ON RING GENERATED BY CHERN 2-FORMS ON $\mathbb{SL}_n/B$

Boris Shapiro[‡], Mikhail Shapiro[*]

[‡] Department of Mathematics, University of Stockholm
S-10691, Sweden, shapiro@matematik.su.se
[*] Department of Mathematics, Royal Institute of Technology
Stockholm, S-10044, mshapiro@math.kth.se

Abstract. In this short note we give an explicit presentation of the ring $\mathcal{A}_n$ generated by the curvature 2-forms of the standard Hermitian linear bundles over $\mathbb{SL}_n/B$ as the quotient of the polynomial ring. The difference between $\mathcal{A}_n$ and $H^*(\mathbb{SL}_n/B)$ reflects the fact that $\mathbb{SL}_n/B$ is not a symmetric space. This question was raised by V. I. Arnold in [Ar].

Sur l'algèbre engendrée par les 2-formes de Chern sur SL_n/B

Résumé: Dans cette note nous donnerons une présentation explicite, en tant que quotient de l'anneau polynomial, de l'anneau A_n engendrée par les 2-formes de courbure des fibrés en droites hermitiens standard sur SL_n/B. La différence entre A_n et $H^*(SL_n/B)$ reflète le fait que SL_n/B n'est pas un espace symmétrique. Cette question a été posée par V. I. Arnold dans [Ar].

Version française abrégée.

Considérons l'espace SL_n/B des drapeaux complets dans $\mathbf{C}^n$, aussi bien que les n fibrés quotient $L_i = E_i/E_{i-1}$, où E_i signifie le fibré tautologique standard de dimension i au-dessus de SL_N/B. Quand on fixe une métrique hermitienne dans $\mathbf{C}^n$ tous les L_i deviennent des fibrés hermitiens. On notera ω_i la 2-forme de courbure de L_i. Il est évident que chaque ω_i est U_n-invariante sur SL_n/B. Dans cette note nous allons présenter l'anneau $A_n = A(\omega_1, \dots, \omega_n)$ sur $\mathbb{Z}$, engendrée par $\omega_1, \dots, \omega_n$, comme un quotient de l'anneau polynomial en n générateurs, et on la comparera ensuite avec $H^*(SL_n/B)$.

Nous étudions aussi l'algèbre $A_{k,n}$ sur $\mathbb{Z}$ engendrée par k parmi les formes ω_i. (Sa structure ne dépend pas du choix particulier de k-tuple.) Le résultat principal est le suivant.

Proposition. *L'anneau $A_{k,n}$ est une anneau graduée, isomorphe à $Z[x_1, \dots, x_k]/I_{k,n}$, où $I_{k,n}$ est un idéal engendré par $2^k - 1$ polynômes de la forme*

$$g^{(n)}_{i_1,\dots,i_j} = (x_{i_1} + \dots + x_{i_j})^{j(n-j)+1}.$$

Ici $\{i_1, \dots, i_j\}$ parcourt l'ensemble des sous-ensembles non vides de $\{1, \dots, n\}$.

La conjecture suivante est un résultat des expériments faits sur l'ordinateur avec le système Macaulay.

Conjecture. *a) La dimension de $A_n = A_{n,n}$, en tant qu'espace vectoriel sur $\mathbb{Z}$, est égale au nombre total des forêts sur n sommets marqués, et il existe une base monomiale naturelle pour A_n, dont les éléments sont indexés par ces forêts.*

b) La dimension totale de $A_{k,n}$ est un polynôme unitaire en n de degré k.

§1. Introduction.

Let $\mathbb{SL}_n/B$ denote the space of complete flags in $\mathbb{C}^n$. One has the obvious sequence of tautological bundles $0 \subset E_1 \subset ... \subset E_n = E$ (where E is the trivial $\mathbb{C}^n$-bundle over $\mathbb{SL}_n/B$) and the corresponding n-tuple of quotient line bundles $L_i = E_i/E_{i-1}$. Fixing some Hermitian metric on the original $\mathbb{C}^n$ one equips every bundle E_i and L_i with the induced Hermitian metric. Let w_i denote the curvature form of the above Hermitian metric on L_i. Each w_i is a $\mathcal{U}_n$-invariant 2-form on $\mathbb{SL}_n/B$ representing $c_1(L_i)$ in $H^2(\mathbb{SL}_n/B)$ (the $\mathcal{U}_n$-action is provided by the chosen metric). Denoting $c_1[L_i]$ by x_i one has that $H^*(\mathbb{SL}_n/B, \mathbb{Z}) = \frac{\mathbb{Z}[x_1,...,x_n]}{(s_1,s_2,...,s_n)}$ where the denominator is the ideal generated by all elementary symmetric functions in variables $x_1, ..., x_n$, see e.g. [Bo]. Since we have the standard representative w_i for each $x_i = c_1(L_i)$ it seems natural to study the $\mathbb{Z}$-ring $\mathcal{A}_n = \mathcal{A}(w_1, ..., w_n)$ generated by all w_i's and compare it to $H^*(\mathbb{SL}_n/B, \mathbb{Z})$. We will also discuss the subring $\mathcal{A}_{k,n} = \mathcal{A}(w_{i_1}, ..., w_{i_k})$ generated by any k of w_i's (the structure of $\mathcal{A}_{k,n}$ does not depend on a particular choice of a k-tuple). The main result of this short note is as follows.

1.1. Proposition. $\mathcal{A}_{k,n}$ is a graded ring isomorphic to $\frac{\mathbb{Z}[x_1,...,x_k]}{I_{k,n}}$ where the ideal $I_{k,n}$ is generated by the set of $2^k - 1$ polynomials of the form

$$g^{(n)}_{i_1,...,i_j} = (x_{i_1} + ... + x_{i_j})^{j(n-j)+1} \tag{1}$$

where $\{i_1, ..., i_j\}$ runs over the set of all nonempty subsets of the set $\{1, ..., n\}$.

1.2. Example. The ring $\mathcal{A}_3 = \mathcal{A}_{3,3}$ is isomorphic to $\frac{\mathbb{Z}[x_1,x_2,x_3]}{I_{3,3}}$ where $I_{3,3}$ is generated by $x_1^3, x_2^3, x_3^3, (x_1+x_2)^3, (x_1+x_3)^3, (x_2+x_3)^3, x_1+x_2+x_3$. The Hilbert series of A_3 is $(1, 2, 3, 1)$. (For comparison, the Hilbert series for $H^*(\mathbb{SL}_3/B)$ is $(1, 2, 2, 1)$.)

1.3. Remark. One has the standard surjective ring homomorphism $\pi : \mathcal{A}_n \to H^*(\mathbb{SL}_n/B, \mathbb{Z})$.

The following conjecture is the result of calculation of the Hilbert series of $\mathcal{A}_n$ for $n \leq 7$, see [SS].

1.4. Conjecture. 1) The total dimension of $\mathcal{A}_n$ (as a $\mathbb{Z}$-vector space) equals the number of forests on n labeled vertices and there exists a natural monomial basis for $\mathcal{A}_n$ whose monomials are enumerated by the above forests. 2) The total dimension of $\mathcal{A}_{k,n}$ is a monic polynomial in n of degree k.

Besides purely aesthetic reasons the study of $\mathcal{A}_n$ is motivated by the fact that many known results for $H^*(\mathbb{SL}_n/B, \mathbb{Z})$ (such as e.g. the existence of a good monomial basis, S_n-module structure, the action of divided difference operators etc) have natural counterparts for $\mathcal{A}_n$. Finally, the study of $\mathcal{A}_n$ seems to be important in understanding the structure of the ring $\mathfrak{A}(\mathbb{SL}_n/B)$ of all $\mathcal{U}_n$-invariant forms on $\mathbb{SL}_n/B$ which recently appeared in the arithmetic intersection theory for flag varieties, comp. [Ta1], [Ta2].

The authors are grateful to V. Arnold for the formulation of the problem and to R. Fröberg for his help with Macaulay which was crucial in formulation of 1.4. Sincere thanks are due to R.Stanley who kindly provided to the first author with some details about counting trees and forests and calculated the generating function for the number of forests on labeled vertices. Finally, the authors want to thank H.Tamvakis for the explanation of his papers [Ta1] and [Ta2] as well as for some relevant references.

§2. Proofs.

On ring $\mathfrak{A}(\mathbb{SL}_n/B)$ of $\mathcal{U}_n$-invariant forms on $\mathbb{SL}_n/B$.

One knows that $\mathbb{SL}_n/B$ has another presentation as a homogeneous space, namely $\mathcal{U}_n/T^n$ where $T^n \subset \mathcal{U}_n$ is the usual torus of diagonal matrices. Let us recall an old result from the general theory of homogeneous spaces, see e.g. [DNF].

2.1. Proposition. The ring of all G-invariant differential forms on a homogeneous space G/H is isomorphic to the exterior algebra $\Lambda_{inv}((\mathfrak{G}/\mathfrak{H})^*)$, i.e. to the algebra of skewsymmetric polylinear functions on $\mathfrak{G}$ which a) vanish on $\mathfrak{H}$ and b) are invariant under the action of internal automorphisms by the elements of H. (Here $\mathfrak{G}$ and $\mathfrak{H}$ are the Lie algebras of G and H resp.)

Take the $\binom{n}{2}$-dimensional vector space V_n of all skew-hermitian matrices of the form

$$V_n = \begin{pmatrix} 0 & a_{1,2} & \dots & \dots & \dots & a_{1,n} \\ -\bar{a}_{1,2} & 0 & a_{2,3} & \dots & \dots & a_{2,n} \\ -\bar{a}_{1,3} & -\bar{a}_{2,3} & 0 & a_{3,4} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\bar{a}_{1,n} & -\bar{a}_{2,n} & \dots & \dots & \dots & 0 \end{pmatrix}. \tag{2}$$

Its matrix entries form the linear space $(\mathfrak{U}_n/\mathfrak{T}^n)^*$ where $\mathfrak{U}_n$ (resp. $\mathfrak{T}^n$) is the Lie algebra of $\mathcal{U}_n$ (resp. of T^n). Let us denote by $(e^{i\lambda_1}, e^{i\lambda_2}, ..., e^{i\lambda_n})$ the diagonal entries of elements in T^n acting by conjugation on V_n. Under this action each entry $a_{i,j}$ above the main diagonal is multiplied by $e^{i(\lambda_i-\lambda_j)}$ and each entry $-\bar{a}_{i,j}$ below the main diagonal is multiplied by $e^{i(\lambda_j-\lambda_i)}$. Introducing the fundamental weights $\alpha_i = \lambda_i - \lambda_{i+1}, i = 1, ..., n-1$ we get that $a_{i,j}$ is multiplied by $e^{i(\alpha_i+\alpha_{i+1}+\ldots+\alpha_{j-1})}$. The expression $\alpha_i+\alpha_{i+1}+\ldots+\alpha_{j-1}$ is called *the multiweight* of the entry $a_{i,j}$. (Under this convention the entry $-\bar{a}_{i,j}$ has the opposite multiweight $-\alpha_i - \alpha_{i+1} - \ldots - \alpha_{j-1}$.) The multiweight of an exterior monomial $\tilde{a}_{i_1,j_1} \wedge \tilde{a}_{i_2,j_2} \cdots \wedge \tilde{a}_{i_r,j_r}$ where each $\tilde{a}_{i_l,j_l}$ is either a_{i_l,j_l} or $-\bar{a}_{i_l,j_l}$ equals to the sum of multiweights of its factors.

2.2. Corollary. The ring $\mathfrak{A}(\mathbb{SL}_n/B)$ is the linear span of all exterior monomials $\tilde{a}_{i_1,j_1} \wedge \tilde{a}_{i_2,j_2} \cdots \wedge \tilde{a}_{i_r,j_r}$ having vanishing multiweight. (In particular, $\mathfrak{A}(\mathbb{SL}_n/B)$ has no degree 1 elements.)

Remark. Note that such monomials with vanishing multiweight can be considered as collections of roots in of $\mathfrak{U}_n$ whose sum vanishes. This notion makes sense for any root system.

2.3. Example. The Hilbert series for $\mathfrak{A}(\mathbb{SL}_3/B)$ is $(1, 0, 3, 2, 3, 0, 1)$. (We assume that $\mathfrak{A}(\mathbb{SL}_n/B)$ contains constants.) Its degree 2 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2}, a_{1,3} \wedge \bar{a}_{1,3}, a_{2,3} \wedge \bar{a}_{2,3}$; degree 3 component is spanned by $a_{1,2} \wedge \alpha_{2,3} \wedge \bar{a}_{1,3}, \bar{a}_{1,2} \wedge \bar{\alpha}_{2,3} \wedge a_{1,3}$; degree 4 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{1,3} \wedge \bar{a}_{1,3}, a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{2,3} \wedge \bar{a}_{2,3}, a_{1,3} \wedge \bar{a}_{1,3} \wedge a_{2,3} \wedge \bar{a}_{2,3}$ and, finally, its degree 6 component is spanned by $a_{1,2} \wedge \bar{a}_{1,2} \wedge a_{1,3} \wedge \bar{a}_{1,3} \wedge a_{2,3} \wedge \bar{a}_{2,3}$. The Hilbert series for $\mathfrak{A}(\mathbb{SL}_4/B)$ is $(1, 0, 6, 8, 21, 24, 32, 24, 21, 8, 6, 0, 1)$.

Recall that *an Eulerian digraph* is a directed graph such that the numbers of coming and leaving edges at each vertex are equal. (We do not allow loops.)

The following proposition is a relatively simple reformulation of 2.2.

2.4. Proposition. The dimension of $\mathfrak{A}(\mathbb{SL}_n/B)$ (as a $\mathbb{Z}$-vector space) equals the number of Eulerian digraphs on n labeled vertices.

On curvature forms.

Using the above desription of $\mathcal{U}_n$-invariant forms on $\mathbb{SL}_n/B$ one can be easily derive the following proposition from the results of [GS], see also [Ta2].

2.5. Proposition. The curvature 2-form w_i, $i = 1, ..., n$ of the tautologic line bundle $L_i = E_i/E_{i-1}$ over $\mathbb{SL}_n/B$ equals (up to a constant factor) to the sum of all entries in the ith row of the following matrix of 2-forms

$$\begin{pmatrix} 0 & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,n} \\ -\gamma_{1,2} & 0 & \gamma_{2,3} & \cdots & \gamma_{2,n} \\ -\gamma_{1,3} & -\gamma_{2,3} & 0 & \cdots & \gamma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1,n} & -\gamma_{2,n} & -\cdots & -\gamma_{n-1,n} & 0 \end{pmatrix} \tag{3}$$

where $\gamma_{i,j} = \alpha_{i,j} \wedge \bar{\alpha}_{i,j}$.

Now we are ready to start proving the main proposition 1.1. Consider the ring $\mathcal{A}(w_{i_1}, ..., w_{i_k})$ generated by $w_{i_1}, ..., w_{i_k}$. Since all w_j are $\mathcal{U}_n$-invariant one has that $\mathcal{A}(w_{i_1}, ..., w_{i_k})$ is a subring of the exterior algebra $\Lambda(\mathbb{C}^{\binom{n}{2}})$.

2.6. Lemma. There exists a natural S_n-action on the set of all permutations $(w_{i_1}, ..., w_{i_n})$ of $w_1, ..., w_n$.

Proof. The elementary transposition τ_i, $i = 1, ..., n-1$ acts on the matrix (3) of 2-forms by the simultaneous interchanging of 1) the ith row with the $(i+1)$st row, 2) the ith column with the $(i+1)$st column and 3) changing sign of $\gamma_{i,i+1}$. One can easily check that this determines the required S_n-action. $\square$

2.7. Corollary. All rings $\mathcal{A}(w_{i_1}, ..., w_{i_k})$ are isomorphic to each other and, in particular, to $\mathcal{A}(w_1, ..., w_k)$.

We denote this class of isomorphic algebras by $\mathcal{A}_{k,n}$.

2.8. Lemma. $\mathcal{A}_{n-1,n}$ is isomorphic to $\mathcal{A}_n = \mathcal{A}_{n,n}$.

Proof. Indeed, one has $w_1 + w_2 + ... + w_n = 0$. $\square$

Definition. Let us call by *the vanishing ideal* $I_{k,n}$ of $\mathcal{A}_{k,n}$ the set of all polynomials $p \in \mathbb{Z}[x_1, ..., x_k]$ which vanish if we substitute the variables $x_1, ..., x_k$ by the curvature forms $w_1, ..., w_k$ resp.

2.9. Lemma. The vanishing ideal $I_{k,n}$ consists of all $p \in \mathbb{Z}[x_1, ..., x_k]$ such that 2^n derivatives $p, \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, ..., \frac{\partial p}{\partial x_k}, \frac{\partial^2 p}{\partial x_1 \partial x_2}, ..., \frac{\partial^2 p}{\partial x_{k-1} \partial x_k}, ..., \frac{\partial^k p}{\partial x_1 \partial x_2 ... \partial x_k}$ belong to $I_{k,n-1}$. (Derivatives are taken with respect to all subsets $\{i_1, ..., i_l\} \subset \{1, ..., n\}$ of indices (no repetitions of indices are allowed) including the empty set.)

PROOF. Notice that $\mathcal{A}_{k,n}$ has a natural module structure over $\mathcal{A}_{k,n-1}$. Indeed, one has $w_i^{(n)} = w_i^{(n-1)} + \gamma_{i,n}$ where the upper index shows the dimension of the initial space, see (3). Therefore for any polynomial $p \in \mathbb{Z}[x_1,...,x_k]$ one has after substitution of the curvature forms

$$p(w_1^{(n)},...,w_k^{(n)}) = p(w_1^{(n-1)}+\gamma_{1,n},...,w_k^{(n-1)}+\gamma_{k,n}) = p(w_1^{(n-1)},...,w_k^{(n-1)})+ p_{x_1}(w_1^{(n-1)},...,w_k^{(n-1)})\gamma_{1,n} + ... + p_{x_k}(w_1^{(n-1)},...,w_k^{(n-1)})\gamma_{k,n} + ...+ p_{x_1,x_2,...,x_k}(w_1^{(n-1)},...,w_k^{(n-1)})\gamma_{1,n}\gamma_{2,n}...\gamma_{k,n}. \quad (4)$$

Since $\gamma_{i,n}^2 = 0$ the occuring monomials in the r.h.s. are square-free and the coefficient at the product $\gamma_{i_1,n}...\gamma_{i_l,n}$ equals $\frac{\partial^l p}{\partial x_{i_1}...\partial x_{i_l}}$. Finally, the condition $p \in I_{k,n}$, i.e. $p(w_1^{(n)},...,w_k^{(n)}) = 0$ is equivalent to vanishing of all polynomial coefficients $p_{x_{i_1},...,x_{i_l}}(w_1^{(n-1)},...,w_k^{(n-1)})$ in the r.h.s of (4). By definition this means that $p_{x_{i_1},...,x_{i_l}}(x_1,...,x_k) \in I_{k,n-1}$. $\square$

Denote $\mathcal{D}_{i_1,...,i_l} = \frac{\partial^l}{\partial x_{i_1}...\partial x_{i_l}}$ and let $V_{i_1,...,i_l,r,m,k}$ be the space of all homogeneous polynomials with integer coefficients of degree $m+r$ in k variables of the form $p = (x_{i_1} + ...x_{i_l})^r f$ where $deg(f) = m$.

2.10. LEMMA. The linear map $\mathcal{D}_{i_1,...,i_l} : V_{i_1,...,i_l,r+l,m+k-l,k} \to V_{i_1,...,i_l,r,m,k}$ is a surjection.

PROOF. Simple linear algebra. $\square$

Proof of the main proposition 1.1.

We use a double induction on $k \le n$, i.e. for a given k we apply induction on n and then cover the first case $\mathcal{A}_{k+1,k+1}$ for $k+1$ using lemma 2.8.

Base of induction. $\mathcal{A}_{1,n} = \frac{\mathbb{Z}[x_1]}{(x_1^n)}$. Indeed, $w_1 = \gamma_{1,n} + \gamma_{1,3} + ...\gamma_{1,n}$ where $\gamma_{1,i}$ are independent commuting variables satisfying $\gamma_{1,i}^2 = 0$. Statement follows.

Step of induction. Assume that 1.1. is proven for all pairs (k',n') where $k' < k$ and for all (k,n') where $n' < n$. Let us show that it holds for the pair (k,n). Notice that all polynomials of the form (1) are contained in $I_{k,n}$. This can be either checked directly or by induction using 2.9. Let us temporarily denote by $\tilde{I}_{k,n}$ the ideal generated by all polynomials in (1). We have to show that $I_{k,n} = \tilde{I}_{k,n}$. By the above remark $\tilde{I}_{k,n} \subset I_{k,n}$. Take any $p \in I_{k,n}$. Using lemma 2.10 we will step by step add to p certain polynomials from $\tilde{I}_{k,n}$ in such a way that the resulting polynomial p_{fin} will have all derivatives entering the formulation of lemma 2.9 vanishing. Consider $\mathcal{D}_{1,...,k}(p)$. By 2.9. it belongs to $I_{k,n-1}$. By the inductive assumption $I_{k,n-1}$ is generated by $g_{i_1,...,i_l}^{(n-1)}$, see (1). Therefore one has $\mathcal{D}_{1,...,k}(p) = \sum_{i_1,...,i_l}(x_{i_1} + ... + x_{i_l})^{j(n-j-1)+1}h_{i_1,...,i_l}$. By lemma 2.10 for each $\phi = (x_{i_1} + ... + x_{i_l})^{j(n-j-1)+1}h_{i_1,...,i_l}$ there exists (but nonunique!) $\psi = (x_{i_1} + ... + x_{i_l})^{j(n-j)+1}H_{i_1,...,i_l}$ such that $\mathcal{D}_{i_1,...,i_l}(\psi) = \phi$. Notice that ψ belongs to $\tilde{I}_{k,n}$ by defintion. Therefore subtracting from p an appropriate polynomial belonging to $\tilde{I}_{k,n}$ we get $\tilde{p} \in I_{k,n}$ such that $\mathcal{D}_{1,...,k}(\tilde{p}) = 0$. Consider now any operator $\mathcal{D}_{1,...,\hat{i},...,k} = \frac{\partial^k}{\partial x_1...\widehat{\partial x_i}...\partial x_k}$ and apply it to $\tilde{p}$. One has that $\mathcal{D}_{1,...,\hat{i},...,k}(\tilde{p})$ does not depend on x_i since $\mathcal{D}_{1,...,k}(\tilde{p}) = 0$. Therefore using the same argument as above we can subtract from $\tilde{p}$ some polynomial belonging to $\tilde{I}_{k,n}$ which does not depend on x_i and such that $\mathcal{D}_{1,...,\hat{i},...,k}$ applied to the resulting difference vanish. Let us apply this procedure for all for all $i = k, k-1, \dots, 1$. Notice that each step does not change vanishing of all $\mathcal{D}_{1,\dots,\hat{j},\dots,k}$, $j < i$ applied to the polynomial obtained on the current step since each time we add a function which does not depend on x_i. Having obtained soe modified p such that all $\mathcal{D}_{1,\dots,\hat{i},\dots,k}$, $i \le k$ applied to it vanish we can proceed with all $\mathcal{D}_{1,\dots,\hat{i},\dots,\hat{j},\dots,k}$ (ordering them e.g. lexicographically). Our assumptions imply that $\mathcal{D}_{1,\dots,\hat{i},\dots,\hat{j},\dots,k}$ applied to our polynomial does not depend on x_i and x_j. Therefore we can subtract a polynomial from $\tilde{I}_{k,n}$ which does not depend on x_i and x_j either and such that $\mathcal{D}_{1,\dots,\hat{i},\dots,\hat{j},\dots,k}$ applied to the resulting difference vanish. Using this procedure consecutively we do not change all previously considered derivatives since the polynomials we subtract depend on different groups of variables. Continuing in the same manner we get some polynomial p_{fin} all derivatives of which mentioned in 2.9 vanish. This means that p_{fin} is a constant. But since constants different from zero do not lie in $I_{k,n}$ we have that p_{fin} equlas 0. The statement follows. $\square$

§3. Some related problems.

Problem 1. Give a presentation of the obvious analog of $\mathcal{A}_n$ for any $\mathbb{SL}_n/P$ generated by the standard $\mathcal{U}_n$-invariant forms representing the Chern classes of the tautological (quotient) bundles.

Remark. Notice that in the case of Grassmanian $G_{k,n}$ the analogous algebra coincides with $H^*(G_{k,n})$ since $G_{k,n}$ is a symmetric space and therefore has no nontrivial left-invariant forms homologous to 0, see e.g. [St].

Problem 2. Determine the S_n-module structure for $\mathcal{A}_n$ and its $\mathbb{SL}_n/P$-analogs.

Problem 3. Calculate the Poincare series for the ring $\mathfrak{A}(\mathbb{SL}_n/P)$ of $\mathcal{U}_n$-invariant forms on $\mathbb{SL}_n/P$.

References

[Ar] V. I. Arnold, *Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect*, Selecta Math. **1** (1995), no. 1, 1–19.

[Bo] A. Borel, *La cohomologie mod 2 des certains espaces homogènes*, Comment. Math. Halv **27** (1953), 165–197.

[DNF] B. Dubrovin, S. Novikov and A. Fomenko, *Modern geometry*, vol. 2, Moscow, Nauka, 1984.

[GS] P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math **123** (1969), 253–302.

[PSS] A. Postnikov, B. Shapiro and M. Shapiro, *Ring of Chern forms on flag varieties and forests*, in preparation.

[MSh] M. Shapiro and A. D. Vainshtein, *Stratification of Hermitian matrices, the Alexander mapping, and the bundle of eigenspaces*, Compt. Rendus **321** (1995), no. 12, 1599–1604.

[St] W. Stoll, *Invariant forms on Grassmann manifolds*, vol. 89, Annals of Math. Studies, Princeton Univ. Press, 1977, pp. 113.

[Ta1] H. Tamvakis, *Bott-Chern forms and arithmetic intersections*, L'Enseignement Math **43** (1997), 33–54, alg-geom/9611005, nov. 96.

[Ta2] H. Tamvakis, *Arithmetic intersection theory on flag varieties*, alg-geom/9611006, submitted to Math. Ann.