as they belong to the same bicharacteristic strip. (Notice that on the bicharacteristic curve other singularities might exist as well). The most basic one among such singularities is a simple self-intersection point on $\mathcal{BC}(t)$ at which two of its smooth local branches intersect transversally, while the lifts of these two local branches to the respective bicharacteristic strip $\mathcal{BS}(t)$ are disjoint. The projections of such self-intersection points from $\mathcal{BC}(t)$ to $\mathbb{C}_z$ were baptized *virtual turning points* in [13], where they were first introduced and studied.

Definition 6 ([13, 44, 45, 37]) *Let P be a differential operator of the WKB-type with the principal symbol $\sigma_0(z, \zeta)$ and assume that its Borel transform P_B is well-defined. Assume additionally that the bicharacteristic strip is non-singular at the turning points. A virtual turning point of P is defined as the z-component of a self-intersection point of a bicharacteristic curve $\mathcal{BC}(t)$. If the self-intersection is associated with the factor $(\zeta - \zeta_j(z)\eta)$ and $(\zeta - \zeta_k(z)\eta)$ of the principal symbol $\sigma(P) = \prod_j (\zeta - \zeta_j(z)\eta)$, then the virtual turning point is said to be of type (j, k).*

If z^* is a virtual turning point of type (j, k), the curve emanating from z^*

$$\Im \left[\int_{z^*}^{z} (\zeta_j(\xi) - \zeta_k(\xi)) d\xi \right] = 0,$$

is called a new Stokes curve of type (j, k).

Remark 7 In the case of ordinary turning points of a linear differential operator of the WKB-type of multiplicities greater than 1 the singularities of solutions propagate along the so-called bicharacteristic chains, as shown in [46], see also [37, Ch.3]. In this case singularities bifurcate along two mutually tangent bicharacteristic curves at a double turning point where the simple characteristic condition is violated.

Observe that an exactly solvable operator can be considered as a linear differential operator with poles at the zeros of $\rho_M(z)$. Virtual turning points of operators with pole-type original turning points have been previously considered in [16]. The authors specifically considered a third order differential operator with a pole at $z = 0$ constructed from the Berk–Nevins–Roberts operator [40] by using a singular coordinate transformation. However it turns out that, in general, the analysis of the operator $\mathcal{L}$ can not be reduced to that of an operator with ordinary turning points by means of a coordinate transformation. Namely, in Section 5 we provide an example showing that such factorization does not exist in a neighborhood of a turning point of a cubic exactly solvable differential operator.

For this reason, when extending the concept of virtual turning points to $\mathcal{L}$ in order to analyze the propagation of singularities of its Borel transform $\mathcal{L}_B$ we follow a different approach suggested in [43, Sect. VII p.240], see also [42] and [47, p.44].

Namely, by [43, Cor.7.2.2], for a linear differential operator with complex coefficients and principal complex symbol $p(z, \zeta), z = (z_1, \dots, z_n), \zeta = (\zeta_1, \dots, \zeta_n)$, the singularities of its solutions propagate along $\mathcal{BS}(t)$ provided that

a) $\nabla_\zeta \Re[p] \neq 0$ and $\nabla_\zeta \Im[p] \neq 0$ are linearly independent,

b) $H_p\overline{p} = 0$,

where $H_p q = \dfrac{1}{\imath}\sum_j \dfrac{\partial p}{\partial \zeta_j}\dfrac{\partial q}{\partial z^j} - \dfrac{\partial p}{\partial z^j}\dfrac{\partial q}{\partial \zeta_j}$ is the Hamiltonian operator and $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right)$.

For a linear differential operator with holomorphic coefficients, condition *b*) is automatically satisfied since

$$H_p\overline{p} = [p, \overline{p}] = 0,$$

where $[p, \overline{p}]$ denotes the Poisson bracket of p and $\overline{p}$, and condition *a*) reduces to $\nabla_\zeta p \neq 0$.

Virtual turning points for the operator $\mathcal{L}$ can not be defined by formally following the approach for ordinary turning points since the expression $\dfrac{\partial \sigma_0}{\partial z}$ has a singularity at each turning point. Therefore the notion of "cognate" singularities in the sense of ordinary turning points does not apply in our case.

Below we describe the Stokes regions for the third order exactly solvable operators and consider the ordered crossings of their Stokes curves following the original approach of Berk–Nevins–Roberts in [40], but without providing the rigorous definition of a virtual turning point. In Theorem 1 b), we describe the singularities of the bicharacteristic strip as the initial step toward the understanding of this concept. We plan to return to this notion for exactly solvable operators in a future publication.

Notation 4 *Let $z = a$ be a turning point of type (j, j') (ordinary or pole-type). Then each segment of the Stokes curve emanating from the point a is labeled by either $(j > j')$ or by $(j < j')$, depending on whether*

$$\Re\left[\int_a^z (w_j(\xi) - w_{j'}(\xi))d\xi\right] > 0$$

or

$$\Re\left[\int_a^z (w_j(\xi) - w_{j'}(\xi))d\xi\right] < 0,$$

cf. [37, Def. 1.2.2 p.22].

Definition 7 (see p.38, [37]) *Consider two Stokes curves of types (j_1, j_2) and (j_2, j_3) and assume that they are crossing at a point C. We say that they define an ordered crossing at C if either $j_1 < j_2 < j_3$ or $j_1 > j_2 > j_3$. Following [40], we introduce a new Stokes curve emanating from C as given by*

$$\Im\left[\int_C^z (\zeta_{j_1}(\xi) - \zeta_{j_3}(\xi))d\xi\right] = 0.$$

(By calling it a "new Stokes curve" we distinguish it from the usual Stokes curves emanating from a usual turning point). We denote by $\mathcal{N}$ the set of all new Stokes curves.

Remark 8 For some linear differential operators for which the rigorous notion of a virtual turning point is available, a new Stokes curve can defined as a Stokes curve emanating from a virtual turning point, as in Definition 6. Since currently for the operator (4), the notion of a virtual turning point is unavailable we will follow the classical definition of new Stokes curves suggested in [40].

Definition 8 (see Def. 1.4.3 p.38, [37]) *We say that the Stokes curve is inert near z_0 if there is no Stokes phenomena, i.e. if there is no discontinuous change of the asymptotic near a point z_0 lying on a Stokes curve. If a Stokes curve is inert near all its points, we simply call it inert.*

In Figures below inert Stokes curves are shown by dotted lines.

2.2 Formulation of the main results

Theorem 1 *Take $z_k \in \mathcal{Z}, z \in \Omega$, and $\rho_M(z) \neq (z-a)^M$. Let $\mathcal{S}_{z_k,j}$ be as in Definition 4, and $\mathcal{N}$ be as in Definition 7. If $\psi_{j,B}(z, y, z_k)$ is the Borel transform of $\eta^{-\frac{1}{2}}\exp(\mathtt{Fp}\int_{z_k}^z S(\zeta,\eta)d\zeta)$, then*

a) the singularities of $\psi_{j,B}(z, y, z_k)$ occur at the points $\left\{(z,y) : y = -\displaystyle\int_{z_k}^z \frac{dt}{\sqrt[M]{\rho_M(t)}}\right\}, l = 1, \ldots, M;$

additionally, $\psi_{j,B}(z, y, z_k)$ is of exponential type when

$$y \in [-y_0(z, z_k), t\Re[-y_0(z, z_k)] + \imath(\Im[-y_0(z, z_k)])], t > 0; z \notin \mathcal{S}_{z_k,j} \cup \mathcal{N};$$

b) for the operator $\mathcal{L}_B$, its set $\mathcal{BC}(t)$ has no singularities other than $\mathcal{Z}$;

c) $\eta^{-\frac{1}{2}} \exp(\mathtt{Fp} \int_{z_k}^{z} S(\zeta, \eta) d\zeta)$ is Borel summable provided that $z \notin \mathcal{S}_{z_k,j} \cup \mathcal{N}$.

Theorem 2 *For $M \geq 2$, take a Stokes curve κ of the equation (4) of type $(1, j)$, emanating from a turning point $z_k \in \mathcal{Z}$ of (4), and going to ∞. Assume that ℓ neither connects z_1 with another turning point nor is a loop connecting z_1 to itself. Then, if two Stokes regions $U_i, i = 1, 2$ share a subset of κ with a chosen orientation on their boundaries, then one of the following two situations occur:*

Case 1. For $(1 > j)$, the Borel sums of the WKB-solutions $\Psi_{1,i}(z, \eta, z_k)$ and $\Psi_{j,i}(z, \eta, z_k)$ on U_i continue analytically from U_1 to U_2 (and in the opposite direction from U_2 to U_1). Moreover,

$$\begin{cases} \Psi_{1,1} = \Psi_{1,2} \\ \Psi_{j,1} = \Psi_{j,2}. \end{cases}$$

Case 2. For $(1 < j)$, one has

$$\begin{cases} \Psi_{1,1} = \Psi_{1,2} \\ \Psi_{j,1} = \Psi_{j,2} + c_j \Psi_{1,2}(z, \eta). \end{cases}$$

Here c_j is the "alien derivative" of $\psi_{1,B}$ in the sense of Ecalle (cf. [26, 48]) whose sign depends on a chosen orientation of κ. (The number k_j satisfies the equation $\Delta_{y=y_0(z,z_k)} \psi_{j,B}(z, y) = c_j \psi_{1,B}(z, y)$).

Corollary 1 *If κ is such that $(1 > j)$ then κ is inert.*

Theorem 3 *For $M = 3$, assume that all three zeros of $\rho_3(z)$ are simple. Then,*

a) for each zero z_k of $\rho_3(z)$, there are three Stokes curves emanating from it and they are of the types $(1, 2), (1, 3)$ and $(2, 3)$. Moreover, the curve $(2, 3)$ is a closed Jordan curve crossing supp$[\mu^{\mathcal{M}}]$ and the curves $(1, 2), (1, 3)$ are the Jordan arcs looping around supp$[\mu^{\mathcal{M}}]$ and connecting z_k with ∞, see Fig. 7 b);

b) for $j = 2, 3$, the Stokes curves $(1, j)$ are inert;

c) the new Stokes curves emanating from the intersections of the (initial) Stokes curves defining ordered crossings are inert.

As a consequence of the geometry of the Stokes complex we obtain a description of supp$[\mu^{\mathcal{M}}]$ for $M = 3$.

Corollary 2 *For $M = 3$, assume that the zeros of ρ_3 are simple. Then,*

$$\text{supp}[\mu^{\mathcal{M}}] = \bigcup_{k=1}^{3} \left(\left\{ \Re \left[\int_{z_k}^{z} (w_1(\xi) - w_2(\xi)) d\xi \right] = 0 \right\} \bigcap \left\{ \Im \left[\int_{z_k}^{z} (w_1(\xi) - w_2(\xi)) d\xi \right] \geq 0 \right\} \right).$$

3 Technical results

To study the geometry of the Stokes complex we need to understand, for each $1 \leq j < j' \leq M$, the structure of the complete analytic function defined by $\int_{z_k}^{z} (w_j(t) - w_{j'}(\xi)) d\xi$, where z_k is an ordinary turning point of (4). Notice that

$$\int_{z_k}^{z} (w_j(\xi) - w_{j'}(\xi)) d\xi = (e^{\frac{2(j-1)\pi\imath}{M}} - e^{\frac{2(j'-1)\pi\imath}{M}}) \int_{z_k}^{z} w_1(\xi) d\xi, \quad z \in \Omega, 1 \leq j < j' \leq M. \tag{21}$$

To construct the Riemann surface we proceed as follows. Take a branch cut in Ω consisting of supp$[\mu^{\mathcal{M}}]$ and a Jordan arc τ connecting ∞ and a point $z_0 \in$ supp$[\mu^{\mathcal{M}}]$. For a small disk $D \subset \Omega$ such that $\tau \cap D = \emptyset$, define the function element

$$b_1(z; z_k) = \begin{cases} \int_{z_k}^{z} w_1(\xi) d\xi, z \in D, \quad \rho_M(z) \neq (z - z_1)^M; \\ \ln(z - z_1), z \in D, \quad \rho_M(z) = (z - z_1)^M. \end{cases} \tag{22}$$

For $z \in \Omega_\tau$, define $b_1(z; z_k)$ as the analytic continuation to Ω_τ of the function element (b_1, D).

Further, denote by $(\mathcal{R}, \rho)$ the Riemann surface of the complete analytic function $\mathcal{F}$ obtained from the function element (b_1, D), where $\rho(\mathcal{R}) = \Omega$ is the projection map and Ω is the base space, cf. [49, Defs. 2.7 p.215 & 5.14 p.232].

Take $\omega \in \operatorname{supp}[\mu^{\mathcal{M}}]$ and a Jordan arc τ connecting ∞ and ω such that $\tau \cap D = \emptyset$, where τ is oriented so that ω is the endpoint. Set

$$\begin{aligned} \mathfrak{B}_0 :=& b_1(\Omega_\tau; z_k) \cup b_1^+(\tau; z_k), \text{ where } b_1^+(z_0; z_k) = \lim_{\substack{z\to z_0 \\ z\in\tau^+}} b_1(z; z_k), z_0 \in \tau, \\ \mathfrak{B}_k :=& \mathfrak{B}_0 + 2k\pi\imath \text{ and } \mathfrak{B} := \bigcup_{k\in\mathbb{Z}} \mathfrak{B}_k, \end{aligned} \tag{23}$$

and the sum is understood in the sense $S + z = \{s + z : s \in S\}$.

Notation 5 *For $\omega \in \operatorname{supp}[\mu^{\mathcal{M}}]$, let τ be a Jordan arc connecting ∞ with ω and oriented towards the endpoint ω. Let $\mathfrak{G} = (b_1, \Omega_\tau)$ be a function element, and $\mathfrak{B}_0$ be as in (23). Denote by $\mathfrak{b}_0$ the component of $\partial\mathfrak{B}_0$ contained in $\partial\mathfrak{B}$.*

Lemma 1 *Let $(\mathcal{R}, \rho)$ be the Riemann surface of the complete analytic function $\mathcal{F}$ with the base Ω, where $\rho(\mathcal{R}) = \Omega$ is the projection map. Then the following properties are valid:*

a) The map

$$\begin{aligned} \mathcal{F} : \mathcal{R} &\longrightarrow \mathbb{C} \\ (z, [\phi]_z) &\longrightarrow \phi(z), \end{aligned}$$

defines a homeomorphism between $\mathfrak{R}$ and $\mathfrak{B}$.

b) The definition of the set $\mathfrak{B}$ given by (23) is independent both of the choice of a point $\omega \in \operatorname{supp}[\mu^{\mathcal{M}}]$ and of the choice of a Jordan arc connecting ω and ∞.

c) If $\rho_M(z) \neq (z - z_1)^M$ and $\mathfrak{b}_0$ is as in Definition 5, then the simply connected $\mathfrak{B}$-region is bounded by the polygonal curve $\partial\mathfrak{B} = \mathfrak{b}_0 + 2k\pi\imath, k \in \mathbb{Z}$ so that $\mathfrak{B}$ is to the right of $\partial\mathfrak{B}$ (where $\partial\mathcal{B}$ is oriented from $-\imath\infty$ to $+\imath\infty$). If $\rho_M(z) = (z - z_1)^M$ then $\mathfrak{B}$ coincides with $\mathbb{C}$.

Proof. a) Let $[b_1]_z$ be a germ of b_1 at $z \in D$. Obviously, any other germ $[\phi]_z$ can be obtained by analytic continuation of $[b_1]_z$. Hence, if $D' \subset \Omega$ is any simply connected subset and (Φ, D') is an element in $[\phi]_z$, then

$$\Phi(z) = \int_\Gamma \frac{d\xi}{\sqrt[M]{\rho_M(\xi)}} + b_1(z; z_k) = 2n\pi\imath + b_1(z; z_k), \tag{24}$$

where Γ is a closed curve encircling both $\operatorname{supp}[\mu^{\mathcal{M}}]$ and z, while the number $n = n(\infty, \Gamma)$ equals the index (i.e. the winding number) of Γ with respect to ∞.

Consider a cut in Ω defined by any Jordan arc ν connecting ∞ and $z_0 \in \operatorname{supp}[\mu^{\mathcal{M}}]$. Using (24), we have that for any two function elements (Φ, Ω_ν) and (Ψ, Ω_ν) contained in the respective germs $[\phi]_z$ and $[\psi]_z$ of the complete analytic function $\mathcal{F}$ obtained from (b_1, D),

$$\Phi(z) = \Psi(z) + 2k\pi\imath, \quad z \in \Omega_\nu, \tag{25}$$

for some $k \in \mathbb{Z}$.

Denote by $B = Im[\mathcal{F}] = \bigcup_{(z,[\phi]_z)\in\Omega\times\mathcal{F}} \mathcal{F}(z, [\phi]_z)$ the image of $\mathcal{R}$ under the map $\mathcal{F}$. Observe that $\mathcal{F}$ is injective. Indeed, let $(z_1, [\phi]_{z_1}), (z_2, [\psi]_{z_2}) \in \mathcal{R}$ be such that $(z_1, [\phi]_{z_1}) \neq (z_2, [\psi]_{z_2})$. We can have three alternatives:

- $z_1 \neq z_2, [\phi]_{z_1} \neq [\psi]_{z_2}$;
- $z_1 \neq z_2$ and $[\phi]_{z_1} = [\psi]_{z_2}$;
- $z_1 = z_2$ and $[\phi]_{z_1} \neq [\psi]_{z_1}$.

Notice that the elements in the second alternative do not belong to the domain of the map $\mathcal{F}$, cf.[49, Def. 2.1 p.214]. For a given pair $z_1, z_2 \in \Omega$, consider a cut in Ω by taking a Jordan arc ν connecting ∞ and a point of $\operatorname{supp}[\mu^{\mathcal{M}}]$ and such that $z_1, z_2 \in \Omega_\nu$.

Suppose that $z_1 \neq z_2, [\phi]_{z_1} \neq [\psi]_{z_2}$, and $\mathcal{F}(z_1, [\phi]_{z_1}) = \mathcal{F}(z_2, [\psi]_{z_2})$. Let us choose $(\Phi, \Omega_\nu) \in [\phi]_{z_1}$ and $(\Psi, \Omega_\nu) \in [\psi]_{z_2}$. Using (25) we obtain

$$\Phi(z_1) = \Phi(z_2) + 2m\pi\imath, \quad m \in \mathbb{Z} \setminus \{0\},$$

hence

$$\int_{[z_1,z_2]_\gamma} \frac{d\xi}{m\sqrt[M]{\rho_M(\xi)}} = 2\pi\imath, \tag{26}$$

where $[z_1, z_2]_\gamma$ is the path connecting the points z_1 and z_2 by the arc γ such that $[z_1, z_2]_\gamma \subset \Omega_\nu$. Therefore, if $[z_2, z_1]_{\gamma'} \subset \Omega$ is a path such that $[z_1, z_2]_\gamma \cup [z_2, z_1]_{\gamma'}$ encloses supp$[\mu^{\mathcal{M}}]$ and satisfies $n(\infty, [z_1, z_2]_\gamma \cup [z_2, z_1]_{\gamma'}) = 1$, then from (26) we get

$$\int_{[z_2,z_1]_{\gamma'}} \frac{d\xi}{\sqrt[M]{\rho_M(\xi)}} = 0.$$

But this is a contradiction, since $\displaystyle\int_{[z_2,z]_{\gamma'}} \frac{d\xi}{\sqrt[M]{\rho_M(\xi)}}$ is a conformal mapping in every open connected subset of Ω.

Assume now that $z_1 = z_2$ and $[\phi]_{z_1} \neq [\psi]_{z_1}$ and take $(\Phi, \Omega_\nu) \in [\phi]_{z_1}$ and $(\Psi, \Omega_\nu) \in [\psi]_{z_1}$. By (25) one has that $\Phi(z_1) \neq \Psi(z_1)$, i.e. $\mathcal{F}(z_1, \phi(z_1)) \neq \mathcal{F}(z_2, \psi(z_2))$. Therefore, we conclude that $\mathcal{F}$ is an injective map.

Now, using the open mapping theorem between complex manifolds, we have that $\mathcal{F}$ is an open map, see [49, Th.6.14 p.238]. Hence, $\mathcal{F}$ is an injective open map and therefore $\mathcal{F}$ is a homeomorphism between $\Omega \times \mathcal{R}$ and B.

We prove now that $B = \mathfrak{B}$. Let τ be a Jordan arc as in (23) and suppose that $a \in B$. Then there exists $(z, [\phi]_z) \in \Omega \times \mathcal{F}$ such that $a = \phi(z)$. On the one hand, assume that $z \in \Omega_\tau$. Picking $(\Phi, \Omega_\tau) \in [\phi]_z$ and using (24) one has

$$\Phi(z) = 2k\pi\imath + b_1(z; z_k),$$

for some $k \in \mathbb{Z}$. Therefore by (23), we get $a \in \mathcal{B}_k$. On the other hand, if $z \in \tau \setminus \{\omega\}$, let $(z_n) \subset \tau^+$ be a sequence converging to z and pick $(\Phi, \Omega_\tau) \in [\phi]_z$. By (24) one has

$$\Phi(z_n) = 2k\pi\imath + b_1(z_n; z_k), \tag{27}$$

for some $k \in \mathbb{Z}$. Since $\lim\limits_{\substack{z_n \to z \\ z_n \in \tau^+}} b_1(z_n; z_k) = b_1(z; z_k)$, the relation (27) gives that $a = 2k\pi\imath + b_1(z; z_k)$. Hence from (23) we obtain $a \in \mathcal{B}_k$ and therefore $B \subset \mathfrak{B}$. And conversely, suppose that $a \in \mathcal{B}$. Then there exists $z \in \Omega$ such that $a = 2k\pi\imath + b_1(z; z_k)$, for some $k \in \mathbb{Z}$. Hence, using (24) one obtains that there exists $(z, [\phi]_z)$ such that $a = \phi(z)$. This completes the proof of a).

The item b) follows immediately from the item a) since $\mathfrak{B}$ is the image of the Riemann surface $\mathfrak{R}$ under the map $\mathcal{F}$.

To settle item c) let us first assume that $\rho_M(z) \neq (z - z_1)^M$, pick $\omega \in$ supp$[\mu^{\mathcal{M}}]$, and let τ be any Jordan arc connecting ∞ and ω. By [3, Lem. 4], any primitive of the function $\dfrac{1}{\sqrt[M]{\rho_M(z)}}$ locally defined in a simple connected domain maps the smooth curve segments of supp$[\mu^{\mathcal{M}}]$ to lines. Now, the analytic continuation of the element (ϕ_1, D) to Ω_τ is a conformal map in Ω_τ. Therefore, the set $\mathfrak{b}_0$ is a polygonal curve and $\phi_1(\Omega_\tau)$ is simply connected. Hence, by item b) and (23), $\mathfrak{B}$ is simply connected as well. On the other hand, one also has that

$$\lim_{\substack{z \to \infty \\ z \in \Omega_\tau}} \Re[\phi_1(z)] = +\infty. \tag{28}$$

Therefore, by traversing $\partial\mathfrak{B}$ from $-\imath\infty$ to $+\imath\infty$ one obtains that $\mathfrak{B}$ is to the right of $\partial\mathfrak{B}$.

Consider now the case $\rho_M(z) = (z - z_1)^M$. Pick an horizontal ray ℓ connecting z_1 and $+\infty + \Im[z_1]\imath$. The analytic continuation of the function given by (22) to Ω_ℓ defines the analytic function $\ln(z - z_1)$, which is a conformal map between Ω_ℓ and $-\pi < \arg z < \pi$. Hence, the assertion follows immediately from item b) and the definition of $\mathfrak{B}_k$ and $\mathfrak{B}$ in (23). □

The properties of the complete analytic function defined by $\int_{z_k}^{z} (w_j(t) - w_{j'}(t))dt$ are as follows.

Lemma 2 *Let z_k be an ordinary turning point of (4), τ be a Jordan arc connecting ∞ and a point $z_0 \in$ supp$[\mu^{\mathcal{M}}]$, and choose a small disk $D \subset \Omega$ such that $\tau \cap D = \emptyset$. Then,*

a) The complete analytic function $\mathcal{F}_{(j,j')}$ obtained using the analytic continuation of a function element $(\int_{z_k}^{z} (w_j(t) - w_{j'}(t))dt, D)$ satisfies the equation

$$\mathcal{F}_{(j,j')} = (e^{\frac{2(j-1)\pi\imath}{M}} - e^{\frac{2(j'-1)\pi\imath}{M}})\mathcal{F}, \quad 1 \leq j < j' \leq M.$$

b) The map

$$\begin{aligned} \mathcal{F}_{(j,j')} : \mathcal{R}_{(j,j')} &\longrightarrow \mathbb{C} \\ (z, [\phi]_z) &\longrightarrow \phi(z), \end{aligned}$$

defines a homeomorphism between $\mathfrak{R}_{(j,j')} = (e^{\frac{2(j-1)\pi\imath}{M}} - e^{\frac{2(j'-1)\pi\imath}{M}})\mathfrak{R}$ *and* $\mathfrak{B}_{(j,j')} = (e^{\frac{2(j-1)\pi\imath}{M}} - e^{\frac{2(j'-1)\pi\imath}{M}})\mathfrak{B}$.

Proof. Item a) is immediate from the relation (21) and item b) follows from a) of Lemma 1. □

The next two lemmas calculate of the angles between the arcs in the support $\text{supp}[\mu^{\mathcal{M}}]$ for $M = 3$. (In bigger generality this was done in [3, p.155]).

Lemma 3 *Let* $\rho_M(z) \neq (z - z_1)^M$*,* e_i *be an edge of* $\text{supp}[\mu^{\mathcal{M}}]$ *with a given orientation, and* $C_+(z), C_-(z)$ *be the limiting values of* $C(w)$ *as* w *approaches* $z \in e_i$ *from the left and from the right sides respectively. Then,*

a) $\mu^{\mathcal{M}}$ *is absolutely continuous with respect to the Lebesgue measure;*

b) the unit tangent vector τ *at* $z \in e_i$ *can be expressed as* $\tau(z) = \dfrac{\overline{C_-(z)} - \overline{C_+(z)}}{|\overline{C_-(z)} - \overline{C_+(z)}|} e^{\imath \frac{\pi}{2}}$*;*

c) if v *is a vertex of* $\text{supp}[\mu^{\mathcal{M}}]$ *such that* $\rho_M(v) \neq 0$ *then the degree of* v *is strictly greater than* 2.

Proof. a) By [3, Lem. 4], $\mu^{\mathcal{M}}$ is the union of finitely many smooth curve segments $e_i, 1 \leq i \leq N$. We will prove that the measure $\mu^{\mathcal{M}}$ is absolutely continuous with respect to the Lebesgue measure on every proper open subarc $(\alpha, \beta)_\gamma \subset e_i$. Consider the orientation of $(\alpha, \beta)_\gamma$ obtained by traversing the arc from α to β. Let τ be a Jordan arc connecting a zero of ρ_M to ∞. Let $b_1(z; z_k), z \in \Omega_\tau$ be defined as in (22). From [3, Th. 2] we have that the Cauchy transform

$$C(z) = \int \frac{d\mu^{\mathcal{M}}(w)}{z - w} \tag{29}$$

of the measure $\mu^{\mathcal{M}}$ satisfies for almost all $z \in \mathbb{C}$, the equation

$$C^M(z) = \frac{1}{\rho_M(z)}.$$

Let

$$U^{\mu^{\mathcal{M}}}(z) = \int \frac{1}{\log|z - w|} d\mu^{\mathcal{M}}(w)$$

be the logarithmic potential of $\mu^{\mathcal{M}}$. Using (22) and (29) notice that

$$U^{\mu^{\mathcal{M}}}(z) = -\Re\left[b_1(z; z_k)\right] + c, \quad z \in \Omega_\tau, \tag{30}$$

where $c \in \mathbb{R}$ is a constant. Define $H(z) = -b_1(z; z_k), z \in \Omega_\tau$ and define H_+ and H_- as the restrictions of H to a neighborhoods of $(\alpha, \beta)_\gamma$ to the left and to the right of $(\alpha, \beta)_\gamma$ respectively. Further denote by $H_+(\omega + 0)$ and by $H_-(\omega + 0)$ the non-tangential limits of H_+ and H_- respectively when $z \to \omega \in (\alpha, \beta)_\gamma$, cf. [50, p. 89-90 Ch. II]. It follows from (30) and [50, Th.1.4 Ch. II] that for every $z_0, z_1 \in (\alpha, \beta)_\gamma$ such that z_0 precedes z_1,

$$\mu^{\mathcal{M}}((z_0, z_1)_\gamma) = \frac{1}{2\pi\imath}\left(H_+(z_1 + 0) - H_+(z_0 + 0) - H_-(z_1 - 0) + H_-(z_0 - 0)\right). \tag{31}$$

Notice that for any edge e_i of $\text{supp}[\mu^{\mathcal{M}}]$, the functions $H_+(z)$ and $H_-(z)$ are of the class C^1 for $z \in \mathring{e}_i$. Therefore, if $r(t), t \in [a, b]$ is a parameterization of the smooth arc defined by e_i, one obtains that $H_+(r(t))$ and $H_-(r(t))$ are absolutely continuous in $[a, b]$. [51, Prop. 3.32] implies that $\mu^{\mathcal{M}}$ is an absolutely continuous measure with respect to the Lebesgue measure dz on the arc $(\alpha, \beta)_\gamma$ which proves a).

b) By the preceding item a), there exists a unique $f \in L^1(\alpha, \beta)$ such that $d\mu^{\mathcal{M}} = f dz$. Hence by the Sokhotski-Plemelj formula [52, (17.2) p.42],

$$f = \frac{1}{2\pi\imath}(C_- - C_+). \tag{32}$$

Since $\mu^{\mathcal{M}}$ is a positive measure, then using (32) one immediately obtains that

$$\frac{1}{2\pi\imath}\int_{(a,b)}(C_-(z)-C_+(z))dz \geq 0,$$

where $(a,b)_\gamma$ is any subarc of $\operatorname{supp}[\mu^{\mathcal{M}}]$. Hence, the tangent vector τ to any smooth subarc $(a,b)_\gamma$ of $\operatorname{supp}[\mu^{\mathcal{M}}]$ at z satisfies the relation

$$\arg[\tau(z)] = \arg[\overline{C_-(z)} - \overline{C_+(z)}] + \frac{\pi}{2} \mod 2\pi, \quad z \in (a,b),$$

which completes the proof of b).

c) Observe that the degree of v is necessarily greater than 1, since otherwise v is a branch point and by our assumption, v is not a zero of ρ_M. Suppose now that the degree of v equals 2 and set $e_1 = (v_1, v), e_2 = (v, v_2)$. By [3, Lem. 4], $\operatorname{supp}[\mu^{\mathcal{M}}]$ is the union of finitely many smooth curve segments and has connected complement. Hence we may assume that the Jordan arc $e_1 \cup e_2$ is not differentiable at v.

Consider a small Jordan curve $\mathcal{C}$ enclosing v and consider the arc $int(\mathcal{C}) \cap (e_1 \cup e_2)$ oriented by traversing it from $int(\mathcal{C}) \cap e_1$ to $int(\mathcal{C}) \cap e_2$. Denote by V_+ the left and by V_- the right sides. Let $C_+(z)$ and $C_-(z)$ be the limiting values of $C(w)$ when w approaches z from the left and from the right sides of the arc respectively. Notice that $C_+(z)$ and $C_-(z)$ are continuous functions in a neighborhood of $z = v$.

By item b),

$$\lim_{\substack{z\to v\\ z\in e_i}} \arg[C_+(z) - C_-(z)] = \arg[\tau_i(v)] + \frac{\pi}{2} \mod 2\pi, \tag{33}$$

where $\tau_i(z)$ is the unit tangent vector at z. On the other hand, the functions $C_+(z)$ and $C_-(z)$ are continuous in a neighborhood of $z = v$. Therefore, from (33) one obtains

$$\arg[\tau_1(v)] - \arg[\tau_2(v)] = 0 \mod 2\pi. \tag{34}$$

By our assumption, the Jordan arc $e_1 \cup e_2$ is not differentiable at v. Thus the relation (34) is impossible, which is a contradiction. This completes the proof that the degree of v is strictly greater than 2. □

Lemma 4 *For $M = 3$, suppose that $\rho_3(z)$ has three distinct roots. Then, $\mu^{\mathcal{M}}$ consists of one vertex and three edges connecting the zeros of $\rho_3(z)$ contained in the convex hull of the zeros of $\rho_3(z)$, see Figure 1. Moreover, all angles between the arcs $\mathfrak{r}_i, i = 1,2,3$ of $\operatorname{supp}[\mu^{\mathcal{M}}]$ at the common intersection point v are equal to $\frac{2\pi}{3}$.*

PROOF. a) Let $\{z_1, z_2, z_3\}$ be the zeros of ρ_3. Using the connectivity of $\operatorname{supp}[\mu^{\mathcal{M}}]$, we can find a Jordan arc connecting z_1 and z_2. Since z_1 and z_2 are branch points, $\operatorname{supp}[\mu^{\mathcal{M}}]$ contains at least one vertex $v \notin \{z_1, z_2\}$ and this vertex can be connected to z_3 through a Jordan arc $\mathfrak{r}_3$. Now, $\mathfrak{r}_3$ does not contain vertices other than v'. Indeed, if $v' \in \mathfrak{r}_3$ is such a vertex, then by item c) of Lemma 3 the degree of v' is strictly greater than 2, hence there exists $z_\omega \in \operatorname{supp}[\mu^{\mathcal{M}}]$ such that $z_\omega \notin \{z_1, z_2, z_3\}$ and z_ω is an extreme point. Consequently, z_ω is also a branch point, which a contradiction. Therefore, $\mathfrak{r}_3$ is an edge of $\operatorname{supp}[\mu^{\mathcal{M}}]$. Using a similar argument one obtains that the Jordan arcs $\mathfrak{r}_1$ and $\mathfrak{r}_2$ connecting z_1, v and z_2, v respectively are edges of $\operatorname{supp}[\mu^{\mathcal{M}}]$. This proves the assertion that $\operatorname{supp}[\mu^{\mathcal{M}}]$ consists of three smooths arcs $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3$ and a vertex v of degree 3. The assertion that $\operatorname{supp}[\mu^{\mathcal{M}}]$ is contained in the interior of the convex hull of the zeros of ρ_3 follows from [3, Th. 3].

Let us show that all angles at the common intersection point v are equal to $\frac{2\pi}{3}$. Consider the orientation in each $\mathfrak{r}_i$ obtained by traversing each arc so that v is the end point. Denote by $C_{i,+}(z)$ and $C_{i,-}(z)$ the limiting values of the Cauchy transform $C(z)$ at $z \in \mathfrak{r}_i$ from the left and from the right respectively. Now, consider a small disk D centered at v and notice that C is a continuous function in the connected component $V_i, i = 1,2,3$ of $D \setminus (D \cap (\mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_3))$, see Figure 1. Set

$$\lim_{\substack{z\to v\\ z\in\mathfrak{r}_i}} C_{i,+}(z) = C_{i,+}, \quad \lim_{\substack{z\to v\\ z\in\mathfrak{r}_i}} C_{i,-}(z) = C_{i,-}.$$

Without loss of generality, we will convey that the edges $\mathfrak{r}_i$ are ordered so that when we wind counterclockwise around v we obtain the sequence $(\ldots \mathfrak{r}_3, \mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_1, \ldots)$ as shown in Figure 1.

By the continuity of C, in each region V_i one has

$$C_{i,-} = C_{i-1,+}. \tag{35}$$

By (35) and b) of Lemma 3, if τ_i is the tangent vector of $\mathfrak{r}_i$ at v, then

$$\arg\left[\frac{\tau_i}{\tau_{i-1}}\right] = \arg\left[\frac{\lim\limits_{\substack{z\to v\\ z\in\mathfrak{r}_i}}(C_{i,+}(z)-C_{i,-}(z))}{\lim\limits_{\substack{z\to v\\ z\in\mathfrak{r}_j}}(C_{i-1,+}(z)-C_{i-1,-}(z))}\right] = \arg\left[\frac{C_{i,+}-C_{i,-}}{C_{i,-}-C_{i-1,-}}\right]. \tag{36}$$

Writing

$$\begin{aligned} C_{i,+}-C_{i,-} &= \rho e^{\frac{\theta_0+2k\pi}{3}\imath} - \rho e^{\frac{\theta_0+2(k+1)\pi}{3}\imath}, \\ C_{i,-}-C_{i-1,-} &= \rho e^{\frac{\theta_0+2(k+1)\pi}{3}\imath} - \rho e^{\frac{\theta_0+2(k+2)\pi}{3}\imath}, \end{aligned} \tag{37}$$

and using (36) and (37) one obtains

$$\left|\arg\left[\frac{\tau_i}{\tau_{i-1}}\right]\right| = \frac{2}{3}\pi.$$

Hence, the angle $\alpha_{i,i-1}$ between the arcs $\mathfrak{r}_i, \mathfrak{r}_{i-1}$ at the vertex v equals

$$\alpha_{i,i-1} = \frac{2\pi}{3}. \tag{38}$$

□

Lemma 5 *For $M \geq 2$, let $U \subset \Omega$ be an open subset, $V \subset \mathbb{C}$, and $\phi(z,\eta) : U \times V \to \mathbb{C}$ be a formal power series of the form $\phi(z,\eta) = \sum_{k=1}^{\infty} \phi_k(z)\eta^{-k}, \phi_k \in \mathcal{H}(U)$. Then $v = e^{\eta\int^z(\phi(t,\eta)+w_j(t))dt}$ is a solution of*

$$v^{(M)} + \sum_{k=1}^{M-1} \frac{\rho_k}{\rho_M} v^{(k)} - \eta^M \frac{v}{\rho_M} = 0, \tag{39}$$

if and only if, ϕ is a solution of

$$F_j(w,\dots,w^{(M-1)},z,\eta) = 0,$$

where

$$F_j(x_0,\dots,x_{M-1},z,\eta) = -\frac{1}{\rho_M(z)} + \sum_{l=1}^{M}\sum_{u=1}^{l}\sum_{c_0,\dots,c_{l-u}\in\pi(l,u)} (x_0+w_j(z))^{c_0}\cdots(x_{l-u}+w_j^{(l-u)}(z))^{c_{l-u}} f_{c_0,\dots,c_{l-u}}(z,\eta).$$

Here $\pi(l,u)$ stands for the set of partitions of l into u summands,

$$\begin{aligned} c_0+\dots+(l-u+1)c_{l-u} &= l \\ c_0+\dots+c_{l-u} &= u, \\ c_0,\dots,c_{l-u} &\geq 0, \end{aligned}$$

and w_j is as in (2). Finally, $f_{c_0,\dots,c_{l-u}}(z,\eta) = \dfrac{l!}{c_0!\cdots c_{l-u}!(1!)^{c_0}\cdots((l-u+1)!)^{c_{l-u}}}\eta^{u-M}\dfrac{\rho_l(z)}{\rho_M(z)}$.

Proof. For any integer $1 \leq k \leq M$, denote by $P_k(y',\dots,y^{(k)})$ the polynomial in the variables $(y',\dots,y^{(k)})$ defined by the relation

$$P_k(y',\dots,y^{(k)}) = e^{-y}\,(e^y)^{(k)}.$$

According to the Faà di Bruno formula [53, Th.A p.137], $P_k(y',\dots,y^{(k)})$ can be expressed as

$$P_k(y',\dots,y^{(k)}) = \sum_{l=1}^{k} \mathbf{B}_{k,l}(y',\dots,y^{(k-l+1)}). \tag{40}$$

The relation (40) and the variable change $v = e^y$ provide that (39) can be expressed as

$$\sum_{k=1}^{M} \frac{\rho_k}{\rho_M} P_k(y', \dots, y^{(k)}) - \frac{\eta^M}{\rho_M} = 0. \tag{41}$$

By multiplying the relation (41) by $\frac{1}{\eta^M}$ and using the expression for P_k, one has that $v = e^y, y' = (w + w_j)\eta$ is a solution to (39) if and only if w is a solution of the Riccati equation

$$\sum_{k=1}^{M} \eta^{-M} \frac{\rho_k}{\rho_M} P_k(\eta(w + w_j), \dots, \eta(w + w_j)^{(k-1)}) - \frac{1}{\rho_M} = 0, \tag{42}$$

where

$$P_k(\eta(w + w_j), \dots, \eta(w + w_j)^{(k-1)}) = \sum_{l=1}^{k} \eta^l \mathbf{B}_{k,l}(w + w_j, \dots, (w + w_j)^{(k-l)}).$$

Changing the variables $x_0 = w, \dots, x_{M-1} = w^{(M-1)}$ and rearranging appropriately, we have that the relation (42) can be equivalently expressed as

$$\sum_{k=0}^{M-1} \eta^{-k} \sum_{l=M-k}^{M} \frac{\rho_l}{\rho_M} \mathbf{B}_{l,M-k}(x_0 + w_j, \dots, x_{l-M+k} + w_j^{(l-M+k)}) - \frac{1}{\rho_M} = 0. \tag{43}$$

Further we get

$$-\frac{1}{\rho_M} + \sum_{k=0}^{M-1} \eta^{-k} \sum_{l=M-k}^{M} \frac{\rho_l}{\rho_M} \mathbf{B}_{l,M-k}(x_0 + w_j, \dots, x_{l-M+k} + w_j^{(l-M+k)}) = -\frac{1}{\rho_M} +$$

$$\sum_{k=0}^{M-1} \sum_{l=M-k}^{M} \sum_{c_0,\dots c_{l-M+k} \in \pi(l,M-k)} \eta^{-k} \frac{\rho_l}{\rho_M} \frac{l!}{c_0! \cdots c_{l-M+k}! 1! \cdots (l-M+k+1)!} (x_0 + w_j)^{c_0} \cdots (x_{l-M+k} + w_j^{(l-M+k)})^{c_{l-M+k}}$$

$$= -\frac{1}{\rho_M} + \sum_{l=1}^{M} \sum_{u=1}^{l} \sum_{c_0,\dots c_{l-u} \in \pi(l,u)} (x_0 + w_j)^{c_0} \cdots (x_{l-u} + w_j^{(l-u)})^{c_{l-u}} \frac{l!}{c_0! \cdots c_{l-u}! 1! \cdots (l-u+1)!} \eta^{u-M} \frac{\rho_l}{\rho_M},$$

where $\pi(l, u)$ stands for the set of partitions of l into u summands,

$$\begin{aligned} 0 &\le c_0 + \cdots + (l - u + 1)c_{l-u} = l \\ 0 &\le c_0 + \cdots + c_{l-u} \le u, \end{aligned}$$

which finishes the proof. □

Let $\zeta, f_n, n \ge 0$ be holomorphic functions defined in a domain $U \subset \mathbb{C}$. The formal power series

$$\exp(\eta\zeta(z)) \sum_{n=0}^{\infty} f_n(z) \eta^{-(n+\frac{1}{2})},$$

is said to be *pre-Borel-summmable in U* if for each compact set $K \subset U$, there exists A_k and C_K such that

$$\sup_K |f_n(z)| < A_K C_K^n \Gamma(1 + n).$$

Lemma 6 *Let $U \subset \Omega$ be a simply connected open set. Then, any WKB-formal solution is pre-Borel-summable in U.*

Proof. For convenience, we will use the parameter $\eta = \frac{1}{\epsilon}$. By [41, Chap.II, Proposition 2.1.2], it suffices to prove the pre-Borel-summability of $\phi(z, \epsilon) = \sum_{k=1}^{\infty} S_k(z)\epsilon^k$ in every compact subset $K \subset U$, where S_k are determined so that ϕ is a formal solution of the Riccati equation

$$F_j\left(x, \frac{dx}{dz}, \frac{d^2x}{dz^2}, \dots, \frac{d^{M-1}x}{dz^{M-1}}, z, \epsilon\right) = 0,$$

with F_j defined as in Lemma 5. Then it can be expressed as

$$F_j(x_0, x_1, \ldots, x_{M-1}, z, \epsilon) = \sum_{m_0+\ldots+m_{M-1}=0}^{N} x_0^{m_0} \cdots x_{M-1}^{m_{M-1}} F_{j,m_0,m_1,\ldots,m_{M-1}}(z, \epsilon),$$

where $F_{j,m_0,m_1,\ldots,m_{M-1}}(z,\epsilon) = \displaystyle\sum_{m=0}^{N_{m_0,m_1,\ldots,m_{M-1}}} F_{j,m_0,m_1,\ldots,m_{M-1};m}(z)\epsilon^m$ is a polynomial in the variable ϵ with coefficients in $\mathcal{H}(U)$ and $N \geq 0$. In particular, $F_{j;m_0,m_1,\ldots,m_{M-1};m}$ is of the Gevrey order 1 in ϵ uniformly in $z \in U$. In other words, there exist nonnegative constants K_1 and K_2 such that

$$|F_{j,m_0,m_1,\ldots,m_{M-1}}(z)| \leq K_1 m!(K_2)^m, \tag{44}$$

for $z \in U$ and $0 \leq m_0 + m_1 + \ldots + m_{M-1} \leq N, 0 \leq m \leq N_{m_0,m_1,\ldots,m_{M-1}}$.

The proof follows from Sibuya's theorem on the Gevrey summability of formal power series; cf. Theorem 7 in Appendix, § 8. (An interested reader should take a look at this material before reading the proof). Using Lemma 5, a straightforward calculation shows that

$$\frac{\partial F_j}{\partial x_{M-1}}\left(\phi, \frac{d\phi}{dz}, \frac{d^2\phi}{dz^2}, \ldots, \frac{d^{M-1}\phi}{dz^{M-1}}, z, \epsilon\right) = \epsilon^{M-1}. \tag{45}$$

Hence, condition (81) of Theorem 7 is satisfied. On the other hand, using (43) we have

$$\begin{aligned}
\frac{\partial F_j}{\partial x_0}\left(\phi, \frac{d\phi}{dz}, \frac{d^2\phi}{dz^2}, \ldots, \frac{d^{M-1}\phi}{dz^{M-1}}, z, \epsilon\right) &= M(\phi + w_j)^{M-1} + \\
&\sum_{k=1}^{M-1} \epsilon^k \sum_{l=M-k}^{M} \frac{\rho_l}{\rho_M} \frac{\mathbf{B}_{l,M-k}(\phi + w_j, \ldots, \phi^{(l-M+k)} + w_j^{(l-M+k)})}{\partial x_0}, \qquad (46)\\
\frac{\partial F_j}{\partial x_1}\left(\phi, \frac{d\phi}{dz}, \frac{d^2\phi}{dz^2}, \ldots, \frac{d^{M-1}\phi}{dz^{M-1}}, z, \epsilon\right) &= \epsilon\frac{M(M-1)}{2}(\phi + w_j)^{M-2} + \\
&\sum_{k=2}^{M-1} \epsilon^k \sum_{l=M-k}^{M} \frac{\rho_l}{\rho_M} \frac{\mathbf{B}_{l,M-k}(\phi + w_j, \ldots, \phi^{(l-M+k)} + w_j^{(l-M+k)})}{\partial x_1}. \qquad (47)
\end{aligned}$$

From (46), (47), (82), and (83) of § 8, we have that $h_1 = 0$ and $h_2 = 1$. Hence we are in the situation of Case A of [54, Th. 1.2.1]. Therefore from (46) we have that the relation (85) of § 8 reduces to

$$\mathcal{T}[y]|_{\epsilon=0} = M w_j^{M-1} y.$$

A straightforward calculation also shows that $m_{h_1} = 0$ and $m_{h_2} = 1$. Hence from (84) of § 8 we obtain

$$\rho_2 = 1. \tag{48}$$

By Theorem 7 we get that the formal series ϕ has the Gevrey order $\max\left(\frac{1}{\rho_2}, s\right)$, and by (44) and (48) this order is equal to 1. Therefore, ϕ is pre-Borel-summable in every compact subset $K \subset U$. Consequently, any WKB-formal solution is also pre-Borel-summable in every compact subset $K \subset U$.

□

Lemma 7 *Given $z^* \notin \mathcal{Z}$, $z_k \in \mathcal{Z}$, let U_{z_k} be a neighborhood of z_k, and*

$$\psi_j(z, \eta, z^*) = \eta^{-\frac{1}{2}} \exp\left(\int_{z^*}^{z} S(\zeta, \eta) d\zeta\right), z \in U_{z_k} \cap \Omega$$

be a WKB-solution of (4)*. Then, there exists $\alpha_n \in \mathbb{Z}_{\geq 0}$ such that,*

$$S_n((z - z_k)^M)(z - z_k)^{\alpha_n} \in \mathcal{H}(U_{z_k}), \quad n \geq 0.$$

PROOF. Using $\eta = \dfrac{1}{\epsilon}$ and letting $u_1 = y, u_2 = \epsilon y', \ldots, u_M = \epsilon^M y^{(M)}$, we have that (5) can be expressed as

$$\epsilon u' = M(z,\epsilon)u, \tag{49}$$

where $u = (u_1, \ldots, u_M)^t$ and

$$\begin{aligned} M(z,\epsilon) &= \det \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ & & \vdots & & 1 \\ -\frac{1}{\rho_M(z)} & -\frac{\rho_1(z)}{\rho_M(z)}\epsilon^{M-1} & \cdots & & -\frac{\rho_{M-1}(z)}{\rho_M(z)}\epsilon \end{pmatrix} \\ &= \sum_{k=0}^{M-1} M_k(z)\epsilon^k. \end{aligned}$$

Notice that $M(z,0)$ is diagonalizable in Ω with the eigenvalues given by $(w_j(z))_{j=1}^M$, where w_j's are defined as in (2); note also that w_j is a solution of the equation $\rho_M(z)w^M - 1 = 0$. Let $\mathcal{M}(U_{z_k})(w_1, \ldots, w_M)$ be the smallest functional field containing $w_1, \ldots, w_M$ and the set of meromorphic functions in U_{z_k}, cf. [55, Th.3 p.512 & Def.1 p.517] & [56, Case I p.315]. Then by [55, Cor.25 p.494] there exists an invertible matrix $Q(z)$ with elements in the field $\mathcal{M}(U_{z_k})(w_1, \ldots, w_M)$ such that

$$M_0(z) = Q(z)\Lambda_0(z)Q^{-1}(z),$$

where $\Lambda_0(z) = diag(w_1(z), \ldots, w_M(z))$.

Hence, if U is a fundamental system for (49), we have that using the substitution $U = QY$ one obtains the matrix equation

$$\epsilon Y' = (\Lambda_0(z) + A(z,\epsilon))Y, \quad z \in U_{z_k} \cap \Omega, \tag{50}$$

where $A(z,\epsilon) = \displaystyle\sum_{k=1}^{M-1} A_k(z)\epsilon^k$ is a matrix polynomial of degree $M-1$ in the variable ϵ, with $A_k(z) = Q(z)M_k(z)Q^{-1}(z)$ and $A_k \in \mathcal{M}(w_1, \ldots, w_M)$.

Using the formal series $\Lambda(z,\epsilon) = \displaystyle\sum_{k=0}^{\infty} \Lambda_k(z)\epsilon^k, P(z,\epsilon) = \sum_{k=0}^{\infty} P_k(z)\epsilon^k, P_0(z) = I$, substituting $Y = PZ$ in (50), and collecting powers of ϵ, we obtain the recurrence relation

$$\Lambda_0 P_r - P_r\Lambda_0 = \sum_{s=0}^{r-1}(P_s\Lambda_{r-s} - A_{r-s}P_s) + P'_{r-1}, r > 0,$$

which can be expressed as

$$\Lambda_0 P_r - P_r\Lambda_0 = \Lambda_r - H_r.$$

Here H_r depends only on P_j, P'_j and Λ_j with $j < r$. Choosing $\Lambda_r = diag(\lambda_{j,j}(r)), \lambda_{j,j}(r) = h_{j,j}(r)$, where $H_r = (h_{j,j}(r))$, we define P_r as the solution of the non-homogeneous Sylvester equation

$$\Lambda_0 P_r - P_r\Lambda_0 = C \tag{51}$$

in the field $\mathcal{M}(U_{z_k})(w_1, \ldots, w_M)$, where C is an anti-diagonal matrix with entries in $\mathcal{M}(U_{z_k})(w_1, \ldots, w_M)$. Notice that the entries of the matrix in the leftt-hand side of (51) are

$$(\lambda_i(0) - \lambda_j(0))p_{i,j}(r),$$

where $P_r = (p_{i,j}(r))$. Using the fact that the eigenvalues of Λ_0 are distinct, we immediately obtain that for each r, there exists a unique solution $P_r \in \mathcal{M}(U_{z_k})(w_1, \ldots, w_M)$.

Therefore, the transformation $Y = PZ$ reduces (50) to

$$\epsilon Z' = \Lambda Z, \tag{52}$$

where $\Lambda(z,\epsilon) = \displaystyle\sum_{k=0}^{\infty} \Lambda_k(z)\epsilon^k, P(z,\epsilon) = \sum_{k=0}^{\infty} P_k(z)\epsilon^k, P_0(z) = I.$

Relations (50) and (52) imply that a formal fundamental system for (49) is given by $U = QPe^{\frac{1}{\epsilon}\int^z \Lambda(t,\epsilon)dt}$. Recalling that $u_1 = y$ and that the matrices Q and Λ have entries in the field $\mathcal{M}(U_{z_k})(w_1, \ldots, w_M)$, the first row of U gives the desired WKB-solution

$$\exp\left(\epsilon^{-1}\int_{z^*}^{z} S_0(\zeta)d\zeta\right)\sum_{n\geq 0}\epsilon^{n+\frac{1}{2}}\psi_n(z).$$

Finally, by Lemma 6 we have that $S(z,\epsilon) = \dfrac{1}{\epsilon}\displaystyle\sum_{k=0}^{\infty} S_k(z)\epsilon^k$ is pre-Borel-summable in every open simply connected subset $U \subset \Omega$, which means that for every $n \geq 0$, one has that $S_n((z-z_k)^M)(z-z_k)^{\alpha_n} \in \mathcal{H}(U_{z_k})$, for some $\alpha_n \in \mathbb{Z}_{\geq 0}$. $\square$

Lemma 8 *Let $z^* \notin \mathcal{Z}$ be a fixed reference point and $z \in \Omega$. Then, the singularities of the Borel transform $\psi_{j,B}(z,y,z^*)$ occur at the points given by* $\left(z, -\displaystyle\int_{z^*}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}}, z^*\right)$.

Proof. Let

$$\psi_{j,B}(z,y,z^*) = \sum_{n\geq 0}\frac{f_n(z,z^*)}{\Gamma(n+\frac{1}{2})}(y+y_0(z,z^*))^{n-\frac{1}{2}}$$

be the Borel transform of ψ_j, where $y_0(z,z^*) = \displaystyle\int_{z^*}^{z} w_j(\zeta)d\zeta$. By Lemma 6, the formal expression given by (12) is an analytic solution of (13) when $z \in \Omega$ and $|y + y_0(z,z^*)| < \delta$, for sufficientlly small $\delta > 0$. We will assume that $\psi_{j,B}(z,y,z^*)$ is holomorphic in a maximal domain $(z,y) \in \Omega \times \Omega_Y, \Omega_Y \subset \mathbb{C}$.

Suppose that the function $\psi_{j,B}(z,y,z^*)$ has a singularity at a point $(z,y) = (z',y'), z' \in \Omega$. We have that (z',y') is in the singular support of $\psi_{j,B}$, considered as a distribution which we denote by u_ψ. From [43, Cor.7.2.2 p.249] and [47, Th.6 p.44 (complex version)], one obtains that the bicharacteristic curve $(z(t), y(t), \zeta(t), \epsilon(t))$ defined by the equations (16)-(20) and emanating from $\left(z', y', \frac{1}{\sqrt[M]{\rho_M(z')}}, 1\right)$ belongs to $\mathcal{WF}(u_\psi)$, which is the *wave front* of u_ψ. A straightforward calculation shows that

$$(z(t), y(t), \zeta(t), \epsilon(t)) = \left(\Psi_l^{-1}(t,z'), -Mt + y', \frac{1}{\sqrt[M]{\rho_M(z(t))}}, 1\right), \tag{53}$$

where Ψ_l^{-1} denotes the inverse of $\Psi(z,z') = \dfrac{1}{M}\displaystyle\int_{z'}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}}, z \in \Omega$. The index l refers to the branch of the chosen root. The relation (53) implies that the y-component of $(z(t), y(t), \zeta(t), \epsilon(t))$ can be expressed as

$$y_l(z) = -\int_{z'}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}} + y'. \tag{54}$$

In particular, we define $y_1(z) := y_0(z,z^*)$. Since $\mathcal{WF}(u_\psi)$ is a closed set (cf. [47, §8 p.41]), we obtain that the point $(z^*, y_j(z^*))$ is in the singular support of the distribution u_ψ.

We want prove that $\lim\limits_{z\to z^*} y_l(z) = 0$. Indeed, by the definition of $y_0(z,z^*)$ and $f_n(z,z^*)$, one obtains

$$\begin{aligned}
&\lim_{z\to z^*} y_0(z,z^*) = 0,\\
&\lim_{z\to z^*} f_0(z,z^*) = 1,\\
&\lim_{z\to z^*} f_n(z,z^*) = 0, \quad n \geq 1.
\end{aligned} \tag{55}$$

Using the expression for $\psi_{j,B}$, and (55), we see that $(z^*, 0)$ is the singular support of the distribution u_ψ. Therefore, by taking $\lim\limits_{z\to z^*}$ in (54) we obtain

$$y' = \int_{z'}^{z^*}\frac{dt}{\sqrt[M]{\rho_M(t)}}.$$

This implies that when $z' \in \Omega$, the y-components of the singularities of the Borel transform $\psi_{j,B}$ with reference point at z^* satisfy $y_l = -\displaystyle\int_{z^*}^{z'}\frac{dt}{\sqrt[M]{\rho_M(t)}}$. $\square$

The following two theorems play an important role in the estimation of the growth of the analytic continuation of the power series from Lemma 9.

Theorem 4 (LeRoy & Lindelöf [57], [58] pp. 340-345, [59]) *For a function $\varphi \in \mathcal{H}(\{z : \Re[z] \geq 0\})$ of exponential type $\sigma < \pi$, the series*

$$f(z) = \sum_{k=0}^{\infty} \varphi(n) z^n$$

admits an analytic continuation to the sector $\mathbb{C} \setminus \{z \in \mathbb{D}, |\arg[z]| \leq 2\sigma\}$. Moreover, $f(z) \to 0$ when $z \to \infty$ in each angular domain $\mathbb{C} \setminus \{z \in \mathbb{D}, |\arg[z]| \leq \frac{\beta}{2}\}, \beta \in (2\sigma, 2\pi)$.

Theorem 5 (Arakelyan [59] Th. 1.1) *For $|z| < 1$ and $\sigma \in [0, \pi)$, a power series $\sum_{k=0}^{\infty} f_n z^n$ admits an analytic continuation to the sector $\mathbb{C} \setminus \{z \in \mathbb{D}, |\arg[z]| \leq \sigma\}$, if and only if there exists a function $\phi \in \mathcal{H}(\{z : \Re[z] > 0\})$ of the inner exponential type at most σ, such that*

$$c_n = \phi(n), \quad n = 0, 1, \ldots$$

To establish the Borel summability of $\psi_j(z, \eta, z^*), z^* \notin \mathcal{Z}$ we introduce the Stokes curves

$$\mathcal{S}_{z*,j} = \bigcup_{j':j' \neq j} \left\{ z \in \Omega : \Im\left[\int_{z^*}^{z} (w_j(\xi) - w_{j'}(\xi)) d\xi \right] = 0 \right\}, \tag{56}$$

relative to z^*.

As observed in [40], for higher order operators a Stokes phenomenon can occur in a neighborhood of the intersection point of Stokes curves starting at two turning points, cf. Definition 4. Consequently, the Borel sum of $\psi_j(z, \eta, z^*), z^* \notin \mathcal{Z}$ may not be well-defined on a new Stokes curve obtained from an ordered crossing point. We consider the ordered crossings of curves in $\mathcal{S}_{z*,j}$ and the Stokes curves emanating from the turning points $z_k \in \mathcal{Z}$ and define the new Stokes curves similarly to the Definition 7. Set $\mathcal{N}_{ext} = \mathcal{N} \cup \mathcal{N}_{z^*}$, where $\mathcal{N}_{z^*}$ is the set of new Stokes curves when z^* is a reference point.

Lemma 9 *Fix $z \in \Omega$ and $z \notin \mathcal{S}_{z*,j} \cup \mathcal{N}_{ext}$. Then, $\psi_{j,B}(z, y, z^*)$ can be analytically continued through the horizontal strip $[-y_0(z, z^*) \pm \imath\delta, t\Re[-y_0(z, z^*)] + \imath(\Im[-y_0(z, z^*)] \pm \delta)], t > 0, |\delta| < \delta_0$. Moreover, $\psi_{j,B}(z, y, z^*) \to 0$ as $y \to \infty$ through the strip, in particular $\psi_{j,B}(z, y, z^*), y \in [-y_0(z, z^*), +\infty - \imath\Im[y_0(z, z^*)])$ is of exponential type.*

Proof. By Lemma 8, for $z \notin \mathcal{S}_{z*,j}$, we see that the solution can be analytically continued in the half-plane $y > -\Re[y_0]$. Hence, the statement that $\psi_{j,B}(z, y, z^*) \to 0$ as $y \to \infty$ through the strip follows immediately from Theorems 4 and 5 applied to $(-y + y_0(z, z^*))^{\frac{1}{2}} \psi_{j,B}(z, -y, z^*) = \sum_{n=0}^{\infty} \frac{f_n(z, z^*)}{\Gamma(n + \frac{1}{2})} (-y + y_0(z, z^*))^n, |-y + y_0(z, z^*)| < \delta$, where $\delta > 0$ is sufficiently large.

□

Proposition 1 *Take $z^* \in \Omega$ and let*

$$\psi_j(z, \eta, z^*) = \exp\left(\eta \int_{z^*}^{z} S_0(\zeta) d\zeta \right) \sum_{n=0}^{\infty} \frac{\phi_n(z)}{\eta^{n+\frac{1}{2}}}, z \in U_{z_k} \cap \Omega$$

be a WKB-solution of (4) with reference point at z^*. Then, ψ_j is summable at z provided that $z \notin \mathcal{S}_{z*,j} \cup \mathcal{N}_{ext}$.

Proof. By definition of the Borel sum, see Definition 3, we observe that $\Psi_j(z, \eta, z^*)$ is well-defined when the integration path $[-y_0(z, z^*), +\infty - \imath\Im[y_0(z, z^*)])$ does not contain a singularity. By Lemma 8, the singularities of $\psi_{j,B}(z, y, z^*)$ are located at the points $\left(z, -\int_{z^*}^{z} \frac{dt}{\sqrt[M]{\rho_M(t)}}, z^* \right)$. Using Lemma 9 we have that if $z \notin \mathcal{S}_{z*,j} \cup \mathcal{N}_{ext}$, the Borel sum of ψ_j is well-defined, which completes the proof. □

4 Proofs of the main results

We start with Theorem 1.

Proof. a) Choose $z^* \in \Omega$, S_n as in Definition 2, and $\mathcal{D}(z_k)$ as in Definition 3. Using Lemma 7, for $n \in \mathcal{D}(z_k)$, we can write

$$S_n(z) = \mathfrak{b}_n(z) + \mathfrak{g}_n(z), \tag{57}$$

where $\mathfrak{b}_n(z) = \sum_{l=0}^{n} a_{n,l}(z-z_k)^{\frac{m_{n,l}}{M}}$ such that $\frac{m_{n,l}}{M} \in \mathbb{Q}_{\leq -1}$, and $\mathfrak{g}_n$ is such that $\mathfrak{g}_n((z-z_k)^M)(z-z_k)^{\beta_n} \in \mathcal{H}(U_{z_k})$, for some integer β_n satisfying $0 \leq \beta_n < M$. By (57), we obtain

$$\int_{z^*}^{z} S(\zeta,\eta)d\zeta = r_j(z,z^*,\eta) + s_j(z^*,\eta), \tag{58}$$

where j is the same index as in ψ_j and

$$\begin{aligned} r_j(z,\eta,z^*) = & \sum_{\substack{n\geq 0: n\in\mathcal{D}(z_k),\\ m_{n,0}<-M}} \eta^{-n-1} \int_{\infty}^{z} \sum_{l=1}^{n} a_{j,n,l}(\zeta - z_k)^{\frac{m_{n,l}}{M}} d\zeta + \sum_{\substack{n=0: n\in\mathcal{D}(z_k),\\ m_{n,0}=-M}} \eta^{-n-1} a_{j,n,0}\ln(z-z_k) + \\ & \sum_{\substack{n\geq 0,\\ n\in\mathcal{D}(z_k)}} \eta^{-n-1}\int_{z^*}^{z} \mathfrak{g}_{j,n}(\zeta)d\zeta + \sum_{\substack{n\geq 0,\\ n\notin\mathcal{D}(z_k)}} \eta^{-n-1}\int_{z^*}^{z} S_n(\zeta)d\zeta, \\ s_j(\eta,z^*) = & -\sum_{\substack{n=0:\\ m_{n,0}<-M}}^{\infty} \eta^{-n-1}\int_{\infty}^{z^*}\sum_{l=1}^{n} a_{j,n,l}(\zeta-z_k)^{\frac{m_{n,l}}{M}} d\zeta + \sum_{\substack{n=0:\\ m_{n,0}=-M}}^{\infty} \eta^{-n-1} a_{j,n,0}\ln(z^*-z_k). \end{aligned}$$

Notice that we have

$$\mathtt{Fp}\int_{z_k}^{z} S(\zeta,\eta)d\zeta = \lim_{z^*\to z_k} r_j(z,\eta,z^*).$$

Fix $z^* \in U_{z_k}\cap\Omega$ and $z \in \Omega$. By Proposition 1, $\psi_j(z,\eta,z^*)$ is Borel summable provided that $z \notin \mathcal{S}_{z*,j}\cup\mathcal{N}_{ext}$. By [60, Th.188 p.237] the formal series $\widetilde{\psi}_j(z,y,z^*) = \eta^{\frac{1}{2}} e^{-\int_{z^*}^{z} S_0(\zeta)d\zeta}\psi_j(z,y,z^*)$ is Borel summable and from [61, Prop.4.109 p.108],

$$\ln(\widetilde{\psi}_j(z,\eta,z^*)) = \widetilde{\psi}_j(z,\eta,z^*) - \frac{(\widetilde{\psi}_j(z,\eta,z^*))^2}{2} + \frac{(\widetilde{\psi}_j(z,\eta,z^*))^3}{3} - \ldots,$$

is also Borel summable for $z \notin \mathcal{S}_{z*,j}\cup\mathcal{N}_{ext}$ and large positive η. Therefore, for $z \notin \mathcal{S}_{z*,j}\cup\mathcal{N}_{ext}$, the formal series of $\ln(\psi_j(z,\eta,z^*))$ is Borel summable. For $z^* \in \Omega$ close to z_k, express the Borel transform $\psi^S_{j,B}$ of $\ln(\psi_j(z,\eta,z^*))$ as

$$\psi^S_{j,B}(z,y,u^{*M}+z_k) = \psi^S_{r,j,B}(z,y,u^{*M}+z_k) + \psi^S_{s,j,B}(y,u^{*M}+z_k) - \frac{1}{2}\frac{\gamma+\ln y}{y}, \tag{59}$$

where u^* belongs to a small punctured neighborhood of 0, γ is the Euler-Mascheroni constant, and $\psi^S_{r,j,B}$ and $\psi^S_{s,j,B}$ are the Borel transforms of the functions r_j and s_j respectively defined in (58). By expanding

$$\begin{aligned} \psi^S_{r,j,B}(z,y,u^{*M}+z_k) &= f(y,z) + \sum_{n\in\mathbb{Z}_{\geq 0}} f_n(y,z)u^{*(n+1)}, \\ \psi^S_{s,j,B}(y,u^{*M}+z_k) &= f_{-1}(y)\ln u^* + \sum_{n\in\mathbb{Z}_{<-1}} f_n(y)u^{*(n+1)}, \end{aligned}$$

and by using the Cauchy integral formula

$$f_n(y,z) = \frac{1}{2(n+1)\pi\imath}\int_{\Gamma(0,\epsilon)} \frac{\frac{d}{d\zeta}\psi^S_{j,B}(z,y,\zeta^M+z_k)}{\zeta^{n+1}}d\zeta,\ n\geq 0,$$

for the Taylor coefficients, we get that the set of singularities of $f_n(y,z)$ and $f(y,z) - \frac{1}{2}\frac{\gamma+\ln y}{y}$ with respect to the variable y is included in the set of singularities of $\psi^S_{j,B}$. Here $\Gamma(0,\epsilon)$ is a small circle of radius ϵ around 0. Therefore, the set of singularities of $\psi^S_{r,j,B}(z,y,z^*) - \frac{1}{2}\frac{\gamma+\ln y}{y}$ in the y variable is included in the set of singularities of $\psi^S_{j,B}(z,y,z^*)$.

Hence, a singularity of the Borel transform $\psi_{r,j,B}(z,y,z^*)$ of $\eta^{-\frac{1}{2}}\exp(r_j(z,\eta,z^*))$ is a singularity of $\psi_{j,B}(z,y,z^*)$ as well. A straightforward calculation shows that $\psi_{r,j,B}(z,y,z^*)$ has the obvious singularity $\left(z, -\int_{z^*}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}}\right)$. On the other hand, by Lemma 8, the singularities of $\psi_{j,B}(z,y,z^*)$ are of the form $\left(z, -\int_{z^*}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}}\right)$, which implies that $\psi_{r,j,B}(z,y,z^*)$ and $\psi_{j,B}(z,y,z^*)$ have singularities in the same set.

Using Theorems 4 and 5, we get

$$\lim_{\substack{y\to+\infty\\ y\in\mathcal{S}}}\psi_{r,j,B}(z,y,z^*) = 0, \tag{60}$$

where $\mathcal{S} = [-y_0(z,z^*)\pm\imath\delta, t\Re[-y_0(z,z^*)] + \imath(\Im[-y_0(z,z^*)]\pm\delta)], t>0, |\delta|<\delta_0$. This implies that $\psi_{r,j,B}(z,y,z^*)$ is also of exponential type when $y\in\mathcal{S}$ and $z\notin\mathcal{S}_{z*,j}\cup\mathcal{N}_{ext}$.

By definition, one has

$$\psi_{j,B}(z,y,z_k) = \lim_{z^*\to z_k}\psi_{r,j,B}(z,y,z^*). \tag{61}$$

To prove that the singularities of $\psi_{j,B}(z,y,z_k)$ occur at $\left\{(z,y) : y = -\int_{z_k}^{z}\frac{dt}{\sqrt[M]{\rho_M(t)}}\right\}$ we need to analyze the limit in (61).

Take a smooth path $\mathfrak{c}$ connecting a fixed point z' and z_k as shown in Figure 2a and let the reference point z^* vary along the arc $\kappa\subset\mathfrak{c}$ so that κ is contained in small neighborhood V_{z_k} of z_k, see Figure 2a.

(a) The neighborhood V_{z_k} and the integration path $\mathfrak{c}$ for the function $\psi_{r,j,B}$.

(b) Region Ω_Y

Figure 2: Regions V_{z_k} and Ω_Y in the z- and the y-spaces respectively.

If $z^*\in\kappa$, we can find small neighborhoods $V_{y_l(z',z^*)}, l=1,\dots,M$ of $y_l(z',z^*) = -\int_{z^*}^{z'}\frac{dt}{\sqrt[M]{\rho_M(t)}}$ such that for each $z^*\in\kappa$, $\psi_{r,j,B}(z',y,z^*)$ is analytic in the variable y in the region $\Omega_Y = \mathbb{C}\setminus\left[\bigcup_{l=1}^{M}V_{y_l(z',z^*)}\cup\gamma_Y\right]$, where γ_Y is the branch cut for $\psi_{r,j,B}$ consisting of Jordan arcs connecting each $y_l(z',z^*)$ with ∞, see Figure 2b. When $\rho_M(z)\neq(z-a)^M$, we can write

$$\psi_{r,j,B}(z',y,z^*) = \sum_{n\geq 0}a_n(z',z^*)y^n, \quad |y|<\delta(z'),$$

where each term $a_n(z',z^*)$ is continuous for $z^*\in\overline{\kappa}$, which implies that $\psi_{r,j,B}(z',y,z^*)$ is also continuous when $(y,z^*)\in\{|y|<\delta_0<\delta(z')\}\times\overline{\kappa}$. Therefore, by the Heine–Cantor theorem, $\psi_{r,j,B}(z',y,z^*)$ is uniformly continuous in $\overline{\kappa}$. Hence, $\psi_{r,j,B}(z',y,z^*)$ is a family of functions depending on the variable z^* which uniformly converges as z^* approaches z_k along κ. By [62, Th. 15.12 p.333 Vol.I], $\psi_{r,j,B}(z',y,z_k)$ is analytic when y varies in compact subsets of Ω_Y. Since we can consider arbitrary small neighborhoods V_{z_k} and $V_{y_l(z',z^*)}, l=1,\dots,M$ then using (61) we obtain that $\psi_{j,B}(z',y,z_k)$ has no singularities other than $\left(z', -\int_{z_k}^{z'}\frac{dt}{\sqrt[M]{\rho_M(t)}}\right)$.

The statement that $\psi_{j,B}(z,y,z_k)$ is of exponential type when

$$y\in[-y_0(z,z_k), t\Re[-y_0(z,z_k)] + \imath(\Im[-y_0(z,z_k)])], t>0; z\notin\mathcal{S}_{z_k,j}\cup\mathcal{N}$$

is immediate from (60), (61), and the definition (56).

b) We first prove that the bicharacteristic curve does not have self-intersections by solving the Hamilton–Jacobi equations defining the bicharacteristic strips, see Definition 5. From (19) we have that $\epsilon = c$. Thus without loss of generality, we can take $\epsilon = 1$. Therefore from (20), one obtains that $\zeta = \frac{1}{\sqrt[M]{\rho_M(z)}}$.

From the preceding item a), the singularities of $\psi_{j,B}(z, y, z_k)$ occur at $\left\{(z, y) : y = -\int_{z_k}^{z} \frac{dt}{\sqrt[M]{\rho_M(t)}}\right\}$. Hence, by [43, Cor. 7.2.2], the singularities of the Borel transform propagate along the bicharacteristic strip emanating from $\left(a, -\int_{z_k}^{a} \frac{dt}{\sqrt[M]{\rho_M(t)}}, \frac{1}{\sqrt[M]{\rho_M(a)}}, 1\right)$, $a \neq z_k$. Equations (16)-(20) reduce to

$$\begin{cases} \frac{dz}{dt} = M \sqrt[M]{\rho_M(z)}, \\ \frac{dy}{dt} = -M, \\ \frac{d\zeta}{dt} = -\frac{\rho_M'(z)}{\rho_M(z)}, \\ z(0) = a, \\ y(0) = \int_{z_k}^{a} \frac{dt}{\sqrt[M]{\rho_M(t)}}, \\ \zeta(0) = \frac{1}{\sqrt[M]{\rho_M(a)}}, \end{cases}$$

which gives $y(t) = -Mt - \int_{z_k}^{a} \frac{dt}{\sqrt[M]{\rho_M(t)}}$ implying

$$\begin{cases} \frac{dz}{dt} = M \sqrt[M]{\rho_M(z)}, \\ \frac{d\zeta}{dt} = -\frac{\rho_M'(z)}{\rho_M(z)}, \\ z(0) = a, \\ \zeta(0) = \frac{1}{\sqrt[M]{\rho_M(a)}}. \end{cases} \tag{62}$$

Now, if $z(t)$ is the solution of (62), it is immediate from the expression for y that the bicharacteristic curve associated to the bicharacteristic strip that emanates from $\left(a, -\int_{z_k}^{a} \frac{dt}{\sqrt[M]{\rho_M(t)}}, \frac{1}{\sqrt[M]{\rho_M(a)}}, 1\right)$ does not have self-intersections. On the other hand, from [63, §13.7 p.311], $z(t)$ does not have other singularities. Finally, notice that the zeros of ρ_M obviously are singular points of $\mathcal{BC}(t)$ which completes the proof.

c) Follows immediately from item a). □

Next we prove Theorem 2. Our strategy follows [37, pp. 5-6] and [27, pp. 24-25]. To understand how the Borel sum $\Psi_{1,1}$ changes when we move from a to b we study the analytic continuation of the Borel transform $\psi_{1,B}(z, y, z_k)$ from $z = a$ to $z = b$ as shown in Figure 3.

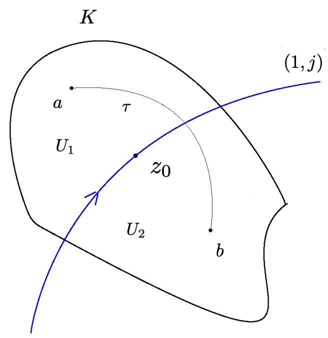


Figure 3

While carrying out the analytic continuation, we deform the integration path to $\tilde{\gamma}$, see Figure 4c. The condition that ℓ does not connect z_1 with another turning point or z_1 with itself implies that we can apply the Cauchy theorem giving

$$\int_{\tilde{\gamma}} e^{-y\eta}\psi_{1,B}(b,y,z_k)dy = \int_{\gamma} e^{-y\eta}\psi_{1,B}(b,y,z_k)dy + \int_{\gamma_0} e^{-y\eta}\psi_{1,B}(b,y,z_k)dy$$
$$= \Psi_{1,2}(b,\eta,z_k) + \int_{\gamma_0} e^{-y\eta}\psi_{1,B}(b,y,z_k)dy. \tag{63}$$

Here the path γ_0 encircles the half-line

$$l_0 = \left\{(z,y) \in \mathbb{C}^2 : \Im[y] = \Im\left[-\alpha_j y_0(z,z_k)\right], \Re[y] > \Re\left[-\alpha_j y_0(z,z_k)\right]\right\},$$

where $\alpha_j = e^{\frac{\pi(j-1)\imath}{M}}$. Hence, the Borel sum $\Psi_{1,1}$ changes by the factor $\int_{\gamma_0}\psi_{1,B}(b,y,z_k)dy$ when z crosses the Stokes curve from a to b. We recall that

$$\Delta_{y=-\alpha_j y_0(z,z_k)}\psi_{1,B}(z,y,z_k) = l_0^+\psi_{1,B}(z,y,z_k) - l_0^-\psi_{1,B}(z,y,z_k) \tag{64}$$

is the alien derivative of $\psi_{1,B}$, and $l_0^{\pm}\psi_{1,B}$ denotes the analytic continuation of $\psi_{1,B}$ from above l_0^+ and below l_0^-.

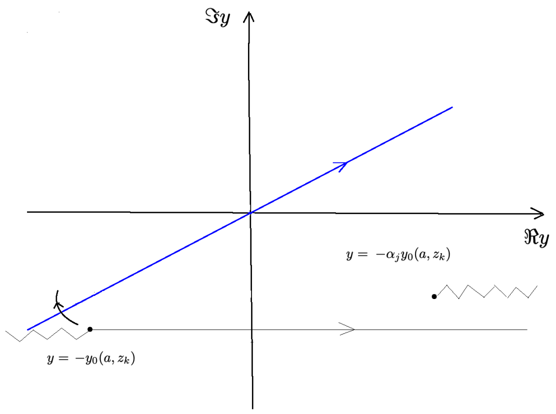
By expanding the second summand of the last expression in (63) as a WKB-solution of (4) we deduce that

$$\Delta_{y=-\alpha_j y_0(z,z_k)}\psi_{1,B}(z,y,z_k) = c_j\psi_{j,B}(z,y,z_k). \tag{65}$$

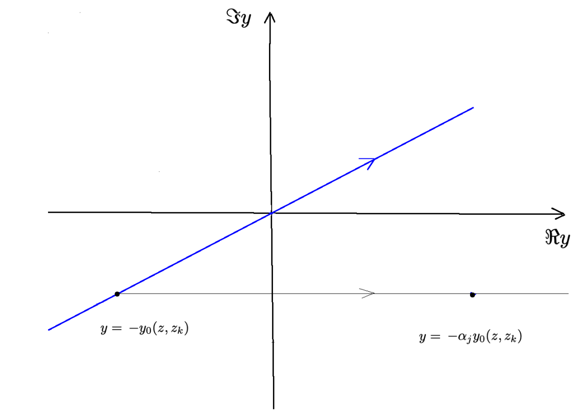
By substituting (64) and (65) in (63) we get,

$$\begin{aligned}\Psi_{1,1}(z,\eta,z_k) &\mapsto \Psi_{1,2}(z,\eta,z_k) + c_j\Psi_{j,2}(z,\eta,z_k),\\ \Psi_{j,1}(z,\eta,z_k) &\mapsto \Psi_{j,2}(z,\eta,z_k),\end{aligned} \tag{66}$$

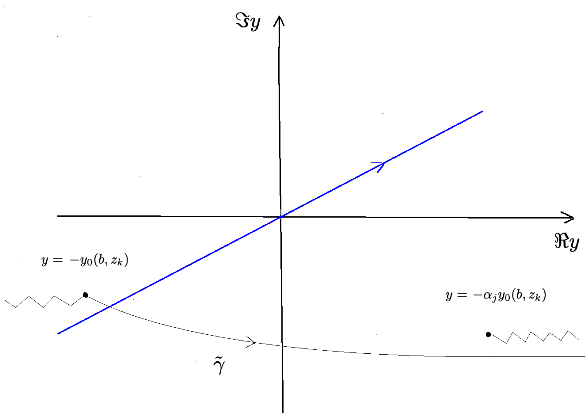
when we cross the Stokes curve.



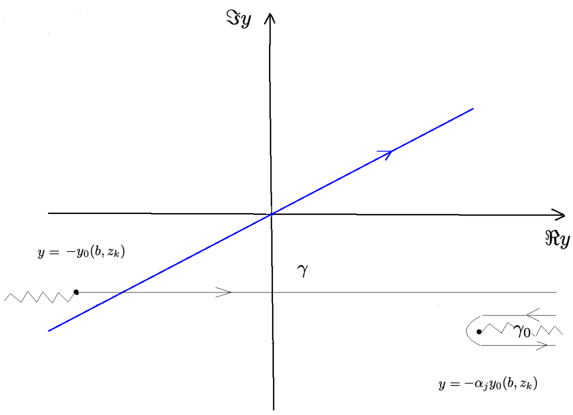
(a) The integration path for the Borel sum of ψ_1 at $z = a$.



(b) Coincidence of the integration paths when z belongs to the Stokes curve.



(c) Deformation of the path to continue $\psi_{1,B}$ analytically.



(d) Decomposition of the path $\tilde{\gamma}$ to obtain the analytic continuation of $\psi_{1,B}$.

Figure 4: Integration paths for the Borel sums (wiggly lines denote the branch cuts for the Borel transforms).

Now, if $K \subset \Omega$ is a compact set as in Figure 3 and $z \in K$, then by [11, Lem. 11], for large enough n, the monic eigenpolynomial of degree n of (1) can be expressed as

$$Q_n^{\mathcal{M}}(z) = \Psi_1\left(z, \sqrt[M]{\lambda_n}, z_k\right) r\left(z, \sqrt[M]{\lambda_n}\right), \tag{67}$$

where λ_n is the eigenvalue associated to the eigenpolynomial $Q_n^{\mathcal{M}}$ of (1). Here we take the branch of the root for which the sequence $\frac{1}{\sqrt[M]{\lambda_n}}$ converges to $\dfrac{1}{n}$ when $n \to \infty$. Since $Q_n^{\mathcal{M}}$ does not have zeros in K for n large enough, we have that r is an analytic function in K. On the other hand, using (67) and (66), when we move from a to b, we obtain

$$\begin{aligned} Q_n^{\mathcal{M}}(z) &\mapsto Q_n^{\mathcal{M}}(z) + c_j \Psi_j\left(z, \sqrt[M]{\lambda_n}, z_k\right) r\left(z, \sqrt[M]{\lambda_n}\right), \\ \Psi_j(z, \eta, z_k) &\mapsto \Psi_j(z, \eta, z_k). \end{aligned}$$

Since $Q_n^{\mathcal{M}}$ is analytic in K we deduce that $c_j = 0$. A similar argument applies when we move from b to a. Thus we get the same connection formula again. The first connection formula when $(1 > j)$ is settled.

Using the same reasoning, for $(1 < j)$ and when z crosses from one region to the other along the curve τ, we obtain the second connection formula whcih completes the proof. □

Corollary 1 follows immediately from Case 1 of the previous theorem.

Finally, let us settle Theorem 3.

PROOF. a) By [3, Th.3], each of the Jordan arcs $\mathfrak{r}_i$ forming $\operatorname{supp}[\mu^{\mathcal{M}}] = \mathfrak{r}_1 \cup \mathfrak{r}_2 \cup \mathfrak{r}_3$ is sent to straight segments by the mapping $\Psi(z) = \int^z w_1(t)dt$. A direct calculation shows that the boundary of the $\mathfrak{B}$-region is a piecewise linear curve $\bigcup_{k=1}^{6} I_k, I_k = [p_k, p_{k+1}]$, where

$$\begin{cases} 0 \equiv p_1 = \lim_{z\to z_1,\ z\in V^+} \Psi(z),\ p_7 = \lim_{z\to z_1,\ z\in V^-} \Psi(z), \\ p_{2i-1} = \Psi(z_i), i = 2, 3 \\ p_{2i} = \lim_{z\to v,\ z\in V_i} \Psi(z),\ i = 1, 2, 3. \end{cases}$$

The curves $V_i, i = 1, 2, 3$ are shown in Figure 5. Notice that

$$p_1 = p_7 + 2\pi\imath. \tag{68}$$

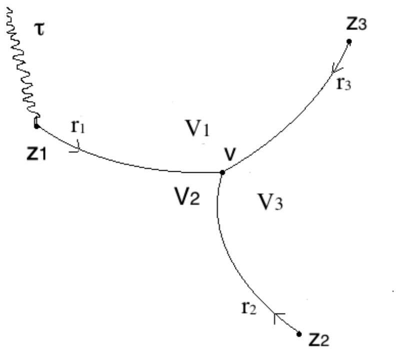


Figure 5: The region $\operatorname{supp}[\mu^{\mathcal{M}}] \setminus \mathfrak{r}$. The wiggly line denotes the branch cut defined by $\mathfrak{r}$

By Lemma 4, the angles at the point v between the arcs $\mathfrak{r}_i, i = 1, 2, 3$ of $\operatorname{supp}[\mu^{\mathcal{M}}]$ are $\frac{2\pi}{3}$. Hence, the interior angles between the line segment $I_k, k = 1, \ldots, 6$ are given by

$$\left(\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}\right). \tag{69}$$

Therefore the lines segments $I_k; k = 1, 3, 5$ are parallel to each other as well as the line segments $I_k; k, k = 2, 4, 6$.

By Definition 4, we have that the Stokes curves of type (j, j') that emanate from z_k are given by

$$\{z \in \Omega : \Im\left[\int_{z_k}^{z} (w_j(\zeta) - w_{j'}(\zeta))d\zeta\right] = 0\}. \tag{70}$$

Hence, from items a) and b) of Lemma 2 if $\kappa = \{z : \Im[z] = 0\}$ we have that

$$\{z \in \Omega : \Im\left[\int_{z_k}^{z} (w_j(\zeta) - w_{j'}(\zeta))d\zeta\right] = 0\} = \rho\left(\mathcal{F}^{-1}_{(j,j')}(\kappa \cap \mathfrak{B}_{(j,j')})\right). \tag{71}$$

In Figure 6 using relations (70), (71), and item c) of Lemma 1, we show the Stokes curves emanating from z_1 in each of the $\mathfrak{B}_{(j,j')}$-regions.

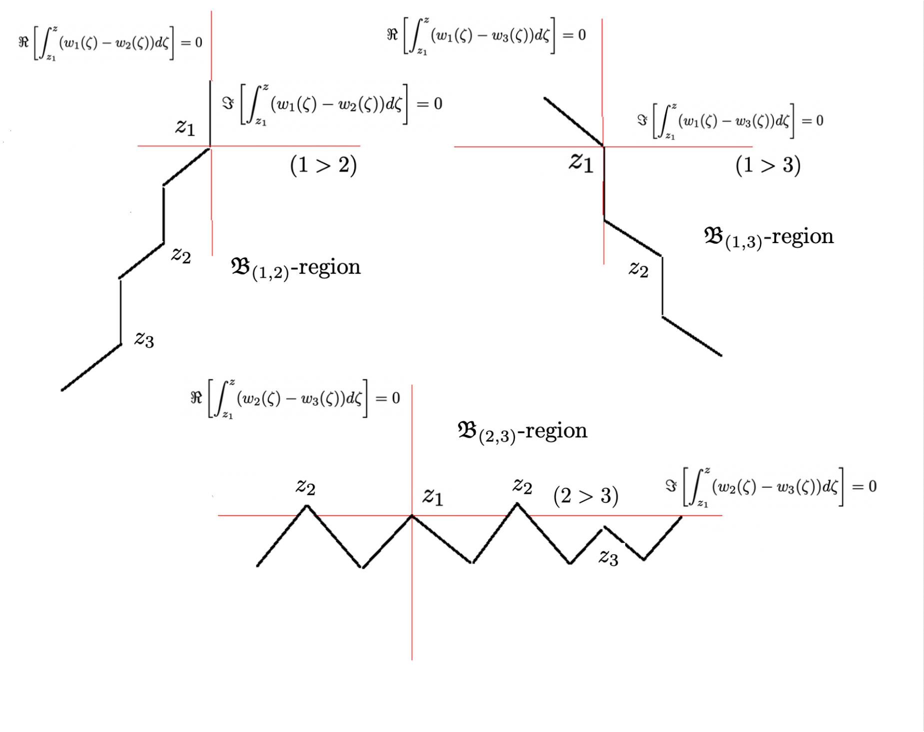
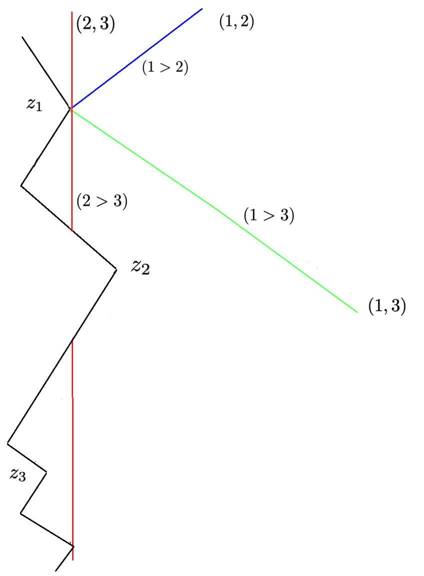
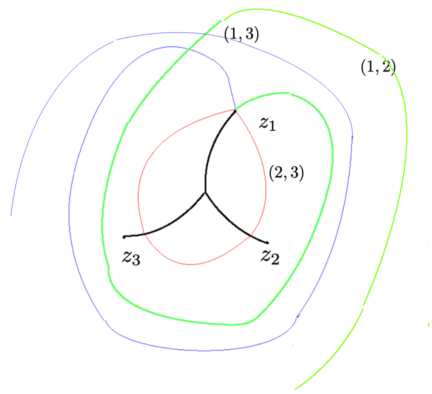


Figure 6: The Stokes curves emanating from z_1 in the $\mathfrak{B}_{(j,j')}$-regions. (Notice that $\mathfrak{B}_{(1,2)} = e^{-\frac{i\pi}{6}}\mathfrak{B}, \mathfrak{B}_{(1,3)} = e^{\frac{i\pi}{6}}\mathfrak{B}, \mathfrak{B}_{(2,3)} = e^{\frac{i\pi}{2}}\mathfrak{B}$).

The set of the Stokes curves emanating from z_1 forms the configuration presented in Figure 7.



(a) Projection of the Stokes curves emanating from z_1 in the $\mathfrak{B}$-region



(b) The Stokes curves emanating from z_1 in the Ω-region. The blue and the green lines continue to ∞ looping around the support μ^M.

Figure 7: The Stokes curves emanating from z_1. (It might happens that for some special polynomials ρ_3, that the red curve passes through the points z_2 and z_3).

A similar argument applies to the remaining roots z_2 and z_3 and we obtain the Stokes complex (i.e. configuration of all the Stokes curves) shown in Figure 8.

b) Follows from Case 1) of Theorem 2.

c) We analyze the ordered crossings to identify the new Stokes curves. Notice that we could have a possible ordered crossing only when $(1,2)$ and $(2,3)$ intersect. However, the labeling for $(2,3)$ as $2<3$ or $3<2$ is not well defined in the Ω-region since this Stokes curve is a loop in this space. For this reason, our analysis is performed in the $\mathfrak{B}$-region.

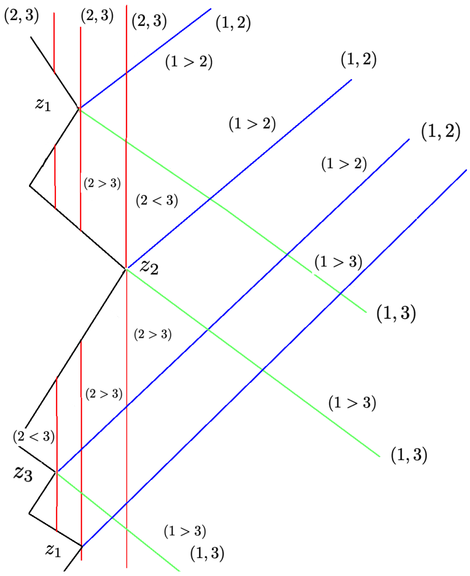
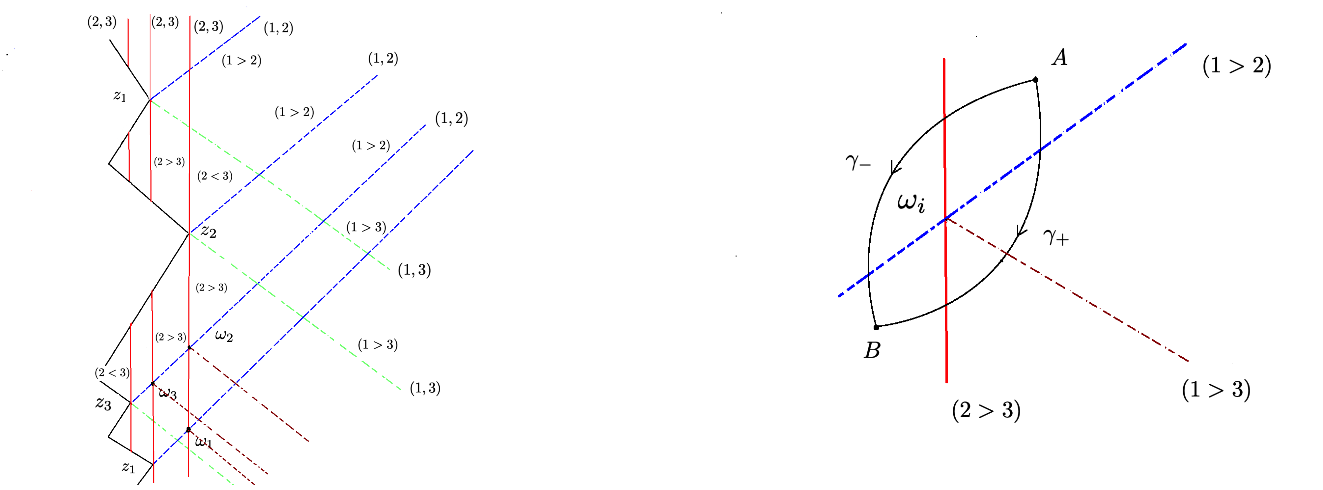


Figure 8: The Stokes curves for $M = 3$ in the $\mathfrak{B}$-region.

The intersection points are shown in Figure 9a.



(a) Ordered crossings and the new Stokes curves (in brown).

(b) The inert new Stokes curve $(1 > 3)$ in a neighborhood of ω_i.

Figure 9: The original Stokes curves and the new Stokes curves. (Dotted lines indicate the inert Stokes curves, see Definition 8).

Now, if ω_i is a crossing point of these curves, it follows from Case 1) of Theorem 2 that $(1, 2)$ is inert. Hence, by moving from the point A to the point B along the paths $\gamma_\pm$, as in Figure 9b, we conclude that $\psi_1 \mapsto \psi_1$, i.e., there is no Stokes phenomenon around the intersections. $\square$

Corollary 2 follows from the relation $\mathfrak{B}_{(1,2)} = e^{-\frac{\imath\pi}{6}}\mathfrak{B}$, see Figure 6.

5 Local structure of a third order exactly solvable differential operator near a turning point

Factorization of some higher order differential operators of the WKB-type near a simple ordinary turning point into lower order differential operators of the same class has been considered in [13, 14, 15]. For some higher order linear ordinary differential operators, the same has been done in [16, 17] near a simple pole-type turning point. In this section we provide an example of an exactly solvable $\mathcal{L}$ which can not be factorized near of a turning point into operators of lower order with analytic coefficients. Therefore in general, the study of the WKB-solutions of exactly solvable operators can not be reduced to the study of lower order differential operators. More precisely, consider

$$\mathcal{L} = (z-a)p(z)\frac{d^3}{dz^3} + p(z)\frac{d^2}{dz^2} + q(z)\frac{d}{dz} - \eta^3, \tag{72}$$

where p and q are polynomials of degree 2 and 1 respectively such that $p(a) \neq 0$ and $\eta > 0$ is a large real number.

Recall the following notion, see [15, Def. 3.3].

Definition 9 *Let $P_0(z,\zeta)$ be the principal symbol of a differential operator P of the WKB-type on an open set $U \subset \mathbb{C}_z$ and let $z_* \in U$ be an ordinary turning point with characteristic value ζ_*. In other words, the system of equations*

$$P_0(z,\zeta) = \partial_\zeta P_0(z,\zeta) = 0$$

has a solution $(z_*,\zeta_*) \in U \times \mathbb{C}_\zeta$ and $P_0(z_*,\zeta)$ does not vanish identically as a function of ζ. The smallest positive integer m such that $\partial_\zeta^m P_0(z,\zeta)$ does not vanish is called the rank of the turning point z_* with the characteristic value ζ_*.

Using the transformation $z-a = (x-a)^2$ we get

$$(z-a)^{-1/2}\mathcal{L} = \mathcal{L}^*,$$

where

$$\mathcal{L}^* = p((x-a)^2+a)\frac{d^3}{dx^3} - \frac{1}{2}q((x-a)^2+a)\frac{d}{dx} - (x-a)\eta^3. \tag{73}$$

Notice that $\eta^{-3}\mathcal{L}^*$ is of the WKB-type and that $x = a$ is an ordinary turning point of rank 3 with characteristic value $\zeta_* = 0$.

Let V_a be a neighborhood of $z = a$ and take the cut in V_a by using the arc of $\operatorname{supp}[\mu^{\mathcal{M}}]$ whose endpoint is a. Pick a branch of $\sqrt{z-a}$ in V_a and set $U_a^+ = \mathcal{T}^+(V_a), U_a^- = \mathcal{T}^-(V_a)$, and $U_a = U_a^- \bigcup U_a^+$, where $\mathcal{T}^\pm(z) = a \pm \sqrt{z-a}$.

Proposition 2 *For every sufficiently small neighborhood of $z = a$, there are no differential operators Q and R of the WKB-type such that*

$$\eta^{-3}\mathcal{L} = QR.$$

Here $Q = \sum_{j\geq 0}\eta^{-j}Q_j(x,\eta^{-1}\frac{d}{dx})$ and $R = \sum_{j\geq 0}\eta^{-j}R_j(x,\eta^{-1}\frac{d}{dx})$ are differential operators of order 1 and order 2 in $\frac{d}{dx}$ respectively, such that

$$Q_0(a,0) \neq 0, \tag{74}$$

$$R_0(x,\zeta) = (\zeta - \zeta_j(x))(\zeta - \zeta_k(x)), \tag{75}$$

where $Q_0(x,\zeta)$ (resp. $R_0(x,\zeta)$) denotes the principal symbol of the operator Q, (resp. R), i.e. $Q_0(x,\frac{\varsigma}{\eta})$ (resp. $R_0(x,\frac{\varsigma}{\eta})$) with $\frac{\varsigma}{\eta}$ denoted by ζ.

Proof. Let $\mathcal{L}^*$ be defined as in (73). A straightforward calculation shows that $x = a$ is an ordinary turning point of rank 3 for $\eta^{-3}\mathcal{L}^*$. By [15, Th. 5.2], we have that for $x \in U_a$,

$$\eta^{-3}\mathcal{L}^* = QR,$$

where $Q = \sum_{j\geq 0}\eta^{-j}Q_j(x,\eta^{-1}\frac{d}{dx})$ and $R = \sum_{j\geq 0}\eta^{-j}R_j(x,\eta^{-1}\frac{d}{dx})$ are the unique WKB-type differential operators in $\frac{d}{dx}$ of order 0 and order 3 respectively. Finally, by considering $(z-a)^{-1/2}\mathcal{L} = \mathcal{L}^*$, we obtain the required result. □

6 Euler-Cauchy equations

Theorem 6 *Let ϵ be a small complex parameter varying in a punctured neighborhood of the origin, and $K \subset \Omega$ be a compact set. Then, for the Euler-Cauchy differential equation*

$$z^M v^{(M)}(z,\eta) + \sum_{k=1}^{M-1} a_k z^k v^{(k)}(z,\eta) - \eta^M v(z,\eta) = 0, \tag{76}$$

there exist M linearly independent WKB-solutions

$$\psi_j = \exp\left[\sum_{k=0}^{\infty} h_{j,k}\epsilon^{k-1} \ln z\right], \varepsilon \in V^*,$$

convergent for all $z \in K$, in a reduced neighborhood of $\varepsilon = 0$, where $h_{j,0} = \sqrt[M]{1}$.

In particular, by the preceding theorem, there is no Stokes phenomenon. Before we prove Theorem 6 we need a preliminary lemma.

Lemma 10 *Let $b_1, \ldots, b_{M-1}$ be complex numbers. Then the algebraic equation*

$$w^M + \sum_{k=1}^{M-1} b_k w^k - b_0 \epsilon^{-M} = 0 \tag{77}$$

has M solutions $w_j(\epsilon) = \sum_{k=0}^{\infty} h_{j,k}\epsilon^{k-1}, j = 1, \ldots, M$ holomorphic in a neighborhood V^* of 0, where $h_{j,0} = \sqrt[M]{b_0}$ and $h_{j,k} \in \mathbb{C}$.

Proof. Multiplying (77) by ϵ and making the variable change $y = w\epsilon$, we obtain the equation

$$F(\epsilon, y) = y^M + \sum_{k=1}^{M-1} b_k(\epsilon^{M-k} y^k) - b_0 = 0. \tag{78}$$

Notice that

$$\begin{cases} F(0, h_{j,0}) = 0 \\ \frac{\partial F}{\partial \epsilon}(0, h_{j,0}) \neq 0, \end{cases}$$

where $h_{j,0} = \sqrt[M]{b_0}$ (i.e. all the roots of b_0). Hence, from the implicit function theorem, there exists a neighborhood V of 0 and M unique analytic functions $y_j(\epsilon) = \sum_{k=0}^{\infty} \epsilon^k h_{j,k}$ such that $y_j(0) = h_{j,0}$ and $F(\epsilon, y_j(\epsilon)) = 0, \forall \epsilon \in V$, see [62, Th 3.11, Vol II]. Taking into account that $y = w\epsilon$, $\epsilon \neq 0$ we complete the proof. $\square$

Next we settle Theorem 6.

Proof. Looking for a solution of equation (76) in the form $v = z^w$ we obtain for w the indicial equation

$$w^M + A_{M-1} w^{M-1} + \ldots + A_1 w - \frac{1}{\epsilon^M} = 0, \tag{79}$$

where $A_k \in \mathbb{C}$.

By Lemma 10 we have that (79) has M solutions $w_j(\epsilon) = \sum_{k=0}^{\infty} h_{j,k}\epsilon^{k-1}$ defined in a reduced neighborhood V^* of 0. Hence, the eqution (76) has M solutions of the form

$$v = z^{\sum_{k=0}^{\infty} h_{j,k}\epsilon^{k-1}}, \ \varepsilon \in V^*,$$

where $h_{j,0} = \sqrt[M]{1}$. By writing the latter expression as

$$v = \exp\left[\sum_{k=0}^{\infty} h_{j,k}\epsilon^{k-1} \ln z\right], \ \varepsilon \in V^*,$$

we obtain M linearly independent convergent WKB-solutions for the equation (76). $\square$

7 Open problems

1. The following questions are very crucial for our considerations.

Problem 1 *Give a formal definition of a virtual turning point for exactly solvable operators.*

Problem 2 *Extend Theorem 1 in the case* $\rho_M = (z-a)^M$. *Is the definition of Stokes curves given by* (14) *appropriate for this case?*

2. The next question is related to b) of Theorem 1 and c) of Theorem 3.

Problem 3 *Does the non-existence of self-intersections on the bicharacteristic curve imply that all new Stokes curves are inert? Consequently, are there no "new turning points" from which "new Stokes curves" emanate?*

3. The following guess is related to Theorem 2.

Conjecture 1 *For a generic equation* (4), *Case 2 never happens.*

4. The last question is the most important in this area of research.

Problem 4 Describe the Stokes complex, i.e. the union of Stokes curves for an arbitrary (non-degenerate) exactly solvable operator (4).

5. Inspired by Theorem 6 and based on some calculations, we have the following guess.

Conjecture 2 *For an arbitrary holomorphic function* ρ_3, *the WKB-solutions of the differential equation*

$$\rho_3(z)v''' + \rho_3'(z)v'' + \left(3\rho_3''(z) - \frac{\rho_3'^2(z)}{\rho_3(z)}\right)v' - \frac{v}{\epsilon^M} = 0, \tag{80}$$

reduce to

$$\psi_j = \exp\left[\frac{1}{\epsilon}\int^z \frac{1}{\sqrt[3]{\rho_3(\xi)}}d\xi\right].$$

8 Appendix. Sibuya's theorem on Gevrey summability of formal power series depending on a parameter.

Suppose that we have:

1) a formal power series $\phi(z,\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \phi_k(z)$ depending on ϵ, where the coefficients ϕ_k are complex–valued functions holomorphic in z in a simply connected domain D_0 of the z–plane;

2) a nontrivial polynomial

$$F(x_0, x_1, \ldots, x_l, z, \epsilon) = \sum_{m_0+\ldots+m_l=0}^{R} x_0^{m_0} \cdots x_l^{m_l} F_{m_0,\ldots,m_l}(z,\epsilon),$$

with coefficients $F_{m_0,\ldots,m_l}(z,\epsilon) = \sum_{m=0}^{\infty} \epsilon^m F_{m_0,\ldots,m_l;m}(z)$ which are formal power series in ϵ with coefficients $F_{m_0,\ldots,m_l;m}(z)$ being complex–valued and holomorphic in z in the domain D_0;

3) a formal power series $F\left(\phi, \frac{d\phi}{dz}, \frac{\phi^2\phi}{dz^2}, \ldots, \frac{d^{N-1}\phi}{dz^{N-1}}, z, \epsilon\right)$ depending on ϵ which is identically equal to zero in the domain D_0;

4) nonnegative numbers s, K_1, and K_2 such that

$$|F_{m_0,\ldots,m_l;m}(z)| \le K_1(m!)^s K_2^m,$$

for $z \in D_0$ and $(m_0, m_1, \ldots, m_l; m) \in \mathbb{N}^{l+2}$ such that $0 \le \sum_{k=0}^{l} m_k \le R$. In other words, $F_{m_0,\ldots,m_l}(z,\epsilon)$ are of the Gevrey order s in ϵ uniformly in $z \in D_0$.

A theorem due to Sibuya (see [54, Th.1.2.1]) provides the Gevrey summability of a formal series in ϵ satisfying the condition 3) uniformly in z on every compact subset of D_0, under some assumptions described below as Cases A, B and C.

To state this theorem, let us assume that

$$\frac{\partial F_j}{\partial x_{N-1}}\left(\phi, \frac{d\phi}{dz}, \frac{\phi^2\phi}{dz^2}, \dots, \frac{d^{N-1}\phi}{dz^{N-1}}, z, \epsilon\right) \neq 0, \tag{81}$$

for some $z \in D_0$ as a formal power series in ϵ.

Define the linear differential operator

$$\mathcal{T}[x] = \sum_{h=0}^{N-1} \frac{\partial^h F_j}{\partial x_h}\left(\phi, \frac{d\phi}{dz}, \frac{d^2\phi}{dz^2}, \dots, \frac{d^{N-1}\phi}{dz^{N-1}}, z, \epsilon\right) D^h x,$$

where $D = \dfrac{d}{dz}$.

For the above operator $\mathcal{T}$, construct a convex polygon as follows.

Set

$$\frac{\partial F_j}{\partial x_h}\left(\phi, \frac{d\phi}{dz}, \frac{d^2\phi}{dz^2}, \dots, \frac{d^{N-1}\phi}{dz^{N-1}}, z, \epsilon\right) = \sum_{m=0}^{\infty} \epsilon^m a_{h,m}(z), \quad h = 0, \dots, N-1.$$

For $h = 0, \dots, N-1$, fix nonnegative integers coefficients $m_h, (h = 0, 1, \dots, N-1)$ defined by the conditions

$$\begin{aligned} a_{h,m}(z) &= 0, \quad h = 0, \dots, m_h - 1, \forall z \in U, \\ a_{h,m_h}(z) &\neq 0 \quad \text{for some} \quad z \in U. \end{aligned} \tag{82}$$

If all $a_{h,m}(z) = 0$ identically in z for all $m \geq 0$, we set $m_k = +\infty$.

Let us consider N points $(h, m_h), h = 0, \dots, N-1$ in the (X, Y)-plane. The convex hull of the set $\mathcal{P} = \bigcup_h \mathcal{P}_h$, where $\mathcal{P}_h = \{(X, Y) : 0 \leq X \leq h, Y \geq m_h\}$ is called the *polygon of the operator $\mathcal{T}$*. In other words, there exist nonnegative integers

$$0 \leq h_1 < h_2 < \dots < h_k = N-1, \tag{83}$$

such that

i) $m_{h_1} \geq 0$,

ii) if we set

$$\rho_\nu = \frac{m_{h_\nu} - m_{h_{\nu-1}}}{h_\nu - h_{\nu-1}}, \nu = 2, \dots, k \tag{84}$$

we have

$$0 < \rho_2 < \dots < \rho_k,$$

iii) $m_h \geq m_{h_1}$, for $0 \leq h \leq h_1$, and

$$\frac{m_{h_\nu} - m_h}{h_\nu - h}, \quad \text{for} \quad h_{\nu-1} < h \leq h_\nu, \quad \text{and} \quad \nu = 2, \dots, k.$$

Now, under the assumption (81), the Cases A, B, and C are described as follows.

Case A: The integer $h_1 = 0$, i.e.

$$\epsilon^{-m_{h_1}} \left|\mathcal{T}[y]\right|_{\epsilon=0} = Q_0(z)y, \tag{85}$$

where $Q_0(z)$ is holomorphic on D_0 and not identically equal to zero.

For Cases B and C, we have $h_1 > 0$, i.e.

$$\epsilon^{-m_{h_1}} \left|\mathcal{T}[y]\right|_{\epsilon=0} = \sum_{j=0}^{h_1} Q_j(z) D^j y,$$

where $Q_0(z), \dots, Q_{h_1}(z)$ are holomorphic in D_0 and $Q_{h_1}(z)$ is not identically vanishing in D_0.

Case B: $Q_{h_1}(z)$ has no zeros in D_0.

Case C: $Q_{h_1}(z)$ vanishes at some point $z \in D_0$.

In this article we are only interested in Case A in which, under the assumption (81), Sibuya's theorem claims the following.

Theorem 7 (Theorem 1.2.1 of [54]) *In Case A, the formal series $\phi(z, \epsilon)$ has Gevrey order* $\max\left(\frac{1}{\rho_2}, s\right)$ *in ϵ uniformly in the variable z belonging to any compact subset of D_0.*

References

[1] T. Bergkvist, H. Rullgård, and B. Shapiro, "On Bochner–Krall orthogonal polynomial systems," *Math. Scand.*, vol. 94, no. 1, pp. 148–154, 2004.

[2] T. Bergkvist, "On asymptotics of polynomial eigenfunctions for exactly solvable differential operators," *J. Approx. Theory*, vol. 149, pp. 151–187, 2007.

[3] T. Bergkvist and H. Rullgård, "On polynomial eigenfunctions for a class of differential operators," *Math. Res. Lett.*, vol. 9, pp. 153–171, 2000.

[4] G. Masson and B. Shapiro, "On polynomial eigenfunctions of a hypergeometric type operator," *Exper. Math.*, vol. 10, pp. 609–618, 2001.

[5] S. Bochner, "Über Sturm-Liouvillesche polynomsysteme," *Math. Z.*, vol. 29, no. 1, pp. 730–736, 1929.

[6] W. N. Everitt, K. H. Kwon, L. L. Littlejohn, and R. Wellman, "Orthogonal polynomial solutions of linear ordinary differential equations," *J. Comput. Appl. Math.*, vol. 133, no. 1–2, pp. 85–109, 2001.

[7] E. Horozov, B. Shapiro, and M. Tater, "In search of higher Bochner theorem." arXiv preprint arXiv:1807.01558, 2018.

[8] K. Kwon, L. Littlejohn, and G. Yoon, "Bochner–Krall orthogonal polynomials," in *Special functions* (C. Dunkl, M. Ismail, and R. Womg, eds.), pp. 181–193, Singapore: World Scientific, 2000.

[9] H. L. Krall, "On orthogonal polynomials satisfying a certain fourth order differential equation," Tech. Rep. 6, the Pennsylvania State College Studies, State College PA, 1940.

[10] A. Turbiner, "Lie algebras and polynomials in one variable," *J Phys A Math Gen*, vol. 25, no. 18, p. L1087, 1992.

[11] J. A. Borrego-Morell, "An asymptotic expansion of eigenpolynomials for a class of linear differential operators," *Stud. Appl. Math*, vol. 151, no. 3, pp. 923–956, 2023.

[12] F. W. J. Olver, *Asymptotics and Special Functions.* Ltd.: A K Peters, 1974.

[13] T. Aoki, T. Kawai, and Y. Takei, "New turning points in the exact WKB analysis for higher–order ordinary differential equations," in *Analyse algébrique des perturbations singuliéres I*, pp. 69–84, Hermann, 1994.

[14] T. Aoki, T. Kawai, and Y. Takei, "Algebraic analysis of singular perturbations-on exact WKB analysis," *Sugaku Expositions*, vol. 8, no. 2, p. 217, 1995.

[15] T. Aoki, T. Kawai, T. Koike, and Y. Takei, "On the exact WKB analysis of operators admitting infinitely many phases," *Adv. Math.*, vol. 181, no. 1, pp. 165–189, 2004.

[16] T. Kawai, T. Koike, and Y. Takei, "On the structure of higher order simple-pole type operators in exact wkb analysis," *Funkcialaj Ekvacioj*, vol. 53, no. 2, pp. 249–276, 2010.

[17] T. Kawai, T. Koike, and Y. Takei, "On the exact WKB analysis of higher order simple-pole type operators," *Adv. Math.*, vol. 228, no. 1, pp. 63–96, 2011.

[18] C. M. Bender and T. Wu, "Anharmonic oscillator," *Phys. Rev.*, vol. 184, no. 5, pp. 1231–1260, 1969.

[19] A. Voros, "The return of the quartic oscillator. The complex WKB method," *Inst. H. Poincaré Sect. A (N.S.)*, vol. 39, no. 3, pp. 211–338, 1983.

[20] H. J. Silverstone, "JWKB connection-formula problem revisited via Borel summation," *Phys. Rev. Lett.*, vol. 55, no. 23, p. 2523, 1985.

[21] J. Ecalle, "Les fonctions résurgentes," *Publ. Math. Orsay, Univ. Paris-Sud*, vol. I,II,III, 1981,1983,and 1985.

[22] J. Ecal1e, "Cinq applications des fonctions résurgentes," *Prepublications d'Orsay, Univ. Paris-Sud*, vol. 84T, no. 62, 1984.

[23] F. Pham, "Resurgence, quantized canonical transformations, and multi-instanton expansions," in *Algebraic analysis*, pp. 699–726, Elsevier, 1988.

[24] B. Candelpergher, J. C. Nosmas, and F. Pham, *Approche de la résurgence.* Hermann, 1993.

[25] H. Dillinger, E. Delabaere, and F. Pham, "Résurgence de Voros et périodes des courbes hyperelliptiques," *Annales de l'institut Fourier*, vol. 43, no. 1, pp. 163–199, 1993.

[26] E. Delabaere and F. Pham, "Resurgent methods in semi-classical asymptotics," *Annales de l'IHP Physique théorique*, vol. 71, no. 1, pp. 1–94, 1999.

[27] T. Kawai and Y. Takei, *Algebraic Analysis of Singular Perturbation Theory*, vol. 227. American Mathematical Soc., 2005.

[28] Y. Takei, "WKB analysis and Stokes geometry of differential equations," in *Analytic, algebraic and geometric aspects of differential equations*, pp. 263–304, Springer, 2017.

[29] G. Nemes, "On the Borel summability of WKB solutions of certain Schrödinger-type differential equations," *J. Approx. Theory*, vol. 265, p. 105562, 2021.

[30] G. Nemes, "On the Borel summability of formal solutions of certain higher-order linear ordinary differential equations," *arXiv:2312.14449*, 2023.

[31] T. Moteki and Y. Takei, "Stokes geometry of higher order linear ordinary differential equations and middle convolution," *Adv. Math.*, vol. 310, pp. 327–376, 2017.

[32] G. Jones and D. Singerman, *Complex functions: an algebraic and geometric viewpoint.* Cambridge University Press, 1987.

[33] Y. Sibuya, "On nonlinear ordinary differential equations containing a parameter," *J. Math. Mech.*, vol. 9, no. 3, pp. 369–397, 1960.

[34] O. P. Misra and J. L. Lavoine, *Transform Analysis of Generalized Functions.* Elsevier, 1986.

[35] T. Aoki, T. Kawai, and Y. Takei, "The Bender–Wu analysis and the Voros theory," in *ICM-90 Satellite Conference Proceedings*, pp. 1–29, Springer, 1991.

[36] W. Wasow, *Linear Turning Point Theory*, vol. 54 of *Applied Mathematical Sciences.* Springer-Verlag, 1985.

[37] N. Honda, T. Kawai, and Y. Takei, *Virtual Turning Points*, vol. 4. Springer, 2015.

[38] T. Koike, "On the exact WKB analysis of second order linear ordinary differential equations with simple poles," *Publ. Res. Inst. Math. Sci.*, vol. 36, no. 2, pp. 297–319, 2000.

[39] M. V. Fedoryuk, *Asymptotic Analysis.* Berlin: Springer–Verlag, 1993.

[40] H. Berk, W. Nevins, and K. Roberts, "New Stokes' line in WKB theory," *J. Math. Phys.*, vol. 23, 1982.

[41] M. Sato, T. Kawai, and M. Kashiwara, "Microfunctions and pseudodifferential equations," in *Hyperfunctions and Pseudo-Differential Equations* (H. Komatsu, ed.), vol. 287 of *Lecture Notes in Mathematics*, (Berlin), pp. 265–529, Springer-Verlag, 1973.

[42] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution theory and Fourier analysis.* Springer, 2nd ed., 1990.

[43] J. F. Duistermaat and L. Hörmander, *Fourier Integral Operators. II.* Springer, 1994.

[44] T. Aoki, T. Kawai, T. Koike, and Y. Takei, "On global aspects of exact WKB analysis of operators admitting infinitely many phases," *Contemp. Math.*, vol. 373, pp. 11–48, 2005.

[45] T. Aoki, N. Honda, T. Kawai, T. Koike, Y. Nishikawa, S. Sasaki, A. Shudo, and Y. Takei, "Virtual turning points—a gift of microlocal analysis to the exact WKB analysis," in *Algebraic Analysis of Differential Equations: from Microlocal Analysis to Exponential Asymptotics Festschrift in Honor of Takahiro Kawai*, pp. 29–43, Springer, 2008.

[46] M. Kashiwara, T. Kawai, and T. Oshima, "Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics, I," *Proc. Japan Acad.*, vol. 50, no. 7, pp. 420–425, 1974.

[47] L. Nirenberg, *Lectures on Linear Partial Differential Equations*, vol. 17. American Mathematical Soc., 1973.

[48] D. Sauzin, "Resurgent functions and splitting problems," *RIMS Kôkyûroku*, vol. 1493, pp. 48–117, 2006.

[49] J. B. Conway, *Functions of One Complex Variable*, vol. I. New York: Springer–Verlag, 2nd ed., 1984.

[50] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, vol. 316. Springer-Verlag, 2013.

[51] G. Folland, *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2nd ed., 1999.

[52] N. I. Muskhelishvili, *Singular Integral Equations.* Groningen-Holland: Noordhoff, 1953. Translated from the Russian by J.R.M. Radok.

[53] L. Comtet, *Advanced combinatorics: The Art of Finite and Infinite Expansions.* Dordrecht–Holland/Boston–USA: D Reidel Publishing Company, 1974.

[54] Y. Sibuya, "Formal power series solutions in a parameter," *J. Differ. Equ.*, vol. 190, no. 2, pp. 559–578, 2003.

[55] D. Dummit and R. Foote, *Abstract Algebra*, vol. 3. Wiley Hoboken, 2004.

[56] J. B. Fraleigh, *A First Course in Abstract Algebra.* Pearson, 8th ed., 2021.

[57] E. Lindelöf, *Le Calcul des Résidus et ses Applications à la Théorie des Fonctions.* Gauthier-Villars, 1905.

[58] P. Dienes, *The Taylor series. An introduction to the theory of functions of a complex variable.* Dover Books on Science S, 1957.

[59] N. U. Arakelyan, "On efficient analytic continuation of power series," *Math. USSR-Sb.*, vol. 52, no. 1, p. 21, 1985.

[60] G. H. Hardy, *Divergent Series.* Clarendon Press, 1949.

[61] O. Costin, *Asymptotics and Borel summability.* CRC press, 2008.

[62] A. I. Markushevich, *Theory of Functions of a Complex Varible.* Englewood Cliffs, N.J.: Prentice-Hall, 1965. Translated from the Russian by R. A. Silverman.

[63] E. Ince, *Ordinary Differential Equations.* Courier Corporation, 1956.