

A Tropical Analog of Descartes' Rule of Signs

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We prove that for any degree d , there exist (families of) finite sequences $\{\lambda_{k,d}\}_{0 \leq k \leq d}$ of positive numbers such that, for any real polynomial P of degree d , the number of its real roots is less than or equal to the number of the so-called essential tropical roots of the polynomial obtained from P by multiplication of its coefficients by $\lambda_{0,d}, \lambda_{1,d}, \dots, \lambda_{d,d}$, respectively. In particular, for any real univariate polynomial $P(x)$ of degree d with a non-vanishing constant term, we conjecture that one can take $\lambda_{k,d} = e^{-k^2}$, $k = 0, \dots, d$. The latter claim can be thought of as a tropical generalization of Descartes' rule of signs. We settle this conjecture up to degree 4 as well as a weaker statement for arbitrary real polynomials. Additionally, we describe an application of the latter conjecture to the classical Karlin problem on zero-diminishing sequences.

1 Introduction

The famous Descartes' rule of signs claims that the number of positive roots of a real univariate polynomial does not exceed the number of sign changes in its sequence of coefficients. In what follows, among other things, we suggest a conceptually new conjectural upper bound on the number of real roots of real univariate polynomial applicable in the situation when Descartes' rule of signs gives a trivial restriction.

Received October 14, 2015; Revised March 1, 2016; Accepted May 5, 2016
Communicated by Misha Sodin

Recall that a sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ of real numbers is called a *multiplier sequence (of the first kind)* if the diagonal operator $T_\lambda: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ defined by $x^k \mapsto \lambda_k x^k$, for $k = 0, 1, \dots$, and extended to $\mathbb{R}[x]$ by linearity, preserves the set of real-rooted polynomials (see, e.g., [6]). To formulate our results, we need to introduce tropical analogs of multiplier sequences.

The following classical notion is borrowed from the Wiman–Valiron theory (see [7]). A non-negative integer k is said to be a *central index* of P if there exists a number $x_k \geq 0$ such that

$$|a_k| x_k^k \geq \max_{i \neq k} |a_i| x_k^i. \tag{1}$$

The next notion is analogous to the central index. A non-negative integer k is called a *dominating index* of a polynomial

$$P(x) = \sum_{i=0}^d a_i x^i$$

if there exists a real number $x_k \geq 0$ such that

$$|a_k| x_k^k \geq \sum_{i \neq k} |a_i| x_k^i. \tag{2}$$

Condition (2) appeared earlier in the context of amoebas (see, e.g., [15]).

Notice that (1) is an analog of (2) if the right-hand side of (2) is interpreted as a tropical sum. We will say that a polynomial P of degree d is *tropically real-rooted* if each integer $k = 0, \dots, d$ is a central index of f .

To relate property (2) to real-rootedness of univariate polynomials, we say that a real-rooted polynomial P is called *strongly real-rooted* if each polynomial obtained by an arbitrary sign change of the coefficients of $P(x)$ is real-rooted as well (see [11]). (The same class of polynomials was called *sign-independently real-rooted* in [11].) In *loc. cit.*, the following statement was proven.

Proposition 1. A real polynomial P of degree d is strongly real-rooted if and only if every integer $k = 0, \dots, d$ is a dominating index of P . □

By the (standard) *tropicalization* of a real polynomial $P(x) = \sum_{i=0}^d a_i x^i$ we mean the tropical polynomial given by:

$$\text{tr}_P(\xi) = \max_{0 \leq i \leq d} (i\xi + \ln |a_i|), \quad \xi \in \mathbb{R}. \tag{3}$$

(In the literature the function $\text{tr}_P(\xi)$ is also referred to as the *Archimedean tropical polynomial* associated with P .) If $a_i = 0$, then the corresponding term in $\text{tr}_P(\xi)$ should be interpreted as $-\infty$, and thus it can be ignored when taking the maximum.

Remark 2. One can describe $\text{tr}_P(\xi)$ using the Newton–Hadamard polygon, an important object in Wiman–Valiron theory (see [7]). This description amounts to the duality between a tropical hypersurface and a convex polyhedral subdivision of its Newton polytope. Define the set A_P of points in the (u, v) -plane corresponding to the monomials of P as

$$A_P = \{(i, -\log |a_i|), 0 \leq i \leq d, a_i \neq 0\}.$$

Let $A_P(u)$ be a piecewise-linear continuous function on $[0, d]$ such that

$$A_P(i) = -\log |a_i|$$

whenever $a_i \neq 0$ and linear on the segment between two consecutive indices corresponding to non-zero coefficients a_i . Denote by $\tilde{A}_P(u)$ the greatest convex minorant of $A_P(u)$ on $[0, d]$. Finally, define the Newton–Hadamard \mathcal{NA}_P polygon of P as

$$\mathcal{NA}_P = \{v \geq \tilde{A}_P(u), 0 \leq u \leq d\}.$$

Observe that k is a central index of P if and only if $(k, -\log |a_k|)$ is a boundary point of \mathcal{NA}_P , and $\text{tr}_P(\xi) = \max_{z \in \mathcal{NA}_P}(\xi, -1) \cdot z$, that is, $\text{tr}_P(\xi)$ is the support function of \mathcal{NA}_P . Alternatively, $\text{tr}_P(\xi)$ is the Legendre transform of $\tilde{A}_P(u)$. \square

Any corner of the graph of $\text{tr}_P(\xi)$, that is, a value of ξ at which its slope changes, is called a *tropical root* of $\text{tr}_P(\xi)$. These are precisely the slopes of the edges of \mathcal{NA}_P . We define *Descartes' multiplicity* of a tropical root ζ of tr_P to be one less than the number of terms of (3) for which the maximum in the right-hand side of (3) is attained at ζ . (Notice that this definition differs from the standard definition of root multiplicity in tropical geometry which is equal to the length of the edge in \mathcal{NA}_P with slope ξ , comp. [14]. This illustrates our focus on real rather than complex-valued polynomials.) With our definition of Descartes' multiplicity of a tropical root, the number of tropical roots of $\text{tr}_P(\xi)$ counted with multiplicities is one less than the number of central indices of P . In particular, the number of tropical roots of $\text{tr}_P(\xi)$ is at most by one less than the number of monomials of P . The latter circumstance is analogous to the fact that the number of real roots of P is at most one less than its number of monomials.

We will now define positive and negative tropical roots of P by using the signs of its coefficients. Let $k_0 \leq k_1 \leq \dots \leq k_m$ be the central indices of P . Consider two sequences $\{\text{sgn}(a_{k_i})\}_{0 \leq i \leq m}$ and $\{\text{sgn}((-1)^{k_i} a_{k_i})\}_{0 \leq i \leq m}$.

Take two consecutive central indices k_{i-1} and k_i of the polynomial P ; to this pair we associate the tropical root $\xi_i = -\ln(a_{i-1}/a_i)/(k_{i-1} - k_i)$ of $\text{tr}_P(\xi)$. If the difference $k_{i-1} - k_i$ is odd, then the pair (k_{i-1}, k_i) contributes a sign alternation in exactly one of the above sequences. In this case, we will say that ξ_i is a *positive* (respectively *negative*) *essential tropical root* of P . If the difference $k_{i-1} - k_i$ is even, then either the pair (k_{i-1}, k_i) does not contribute a sign alternation in any of the sequences $\{\text{sgn}(a_{k_i})\}_{0 \leq i \leq m}$ and $\{\text{sgn}((-1)^{k_i} a_{k_i})\}_{0 \leq i \leq m}$, or it contributes a sign alternation in both. In the former case we will say that ξ_i is a *non-essential tropical root* of P , and in the latter case we will say that ξ_i is a *positive–negative essential tropical root* of P . By the number of *positive essential tropical roots of P* we mean the sum of the numbers of positive and positive–negative tropical roots of P . Analogously, by the number of *negative essential tropical roots of P* we mean the sum of the numbers of negative and positive–negative tropical roots of P . Finally by the total number of *essential tropical roots of P* we call the sum of the latter two numbers.

It is easy to see that the number of essential tropical roots of P is at most d .

Example 3. Consider $P_1(x) = 1 + x^2$. The central indices of P_1 are $k_0 = 0$ and $k_1 = 2$. As $\ln |a_1| = \ln |0| = -\infty$, the polynomial P_1 has (with our definition of Descartes' multiplicity) exactly one simple tropical root. To count the number of positive and negative tropical roots of P_1 , we need to count the number of sign alternations in the sequences $\{1, 1\}$ and $\{1, (-1)^2\} = \{1, 1\}$, respectively. That is, the number of essential tropical roots of P is equal to 0.

Consider now the polynomial $P_2(x) = 1 - x^2$. Similarly to P_1 , the polynomial P_2 has one tropical root. However, to count the number of positive and negative tropical roots of P_2 we count the number of sign alternations in the sequences $\{1, -1\}$ and $\{1, -(-1)^2\} = \{1, -1\}$, respectively. That is, the number of essential tropical roots of P_2 is equal to 2. \square

As the definitions of the central and the dominating indices only depend on the moduli $|a_i|$, for $i = 0, \dots, d$, they immediately extend to complex-valued polynomials. However, below we restrict ourselves only to real polynomials and positive sequences λ .

A sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ is called *log-concave* if $\lambda_k^2 \geq \lambda_{k-1} \lambda_{k+1}$ for all k . In [11] using discriminant amoebas, it is proven that the diagonal operator $T_\lambda: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ preserves the set of strongly real-rooted polynomials if and only if λ is log-concave. For

this reason, log-concave sequences were called *multiplier sequences of the third kind* in *loc. cit.* We prefer to refer to log-concave sequences λ as *tropical multiplier sequences*.

Definition 4. A positive sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ is said to be a *central (resp. dominating) index preserver* if, for each polynomial P , the set of central (resp. dominating) indices of P is a subset of the set of central (resp. dominating) indices of the polynomial $T_\lambda[P]$. \square

Our first result is as follows.

Theorem 5. For positive sequences λ , the following three conditions are equivalent:

- (1) λ is log-concave, that is, λ is a tropical multiplier sequence.
- (2) λ is a central index preserver.
- (3) λ is a dominating index preserver. \square

In particular, Theorem 5 provides an alternative (and elementary) way to settle [11, Theorem 1] as requested in Problem 2 of *loc. cit.*

Corollary 6 (Theorem 1 in [11]). A positive sequence λ preserves the set of strongly real-rooted polynomials if and only if it is log-concave. \square

In what follows, we will need a slightly more general definition of a tropicalization of P . Given an arbitrary triangular sequence $\lambda = \{\lambda_{k,j}\}_{0 \leq k \leq d, d \in \mathbb{N}}$ of positive numbers, and a univariate polynomial $P(x) = \sum_{i=0}^d a_i x^i$ of any degree d , we define its λ -tropicalization as

$$\text{tr}_P^\lambda(\xi) = \max_{0 \leq k \leq d} (k\xi + \ln |a_k| + \ln \lambda_{k,d}), \quad \xi \in \mathbb{R}. \quad (4)$$

Remark 7. Here is another description of $\text{tr}_P^\lambda(\xi)$. Let $\Theta_d(u)$ be a continuous piecewise-linear function on $[0, d]$, linear on intervals $[k, k+1]$ for $k = 0, \dots, d-1$ and such that $\Theta_d(k) = -\log \lambda_{k,d}$ for $k = 0, \dots, d$. Define $\tilde{A}_P^\lambda(u)$ as the greatest convex minorant of $A_P(u) + \Theta_d(u)$. Then tr_P^λ is the Legendre transform of $A_P^\lambda(u)$, see Remark 2. \square

Definition 8. A finite sequence $\{\lambda_{k,d}\}_{0 \leq k \leq d}$, of positive numbers is called a *degree d (positive) real-to-tropical root preserver* if for any polynomial P of degree d (with positive coefficients), the number of essential tropical roots of (4) is greater than or equal to the number of non-zero real roots of P . A triangular sequence $\lambda = \{\lambda_{k,j}\}_{0 \leq k \leq j, j \in \mathbb{N}}$ is called a

(positive) real-to-tropical root preserver if, for each d , its finite subsequence $\{\lambda_{k,d}\}_{0 \leq k \leq d}$, is a degree d (positive) real-to-tropical root preserver. \square

We recall that the *recession cone* of a set $X \subset \mathbb{R}^{d+1}$ is the largest pointed (i.e., including the origin) cone $C \subseteq \mathbb{R}^{d+1}$ such that if $x \in X$, then $x + c \in X$ for all $c \in C$. Our main result is as follows.

Theorem 9. The set $\Lambda_d \subset \mathbb{R}_+^{d+1}$ (respectively $\Lambda_d^+ \subset \mathbb{R}_+^{d+1}$) of all degree d (positive) real-to-tropical root preservers $\{\lambda_{k,d}\}_{0 \leq k \leq d}$ is a non-empty closed full-dimensional subset of \mathbb{R}_+^{d+1} . Moreover, the recession cone of its logarithmic image $\text{Ln}(\Lambda_d)$ (respectively $\text{Ln}(\Lambda_d^+)$) coincides with the cone of all concave sequences of length $d + 1$. (Here for any $\Omega \subset \mathbb{R}_+^k$, by $\text{Ln}(\Omega)$ we mean the set in \mathbb{R}^k obtained by taking natural logarithms of points from Ω coordinatewisely.) \square

Theorem 9 shows that there exist large families of real-to-tropical root preservers in each degree, and therefore large families of real-to-tropical root preserving triangular sequences.

First we show that, if $\lambda = \{\lambda_{k,d}\}_{0 \leq k \leq d}$ is sufficiently log-concave, then λ is a degree d real-to-tropical root preserver:

Theorem 10. Assume that a sequence $\lambda = \{\lambda_{k,d}\}_{0 \leq k \leq d}$ of positive numbers satisfies the condition:

$$\log \frac{\lambda_{k,d}^2}{\lambda_{k-1,d}\lambda_{k+1,d}} > 2\Delta_d := \frac{d^2}{4} \log 36d + (d + 1) \log d + \log 4, \quad 1 \leq k \leq d - 1. \quad (5)$$

Then, for any real polynomial P , the number of positive (negative) tropical roots of tr_p^λ is greater than or equal to the number of positive (negative) roots of P . In particular, λ is a real-to-tropical root preserver. \square

Next we show that to be a real-to-tropical root preserver, the sequence $\lambda = \{\lambda_{k,d}\}_{0 \leq k \leq d}$ should be sufficiently log-concave.

Theorem 11. There exists $c > 0$ with the following property. Assume that for some $k < d - 100$

$$\log \frac{\lambda_{j,d}^2}{\lambda_{j-1,d}\lambda_{j+1,d}} < 2c, \quad j = k, \dots, k + 100. \quad (6)$$

Then there exists a polynomial P of degree d with positive coefficients such that tr_P^λ has three tropical roots, and P has four negative roots. In particular, $\{\lambda_{k,d}\}_{0 \leq k \leq d}$ cannot be a degree d (positive) real-to-tropical root preserver. \square

In this direction, we present the following tantalizing conjecture. Consider the sequence λ^\dagger given by

$$\lambda_k^\dagger := e^{-k^2}, \quad k = 0, 1, \dots$$

We will denote by $\text{tr}_P^\dagger(\xi)$ the corresponding tropical polynomial associated with any real polynomial P , that is,

$$\text{tr}_P^\dagger(\xi) = \max_{0 \leq k \leq d} (k\xi + \ln |a_k| - k^2), \quad \xi \in \mathbb{R}. \quad (7)$$

Conjecture 12 (Conjectural tropical analog of Descartes' rule of signs). For any real univariate polynomial $P(x)$, the number of its positive (negative) roots does not exceed the number of positive (negative) essential tropical roots of $\text{tr}_P^\dagger(\xi)$. \square

We have the following partial result supporting Conjecture 12.

Proposition 13. Conjecture 12 holds for $d \leq 4$. \square

Remark 14. Conjecture 12 can be partially explained by the following argument. If an infinite real-to-tropical root preserving sequence $\{\lambda_k\}_{k=0}^\infty$ exists, then it must necessarily be close to the form $\{e^{-\alpha k^2}\}$, for some positive α . Indeed, Theorems 10 and 11 suggest that, for a sequence to be a real-to-tropical root preserver for any degree d , the expression $\log \frac{\lambda_j^2}{\lambda_{j-1}\lambda_{j+1}}$ should be bounded from below by a certain positive constant. The operation $Q \rightarrow x^d Q(x^{-1})$ acts on \mathcal{NA}_Q by reflection with respect to $u = d/2$ while preserving the number of positive and negative roots. This additionally suggests that $\log \frac{\lambda_j^2}{\lambda_{j-1}\lambda_{j+1}}$ should be the same for $j = 1$ and for $j = d - 1$, that is, independent of j . Finally, the choice $\alpha = 1$ is additionally supported by our numerical experiments. \square

Besides the fact that Conjecture 12 looks quite appealing, it might also shed light on possible extensions of the classical Newton inequalities from the case of real-rooted polynomials to the case of polynomials with a non-maximal number of real roots and positive coefficients. Additionally, (if settled) it also gives interesting consequences in the classical Karlin problem on zero-diminishing sequences (see [8] and Section 5).

2 Introductory Results and Theorem 9

We will begin with the following statement. Given a sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$, define its *symbol* as the formal series $S_\lambda(x) := \sum_{k=0}^\infty \lambda_k x^k$. Define the d th truncation $S_\lambda^{(d)}(x)$ as

$$S_\lambda^{(d)}(x) := \sum_{k=0}^d \lambda_k x^k.$$

Lemma 15. A positive sequence λ is log-concave if and only if, for each d , the d th truncation $S_\lambda^{(d)}(x)$ is a tropically real-rooted polynomial. \square

Proof of Lemma 15. Assume first that λ is log-concave. For each $m \geq 1$, set $x_m := \sqrt{\lambda_{m-1}/\lambda_{m+1}}$. Then,

$$\frac{x_{m+1}}{x_m} = \frac{\lambda_m}{\sqrt{\lambda_{m-1}\lambda_{m+1}}} \frac{\lambda_{m+1}}{\sqrt{\lambda_m\lambda_{m+2}}} \geq 1,$$

so that $\{x_m\}_{m=1}^\infty$ is a non-decreasing sequence of positive real numbers. Furthermore,

$$\frac{\lambda_m x_m^m}{\lambda_{m-1} x_m^{m-1}} = \frac{\lambda_m x_m^m}{\lambda_{m+1} x_m^{m+1}} = \frac{\lambda_m}{\sqrt{\lambda_{m-1}\lambda_{m+1}}} \geq 1.$$

Since both binomials $\lambda_k x^k - \lambda_{k+1} x^{k+1}$ and $\lambda_k x^k - \lambda_{k-1} x^{k-1}$ have exactly one positive real root, we conclude that $\lambda_k x_m^k \geq \lambda_{k+1} x_m^{k+1}$ if $k \geq m$ and that $\lambda_k x_m^k \geq \lambda_{k-1} x_m^{k-1}$ if $k \leq m$. Hence,

$$\lambda_m x_m^m \geq \max_{k \neq m} \lambda_k x_m^k.$$

For the converse, assume that λ is not log-concave. That is, there exists an index m for which $\lambda_m^2 < \lambda_{m-1}\lambda_{m+1}$. Then, for $x \geq 0$,

$$\lambda_m x^m < \sqrt{\lambda_{m-1} x^{m-1} \lambda_{m+1} x^{m+1}} \leq \max(\lambda_{m-1} x^{m-1}, \lambda_{m+1} x^{m+1}).$$

In particular, m is not a central index of $S_\lambda(x)$. \blacksquare

Proof of Theorem 5. Let us first prove that a sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ is log-concave if and only if it is a central index preserver. Assume first that λ is log-concave. Let m be a central index of P , and let $x_m \geq 0$ be such that

$$a_m x_m^m \geq \max_{k \neq m} a_k x_m^k.$$

By Lemma 15 we can find ζ_m such that

$$\lambda_m \zeta_m^m \geq \max_{k \neq m} \lambda_k \zeta_m^k.$$

Then, for all k ,

$$\lambda_m a_m (z_m \zeta_m)^m = \lambda_m x_m^m a_m \zeta_m^m \geq \lambda_k x_m^k a_k \zeta_m^k.$$

Hence, m is a central index of $T_\lambda[P]$. For the converse, it suffices to consider the sequence of polynomials $1 + x + \dots + x^d$, which are tropically real-rooted for all d , and use Lemma 15.

Let us now prove that λ is log-concave if and only if it is a dominating index preserver. Assume first that λ is log-concave, and let ζ_m be as in the proof of Theorem 5. Let m be a dominating index of P , and let x_m be such that

$$a_m x_m^m \geq \sum_{k \neq m} a_k x_m^k.$$

Then,

$$\lambda_m a_m (x_m \zeta_m)^m \geq \sum_{k \neq m} \lambda_m \zeta_m^m a_k x_m^k \geq \sum_{k \neq m} \lambda_k \zeta_m^k a_k x_m^k,$$

implying that m is a dominating index of $T_\lambda[P]$. For the converse, assume that $\lambda_m^2 < \lambda_{m-1} \lambda_{m+1}$, and consider the action of T_λ on the trinomial $x^{m-1} + 2x^m + x^{m+1}$. ■

Using Lemma 15, we can rephrase Theorem 5 in a manner similar to the classical result of Pólya and Schur [13]. Given a sequence λ of real numbers, we say that its symbol $S_\lambda(x)$ is *tropically real-rooted* if, for each $d = 0, 1, \dots$, the d th truncation $S_\lambda^{(d)}(x)$ is tropically real-rooted.

Corollary 16. A positive sequence λ is a dominating index and central index preserver if and only if its symbol $S_\lambda(x)$ is tropically real-rooted. □

Proof of Proposition 1. To prove the *only if*-part, consider the polynomial

$$Q(x) = |a_k| x^k - \sum_{i \neq k} |a_i| x^i,$$

for some $1 \leq k \leq d - 1$. Notice that Q is obtained from P by flipping signs of the coefficients and hence, by assumption, Q is real-rooted. In particular, Q has exactly two

positive roots counted with multiplicity. (This fact follows from the observation that Q has all d roots real of which at most $d - 2$ are negative by Descartes's rule of signs.) Let x_k be the mean value of the positive roots of Q . Then,

$$|a_k|x_k^k - \sum_{i \neq k} |a_i|x_k^i \geq 0,$$

with equality if and only if Q has a positive root of multiplicity 2. In particular, k is a dominating index of P .

For the if -part, choose arbitrary signs of the coefficients of P . We note that condition (2) implies that

$$\text{sgn}(P(x_k)) = \text{sgn}(a_k x_k^k) = \text{sgn}(a_k),$$

for $x > 0$. Using Descartes' rule of signs, we conclude that the number of positive roots of P is equal to the number of sign changes in the sequence $\{a_i\}_{0 \leq i \leq d}$. Similarly, the number of negative roots of P is equal to the number of sign changes in the sequence $\{(-1)^i a_i\}_{0 \leq i \leq d}$. As $a_i \neq 0$ for each i , these two numbers sum up to d , implying that $P(x)$ is real-rooted. Since the signs of the coefficients were chosen arbitrary, we are done. ■

Proof of Corollary 6. It follows from Proposition 1 that a positive sequence λ preserves the set of strongly real-rooted polynomials if and only if it preserves dominating indices. Additionally, it follows from Theorem 5 that a positive sequence λ preserves dominating indices if and only if it is log-concave. ■

Proof of Theorem 9. As we are only concerned with the number of (real) roots of the polynomial P , we can consider P up to a non-vanishing scalar, that is, we identify P with its coefficient vector $(a_0 : \dots : a_d) \in \mathbb{R}\mathbb{P}^d$. (This implies that the signs of the coefficients are not well-defined. However, as sign-alternations between coefficients are well-defined this introduces no ambiguity in the above-defined concepts.)

Let us first show that the set Λ_d is non-empty. Let $\lambda = \{\lambda_k\}_{0 \leq k \leq d}$ be a finite positive strictly log-concave sequence. By Lemma 15 we have that $S_\lambda^{(d)}(x)$ is tropically real-rooted. Moreover, it follows from the proof of Lemma 15 and the strict log-concavity that all the tropical roots of $S_\lambda^{(d)}(x)$ are of Descartes' multiplicity 1.

Firstly, for each $P \in \mathbb{R}\mathbb{P}^d$, we claim that there exists a positive number $s = s(P)$ such that $\text{tr}_P^{\lambda^s}(\xi)$ has at least as many distinct negative tropical roots as the number of negative roots of P . Here, λ^s denotes the sequence $\{\lambda_k^s\}_{0 \leq k \leq d}$. To prove this, notice first

that, by using the change of variables $\xi \mapsto s\xi$, the number of negative tropical roots of

$$\mathrm{tr}_P^{\lambda^s}(\xi) = \max_{0 \leq k \leq d} (k\xi + \ln |a_k| + s \ln \lambda_k)$$

is equal to the number of negative tropical roots of the tropical polynomial

$$\max_{0 \leq k \leq d} \left(s \left(k\xi + \frac{\ln |a_k|}{s} + \ln \lambda_k \right) \right).$$

Since the factor s does not change which term is maximal, the number of negative tropical roots of $\mathrm{tr}_P^{\lambda^s}$ is equal to the number of negative tropical roots of

$$\max_{0 \leq k \leq d} \left(k\xi + \frac{\ln |a_k|}{s} + \ln \lambda_k \right).$$

Note that

$$\lim_{s \rightarrow \infty} \frac{\ln |a_k|}{s} = \begin{cases} 0, & \text{if } a_k \neq 0, \\ -\infty, & \text{if } a_k = 0. \end{cases}$$

Hence, for s sufficiently large, the number of negative tropical roots of $\mathrm{tr}_P^{\lambda^s}$ is equal to the number of negative tropical roots of

$$\max_{a_k \neq 0} (k\xi + \ln \lambda_k).$$

Since the sequence λ is log-concave, it follows from Lemma 15 that each for k with $a_k \neq 0$ the k th term is dominating for some ξ_k . In particular, the number of negative tropical roots of the latter polynomial is equal to the Descartes' bound on the maximal number of negative roots of P .

Secondly, we claim that $s = s(P)$ can be chosen in such a way that there exists a neighborhood $N(P) \subset \mathbb{RP}^d$ of P such that, for each $Q \in N(P)$, the number of negative essential tropical roots of $\mathrm{tr}_Q^{\lambda^s}$ is not less than the number of negative roots of Q . Consider first the case $a_0 \neq 0$. Then, there is a neighborhood $N_1(P)$ of P such that the number of negative roots of $Q \in N_1(P)$ is at most equal to the number of negative roots of P . Since all negative tropical roots of $\mathrm{tr}_P^{\lambda^s}$ are distinct, there is a neighborhood $N_2(P)$ such that the number of negative tropical roots of $\mathrm{tr}_P^{\lambda^s}$ is equal to the number of negative tropical roots of $\mathrm{tr}_Q^{\lambda^s}$, for all $Q \in N_2(P)$. (If P has some vanishing coefficients, then $N_2(P)$ can be chosen so that the corresponding indices are not central indices of Q , for any $Q \in N_2(P)$.) In this case we can take $N(P) = N_1(P) \cap N_2(P)$. Complementarily, consider the case $a_0 = 0$. For each polynomial Q , let Q' denote the polynomial obtained by removing the constant

term of Q . Using an inductive argument, we can choose a neighborhood $N(P)$ of P such that, for each $Q \in N(P)$, the number of negative tropical roots of $\text{tr}_Q^{\lambda^s}$ is not less than the number of negative roots of Q . Notice that for the first non-zero coefficient a_k of P its index k is a central index of P . Indeed, after division by x_k , which does not change the set of central indices, this corresponds to the constant term. If $(-1)^k a_k$ is positive, then the number of negative real roots of P increases by 1 if a_0 is perturbed by a small negative number, and similarly the number of negative tropical roots is increased by 1, and vice versa.

Finally, to see that Λ_d is non-empty, we note that \mathbb{RP}^d is compact. Therefore, the open covering $\cup_{P \in \mathbb{RP}^d} N(P)$ of \mathbb{RP}^d has a finite subcovering $\mathbb{RP}^d \subset N(P_1) \cup \dots \cup N(P_M)$. Let $s^* = \max_{1 \leq i \leq M} s(P_i)$. Since $\lambda^{s^* - s(P_i)}$ is log-concave, it is a central index preserver by Theorem 5. Hence, we conclude that $\lambda^{s^*} \in \Lambda_d$.

Let us now prove that the recession cone C of $\text{Ln}(\Lambda_d)$ is equal to the set of log-concave sequences of length $d + 1$. The fact that the latter set is contained in C follows immediately from Theorem 5, as each log-concave sequence is a central index preserver. Conversely, if λ is not log-concave, then the d th truncation $S_\lambda^{(d)}$ of its symbol is not tropically real-rooted. Let P be a tropically real-rooted polynomial, and let λ^* be a log-concave sequence. By a similar argument as above, we can conclude by letting s tend to infinity, that the tropical polynomial

$$\text{tr}_P^{\lambda^* \lambda^s}(\xi) = \max_{0 \leq k \leq d} (k\xi \ln |a_k| + \ln \lambda_k^* + s \ln \lambda_k)$$

is not tropically real-rooted. Hence, λ is not contained in the recession cone of the set $\text{Ln}(\Lambda_d)$.

The remaining statements of Theorem 9 follow easily from the above facts. ■

3 Theorems 10 and 11

To settle Theorem 10, recall the following statement proved in, for example, [10].

Lemma 17. For a given real polynomial P and real $x \neq 0$, assume that all tropical roots of tr_P are more than $\log 3$ away from $\log |x|$. Let k be the central index corresponding to x . Then k is a dominating index and, in particular, $P(x) \neq 0$. □

Proof. The function $\tilde{A}_P(u)$ defining the Newton–Hadamard polygon of P is convex. Therefore, its slopes form an increasing sequence. The condition on the tropical roots of tr_P means that the slopes of the edges of $\tilde{A}_P(u)$ are smaller than $-\log 3 + \log x$, for

$u < k$ and greater than $\log 3 + \log x$, for $u > k$. In other words,

$$\frac{-\log |a_k| + \log |a_j|}{k-j} < \log |x| - \log 3, \quad \text{for } j < k,$$

and

$$\frac{-\log |a_j| + \log |a_k|}{j-k} > \log |x| + \log 3, \quad \text{for } j > k.$$

The first inequality means that $|a_j x^j| < 3^{j-k} |a_k x^k|$ for all $j < k$. Therefore,

$$\sum_{j < k} |a_j x^j| < \frac{1}{2} |a_k x^k|.$$

Similarly,

$$\sum_{j > k} |a_j x^j| < \frac{1}{2} |a_k x^k|,$$

and the claim follows. ■

Corollary 18. Let P be a polynomial of degree d and assume that every integer $k = 0, \dots, d$ is a central index of tr_P . Assume that all tropical roots of tr_P are simple and separated from one another by more than $2 \log 3$. Then P is strongly real-rooted. □

Proof. Indeed, for $x = \sqrt{a_{k-1}/a_{k+1}}$ the conditions of Lemma 17 are satisfied. So k is a dominating index and the claim follows from Proposition 1. ■

Our proof of Theorem 10 requires two steps. During the first step, we prove (see Lemma 21) that if a polynomial $P = \dots + a_m x^m + \dots + a_n x^n + \dots$ is a small perturbation of a polynomial $\tilde{P} = a_m x^m + \dots + a_n x^n$ with positive coefficients, then P has no roots in a positive interval containing exponentials of all tropical roots of $\text{tr}_{\tilde{P}}(\xi)$,

During the second step, we group the tropical roots of $\text{tr}_P(\xi)$ into several clusters of closely located roots. Each cluster corresponds to an interval of the positive axis, and the monomials corresponding to tropical roots in other clusters are insignificant on this interval. We treat each interval separately using a generalization of Rolle's theorem presented in Lemma 22. Namely, we find first-order linear differential operators L_k which

- (a) decrease the number of positive tropical roots of tr_P by 1;
- (b) decrease the number of roots of P on the corresponding interval by at most 1; and
- (c) have a controllable effect on the magnitude of the coefficients of P .

After several applications of different L'_k s, we arrive at the situation covered by the first step described above, and conclude that the number of positive roots of P on this interval does not exceed the number of positive tropical roots in its cluster. A similar fact holds for the negative roots as well.

Lemma 19. Given a real polynomial P , let $U = [\alpha', \alpha'']$ be a real interval such that

- (1) tr_P has a unique tropical root $\alpha \in U$ corresponding to two monomials $a_m x^m$ and $a_n x^n$, $m < n$, that is, $\alpha = \frac{\log |a_n| - \log |a_m|}{n - m}$;
- (2) α', α'' are located more than $\log 4$ away from all tropical roots of tr_P ;
- (3) for all $l, m < l < n$,

$$\log |a_l| \leq v(l) - \log d - \log 4, \tag{8}$$

where $v(u) = \alpha u + \beta$ is the linear function whose graph passes through $(m, \log |a_m|)$ and $(n, \log |a_n|)$.

Then P has the same number of real roots on the interval $[e^{\alpha'}, e^{\alpha''}]$ as the binomial $a_m x^m + a_n x^n$, and the same holds on the interval $[-e^{\alpha'}, -e^{\alpha''}]$. □

Proof. The sum $\sum_{k < m} |a_k x^k|$ is $< \frac{1}{3} |a_m x^m|$ on $\{x \in \mathbb{C}, \log |x| > \alpha'\}$, comp. the proof of Lemma 17. Similarly, $\sum_{k > n} |a_k x^k| < \frac{1}{3} |a_n x^n|$ on $\{x \in \mathbb{C}, \log |x| < \alpha''\}$. Also, $|\sum_{m < k < n} a_k x^k| \leq \frac{1}{4} (|a_m x^m| + |a_n x^n|)$ on $\{x \in \mathbb{C}, \alpha' \leq \log |x| \leq \alpha''\}$.

Consider the case $I = [e^{\alpha'}, e^{\alpha''}]$; the case of $I = [-e^{\alpha'}, -e^{\alpha''}]$ is treated similarly. Assume first that $a_n x^n$ and $a_m x^m$ have the same signs on I . This means that their sum dominates the sum of all other terms. Thus, P has no zeros on I at all.

If the signs are different, choose a curvilinear rectangle Π containing I and bounded by $\{\log |x| = \alpha'\}$, $\{\log |x| = \alpha''\}$, and $\{\arg x = \pm \pi / (n - m)\}$. The inequalities above imply that $a_m x^m$ dominates the sum of all other terms on $\{\log |x| = \alpha'\}$. Similarly, $a_n x^n$ dominates the sum of all other terms on $\{\log |x| = \alpha''\}$.

Moreover, the sum $a_m x^m + a_n x^n$ dominates the sum of all other terms on $\{\log |x| \in U, |\arg x| = \pi / (n - m)\}$ as the arguments of $a_m x^m$ and $a_n x^n$ are equal there. In other words, the increment of the argument of P on the boundary of Π is the same as that of $a_m x^m + a_n x^n$. Therefore P has a unique root in Π , which is necessarily real. ■

Corollary 20. Assume that the tropical roots of tr_P are at least $2 \log 4$ apart from one another. Assume also that, for any l lying between two consecutive central indices m, n ,

inequality (8) is satisfied. Then, the number of positive (resp. negative) roots of P is equal to the number of positive (resp. negative) tropical roots of P . □

To take into account the signs of tropical roots, we will need a more refined version of Lemma 19.

Lemma 21. Given a real polynomial P , let $m < n$ be its two central indices with $a_m, a_n > 0$. Let $U = [\alpha', \alpha'']$ be a real interval such that

- (1) the central index of any $u \in U$ lies in $[m, n]$ and U is more than $\log 4$ away from the tropical roots of tr_P corresponding to the edges of $\tilde{A}_P(u)$ lying outside of $[m, n]$, and
- (2) for all $l, m < l < n$, we have that either $a_l > 0$ or

$$-\log |a_l| \geq v(l) + \log d + \log 4, \tag{9}$$

where $v(u) = \alpha u + \beta$ is the linear function whose graph joins the vertices $(m, -\log |a_m|)$ and $(n, -\log |a_n|)$ of \mathcal{NA}_P .

Then, P has no roots on $I = [e^{\alpha'}, e^{\alpha''}]$. □

Proof. Take $x \in I$. As before, the sum $\sum_{k < m} |a_k x^k|$ is at most $\frac{1}{3} a_m x^m$ on I , as in the proof of Lemma 17. Similarly, $\sum_{k > n} |a_k x^k| < \frac{1}{3} a_n x^n$ on I . Also, $\sum'_{m < k < n} |a_k| x^k \leq \frac{1}{4} (a_m x^m + a_n x^n)$ on I , where the sum is taken over all monomials with negative coefficients. Therefore, P is positive on I . ■

3.1 Generalized Rolle’s theorem

For a given non-negative integer k , define the differential operator L_k :

$$L_k \left(\sum a_j x^j \right) := \sum (j - k) a_j x^j.$$

One can easily check that the latter definition is equivalent to

$$L_k(P) := x^{k+1} (x^{-k} P)'$$

The following version of Rolle’s theorem immediately follows from the second definition of L_k .

Lemma 22. Let $I \subset \mathbb{R}_+$ be some interval, then

$$\#\{x \in I, L_k(P(x)) = 0\} \geq \#\{x \in I, P(x) = 0\} - 1. \quad \square$$

One can define a natural tropical counterpart l_k of L_k as

$$l_k(\{\epsilon_j\}_{j=0}^n) = \{\text{sgn}(j - k)\epsilon_j\}_{j=0}^n,$$

where $\{\epsilon_j\}_{j=0}^n$ is any sequence of real numbers. Evidently, the number of sign changes in $\{\epsilon_j\}$ differs from that in $l_k(\{\epsilon_j\})$ by at most 1.

Let α_k be tropical roots of tr_p in the increasing order and let U be a connected component of the ρ -neighborhood of $\{\alpha_k\}$, with $\rho = \log 36d$.

Denote by $[m, n]$ the maximal interval such that the restriction of \tilde{A}_p to it has edges with slopes equal to the tropical roots of tr_p lying in U . (We can assume that $n > m + 1$ since the case $n = m + 1$ is covered by Lemma 19.)

We choose a sequence $\lambda = \{\lambda_{k,d}\}_{k=0}^d$ such that

$$\log(\lambda_{k-1,d}^{-1} \lambda_{k,d}^2 \lambda_{k+1,d}^{-1}) = 2\Delta_d := \frac{d^2}{4} \log 36d + (d + 1) \log d + \log 4, \quad 1 \leq k \leq d - 1. \quad (10)$$

Let $q_k = (n_k, -\log |a_{n_k}| - \log \lambda_{n_k})$, $k = 0, \dots, N$, be the vertices of \tilde{A}_p^λ on the interval $[m, n]$ in the increasing order. Note that $n_0 = n$, $n_N = m$. Let $\alpha_a < \alpha_{a+1} < \dots < \alpha_b$ be the tropical roots of tr_p lying in U .

Denote by $\Sigma_U = \{\text{sgn}(a_{n_k})\}$ the sequence of signs of a_{n_k} . Choose a sequence $\{m_j\}_{j=1}^M$, $m_j \in \{n_k\}_{k=1}^{N-1}$, such that

- (i) $l_{m_1} \dots l_{m_M}(\Sigma_U)$ has no sign changes; and
- (ii) M is equal to the number of sign changes of Σ_U .

We can assume that $n > m_1 > \dots > m_{M-1} \geq m_M > m$.

Proposition 23. The polynomial $Q = L_{m_1} \dots L_{m_M}(P)$ has no roots in e^U . □

Proof. Without loss of generality we can assume $a_n > 0$. Moreover, by rescaling of x and multiplication of P by a constant, we can assume that $a_n = |a_m| = 1$.

Let $Q = \sum_{j=0}^d b_j x^j$, $b_j = a_j \prod_{k=1}^M (j - m_k)$. We claim that Q satisfies conditions of Lemma 21.

Let us start with the first condition of Lemma 21. Let $l < m$ and

$$\kappa_{l,m}^Q = \frac{-\log |a_l| - \sum_{k=1}^M \log |l - m_k| + \log |a_m| + \sum_{k=1}^M \log |m - m_k|}{l - m}$$

be the slope of the segment joining the two points on the graph of A_Q corresponding to the monomials of degree l and m . We have

$$\kappa_{l,m}^Q = \kappa_{l,m}^P + \frac{1}{m-l} \sum_{k=1}^{k_U-1} \log \frac{|l-m_k|}{|m-m_k|}, \tag{11}$$

where

$$\kappa_{l,m}^P = \frac{-\log |a_l| + \log |a_m|}{l-m}$$

is the slope of the segment joining the two points on the graph of A_P corresponding to the monomials of degree l and m .

Elementary computations show that

$$\frac{1}{m-l} \log \frac{|m_k-l|}{|m_k-m|} = \frac{1}{m_k-m} (t^{-1} \log(1+t)) \leq \frac{1}{m_k-m}, \quad t = \frac{m-l}{m_k-m} > 0,$$

as the function $t^{-1} \log(1+t)$ is monotone decreasing.

Therefore, the last sum in (11) is bounded from above by $(2 + \log d)$; thus

$$\kappa_{l,m}^Q \leq \kappa_{l,m}^P + 2 + \log d \leq \alpha_{a-1} + 2 + \log d.$$

This means that $\kappa_{l,m}^Q$ is more than $\log 4$ away from U , as α_{a-1} is at least ρ away from U and $\rho = \log 36d > 2 + \log d + \log 4$.

Similarly, $\kappa_{l,n}^Q \geq \alpha_{b+1} - 2 - \log d$, for $l > n$. This means that all slopes of \tilde{A}_Q to the left or to the right of $[m, n]$ are more than $\log 4$ away from U , that is, Q satisfies the first assumption of Lemma 21.

To prove the second assumption, we use the following elementary statement.

Lemma 24. Let $\phi(u)$ be a continuous convex piecewise-linear function on $[m, n]$ which is linear on each segment $[k, k+1]$, $k \in \mathbb{Z}$; we denote by μ_k its slope on the latter interval. Assume additionally that $\phi(m) = \phi(n) = 0$. Then,

- (1) if $0 \leq \mu_{k+1} - \mu_k \leq 2C$, then $\phi(u) \geq -C(m-n)^2/4$; and
- (2) if $0 \leq \mu_{k+1} - \mu_k = 2\Delta_d$, then $\phi(k) \leq -(n-m-1)\Delta_d$ for all $m < k < n$, $k \in \mathbb{Z}$. □

Corollary 25.

$$-\log |a_l| \geq -\frac{d^2}{4} \log 36d, \quad m \leq l \leq n. \tag{12}$$

□

Proof. By definition of U we have $0 \leq \alpha_{j+1} - \alpha_j \leq 2 \log 36d$, for $a \leq j \leq b - 1$. Therefore, the restriction of \tilde{A}_P to the segment $[m, n]$ satisfies assumptions of the first claim of Lemma 24. ■

Corollary 26. Choose $l \in [m, n]$, $l \in \mathbb{Z}$, and $l \notin \{n_k\}$. Then

$$-\log |a_l| \geq -\frac{d^2}{4} \log 36d + \Delta_d,$$

where Δ_d is the same as in Theorem 10. □

Proof. Condition $l \notin \{n_k\}$ means that $-\log |a_l| - \log \lambda_{l,d} > \alpha l + \beta$, where α, β are chosen in such a way that $\alpha m + \beta = -\log \lambda_{m,d}$ and $\alpha n + \beta = -\log \lambda_{n,d}$. Therefore

$$-\log |a_l| \geq -\Theta_d(u) + \alpha u + \beta,$$

and the bound follows from the second claim of Lemma 24 applied to $\phi(u) = \Theta_d(u) - \alpha u - \beta$. ■

Now, $\log |b_l| = \log |a_l| + \sum \log |m_k - l| \leq \log |a_l| + d \log d$. Therefore,

$$-\log |b_l| \geq -\frac{d^2}{4} \log 36d + \Delta_d - d \log d \geq \log d + \log 4.$$

Recall that we choose a rescaling such that $|a_m| = |a_n| = 1$. This fact implies that both $\log |b_m|$ and $\log |b_n|$ are positive, and the linear function $v(l)$ defined for Q as in the second condition of Lemma 21 is negative on $[m, n]$. Therefore Q satisfies the second condition of Lemma 21 as well, which finishes the proof of Proposition 23. ■

Corollary 27. Let M be the number of sign changes in $\{a_{n_k}\}$, where $\{n_k\}$ are central indices of tr_P^λ on the interval $[m, n]$. Then, P has at most M roots on e^U . □

Proof. Follows from Proposition 23 and Lemma 22. ■

Proof of Theorem 10. Applying Corollary 27 to each connected component of the $(\log 36d)$ -neighborhood of the set of tropical roots of tr_P (and using Lemma 17 outside of it), we see that the number of positive roots of P does not exceed the number of positive tropical roots of tr_P^λ .

Changing $P(x)$ to $P(-x)$, we get the same statement for the negative roots. In particular, we conclude that $\{\lambda_{k,d}\}$ defined in (10) is a real-to-tropical root preserver. ■

To prove Theorem 11, we need an auxiliary statement.

Lemma 28. There exists a polynomial R of degree 100 with four simple negative roots, whose leading and constant coefficients are equal to 1 and the remaining coefficients are non-negative and strictly < 1 . \square

Proof of Lemma 28. Set $Q_1(x) = x + 1$ and define $Q_{k+1}(x) = Q_k(x)(x^n + 1)$, $k = 2, 3, \dots$, where n is the smallest odd number greater than $\deg Q_k$. Note that

- (1) all coefficients of Q_k are either 1 or 0, and
- (2) $Q_k(x)$ is divisible by $(x + 1)^k$.

Take $Q_4(x^5)$ (which has a root of multiplicity 4 at -1), add some small positive multiple of $(x + 1)^3$ to split of a simple real root from the four-tuple root at -1 , then add an even smaller positive multiple of $(x + 1)^2$ to split of another simple root from -1 , and then add an even smaller multiple of $x + 1$ to split of the third simple root. (Note that $Q_4(x^5)$ has no monomials of degrees 1–3.)

The resulting perturbation \tilde{Q}_4 has four negative roots, is of degree 100, has a leading term equal to 1, the constant term $a_0 > 1$, and all the remaining coefficients at most 1. (All of them are equal to either 0 or 1 except in degrees 1–3, where they are small positive numbers). Define $R = a_0^{-1} \tilde{Q}_4(a_0^{1/100} x) = x^{100} + \dots + 1$, with all other coefficients non-negative and smaller than $a_0^{-1/100}$. \blacksquare

Proof of Theorem 11. Starting with the above polynomial R , we construct a polynomial P with four negative roots and with only three tropical roots. Note that

$$A_R(u) \geq \tilde{A}_R(u) \equiv 0, \quad \text{for } 0 \leq u \leq 100,$$

with equality, for $u = 0$ and 100 only.

Choose $c > 0$ in Theorem 11 such that $A_R(u) \geq cu(100 - u)$, for $0 \leq u \leq 100$. Inequality (6) implies that $\Theta_d(u)$ is almost flat on the interval $[k, k + 100]$, see Remark 7. More exactly, there exists a linear function $\ell(u)$ such that,

$$\Theta_d(u) \geq \ell(u) - cu(100 - u), \quad k \leq u \leq k + 100,$$

with equality for $u = k, k + 100$. Therefore $A_{x^{kR}}(u) + \Theta_d(u) \geq \ell(u)$ for $0 \leq u \leq 100$, with equality for $u = k, k + 100$ (i.e., lies below its chord on $[k, k + 100]$). Therefore $A_{x^{kR}}^\lambda(u)$ is linear, and $\text{tr}_{x^{kR}}^\lambda(\xi)$ has just one tropical root.

Now, choose $\delta > 0$ so small that $P = \delta(x^d + 1) + x^k R$ still has four negative simple roots. Then $\text{tr}_P^\lambda(\xi)$ has at most three tropical roots, since only two extra monomials were added. The latter choice of P settles Theorem 11. ■

4 Proposition 13

We start with some explicit information about Λ_d and Λ_d^+ for small d , comp. Theorem 9.

Lemma 29.

- (1) For $d = 1$, $\Lambda_1^+ = \Lambda_1 = \mathbb{R}_+$;
- (2) For $d = 2$, $\Lambda_2^+ = \Lambda_2 = \{\lambda \mid 4\lambda_1^2 \geq \lambda_0\lambda_2\}$. □

Proof. (1) It is enough to consider only fully supported polynomials P . Then, by normalization, we can assume that $a_0 = a_1 = 1$. For $d = 1$, there is nothing to prove.

(2) For $d = 2$, consider $P(x) = 1 + x + ax^2$. Then, $P(x)$ has two real roots if and only if $a \leq \frac{1}{4}$. If $a < 0$, then $\text{tr}_P^+(\xi)$ has two essential tropical roots for all a . Thus, it suffices to consider only the case $a > 0$. We need to compare the above inequality to the condition that the tropical polynomial

$$\text{tr}_P^\lambda(\xi) = \max(\ln \lambda_0, \xi + \ln \lambda_1, 2\xi + \ln a + \ln \lambda_2)$$

has two tropical roots. One can easily check that this happens if and only if $\lambda_1^2 \geq a\lambda_0\lambda_2$. This inequality holds for all $0 \leq a \leq \frac{1}{4}$ if and only if $4\lambda_1^2 \geq \lambda_0\lambda_2$. Clearly, the latter inequality is also necessary and sufficient if we restrict ourselves to polynomials with positive coefficients. ■

Lemma 30. For $d = 4$, Λ_4^+ contains the set defined by the system of inequalities:

$$\begin{cases} 2\lambda_1^2 \geq \lambda_0\lambda_2, & 9\lambda_2^2 \geq 4\lambda_1\lambda_3, & 2\lambda_3^2 \geq \lambda_2\lambda_4, \\ 2(\sqrt[4]{3} - 1)\lambda_1^4 \geq \sqrt[4]{3}\lambda_0^3\lambda_4, & 2(\sqrt[4]{3} - 1)\lambda_3^4 \geq \sqrt[4]{3}\lambda_0\lambda_4^3. \end{cases} \tag{13} \quad \square$$

Proof. As we consider only P with positive coefficients, we can without loss of generality restrict ourselves to the case $a_0 = a_4 = 1$, that is,

$$P(x) = 1 + a_1x + a_2x^2 + a_3x^3 + x^4.$$

We compare the appearance of its real roots with the appearance of tropical roots of the tropical polynomial

$$\text{tr}_P^\lambda(\xi) = \max(\ln \lambda_0, \xi + \ln a_1 + \ln \lambda_1, 2\xi + \ln a_2 + \ln \lambda_2, 3\xi + \ln a_3 + \ln \lambda_3, 4\xi + \ln \lambda_4),$$

where $\lambda_0, \dots, \lambda_4$ are variables. For real-rooted polynomials, we obtain the inequalities:

$$8\lambda_1^2 \geq 3\lambda_0\lambda_2, \quad 9\lambda_2^2 \geq 4\lambda_1\lambda_3, \quad 8\lambda_3^2 \geq 3\lambda_2\lambda_4.$$

Let us now consider polynomials $P(x)$ with exactly two real roots. When decreasing a_1, a_2 , and a_3 simultaneously, one can only decrease the number of essential tropical roots. Therefore, it suffices to prove the statement for polynomials $P(x)$ with a real double root only. With our normalization, such a polynomial can be written as

$$\begin{aligned} P(x) &= (r+x)^2(r^{-2}+sx+x^2) \\ &= 1 + (2r^{-1}+sr^2)x + (r^{-2}+2sr+r^2)x^2 + (2r+s)x^3 + x^4. \end{aligned}$$

Associated tropical polynomials are of the form

$$\begin{aligned} \text{tr}_P(\xi) &= \max(\ln \lambda_0, \xi + \ln(2r^{-1}+sr^2) + \ln \lambda_1, \\ &\quad 2\xi + \ln(r^{-2}+2sr+r^2) + \ln \lambda_2, \\ &\quad 3\xi + \ln(2r+s) + \ln \lambda_3, \quad 4\xi + \ln \lambda_4). \end{aligned}$$

We will split our consideration into two cases. If $r \leq 1$, then we will require that the first-order term dominates the even order terms at some point. If $r \geq 1$, we will require that the third-order term dominates the even order terms at some point. In the first case, we take the point

$$\xi_1 = -\ln(2r^{-1}+sr^2) - \ln \lambda_1 + \ln \lambda_0$$

and obtain the inequalities

$$\frac{\lambda_1^2}{\lambda_0\lambda_2} \geq \frac{1+2sr^3+r^4}{(2+sr^3)^2} \quad \text{and} \quad \frac{\lambda_1^4}{\lambda_0^3\lambda_4} \geq \frac{r^4}{(2+sr^3)^4}.$$

Since we require the coefficients of P to be positive, it is sufficient that these inequalities are valid for all $0 < r \leq 1$ and $s \geq -\frac{2}{4/3}$. We obtain

$$\sup_{r,s} \frac{1+2sr^3+r^4}{(2+sr^3)^2} = \sup_r \frac{1}{3-r^4} = \frac{1}{2},$$

and

$$\sup_{r,s} \frac{r}{2 + sr^3} = \sup_r \frac{r}{2 - \frac{2}{\sqrt[4]{3}}r^3} = \frac{\sqrt[4]{3}}{2(\sqrt[4]{3} - 1)}.$$

Thus, in case $r \leq 1$ we get

$$2\lambda_1^2 \geq \lambda_0\lambda_2 \quad \text{and} \quad 2(\sqrt[4]{3} - 1)\lambda_1^4 \geq \sqrt[4]{3}\lambda_0^3\lambda_4.$$

By symmetry, for $r \geq 1$, we obtain the inequalities

$$2\lambda_3^2 \geq \lambda_2\lambda_4 \quad \text{and} \quad 2(\sqrt[4]{3} - 1)\lambda_3^4 \geq \sqrt[4]{3}\lambda_0\lambda_4^3.$$

Altogether, we derived the system (13). ■

Proof of Proposition 13. Up to degree 3, the statement is covered by Lemma 29, as there is nothing to prove in the case of a cubic polynomial with one real root. The case of degree 4 follows immediately from Lemma 30. ■

5 Application to Zero-Diminishing Sequences

We start with the following standard definition (see, e.g., [3, 4]).

Definition 31. A sequence $\Gamma = \{\lambda_k\}_{k=0}^d$ of real numbers is called a *complex zero decreasing sequence in degree d* (a CZDS in degree d , for short) if, for any polynomial $P = a_0 + a_1x + \dots + a_dx^d$ with real coefficients, the polynomial $T_\lambda(P) = \lambda_0a_0 + \lambda_1a_1x + \dots + \lambda_da_dx^d$ has no more non-real roots than P .

A sequence $\Gamma = \{\lambda_k\}_{k=0}^\infty$ of real numbers is called a CZDS if for every $d \in \mathbb{N}$ the sequence $\Gamma = \{\lambda_k\}_{k=0}^d$ is a CZDS in degree d . □

Laguerre's classical result from 1884 gives the best so far recipe how to generate such sequences. Namely,

Theorem 32 (p. 116 of [9]). For any real polynomial $f(z)$ with all strictly negative roots, the sequence $\{f(n)\}$, $n = 0, 1, \dots$ is a CZDS. □

On p. 382 of his well-known book [8], Karlin posed the problem of characterizing the inverses of CZDS which are called *zero-diminishing sequences*. This problem is sometimes referred to as the Karlin problem. (In [3] the authors initially claimed that they

have solved Karlin's problem, but later they discovered a mistake in the presented solution.) Substantial information about CZDS can be found in Section 4 of [5] and a number of earlier papers. Several interesting attempts to find the converse of Laguerre's theorem and to solve the Karlin problem were carried out over the years, the most successful of them apparently being [2] and [1]. (For the history of the subject consult [3] and [12].) But in spite of some hundred and thirty years passed since the publication of [9] and certain partial progress, satisfactory characterization of the sets of all CZDSs and/or of all zero-diminishing sequences is still unavailable at present. In particular, it is still unknown whether the rapidly decreasing sequence $\{e^{-k^\alpha}\}_{k=0}^\infty$ with $\alpha > 2$ is a CZDS.

We will now illustrate how the theory developed in this paper can be applied to obtain new results regarding CZDS.

Theorem 33. Let $\lambda^* = \{\lambda_{k,j}^*\}_{0 \leq k \leq j \in \mathbb{N}}$ be a triangular real-to-tropical root preserver. Let $\lambda = \{\lambda_k\}_{k=0}^d$ be a sequence of positive numbers. If the set of dominating indices of the polynomial

$$Q_d(x) = \sum_{k=0}^d \frac{\lambda_k}{\lambda_{k,d}^*} x^k$$

is equal to $\{0, 1, \dots, d\}$, that is, $Q_d(x)$ is strongly real rooted, then λ is a CZDS in degree d .

In particular, if any initial segment $\{\lambda_k\}_{k=0}^d$ of a sequence $\{\lambda_k\}_{k=0}^\infty$ satisfies this condition, then $\{\lambda_k\}_{k=0}^\infty$ is a CZDS. \square

Proof. Consider a polynomial $P(x) = \sum_{i=0}^d a_i x^i$, and its image

$$T_\lambda[P] = \sum_{i=0}^d \lambda_i a_i x^i = \sum_{i=0}^d \frac{\lambda_i}{\lambda_{i,d}^*} \lambda_{i,d}^* a_i x^i$$

under the action of the operator T_λ . Since λ^* is a triangular real-to-tropical root preserver, the number of essential tropical roots of the polynomial

$$R(x) = \sum_{i=0}^d \lambda_{i,d}^* a_i x^i$$

is at least equal to the number of real roots of P . Let $0 = k_0 < k_1 < \dots < k_m = d$ be the central indices of $R(x)$, and let $x_0, \dots, x_m > 0$ be such that the central index k_j is

dominating at x_j , that is,

$$\lambda_{j,d}^* |a_j| x_j^j \geq \max_{i \neq j} \lambda_{i,d}^* |a_i| x_j^i. \tag{14}$$

Since each k_j is a dominating index of the polynomial $Q_d(x)$, we can find points Y_1, \dots, Y_m such that

$$\frac{\lambda_j}{\lambda_{j,d}^*} Y_j^j \geq \sum_{i \neq j} \frac{\lambda_i}{\lambda_{i,d}^*} Y_j^i. \tag{15}$$

Inequalities (14) and (15) imply that

$$\lambda_j |a_j| (x_j Y_j)^j = \frac{\lambda_j}{\lambda_{j,d}^*} Y_j^j \lambda_{j,d}^* |a_j| x_j^j \geq \sum_{i \neq j} \frac{\lambda_i}{\lambda_{i,d}^*} Y_j^i \lambda_{i,d}^* |a_i| x_j^i = \sum_{i \neq j} \lambda_i |a_i| (x_j Y_j)^i.$$

Thus, each k_j is a dominating index of $T_\lambda[P]$. In particular, the number of real roots of $T_\lambda[P]$ is at least equal to the number of essential tropical roots of $R(x)$, which in turn is at least equal to the number of real roots of P . ■

Theorem 34. Assume that the sequence $\{e^{-k^2}\}_{k=0}^\infty$ is a real-to-tropical root preserver. Then, the sequence $\{e^{-k^\alpha}\}_{k=0}^\infty$ is a CZDS for all $\alpha \geq 3$. □

Proof. For the corresponding polynomial $Q_d(x) = \sum_{k=0}^d e^{-k^\alpha + k^2} x^k$ the tropical roots are $\gamma_k = 2k - 1 + (k - 1)^\alpha - k^\alpha$. We see that

$$\gamma_k - \gamma_{k+1} = -2 + (k - 1)^\alpha + (k + 1)^\alpha - 2k^\alpha > -2 + \alpha(k - 1)k^{\alpha-2}$$

as soon as $\alpha > 3$. Already for $\alpha > 2.608\dots$ and $k \geq 1$, the latter expression is bigger than $2 \log 3$. Therefore, Corollary 18 implies that $Q_d(x)$ is a strongly real rooted for any $\alpha > 3$. Then, Theorem 33 implies the result. ■

Remark 35. The lower bound $\alpha \geq 3$ for the sequence $\{e^{-k^\alpha}\}_{k=0}^\infty$ to be a CZDS is apparently not sharp. In particular, computer experiments show that conclusion of Theorem 33 holds for $\alpha > 2.437623\dots$. But since we do not currently see how to prove Conjecture 12, we were not trying to get the optional lower bound with the help of Theorem 33. □

Funding

This work was supported by the Minerva foundation with funding from the Federal German Ministry for Education and Research.

Acknowledgment

B.S. wants to thank Professor A. Eremenko of Purdue University and Professor V. P. Kostov of Université de Nice for discussions.

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