VARIATION ON A THEME OF KAC-MURDOCK-SZEGÖ AND KUIJLAARS-VAN ASSCHE-TILLI. PROBABILISTIC APPROACH

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ABSTRACT. This paper initiates the study of sequences of complex Jacobi matrices of increasing sizes which satisfy a natural condition of stabilization of their entries. We discuss some natural generalizations of the well-known results by Kac-Murdock-Szegö [18], Kuijlaars - Van Assche [20], and Tilli [33, 34, 35] which provide a description of the asymptotic distribution of eigenvalues for sequences of real Jacobi matrices satisfying the same stabilization condition, see also [8]. Below we introduce a probabilistic set-up in the latter problem and prove that the asymptotic spectral measure of thus obtained random sequences converges almost surely to the measure obtained by averaging of a natural 1-parameter family of arcsine measures supported on straight intervals lying in $\mathbb{C}$. (In the case of real Jacobi matrices the prototype of the latter 1-parameter family of arcsine measure has served as the major technical tool of the foundational paper [20].)

1. INTRODUCTION

Asymptotic spectral theory of real infinite Jacobi matrices is a vast area of mathematics going back to the 1920’s. It has been developed by a number of famous mathematicians including I. M. Gelfand, Yu. M. Berezanskii, B. Simon, M. Kac, N. Levinson, B. Levitan, P. van Moerbeke, etc.

A similar, but more complicated problem of spectral asymptotics of complex infinite Jacobi matrices has also attracted substantial attention over the years. In particular, recent developments in this area have been stimulated by the $PT$-symmetric quantum mechanics, see e.g. [14].

Our interest in the asymptotic spectral theory of complex Jacobi matrices has two main sources: spectral asymptotics of families of orthogonal polynomials satisfying three-term recurrent relations with variable coefficients [20] and spectral theory of quasi-exactly solvable potentials for the Schrödinger equations, [28], [27], [29], [30].

The basic object of our study is as follows. Consider a double indexed sequence of monic polynomials \( \{P_{k,n}(z)\} \), \( n \in \mathbb{N} \) and \( 1 \leq k \leq n \), satisfying a three-term recurrence relation with varying coefficients of the form:

\[
P_{k,n}(z) = (z - b_{k,n})P_{k-1,n}(z) - a_{k,n}P_{k-2,n}(z),
\]

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together with the standard initial conditions: \( P_{-1,n} = 0 \) and \( P_{0,n} = 1 \). Here \( \{a_{k,n}\} \) and \( \{b_{k,n}\} \) are double-indexed sequences of real or complex numbers with some reasonable asymptotic behaviour.

**Problem.** Describe the asymptotic root distribution of the diagonal polynomial sequence \( \{P_{n,n}(z)\} \).

To the best of our knowledge, the first fundamental result in this area have been obtained by A. Kuijlaars and W. Van Assche in [20]. Namely, assume that in the above notation, all \( b_{k,n} \) are real, \( a_{k,n} \) are non-negative and, additionally, have the property that

\[
\lim_{k/n \to \tau} a_{k,n} = a(\tau) \geq 0, \quad \tau \in [0,1] \quad \text{and} \quad \lim_{k/n \to \tau} b_{k,n} = b(\tau), \quad \tau \in [0,1]
\]

for some continuous functions \( a(\tau) \geq 0 \) and \( b(\tau) \). Then the density of the asymptotic root distribution for the diagonal polynomial sequence \( \{P_{n,n}(z)\} \) can be obtained by averaging over \( \tau \in [0,1] \) the family of the standard asymptotic root densities for the recurrences:

\[
Q_{k,\tau}(z) = (z - b(\tau))Q_{k-1,\tau}(z) - a(\tau)Q_{k-2,\tau}(z),
\]

with constant coefficients. More exactly, the following statement holds.

**Theorem A** (A. Kuijlaars–W. Van Assche). In the above notation and under the assumption (1.2), the density \( \rho(x) \) of the asymptotic root-counting measure for the diagonal sequence \( \{P_{n,n}(z)\} \) is given by

\[
\rho(x) = \frac{1}{\pi} \int_0^1 \omega_1^\tau \left[ b(\tau) - 2a(\tau) + 2\sqrt{a(\tau)} \right] d\tau,
\]

where \( \omega_{[\alpha,\beta]}(x) := \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} \), \( \alpha \leq x \leq \beta \).

Somewhat similar results in case of certain special recurrences of length 4 were later obtained in [9].

In the present paper we generalize the above set-up to the situation, when \( a(\tau) \) and \( b(\tau) \) are piecewise continuous complex-valued functions.

In order to get a situation where the appropriately chosen \( \mu \) will coincide with \( \mathcal{M} \) for complex-valued functions \( a(\tau) \) and \( b(\tau) \), let us introduce a randomized version of the original problem.

As above, let us fix two piecewise continuous complex-valued functions \( a(\tau), b(\tau), \tau \in [0,1] \).

For a given positive integer \( n \), sample independent random variables \( \tau_1, \tau_2, \ldots, \tau_n \) uniformly distributed on \([0,1] \). Let \( \Theta_n = \{0\leq \hat{\tau}_1 \leq \hat{\tau}_2 \leq \cdots \leq \hat{\tau}_n \leq 1\} \) be the reordering of the latter sample according to the values of the random variables. To the ordered sample \( \Theta_n \), let us associate the following finite segment of a random three-term recurrence:

\[
P_{k,\Theta_n}(z) = (z - b(\hat{\tau}_i))P_{k-1,\Theta_n}(z) - a(\hat{\tau}_i)P_{k-2,\Theta_n}(z), \quad 1 \leq i \leq n
\]

satisfying the standard initial conditions: \( P_{-1,\Theta_n} = 0 \) and \( P_{0,\Theta_n} = 1 \). In other words, we are choosing a random subdivision \( \Theta_n \) of \([0,1] \) of uniformly distributed points (as opposed to the above deterministic subdivision \( T_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \)) and then, we assign recurrence coefficients as the values of the functions \( a(\tau) \) and
Figure 1. Illustration of the probabilistic set-up. Measure $\mathcal{M}$ and a sample of measure $\mu_{\Theta_n}$ for $b(\tau) = -3 - 5\tau - 4\tau^2 - \tau^3 - \tau^4 + \tau^5 - 4\tau^6 - 3\tau^7 + 3\tau^8 + 5\tau^9 + 3\tau^{10} - 2\tau^{11} - 3\tau^{13} + 4\tau^{14}$ and $a(\tau) = (4i\tau - 4i\tau^2 - i\tau^3)^2$.

$b(\tau)$ at the points of $\Theta_n$. Denote by $\mu_{\Theta_n}$ the (random) root-counting measure of the random polynomial $P_{n,\Theta_n}$.

The main result of the second part of this paper is as follows.

**Theorem 1.** Under the above assumptions, the random sequence of measures $\{\mu_{\Theta_n}\}$ converges almost surely to the measure $\mathcal{M}$.

Theorem 1 is illustrated in Fig. 1 below.

**Remark 1.** Almost all conjectures and results of the present paper have straightforward generalizations to the case of recurrence relations with variable coefficients whose length is larger than three. We plan to return to this project in the future.

**Remark 2.** The set-up of the present paper can be extended to the situation when besides the above functions $a(\tau)$ and $b(\tau)$, one additionally chooses a piecewise continuous probability density $\nu(\tau) \geq 0, \tau \in [0, 1]$. Then one can define the $n$-th deterministic subdivision $T_n^\nu$ of $[0, 1]$ with respect to the density function $\nu$ as follows. The nodes $t_0 = 0 < t_1 < t_2 < \cdots < t_n < 1 = t_{n+1}$ of the $n$-th $\nu$-subdivision $T_n^\nu$ are given by the condition:

$$\int_{t_{i-1}}^{t_i} d\nu = \frac{1}{n+1}, \ i = 1, \ldots, n.$$

One can similarly define the random subdivision $\Theta_n^\nu$ by sampling points independently according to $\nu$ and reordering the result. However, this more general case of a non-negative piecewise continuous probability distribution $\nu(\tau)$ can be reduced to the above special case $\nu(\tau) \equiv 1$ by the change of variables $\tilde{\tau} = \int_0^\tau \nu(t)dt$ and, therefore, it does not require special consideration. On the other hand, in case when $\nu$ contains point masses might need additional care.

The structure of the paper is as follows. In § 2 we prove Theorem 1. In § 3 we discuss and present some surprising examples of the deterministic set-up to the latter problem. Finally, in the appendix we give a proof of Theorem B which is important for the deterministic approach, see below.
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2. Proofs

Let \( a, b \in C([0,1]) \) be complex-valued functions. With a partition \( \Delta_n = \{0 \leq t_1 < t_2 < \cdots < t_n \leq 1\} \) of the interval \([0,1]\), we associate the sampling Jacobi matrix

\[
J(\Delta_n) := \begin{pmatrix}
b(t_1) & a(t_1) & & \\
a(t_1) & b(t_2) & a(t_2) & \\
& a(t_2) & b(t_3) & a(t_3) \\
& & & \ddots & \ddots & \ddots \\
& & & & a(t_{n-2}) & b(t_{n-1}) & a(t_{n-1}) \\
& & & & & a(t_{n-1}) & b(t_n)
\end{pmatrix}.
\]

Recall that the norm of the partition \( \Delta_n \) is defined as

\[
\|\Delta_n\| := \max_{0 \leq i \leq n} (t_{i+1} - t_i)
\]

where \( t_0 := 0 \) and \( t_{n+1} := 1 \).

**Definition 1.** For a sequence of subsets \( \{A_n\} \subset \mathbb{C} \), we define two sets of limit points:

\[
\Lambda_\sigma^{(w)} := \{ \lambda \in \mathbb{C} \mid \liminf_{n \to \infty} \text{dist}(\lambda, A_n) = 0 \}
\]

and

\[
\Lambda_\sigma^{(s)} := \{ \lambda \in \mathbb{C} \mid \lim_{n \to \infty} \text{dist}(\lambda, A_n) = 0 \}.
\]

Clearly, \( \Lambda_\sigma^{(s)} \subset \Lambda_\sigma^{(w)} \). The main aim of this section is to show that for any sequence of partitions \( \Delta_n \) of the interval \([0,1]\), such that

\[
\lim_{n \to \infty} \|\Delta_n\| = 0,
\]

the inclusion

\[
\Lambda_\sigma^{(w)}(J(\Delta_n)) \subset \bigcup_{0 \leq t \leq 1} [b(t) - 2a(t), b(t) + 2a(t)]
\]

holds true. Here \([w,z] \) denotes the line segment in \( \mathbb{C} \) connecting the complex numbers \( w \) and \( z \).

2.1. Proof of Theorem 1.

2.1.1. The case of piecewise constant functions \( a \) and \( b \). In this subsection functions \( a, b : [0,1] \to \mathbb{C} \) with

\[
\text{Ran}(a) = \{a_1, \ldots, a_n\} \quad \text{and} \quad \text{Ran}(b) = \{b_1, \ldots, b_n\},
\]

for some \( n \in \mathbb{N} \), are considered. One expects that

\[
\Lambda_\sigma^{(w)}(J(\Delta_n)) = \bigcup_{i=1}^{n} [b_i - 2a_i, b_i + 2a_i]
\]

(THOUGH WE HAVE NO PROOF FOR THIS ASSERTION).

Let us denote by \( J_n(c,d) \) the \( n \times n \) Jacobi matrix with constant diagonal sequence \( c \in \mathbb{C} \) and constant off-diagonal sequence \( d \in \mathbb{C} \). First, recall some well
known properties of Chebyshev polynomials of the second kind. They are defined recursively by the recurrence rule

\[ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n \geq 1 \]

with initial setting \( U_0(x) = 1 \) and \( U_1(x) = 2x \). Consequently,

\[ (2.1) \quad U_n(x) = \det J_n(2x, 1), \quad \forall n \in \mathbb{N}. \]

Further, one has

\[ \lim_{n \to \infty} \frac{U_n(z)}{U_{n+1}(z)} = \frac{1}{z + \sqrt{z^2 - 1}}, \quad \forall z \notin [-1, 1], \]

and the convergence is local uniform in \( \mathbb{C} \setminus [-1, 1] \). Here we always take the branch of the square root such that \( \sqrt{z} > 0 \) for \( z > 0 \). The above limit relation admits a slight generalization in the form

\[ (2.2) \quad \lim_{n \to \infty} \frac{U_{f(n)}(z)}{U_{f(n+1)}(z)} = \frac{1}{(z + \sqrt{z^2 - 1})^{\ell}}, \quad \forall z \notin [-1, 1], \]

providing \( f : \mathbb{N} \to \mathbb{N} \) is strictly increasing function such that

\[ \lim_{n \to \infty} (f(n + 1) - f(n)) = \ell \in \mathbb{N}. \]

The convergence in (2.2) is again local uniform in \( \mathbb{C} \setminus [-1, 1] \).

Let us denote a complex \( n \times n \) Jacobi matrix by

\[ J_n := \begin{pmatrix} d_1 & c_1 & \cdots & \cdots & c_{n-1} & d_{n-1} & c_n & d_n \\ c_1 & d_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ c_{n-2} & d_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & \cdots & \cdots & \cdots & \cdots & d_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \end{pmatrix} \]

Further, for \( 1 \leq i \leq j \leq n \), let \( J[i,j] \) stands for the \( (j - i + 1) \times (j - i + 1) \) submatrix of \( J_n \) whose rows are composed from \( i \)th to \( j \)th row of \( J \) (and the same for columns), i.e.,

\[ (J[i,j])_{k,l} = (J_n)_{i+k-1,i+l-1}, \quad \text{for } k, l \in \{1, \ldots, j-i+1\}. \]

The following formula is needed in the forthcoming part. Let \( n \geq 3 \) and take arbitrary \( i \in \mathbb{N} \) such that \( 1 < i < n \), then it holds

\[ (2.3) \quad \det J_n = \det (J[1, i]) \det (J[i+1, n]) - c_i^2 \det (J[1, i-1]) \det (J[i+2, n]) \]

where we use the convention that \( \det J[k, k-1] := 1 \) for any \( k \in \mathbb{N} \). The verification of this formula is a matter of simple linear algebra and determinant manipulations (details can be provided).

Let \( m \in \mathbb{N} \) and \( n_1, \ldots, n_m \in \mathbb{N} \) be given. Consider a Jacobi matrix of the form

\[ (2.4) \quad A = \bigoplus_{i=1}^{m} J_{n_i}(a_i, b_i) + \sum_{i=1}^{m-1} x_i \left( e_{n_i} e_{n_i+1}^T + e_{n_i+1} e_{n_i}^T \right) \]

where \( a_1, \ldots, a_m, b_1, \ldots, b_m, x_1, \ldots, x_{m-1} \in \mathbb{C} \). In the following, we will derive a formula for the characteristic polynomial of \( A \) in terms of Chebyshev polynomials, which might be a formula of independent interest. However, before we do that, we have to introduce some notation.
Definition 2. For \( m \in \mathbb{N} \), define the set of multiindices \( \mathcal{M}(m) \subset \{0,1,2\}^m \) recursively as

\[
\mathcal{M}(1) := \{(0)\} \quad \text{and} \quad \mathcal{M}(m) := \mathcal{M}_0(m) \cup \mathcal{M}_1(m)
\]

where

\[
\mathcal{M}_0(m) = \{(0, \nu_1, \ldots, \nu_{m-1}) \mid \nu \in \mathcal{M}(m-1)\}
\]

and

\[
\mathcal{M}_1(m) = \{(1, \nu_1 + 1, \ldots, \nu_{m-1}) \mid \nu \in \mathcal{M}(m-1)\}.
\]

Example 1. First five sets of multi-indices are as follows:

\[
\mathcal{M}(1) = \{(0)\}, \quad \mathcal{M}(2) = \{(0,0), (1,1)\},
\]

\[
\mathcal{M}(3) := \{(0,0,0),(0,1,1),(1,1,0),(1,2,1)\},
\]

\[
\mathcal{M}(4) := \{(0,0,0,0),(0,0,1,1),(0,1,1,0),(0,1,2,1),
\quad (1,1,0,0),(1,1,1,1),(1,2,1,0),(1,2,2,1)\},
\]

and

\[
\mathcal{M}(5) := \{(0,0,0,0,0),(0,0,0,1,1),(0,0,1,1,0),(0,0,1,2,1),
\quad (0,1,1,0,0),(0,1,1,1,1),(0,1,2,1,0),(0,1,2,2,1),
\quad (1,1,0,0,0),(1,1,0,1,1),(1,1,1,1,0),(1,1,1,2,1),
\quad (1,2,1,0,0),(1,2,1,1,1),(1,2,2,1,0),(1,2,2,2,1)\}.
\]

Note that \(|\mathcal{M}(m)| = 2^{m-1}\) and the order of \(\nu \in \mathcal{M}(m)\), i.e., the number \(|\nu| = \nu_1 + \cdots + \nu_m\) is always an even number between 0 and \(2m - 2\). In addition, for \(0 \leq k \leq m - 1\), one has

\[
|\{\nu \in \mathcal{M}(m) \mid |\nu| = 2k\}| = \binom{m-1}{k}.
\]

Definition 3. Let \( m \geq 2 \). We define the map \( \text{red} : \mathcal{M}(m) \to \mathcal{M}(m-1) \) as

\[
\text{red}(\nu) := \begin{cases} 
(\nu_2, \ldots, \nu_m), & \text{if } \nu_1 = 0, \\
(\nu_2 - 1, \ldots, \nu_m), & \text{if } \nu_1 = 1.
\end{cases}
\]

Further, \( \alpha : \mathcal{M}(m) \to \{0,1\}^{m-1} \) is given by

\[
\alpha_k(\nu) := \left[\text{red}^{k-1}(\nu)\right]_1, \quad \forall k \in \{1, \ldots, m-1\}
\]

where \(\text{red}^k\) equals the composition \(\text{red} \circ \text{red} \cdots \circ \text{red} \) \( k \)-times and \(\text{red}^0 \) := id.

Recall that, for \( m \geq 2 \), the set \( \mathcal{M}(m) \) decomposes into two disjoint sets \( \mathcal{M}_0(m) \) and \( \mathcal{M}_1(m) \) given in Definition 2, and these are determined by the first element the respective multiindex. Then \(\text{red}(\nu)\) identifies the ancestor of \(\nu\) in the sense that if the recurrence rule from the Definition 2 is applied to \(\text{red}(\nu)\), one arrives at \(\nu\). Vector \((\text{red}^0(\nu), \text{red}^1(\nu), \ldots, \text{red}^{m-2}(\nu))\) then records all the ancestors of \(\nu\) going along the tree structure of \(\mathcal{M}(m), \mathcal{M}(m-1), \ldots, \mathcal{M}(2)\) to its root. The vector \(\alpha(\nu)\) then makes the same job indicating the membership in \(\mathcal{M}_0\) or \(\mathcal{M}_1\) at each level.

Example 2. Take \( \nu = (1,2,1,1,2,1) \in \mathcal{M}(6) \). Then

\[
\text{red}(\nu) = (1,1,1,2,1), \quad \text{red}^2(\nu) = (0,1,2,1), \quad \text{red}^3(\nu) = (1,2,1), \quad \text{red}^4(\nu) = (1,1), \quad \text{and} \quad \alpha(\nu) = (1,1,0,1,1).
Note that $|\alpha(\nu)| = |\nu|/2$. Now, we are ready to formulate the following statement.

**Proposition 4.** Let $m \in \mathbb{N}$, $n_1, \ldots, n_m \geq 2$ and Jacobi matrix $A$ is given as in (2.4). Then it holds

$$\det(A) = \sum_{\nu \in \mathcal{M}(m)} (-1)^{|\nu|} \left( \prod_{i=1}^{m-1} x_i^{2\alpha_i(\nu)} \right) \prod_{j=1}^{m} a_{j}^{\nu_j-\nu_j} U_{n_j-\nu_j} \left( \frac{b_j}{2a_j} \right)$$

(after obvious cancellations, the RHS is well defined even if $a_j = 0$ for some $j \in \{1, \ldots, m\}$).

**Proof.** For the sake of simplicity, we temporarily denote $d_{\nu_j} := \det J_{n_j}(a_j, b_j)$. Taking into account (2.1), identity (2.5) can be reformulated as

$$\det(A) = \sum_{\nu \in \mathcal{M}(m)} (-1)^{|\nu|} \left( \prod_{i=1}^{m-1} x_i^{2\alpha_i(\nu)} \right) \prod_{j=1}^{m} d_{n_j-\nu_j}.$$  

The proof proceeds by mathematical induction in $m \in \mathbb{N}$. The case $m = 1$ yields $\det A = d_{n_1}$ which is clearly true since $A = J_{n_1}(a_1, b_1)$.

Let $m \geq 2$ and assume the formula (2.6) holds true for $m - 1$. Since $\mathcal{M}(m)$ decomposes into two disjoint sets $\mathcal{M}_0(m)$ and $\mathcal{M}_1(m)$ by Definition 2, we may split the sum on the RHS of (2.6) correspondingly and we get

$$\sum_{\nu \in \mathcal{M}(m)} (-1)^{|\nu|} \left( \prod_{i=1}^{m-1} x_i^{2\alpha_i(\nu)} \right) \prod_{j=1}^{m} d_{n_j-\nu_j} = \sum_{(\nu_2, \ldots, \nu_m) \in \mathcal{M}(m-1)} (-1)^{\nu_2 + \ldots + \nu_m} \left( \prod_{i=2}^{m-1} x_i^{2\alpha_i(\nu)} \right) d_{n_1} \prod_{j=2}^{m} d_{n_j-\nu_j}$$

$$- \sum_{(\nu_2, \ldots, \nu_m) \in \mathcal{M}(m-1)} (-1)^{\nu_2 + \ldots + \nu_m} x_1^2 \left( \prod_{i=2}^{m-1} x_i^{2\alpha_i(\nu)} \right) d_{n_1-1} \prod_{j=2}^{m} d_{n_j-\nu_j}$$

$$= d_{n_1} \sum_{\tilde{\nu} \in \mathcal{M}(m-1)} (-1)^{|\tilde{\nu}|} \left( \prod_{i=1}^{m-2} x_i^{2\alpha_i(\tilde{\nu})} \right) \prod_{j=1}^{m-1} d_{n_{j+1}-\tilde{\nu}_j}$$

$$- x_1^2 d_{n_1-1} \sum_{\tilde{\nu} \in \mathcal{M}(m-1)} (-1)^{|\tilde{\nu}|} \left( \prod_{i=1}^{m-2} x_i^{2\alpha_i(\tilde{\nu})} \right) d_{n_{2}-1-\tilde{\nu}_1} \prod_{j=2}^{m-1} d_{n_{j+1}-\tilde{\nu}_j}.$$ 

By applying the induction hypothesis we obtain

$$\sum_{\nu \in \mathcal{M}(m)} (-1)^{|\nu|} \left( \prod_{i=1}^{m-1} x_i^{2\alpha_i(\nu)} \right) \prod_{j=1}^{m} d_{n_j-\nu_j} = d_{n_1} \det A[n_1 + 1, |n|] - x_1^2 d_{n_1-1} \det A[n_1 + 2, |n|]$$

where $|n| = n_1 + \cdots + n_m$. However, the RHS of the last equation coincides with $\det A$ as it follows from (2.3) which concludes the induction step. \[\square\]
Lemma 5. Let \( m \in \mathbb{N} \) and the formal polynomial \( F(f_1, \ldots, f_m) \) in indeterminates \( x_1, \ldots, x_{m-1}, f_1, \ldots, f_m \) be defined as
\[
F(f_1, \ldots, f_m) = \sum_{\nu \in \mathcal{M}(m)} (-1)^{|\nu|} \prod_{i=1}^{m-1} x_i^{2\alpha_i(\nu)} \prod_{j=1}^{m} f_j^{\nu_j}.
\]
Then it holds
\[
F(f_1, \ldots, f_m) = \prod_{j=1}^{m-1} (1 - x_j^2 f_j f_{j+1}).
\]

Proof. By using the decomposition of \( \mathcal{M}(m) \) into \( \mathcal{M}_0(m) \cup \mathcal{M}_1(m) \) and following the similar steps as in the proof of Proposition 4, one verifies the recurrence
\[
F(f_1, \ldots, f_m) = (1 - x_j^2 f_j f_2) F(f_1, \ldots, f_{m-1}), \quad \text{for } m \geq 2.
\]

By iterating this rule and noticing that \( F(f_1) = 1 \), one obtains the formula from the statement. \( \square \)

Proposition 6. Let \( m \in \mathbb{N} \) and for all \( j \in \{1, \ldots, m\} \), \( n_j : \mathbb{N} \to \mathbb{N} \) be such that \( n_j(n) \to \infty \), as \( n \to \infty \). Denote by \( A_n \) the matrix of the form (2.4) with \( n_j \) being replaced by \( n_j(n) \) for all \( j \in \{1, \ldots, m\} \). Then for all \( z \notin \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j] \), one has
\[
\lim_{n \to \infty} \frac{\det(A_n - z)}{\prod_{j=1}^{m} a_j^{n_j(n)} U_{n_j(n)} \left( \frac{b_j - z}{2a_j} \right)} = \prod_{j=1}^{m-1} \left(1 - x_j^2 f_j(z) f_{j+1}(z) \right)
\]
where
\[
f_j(z) = \frac{2}{b_j - z + \sqrt{(b_j - z)^2 - 4a_j^2}}.
\]

In addition, the convergence in (2.7) is local uniform in \( z \in \mathbb{C} \setminus \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j] \).

Proof. Let us temporarily denote
\[
\rho_j[k, l](z) := \frac{a_j^k U_k \left( \frac{z-b_j}{2a_j} \right)}{a_j^l U_l \left( \frac{z-b_j}{2a_j} \right)}, \quad k, l \in \mathbb{N}.
\]

With the aid of (2.2) one verifies that for all \( p \in \mathbb{N} \),
\[
\lim_{n \to \infty} \rho_j[n_j(n) - p, n_j(n)](z) = f_j^p(z)
\]
locally uniformly in \( z \in \mathbb{C} \setminus \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j] \). By using the identity (2.5), we arrive at the equality
\[
\det(A_n - z) = \prod_{j=1}^{m} a_j^{n_j(n)} U_{n_j(n)} \left( \frac{b_j - z}{2a_j} \right) \times \sum_{\nu \in \mathcal{M}(m)} (-1)^{|\nu|} \prod_{i=1}^{m-1} x_i^{2\alpha_i(\nu)} \prod_{j=1}^{m} \rho_j[n_j(n) - \nu_j, n_j(n)](z)
\]

Since
\[
U_k \left( \frac{b_j - z}{2a_j} \right) \neq 0, \quad \forall z \notin \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j], \quad \forall k \in \mathbb{N}_0,
\]
we may divide by the product and, taking into account (2.8), send $n \to \infty$ in (2.9) getting
\[
\lim_{n \to \infty} \frac{\det(A_n - z)}{\prod_{j=1}^{m} a_{n(j)}^j U_{n(j)}(b_j - z/2a_j)} = \left( \prod_{j=1}^{m-1} \left( \frac{m-1}{j} \right)^{\frac{2}{\nu_j}} \prod_{j=1}^{m} f_j'(z) \right)
\]
locally uniformly in $z \in \mathbb{C} \setminus \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j]$. Now, it suffices to apply Lemma 5 to conclude the proof. \hfill \Box

For the purpose of the following corollary we denote
\[
X_m := \bigcup_{j=1}^{m-1} \{ z \in \mathbb{C} \mid x_j^2 f_j(z) f_{j+1}(z) = 1 \}.
\]
Note that $|X_m| < \infty$, since the condition $x_j^2 f_j(z) f_{j+1}(z) = 1$ can be turned out into a polynomial equation in $z$.

**Corollary 1.**
\[
A_{(w)}^{(u)} \subset X_m \cup \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j].
\]

**Proof.** If
\[
z \notin X_m \cup \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j],
\]
then we have the relation (2.7) with the RHS equal to a nonzero number. Consequently, $\det(A_n - z) \neq 0$, for all $n$ sufficiently large and therefore $z \notin A_{(w)}^{(u)}. \hfill \Box

**Proposition 7.** Let $m \in \mathbb{N}$ and for all $j \in \{1, \ldots, m\}$, $n_j : \mathbb{N} \to \mathbb{N}$ be such that
\[
\lim_{n \to \infty} n_j(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} n_j(n+1) - n_j(n) = \ell_j \in \mathbb{N}.
\]
Denote by $A_n$ the matrix of the form (2.4) with $n_j$ being replaced by $n_j(n)$ for all $j \in \{1, \ldots, m\}$. Then one has
\[
\lim_{n \to \infty} \frac{\det(A_n - z)}{\det(A_{n+1} - z)} = \prod_{j=1}^{m} \left( \frac{2}{z - b_j + \sqrt{(z - b_j)^2 - 4a_j^2}} \right)^{\ell_j},
\]
(2.10)
for all
\[
z \in \mathbb{C} \setminus X_m \bigcup \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j]
\]
and the convergence is local uniform on this set.

**Proof.** Using (2.2) one verifies that
\[
\lim_{n \to \infty} a_{f(n)} U_{f(n)}(\frac{z - b}{2a}) = \left( \frac{2}{z - b + \sqrt{(z - b)^2 - 4a^2}} \right)^{\ell}, \quad \forall z \notin [b - 2a, b + 2a],
\]
for any $a, b \in \mathbb{C}$ and $f : \mathbb{N} \to \mathbb{N}$ such that
\[
\lim_{n \to \infty} f(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} f(n+1) - f(n) = \ell \in \mathbb{N}.
\]
The convergence is still local uniform in $z \in \mathbb{C} \setminus [b - 2a, b + 2a]$. The rest then immediately follows from (2.7). \hfill \Box
Recall the arcsine measure $\omega_{a,b}$ is an absolutely continuous measure supported on $[b - 2a, b + 2a]$ with density
\[
\frac{d\omega_{a,b}}{dx}(x) = \frac{1}{\pi \sqrt{(b + 2a - x)(x - b + 2a)}}.
\]
If $a = 0$, then $\omega_{a,b}$ is to be identified with the Dirac point mass at $b$, $\omega_{0,b} = \delta_b$.

Recall also that the logarithmic potential $U^\mu$ of a compactly supported measure $\mu$ is defined as
\[
U^\mu(z) = \int_{\mathbb{C}} \log |z - y|d\mu(y).
\]
In case of the arcsine measure, one has
\[
U^{\omega_{a,b}}(z) = \log 1 \quad \forall z \not\in [b - 2a, b + 2a].
\]

**Proposition 8.** Let $m \in \mathbb{N}$ and for all $j \in \{1, \ldots, m\}$, $n_j : \mathbb{N} \to \mathbb{N}$ be such that
\[
\lim_{n \to \infty} n_j(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} n_j(n + 1) - n_j(n) = \ell_j \in \mathbb{N}.
\]
Denote by $A_n$ the matrix of the form (2.4) with $n_j$ being replaced by $n_j(n)$ for all $j \in \{1, \ldots, m\}$. Then
\[
\lim_{n \to \infty} \mu_n^{(m)}(z) = \sum_{j=1}^m \ell_j \omega_{a_j,b_j}
\]
where $\mu_n^{(m)}$ stands for the eigenvalue-counting measure of $A_n$, i.e.,
\[
\mu_n^{(m)} = \sum_{\lambda \in \sigma(A_n)} \frac{1}{\nu_a(\lambda)} \delta_{\lambda}
\]
and $\nu_a(\lambda)$ is the algebraic multiplicity of the eigenvalue $\lambda$.

**Proof.** The proof relies on the result from the potential theory which implies the validity of (2.12) if the corresponding logarithmic potentials converges almost everywhere in $\mathbb{C}$, i.e.,
\[
\lim_{n \to \infty} U^{\mu_n^{(m)}}(z) = U^{\mu^{(m)}}(z) \quad \forall z \in \mathbb{C},
\]
where we denote
\[
\mu^{(m)} = \sum_{j=1}^m \ell_j \omega_{a_j,b_j}.
\]
By using (2.10), one immediately deduces
\[
\lim_{n \to \infty} U^{\mu_n^{(m)}}(z) = \lim_{n \to \infty} \frac{\log |\det(A_n - z)|}{N(n)} = \sum_{j=1}^m \ell_j \log \frac{1}{2} \quad \forall z \in \mathbb{C} \setminus X_m \cup \bigcup_{j=1}^m [b_j - 2a_j, b_j + 2a_j].
\]
where $N(n) := n_1(n) + \cdots + n_m(n)$. In addition, taking into account (2.11), one observes the RHS of the last equation is equal to $U^{\mu^{(m)}}(z)$, for all
\[
z \in \mathbb{C} \setminus X_m \cup \bigcup_{j=1}^m [b_j - 2a_j, b_j + 2a_j].
\]
The excluded set is clearly of the Lebesgue measure zero and thus (2.13) is verified. \qed
3. Mystery of deterministic approach

Take two piecewise continuous complex-valued functions $a(\tau), b(\tau)$, $\tau \in [0, 1]$. Analogously to (1.1), consider the recurrence

\begin{equation}
 P_{k,n}(z) = (z - b(k/n))P_{k-1,n}(z) - a(k/n)P_{k-2,n}(z), \quad 1 \leq i \leq n,
\end{equation}

with the standard initial conditions: $P_{-1,n} = 0$ and $P_{0,n} = 1$.

Define the measure $\mathcal{M}$ as given by:

\begin{equation}
 \mathcal{M} := \mathcal{M}^{a,b} = \int_0^1 \kappa(\tau) d\tau,
\end{equation}

where $\kappa(\tau)$ is the arcsine measure supported on the straight segment

$I(\tau) := [b(\tau) - 2\sqrt{a(\tau)}, b(\tau) + 2\sqrt{a(\tau)}]$ in the complex plane. The support of $\mathcal{M}$ coincides with the union $\Omega := \bigcup_{\tau \in [0,1]} I(\tau)$, see Fig. 2.

Problem 1. Denoting by $\mu_n$ the root-counting measure of $P_{n,n}$, does there exist a weak limit $\mu := \lim_{n \to \infty} \mu_n$? If $\mu$ exists, how is it related to the above measure $\mathcal{M}$?

Remark 3. Observe that Theorem A of Kuijlaars – Van Assche claims the coincidence of $\mu := \lim_{n \to \infty} \mu_n$ with $\mathcal{M}$ under the assumptions of this theorem. However, for almost any pair of complex-valued $a(\tau)$ and $b(\tau)$,

$\mu \neq \mathcal{M}$.

To the best of our knowledge, the first example of this phenomenon has observed and rigorously motivated in [28] and [27]. Nevertheless, although $\mu$ (if it exists) does not coincide with $\mathcal{M}$, they enjoy the following important property.

Theorem B. Any weakly converging subsequence of measures $\{\mu_n\}$ converges to a measure whose potential coincides with that of $\mathcal{M}$ in a neighbourhood of infinity.

Remark 4. Theorem B was formulated to the first author by A. Kuijlaars around 2010 as a result which follows from the technique developed in [20]. It has been used in e.g. [28] with a reference to [20]. Since we strongly believe that it has to be better known, we decided to include a proof of its generalization to recurrence relations of arbitrary length in the Appendix of the present paper, see § 4.

To formulate our main conjecture, we need some notation. Consider a finite positive Borel measure $\Xi$ supported on a bounded domain $\Omega \subset \mathbb{C}$ with a piecewise smooth boundary and whose density is a continuous function in $\Omega$.

Definition 1. By a mother body measure $\mu_\Xi$ of $\Xi$, we mean a positive measure such that

(i) its support $S := \operatorname{supp} \mu_\Xi$ belongs to $\Omega$ and consists of finitely many compact real-analytic curves and finite many points;
(ii) the logarithmic potential of $\mu_\Xi$ coincides with that of $\Xi$ in the complement $\mathbb{C} \setminus \Omega$;
(iii) the set $\mathbb{C} \setminus S$ is path-connected.
Figure 2. Illustration of the deterministic set-up. Measures $\mathcal{M}$ and $\mu$ in the cases:

(i) $b(\tau) = -\tau^3 - 3i\tau^2$ and $a(\tau) = -\frac{(\tau-1)(\tau-i)(\tau-(3+2i))(\tau-3i)}{10}$ (left),

(ii) $b(\tau) = -\tau^3 - 3i\tau^2$ and $a(\tau) = -\frac{(\tau-1)(\tau-i)(\tau-3)}{5}$ (central),

(iii) $b(\tau) = -\tau + (3 + i)$ and $a(\tau) = -\frac{(\tau+4-i)(\tau-3+2i)}{5}$ (right).

Remark 5. The notion of a mother body of a solid domain or, more generally, of a positive Borel measure was discussed during the last decades both in geophysics and mathematics, see e.g. [31], [24], [16], [38]. It was apparently pioneered in the 1960’s by a Bulgarian geophysicist D. Zidarov [38] and later mathematically developed by B. Gustafsson [16]. Although a number of interesting results about mother bodies was obtained in several special cases, [24], [16], [38] there is still no consensus about its appropriate general definition. In particular, no general existence and/or uniqueness results are known at present. The above definition is only one of possible natural versions.

The initial observation motivating the deterministic part of the present paper was that in a specific case considered in [28] and [27] the support of the asymptotic root-counting measure of the diagonal family $\{P_{n,n}(z)\}$ consisted of a finite number of compact real-analytic curves in $\mathbb{C}$. After this initial observation the first author made extensive computer experiments with a large number of different limiting functions $a(\tau)$ and $b(\tau)$ and always obtaining one-dimensional support of the asymptotic root-counting measure of $\{P_{n,n}\}$.

Conjecture 1. For any pair of piecewise continuous complex-valued analytic functions $a(\tau)$ and $b(\tau)$, $\tau \in [0,1]$, the asymptotic root-counting measure $\mu := \mu^{a,b}$ of the diagonal sequence $\{P_{n,n}(z)\}$ exists and is a mother body measure for $\mathcal{M} := \mathcal{M}^{a,b}$.

Pictures in Fig. 2 and 3 show the distinction between the supports of $\mathcal{M}$ ad $\mu$ for several concrete cases of $a(\tau)$ and $b(\tau)$.

4. Appendix. Generalization of the equipotentiality result of Kuilaars-Van Assche

We adopt the notation used in [20]:

$$\lim_{n/N \to t} X_{n,N} = X$$
which denotes that for the doubly indexed sequence it holds that

$$\lim_{j \to \infty} X_{n_j, N_j} = X,$$

for any \(\{n_j\}_{j \in \mathbb{N}}, \{N_j\}_{j \in \mathbb{N}} \subseteq \mathbb{N}\) such that \(N_j \to \infty\) and \(n_j/N_j \to t\), as \(j \to \infty\).

Additionally, we occasionally add the meaning of the limit sense. For example,

$$\text{w-lim}_{n/N \to t} \mu_{n, N} = \mu$$
expresses the limit of the double indexed sequence of measures \( \mu_{n,N} \) converging to \( \mu \) in the weak* topology. Similarly,

\[
\text{s-lim}_{n/N \to t} A_{n,N} = A
\]

stands for the limit of the double indexed sequence of bounded operators \( A_{n,N} \) converging to \( A \) strongly.

In what follows we will need some well-known facts from the theory of linear operators.

**Claim 1.**

(i) For any closed operator \( T \) acting on a Banach space,

\[
\| (T - z)^{-1} \| \geq \frac{1}{\text{dist}(z, \sigma(T))}, \quad \forall z \in \rho(T).
\]

(ii) On the other hand, for a bounded operator \( B \), one has

\[
\| (B - z)^{-1} \| \leq \frac{1}{\text{dist}(z, D_\|B\|)}, \quad \forall z, \ |z| > \|B\|,
\]

where \( D_\|B\| = \{ z \in \mathbb{C} \mid |z| \leq \|B\| \} \). (Notice that \( \sigma(B) \subset D_\|B\| \).)

Now take two sequences \( \{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 0} \) of complex numbers such that \( a_n \neq 0, \forall n \geq 1 \). Let \( J_n \) denote the \((n + 1) \times (n + 1)\) Jacobi matrix of the form

\[
J_n := \begin{pmatrix}
  b_0 & a_1 & & & \\
  a_1 & b_1 & a_2 & & \\
  & a_2 & b_2 & a_3 & \\
  & & \ddots & \ddots & \ddots \\
  & & & a_{n-1} & b_{n-1} & a_n \\
  & & & & a_n & b_n
\end{pmatrix}.
\]

Recall that the polynomial sequence \( \{p_n\}_{n \geq 0} \) determined by three-term recurrence

\[
a_n p_{n-1}(x) + b_n p_n(x) + a_{n+1} p_{n+1}(x) = x p_n(x), \quad n \geq 0,
\]

with the initial conditions \( p_{-1} = 0 \) and \( p_0 = 1 \) is related to \( J_n \) as follows

\[
p_n(x) = \left( \prod_{k=1}^{n} \frac{1}{a_k} \right) \det(x - J_{n-1}), \quad n \geq 1.
\]

**Lemma 9.** [20, Lem. 2.2] Let \( \|J_n\| \leq M \), then

\[
\left| \frac{p_n(z)}{a_{n+1} p_{n+1}(z)} \right| \leq \frac{1}{\text{dist}(z, D_\|B\|)} = \frac{1}{|z| - M}, \quad \forall z \in \mathbb{C}, |z| > M.
\]

On the other hand, one has

\[
\left| \frac{p_n(z)}{a_{n+1} p_{n+1}(z)} \right| \geq \frac{1}{2|z|} \quad \forall z \in \mathbb{C}, |z| > 3M.
\]

**Proof.** Simple linear algebra gives

\[
\frac{p_n(z)}{a_{n+1} p_{n+1}(z)} = \frac{\det(z - J_{n-1})}{\det(z - J_n)} = \langle e_n, (z - J_n)^{-1} e_n \rangle,
\]
where $e_n$ stands for the $(n+1)$-th vector of the standard basis of $\mathbb{C}^{n+1}$. Consequently,
\[
\left| \frac{p_n(z)}{a_{n+1}p_{n+1}(z)} \right| = |\langle e_n, (z-J_n)^{-1}e_n \rangle| \leq \|z-J_n\|^{-1} \leq \frac{1}{\text{dist}(z, D_{\|J_n\|})} \leq \frac{1}{\text{dist}(z, D_M)},
\]
whenever $|z| > M$.

One the other hand,
\[
|\langle e_n, (z-J_n)^{-1}e_n \rangle| = \frac{1}{|z|} \left| 1 + \langle e_n, \sum_{k=1}^{\infty} z^{-k} J_n^k e_n \rangle \right| \geq \frac{1}{|z|} \left( 1 - \frac{\|J_n\|}{|z| - \|J_n\|} \right) \geq \frac{1}{2|z|},
\]
whenever $|z| \geq 3M$.

Observe that if $R_n \in \mathbb{C}^{n+1,n+1}$ is the permutation matrix determined by equations $R_n e_k = e_{n-k}$, $\forall k \in \{0, 1, \ldots, n\}$, then
\[
R_n J_n R_n = \begin{pmatrix}
  b_0 & a_{n,N} & a_{n-1} & a_{n-2} & \cdots & a_1 \\
  a_{n,N} & b_{n-1} & a_{n-1} & a_{n-2} & \cdots & a_0 \\
  a_{n-1} & b_{n-2} & b_{n-1} & a_{n-2} & \cdots & b_1 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_2 & b_1 & a_1 & b_0 & \cdots & \ddots \\
  a_1 & b_0 & \cdots & \cdots & \cdots & \ddots \\
  b_0 & a_{n,N} & a_{n-1} & a_{n-2} & \cdots & a_1 \\
\end{pmatrix},
\]
det$(J_n - z) = \det(R_n J_n R_n - z)$, and
\[
\langle e_n, (J_n - z)^{-1} e_n \rangle = \langle e_0, (R_n J_n R_n - z)^{-1} e_0 \rangle, \quad \forall z \in \rho(J_n).
\]
If convenient, we identify matrix $R_n$ with the operator $R_n \oplus 0$ acting on $\ell^2(N_0)$.

In what follows, $\{a_{n,N} \mid n, N \in \mathbb{N}\} \subset \mathbb{C}$, $\{b_{n,N} \mid n \in \mathbb{N}_0, N \in \mathbb{N}\} \subset \mathbb{C}$, and
\[
J(N) = \begin{pmatrix}
  b_{0,N} & a_{1,N} & a_{2,N} & a_{3,N} & \cdots \\
  a_{1,N} & b_{1,N} & a_{2,N} & a_{3,N} & \cdots \\
  a_{2,N} & b_{2,N} & a_{3,N} & \cdots & \ddots \\
  \vdots & \vdots & \vdots & \ddots & \cdots \\
  a_{n,N} & b_{n,N} & a_{n+1,N} & a_{n+2,N} & \cdots \\
\end{pmatrix}
\]
stands for the semi-infinite (complex) Jacobi matrix. Finally, $P_n \in \mathcal{B}(\ell^2(N_0))$ denotes the orthogonal projection on span$\{e_0, \ldots, e_n\}$.

**Proposition 10.** Assume that
\[
(4.1) \quad \lim_{n/N \to t} a_{n,N} = A \quad \text{and} \quad \lim_{n/N \to t} b_{n,N} = B.
\]
and assume furthermore that there exists $\epsilon > 0$ such that
\[
(4.2) \quad \sup \{ \| P_n J(N) P_n \| \mid |n/N - t| < \epsilon \} < \infty.
\]
Then
\[
s-lim_{n/N \to t} R_n J(N) R_n = J(A, B)
\]
where $J(A, B) \in \mathcal{B}(\ell^2(N_0))$ stands for the Jacobi matrix with constant diagonal $B$ and constant off-diagonal $A$. 
**Proof.** Recall an easily verifiable statement: Let $X$ be a Banach space, $B \in B(X)$ and $B_n$ a uniformly bounded sequence of operators acting on $X$. If $B_n \varphi \to B \varphi$, as $n \to \infty$, for all $\varphi$ from a dense subset of $X$, then $\text{s-lim}_{n \to \infty} B_n = B$.

Take arbitrary $\{n_j\}, \{N_j\} \subset \mathbb{N}, N_j \to \infty$ and $n_j/N_j \to t$, for $j \to \infty$. Let us denote temporarily $\hat{J}_j = R_{n_j} J(N_j) R_{n_j}$. Assumption (4.2) guarantees the existence of $j_0 \in \mathbb{N}$ such that

$$\sup_{j \geq j_0} \|P_{n_j} J(N_j) P_{n_j}\| = \sup_{j \geq j_0} \|\hat{J}_j\| < \infty.$$ 

Hence, taking into account the above statement, it suffices to verify that

$$\lim_{j \to \infty} \hat{J}_j e_n = J(A, B) e_n, \quad \forall n \in \mathbb{N}_0.$$

Take $n \in \mathbb{N}_0$ and $j > n$, then

$$\|\hat{J}_j e_n - J(A, B) e_n\|^2 = |a_{n_j - n + 1, N_j} - A|^2 + |b_{n_j - n, N_j} - B|^2 + |a_{n_j - n, N_j} - A|^2 \to 0,$$

for $j \to \infty$, by assumption (4.1). If $n = 0$, set $a_{n_j + 1, N_j} = 0$ in the above equation. All in all, the claim is verified. \hspace{1cm} \Box

It is again quite easy to see that for any $B_n, B \in B(X)$ such that $\sup_n \|B_n\| \leq M$ and $\text{s-lim}_{n \to \infty} B_n = B$, one has

$$\text{s-lim}_{n \to \infty} (B_n - z)^{-1} = (B - z)^{-1}, \quad \forall z \notin D_M$$

and the convergence is local uniform in $z$. Indeed, since

$$(B - z)^{-1} - (B_n - z)^{-1} = (B_n - z)^{-1}(B_n - B)(B - z)^{-1}$$

one obtains

$$\|(B - z)^{-1} \varphi - (B_n - z)^{-1} \varphi\| \leq \frac{1}{|z| - M} \|(B_n - B)(B - z)^{-1} \varphi\|,$$

from which the strong convergence of resolvents follows. The local uniformness follows, for example, from the Mantel’s theorem.

The next statement is in fact a corollary of Proposition 10, however, it is also a complex generalization of [20, Thm. 2.1]. Therefore we formulate it as a proposition.

**Proposition 11.** [20, Thm. 2.1] Let the assumptions (4.1) and (4.2) hold. Denote the value of the supremum in (4.2) by $M$. Then

$$\lim_{n/N \to t} \langle e_n, (z - P_n J(N) P_n)^{-1} e_n \rangle = \frac{2}{z - B + \sqrt{(z - B)^2 - 4A^2}}$$

locally uniformly in $\mathbb{C} \setminus D_M$.

**Remark 12.** Note that by (4.1),

$$M = \sup \{\|P_n J(N) P_n\| \mid |n/N - t| < \epsilon \} \geq |B| + 2|A|.$$

**Proof.** It follows from Proposition 10 that

$$\text{s-lim}_{n/N \to t} (z - R_n J(N) R_n)^{-1} = (z - J(A, B))^{-1}$$

locally uniformly in $z \notin D_M$. Hence,

$$\lim_{n/N \to t} \langle e_n, (z - R_n J(N) R_n)^{-1} e_n \rangle = \lim_{n/N \to t} \langle e_0, (z - R_n J(N) R_n)^{-1} e_0 \rangle = \langle e_0, (z - J(A, B))^{-1} e_0 \rangle$$
locally uniformly in $z \notin D_M$. It is a standard result that
\[
\langle \epsilon_0, (z - J(A,B))^{-1} \rangle = \frac{2}{z - B + \sqrt{(z - B)^2 - 4A^2}},
\]
for all $z \notin [B - 2A, B + 2A]$ (a line segment in $\mathbb{C}$). To verify that one can show that
\[
\lim_{n \to \infty} s \lim P_n J(A,B) P_n = J(A,B)
\]
together with the formula
\[
\langle \epsilon_0, (z - J_n(A,B))^{-1} \rangle = \frac{U_n(z - B)}{A U_{n+1}(z - B)}, \quad \forall z \notin [B - 2A, B + 2A]
\]
where $J_n(A,B)$ stands for the $(n+1) \times (n+1)$ truncation of $J(A,B)$, i.e., $J(A,B) = J_n(A,B) \oplus 0$, and
\[
U_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}
\]
are Chebyshev polynomials of the second kind. \qed

Recall that the main result of paper [20] in Theorem 1.4. which claims the following.

**Theorem 13.** [20, Thm. 1.4] If \( \{a_{n,N} \mid n, N \in \mathbb{N}\} \subset \mathbb{R}_+, \{b_{n,N} \mid n \in \mathbb{N}, N \in \mathbb{N}\} \subset \mathbb{R} \) and \( \{p_{n,N} \mid n \in \mathbb{N}, N \in \mathbb{N}\} \) associated family of orthonormal polynomials. Further let non-negative \( a \in C(\mathbb{R}_+) \) and real \( b \in C(\mathbb{R}_+) \) be given, such that
\[
\lim_{n/N \to t} a_{n,N} = a(t) \quad \text{and} \quad \lim_{n/N \to t} b_{n,N} = b(t),
\]
for all \( t > 0 \). Then for the family of \( \{\nu_{n,N} \mid n \in \mathbb{N}, N \in \mathbb{N}\} \) of root-counting measures of polynomials \( \{p_{n,N} \mid n \in \mathbb{N}, N \in \mathbb{N}\} \), one has
\[
\text{w-lim}_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \omega_{[b(s) - 2a(s), b(s) + 2a(s)]} ds
\]
where \( \omega_{[a,b]} \) is absolutely continuous measure supported on \( [a, b] \) with density
\[
\frac{d\omega_{[a,b]}}{dt} = \frac{1}{\pi \sqrt{(\beta - t)(t - \alpha)}}.
\]
if \( \alpha < \beta \). If \( \alpha = \beta \), \( \omega_{[a,b]} = \delta_{[\alpha]} \).

For \( t > 0 \), let us denote
\[
\sigma(t) = \frac{1}{t} \int_0^t \omega_{[b(s) - 2a(s), b(s) + 2a(s)]} ds.
\]
In the proof of Theorem 13, authors prove that
\[
\lim_{n/N \to t} U^{\nu_{n,N}}(z) = U^\sigma(t)(z)
\]
locally uniformly in certain neighborhood of complex \( \infty \) (i.e., for \( |z| > M \)), where \( U^\mu \) denotes the logarithmic potential of the (compactly supported) Borel measure \( \mu \). Under the assumptions of Theorem 13, supports of all measures \( \nu_{n,N} \), for all \( n, N \) such that \( n/N \) is close to \( t \), are included in a real interval \([-M,M]\). This implies that the limit relation (4.3) holds true for all \( z \notin [-M,M] \) and hence for almost all \( z \in \mathbb{C} \) (w.r.t. the Lebesgue measure). Under these conditions one can
show (following standard methods of Potential Theory - Widom’s lemma) the weak convergence

\[
\text{w-lim}_{n/N \to t} \nu_{n,N} = \sigma(t).
\]

However, in the general case of complex sequences \(a_{n,N}\) and \(b_{n,N}\), one can get only the relation (4.3) outside a ball, \(|z| > M\), and not for almost all \(z \in \mathbb{C}\). This however does not imply the weak convergence (4.4). To our best knowledge nor the existence of the weak limit is guaranteed (only a subsequence, by Helly’s theorem). Thus, one can not expect the validity of Theorem 13 in the complex setting. However, one can get at least the following.

**Proposition 14.** Let \(a \in C([0, \infty))\) and \(b \in C([0, \infty))\) be complex-valued functions and

\[
\lim_{n/N \to t} a_{n,N} = a(t) \quad \text{and} \quad \lim_{n/N \to t} b_{n,N} = b(t),
\]

for all \(t > 0\). Then if the weak limit of root-counting measures \(\nu = \text{w-lim}_{n/N \to t} \nu_{n,N}\) exists, then \(\nu\) and \(\sigma(t)\) are equipotential measures, i.e., their logarithmic potentials coincide outside the union of their supports.

**Remark 15.** Note the functions \(a\) and \(b\) are assumed to be continuous in 0 (from the right). This additional condition simplifies the proof considerably and we do not aim here to achieve a full generality.

**Proof.** Note that coefficients \(a_{n,N}\) and \(b_{n,N}\) are uniformly bounded if \(n/N\) is restricted to a compact subsets of \([0, \infty)\), as it follows from the assumptions. Take \(t > 0\) and \(0 < \epsilon < t\), then

\[
\sup\{|b_{n,N}| \mid |n/N - t| \leq \epsilon\} + 2 \sup\{|a_{n,N}| \mid |n/N - t| \leq \epsilon\} < \infty.
\]

Denote by \(J_n(N)\) the \((n+1) \times (n+1)\) truncation of \(J(N)\). Consequently, condition (4.2) is fulfilled and let us denote the uniform bound of operators \(J_n(N)\), for \(|n/N - t| \leq \epsilon\), by \(M\).

The following part proceeds analogously as the proof of [20, Thm. 1.4]. Since

\[
\frac{\det (z - J_{n-1}(N))}{\det (z - J_n(N))} = \langle e_n, (z - J_n(N))^{-1} e_n \rangle,
\]

one has

\[
\det (z - J_n(N)) = \prod_{k=0}^{n} \frac{1}{\langle e_k, (z - J_k(N))^{-1} e_k \rangle}, \quad |z| > M.
\]

Thus,

\[
U^{n,N}(z) = \frac{1}{n+1} \log |\det (z - J_n(N))| = -\frac{1}{n+1} \sum_{k=0}^{n} \log |\langle e_k, (z - J_k(N))^{-1} e_k \rangle|,
\]

or equivalently

\[
U^{n,N}(z) = -\int_{0}^{1} \log |\langle e_{[ns]}, (z - J_{[ns]}(N))^{-1} e_{[ns]} \rangle| \, ds, \quad |z| > M.
\]
As $n/N \to t$, one has $|sn|/N \to st$. Hence, by Proposition 11,
\[
\lim_{n/N \to t} \langle e_{[n]} \mid (z - J_{[n]}(N))^{-1} e_{[n]} \rangle = \frac{2}{z - b(st) + \sqrt{(z - b(st))^2 - 4a(st)^2}}.
\]
Further, by Lemma 9, one has
\[
\frac{1}{2|z|} \leq |\langle e_{[n]} \mid (z - J_{[n]}(N))^{-1} e_{[n]} \rangle| \leq \frac{1}{|z| - M},
\]
for $|z| > 3M$. Consequently, the Lebesgue’s dominated convergence theorem applies and we get
\[
\lim_{n/N \to t} U_{n,N}^\nu(z) = \int_0^1 \log \left| \frac{z - b(st)}{2} + \sqrt{\left(\frac{z - b(st)}{2}\right)^2 - a(st)^2} \right| ds
\]
\[
= \frac{1}{t} \int_0^t \log \left| \frac{z - b(s)}{2} + \sqrt{\left(\frac{z - b(s)}{2}\right)^2 - a(s)^2} \right| ds
\]
for $|z| > 3M$. The function in the last integral is known to coincide with the logarithmic potential of $\omega|b(s) - 2a(s), b(s) + 2a(s)|$ at $z$. All in all, we obtained
\[
U^\nu(z) = \frac{1}{t} \int_0^t U_{w|b(s) - 2a(s), b(s) + 2a(s)}^\nu(z) ds = U^{\sigma(t)}(z),
\]
for $|z| > 3M$. By the harmonicity of logarithmic potentials $U^\mu$ outside the support $\mu$ and Identity principle for harmonic functions the last equality can be extended to all $z \notin (\text{supp} \nu \cup \text{supp} \sigma(t))$. \qed

References


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