THE TRANSLATION GEOMETRY OF PÓLYA'S SHIRES

BORIS SHAPIRO, GUILLAUME TAHAR, AND SANGSAN WARAKKAGUN

Abstract. In his shire theorem, G. Pólya proves that the zeros of iterated derivatives of a meromorphic function in the complex plane accumulate on the union of edges of the Voronoi diagram of the poles of this function. Recasting the local arguments of Pólya into the language of translation surfaces, we prove its generalisation describing the asymptotic distribution of zeros of iterations of a meromorphic function on a compact Riemann surface under the action of a differential operator of the form $f \mapsto \frac{df}{\omega}$, where ω is a given meromorphic 1-form. The accumulation set of these zeros is the union of edges of a generalized Voronoi diagram defined jointly by the initial function f and the singular flat metric induced ω on the Riemann surface.

1. Introduction

1.1. **Short historical account.** The classical shire theorem of G. Pólya claims that for a meromorphic function f with the set S of its poles, the zeros of its iterated derivatives $f^{(n)}$ asymptotically accumulate when $n \to \infty$ along the edges of the Voronoi diagram associated with S, see [11]. An illustration of this famous result is shown in Figure 1.

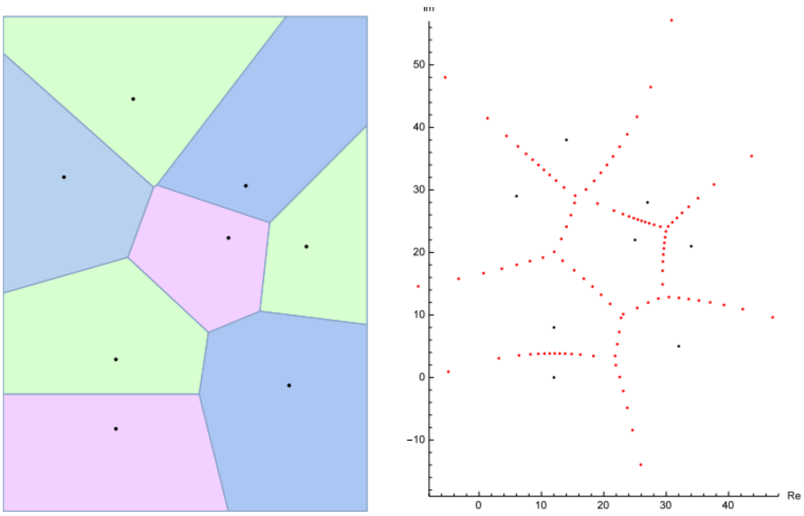

Figure 1. The Voronoi diagram of the roots of a polynomial f of degree 8 (left), and zeros of $(1/f)^{(15)}$ (right).

Several prominent mathematicians including N. Wiener, E. Hille, R. P. Boas have continued Pólya's line of study soon after the publication of the theorem, see references in [15]. A number of articles extending and generalising the original shire theorem appeared over the years, see e.g. [7, 8, 9, 16, 17, 19].

Date: February 3, 2024.

Key words and phrases. Linear differential operator, Translation surface, Singular flat metric, Voronoi diagram.

In [22], Weiss provides a generalization of Pólya's classical theorem to automorphic functions on the half-plane. There are two natural ways to generalize the geometry of flat tori to surfaces of higher genus:

- hyperbolic surfaces (the metric still has constant curvature but it is not flat anymore);
- translation surfaces (the metric is still flat but now it has conical singularities).

The approach of Weiss corresponds to hyperbolic surfaces. In contrast, we study in the present paper a class of linear differential operators corresponding to the complex-analytic data defining a translation structure (see [**?**] for background on translation surfaces).

More recently, using the circle of ideas contained in Pólya's theorem, several publications have concentrated on the weak limits of the root-counting measures for zeros of $f^{(n)}$. In particular, Ch. Hägg and R. Bøgvad obtained a measure-theoretic refinement of Pólya's shire theorem for rational functions, see [3, 4]. Using currents they also proved a similar result for Voronoi diagrams associated with generic hyperplane arrangements in $\mathbb{C}^m$.

Later Ch. Hägg extended the result of [3] by considering meromorphic functions of the form $f = Re^U$ where R is a rational function with at least 2 distinct zeros and U is a non-constant polynomial, see [10]. The class of such functions coincides with the class of meromorphic functions that are quotients of two entire functions of finite order, each having a finite number of zeros, see [20].

In 2021 V. Keo extended the results of [3] and [10] by studying a particular case of meromorphic functions $f(z) = 1/(1 - e^z)$ having an infinite number of poles and whose iterated derivatives are related to Eulerian polynomials, see [12]. In addition, V. Keo considered iterations of rational functions under the action of differential operators of the form $\mathcal{D} = g(z)\dfrac{\partial}{\partial z}$ where $g(z)$ is a polynomial in z which is closely related to the topic of the present paper. He mainly studied a particular case of $z\dfrac{\partial}{\partial z}$ and formulated some new conjectures.

1.2. **Our set-up.** In this paper we generalise the classical Pólya shire theorem to the case of meromorphic functions on compact Riemann surfaces. Namely, let X be a compact Riemann surface with a fixed meromorphic 1-form ω. We associate to the pair (X, ω) the linear differential operator T_ω acting on meromorphic functions on X as

$$T_\omega : f \to \frac{df}{\omega} \tag{1.1}$$

Now given a meromorphic function f on X, we are interested in the asymptotic of zeros for the sequence of meromorphic functions $(f_n)_{n\in\mathbb{N}}$ defined inductively by

$$f_0 = f,\ f_{n+1} = T_\omega f_n = T_\omega^{n+1} f,\ n \geq 1.$$

Definition 1.1. For a meromorphic function f on a Riemann surface X and any operator T acting on the space of meromorphic functions, the *limit set* $\mathcal{L}(T, f) \subseteq X$ is defined as the set of points $z \in X$ such that any open neighborhood of z in X contains a zero of $T^n(f)$ for infinitely many n.

In the classical theorem of Pólya, the limit set coincides with a Voronoi diagram associated with the set of the poles of the initial function. In our generalized settings, the Voronoi diagram is defined with respect to the so-called *principal polar locus*.

Definition 1.2. Consider a (non-vanishing identically) meromorphic 1-form ω and a fixed meromorphic function f on a Riemann surface X. The *principal polar locus* $\mathcal{PPL}(\omega, f)$ of pair (ω, f) is the subset of X containing:

- the poles of f that are not poles of ω;
- the zeros of ω where f is not locally factorized by ω (see Definition 1.3).

Definition 1.3. Given a point z_0 on a Riemann surface X, we say that a meromorphic function f is *locally factorized* by a holomorphic 1-form ω if there exist:

- a neighborhood U of z_0 in X,
- a holomorphic function ϕ defined on U;
- a holomorphic function g defined on a neighborhood of $\phi(z_0)$ in $\mathbb{C}$

such that $\omega = d\phi$ and $f = g \circ \phi$ in U.

Remark 1.4. In the classical Pólya's shire theorem, the Riemann surface is the extended complex plane $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, $\omega = dz$ and $\mathcal{PPL}(\omega, f)$ is the set of the affine poles of f (i.e. poles different from ∞).

The main result of our paper is as follows.

Theorem 1.5. *Consider a (non-vanishing identically) meromorphic* 1*-form ω on a compact Riemann surface X, its associated differential operator T_ω given by* (1.1)*, and any meromorphic function f on X such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. Then the following two facts are valid:*

(i) the limit set $\mathcal{L}(T_\omega, f)$ coincides with the Voronoi diagram $\mathcal{V}_{\omega,f}$ defined by $\mathcal{PPL}(\omega, f)$, see Definition 2.12 below;

(ii) the asymptotic root-counting measure of $(T_\omega)^n f$ when $n \to +\infty$ is proportional to the Cauchy measure of the Voronoi diagram $\mathcal{V}_{\omega,f}$, see Definition 2.14.

1.3. **Organization of the paper.**

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Acknowledgements. The first author wants to acknowledge the financial support of his research provided by the Swedish Research Council grant 2021-04900. He is sincerely grateful to Beijing Institute for Mathematical Sciences and Applications for the hospitality in Fall 2023 where the major part of this research has been carried out. He is sincerely grateful to his friend and colleague Rikard Bøgvad for many discussions of the area. The second author wants to thank Vincent Delecroix and Dmitry Novikov for valuable remarks and discussions.

2. Preliminary notions and results

2.1. **Growth of pole orders under iteration of T_ω.**

Unless a function f has a very specific form, one expects it to develop poles at the zeros of ω under the iterations of the operator $T_\omega : f \mapsto \frac{df}{\omega}$. The following lemma characterizes completely this case.

Lemma 2.1. *We consider a nonzero meromorphic* 1*-form ω and a meromorphic function f on a Riemann surface X. For any point $z_0 \in X$, the following statements are equivalent:*

- *no function in sequence $f, T_\omega f, T_\omega^2 f, \ldots$ has a pole in z_0;*
- *f is locally factorized by ω in z_0 (see Definition 1.3).*

Proof. Up to local biholomorphic change of variable, we can assume that $z_0 = 0$ and $\omega = z^m dz$ for some $m \in \mathbb{N}$. For an arbitrary locally defined holomorphic function f written as a series in coordinate z, we have

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

$T_\omega f$ then writes as a Laurent series:

$$T_\omega f = \sum_{k=0}^{+\infty} a_k k z^{k-m-1}.$$

It follows immediately that if no function in sequence $f, T_\omega f, (T_\omega)^2 f, \dots$ has a pole in 0, then $a_k = 0$ for any $k \notin (m+1)\mathbb{Z}$.
Introducing local primitive $\phi = \frac{z^{m+1}}{m+1}$ of ω, we obtain

$$f(z) = \sum_{k=0}^{+\infty} a_{k(m+1)} (m+1)^k \phi^k.$$

Therefore, in a neighborhood of 0, f factorizes as $g \circ \phi$ where g is a holomorphic function defined in a neighborhood of 0 (given by the series above.

Conversely, for any meromorphic function f locally factorizing as $g \circ \phi$, direct computation proves that for any k, we have:

$$T_\omega^k f = g^{(k)} \circ \phi.$$

Therefore no iterate $T_\omega^k f$ of function f has a pole in z_0. □

The following statement establishes a dichotomy between points belonging to the principal polar locus (see Definition 1.2) for which the orders of the poles grow linearly and the complement to the principal polar locus for which the total order of the poles remains bounded. In particular, for the total order of poles (and thus the total number of zeros) to grow infinitely, the principal polar locus must be non-empty.

Proposition 2.2. *We consider a nonzero meromorphic* 1*-form* ω *and a meromorphic function* f *on a compact Riemann surface* X*. Then there is a bound* $M > 0$ *such that:*

- *for any* $k > M$ *and any element* p *of* $\mathcal{PPL}(\omega, f)$*,* p *is a pole of order* $\alpha_p + k(d_p + 1)$ *of* $T_\omega^k f$ *(*d_p *is the order of* ω *in* p *and* α_p *is some constant);*
- *for any* $k > M$*, the total order of the poles of* $T_\omega^k f$ *outside* $\mathcal{PPL}(\omega, f)$ *is constant. All of these poles are also simple poles of* ω*.*

Proof. A reasoning by iteration shows that for any $k \in \mathbb{N}$, any pole of $(T_\omega)^k f$ is either a pole of f or a zero of ω. Since X is compact, We have finitely many points to examine.
We first consider the case of a point p that is a pole of order m of f and a pole of order $-d_p$ of ω. Direct computation shows that p is a singularity of order $-m-1-d_p$. Therefore, if p is a simple pole of ω, p remains a pole of order m of $(T_\omega)^k f$ for any $k \in \mathbb{N}$. In contrast, if p is a pole of order at least two of ω, then $(T_\omega)^k f$ is holomorphic in p provided k is large enough. In both cases, p does not belong to $\mathcal{PPL}(\omega, f)$.

Now, we consider the case of a point p that is a zero of ω where f is locally factorized by ω. If p is not a pole of f, then Lemma 2.1 proves that p is not a pole of any function of sequence $f, T_\omega f, (T_\omega)^2 f, \dots$. Thus, we have proved that after finitely many steps, the total order of the poles of $(T_\omega)^k f$ outside $\mathcal{PPL}(\omega, f)$

stabilizes to the total order of the poles of f at simple poles of ω.

For any point p of $\mathcal{PPL}(\omega, f)$, it follows from Lemma 2.1 that there exists $k_p \in \mathbb{N}$ such that $(T_\omega)^{k_p} f$ has a pole in p. Then direct computation proves that the order of the pole increases by $d_p + 1$ for each iteration of differential operator T_ω. This ends the proof. $\square$

2.2. **Translation structures.** On a (possibly open) Riemann surface X, a nonzero meromorphic 1-form ω defines a geometric structure as follows. We denote by:

- X^* the surface punctured at the poles of ω;
- X^{**} the surface punctured at the zeros and the poles of ω.

Local primitives of differential ω are locally injective on X^{**}. They form an atlas of local biholomorphisms to $\mathbb{C}$. Since two local primitives of the same differential are defined up to the addition of a constant, transition maps between two distinct charts of the atlas are *translations* of the complex plane. Therefore, ω endows Riemann surface X with a *translation structure.* Pair (X, ω) is a *translation surface* (see [**?**] for general background on translation surfaces).

Differential dz endows the complex plane $\mathbb{C}$ with the standard Euclidean metric $|dz|$. Any chart ϕ of the translation atlas (in other words a local primitive of ω) conjugates the standard differential dz of the complex plane with differential ω defined on X: we have $\phi^* dz = \omega$. Consequently, we can think about a translation surface as formed by pieces of the standard flat plane glued together along translations.

Punctured surface X^{**} is thus endowed with a flat metric $|\omega|$. The latter extends naturally to zeros of ω. A neighborhoods of a zero of ω of order k is mapped (by any local primitive) to the complex plane by a ramified cyclic cover of degree $k+1$. It follows that the flat metric $|\omega|$ extends to such a zero as a conical singularity of angle $2(k+1)\pi$. Punctured surface X^* is an Euclidean surface with conical singularities (see [21]).

Remarkably, as soon as a meromorphic differential defined on a compact Riemann surface has poles, the metric structure it defines on the surface punctured at the poles is noncompact.

Lemma 2.3. *Let X be a compact Riemann surface endowed with a nonzero meromorphic 1-form ω. Then punctured surface X^* is a complete metric space for the singular flat metric $|\omega|$.*

Proof. We have to prove that any Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ of X^* converges to some limit point in X^*. Since X^* is locally compact and ω has finitely many poles, the question reduces to the only case where $(z_n)_{n\in\mathbb{N}}$ converges in X to a pole z_∞ of ω. We will prove that no such sequence is a Cauchy sequence.

Let $k \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$ be respectively the order and the residue of ω in pole z_∞. Up to a biholomorphic change of variable, we can assume that $z_\infty = 0$ and normalize ω to $(\frac{\lambda}{z} + \frac{1}{z^k})dz$ if $k \geq 2$ and $\frac{\lambda dz}{z}$ if $k = 1$ (see [5] for details on local models for poles in translation surfaces).

Up to taking a subsequence, we assume that sequence $(z_n)_{n\in\mathbb{N}}$ belongs to the domain of a unique chart $\phi : z \mapsto \lambda ln(z) - \frac{1}{(k-1)z^{k-1}}$ if $k \geq 2$ and $\phi : z \mapsto \lambda ln(z)$ if $k = 1$. Sequence $(z_n)_{n\in\mathbb{N}}$ being a Cauchy sequence with respect to metric $|\omega|$ precisely means that sequence $(\phi(z_n))_{n\in\mathbb{N}}$ is a Cauchy sequence of the flat plane and thus converges to some point of the plane. It is therefore disjoint from any small enough neighborhood of z_∞. This ends the proof. $\square$

2.3. **Locally isometric immersions of branched disks.** In order to generalize Voronoi diagrams to meromorphic functions defined on a translation surface, we introduce a class of probing objects containing conical singularities.

Definition 2.4. A *branched disk* (Δ, π_Δ) is a connected open Riemann surface endowed with a surjective holomorphic map $\pi_\Delta : \Delta \to \mathbb{D}$ where $\mathbb{D}$ is a centered open disk of the usual complex plane.
Δ is endowed with a singular metric obtained by the pullback of the standard Euclidean metric by the covering map π_Δ.
We refer to the fiber over the center of disk $\mathbb{D}$ by the covering map as the *central fiber* of Δ. We define the *radius* of Δ as the radius of disk $\mathbb{D}$ in the plane.

Provided that the ramification index of any point of Δ is finite, a branched disk is a translation surface (with boundary) where every chart of the atlas factorizes through π_Δ. In the following, we prove that locally isometric immersions of branched disks in a translation surface define calibrated neighborhoods of any point.

Lemma 2.5. *In a compact Riemann surface X endowed with a nonzero meromorphic 1-form ω, for any point z that is not a pole of ω and any radius $r > 0$, there is a unique (up to isomorphism) immersion $\Psi_{r,z}$ of a branched disk (Δ, π_Δ) such that:*

- *$\Psi_{r,z}$ is a local isometry between Δ and X (for the flat metric defined by ω);*
- *the radius of the branched disk is r;*
- *z belongs to the image of the central fiber of Δ ($\Psi_{r,z}$ is centered at z).*

Proof. We identify z with a point $\tilde{z}$ of the universal cover $\tilde{X^*}$ of the surface punctured at the poles of ω. Let ϕ be a local primitive ϕ of ω defined in a neighborhood of z and such that $\phi(z) = 0$. Primitive ϕ lifts to $\tilde{X^*}$ and extends by analytic continuation to a univalued holomorphic function $\tilde{\phi}$ on $\tilde{X^*}$.

For any $r > 0$, we denote by $\mathbb{D}_r$ the open centered disk of radius $r > 0$ in the complex plane. Preimage $\tilde{\phi}^{-1}\mathbb{D}_r$ is an open Riemann surface endowed with a holomorphic projection to $\mathbb{D}_r^\circ$. We denote by $\mathcal{A}_r$ the connected component of $\tilde{\phi}^{-1}\mathbb{D}_r^\circ$ containing $\tilde{z}$. Then $\mathcal{A}_r$ endowed with the restriction of $\tilde{\phi}$ is a branched disk. Projection of $\tilde{X^*}$ to X^* provides the locally isometric immersion of the branched disk.

Conversely, any locally isometric immersion $\Psi_{r,z}$ of a branched disk into X^* lifts to $\tilde{X^*}$. It is immediate that its image should coincide with $\mathcal{A}_r$. Two such locally isometric immersions are therefore conjugated by an automorphism of the branched disk. □

In the classical settings of the complex plane endowed with its Euclidean metric, Voronoi diagrams can be defined in terms of maximal disk embeddings whose image is disjoint from the set of the singularities. We introduce a similar notion of locally isometric immersions of branched disks.

Definition 2.6. We consider a compact Riemann surface X endowed with a nonzero meromorphic 1-form ω and a meromorphic function f. For any point z_0 that is not a pole of ω, the *critical radius* r_{z_0} of z_0 is the infimum of radii $r \in \mathbb{R}^+$ such that the image of the (unique up to isomorphism, see Lemma 2.5) locally isometric immersion centered at z_0 of branched disk of radius r is disjoint from $\mathcal{PPL}(\omega, f)$.

We refer to a locally isometric immersion centered at z_0 of branched disk of radius r_{z_0} as a *critical immersion*.

In a locally isometric immersion of a branched disk whose image is disjoint from the principal polar locus, the pullback of meromorphic function f to the branched disk (Δ, π_Δ) is compatible with projection π_Δ.

Lemma 2.7. *We consider a compact Riemann surface X endowed with a nonzero meromorphic 1-form ω and a meromorphic function f. Let z_0 be a point of X that is not a pole of ω and does not belong to the principal polar locus $\mathcal{PPL}(\omega, f)$. We denote by r_{z_0} the (nonzero) critical radius of z_0.*

For any radius $r \leq r_{z_0}$ and any locally isometric immersion Ψ centered at z_0 of a branched disk (Δ, π_Δ) of radius r, there is a holomorphic function $\tilde{f}$ defined on $\mathbb{D}_r$ such that the following diagram commutes:

$$\begin{array}{ccc} \Delta & \xrightarrow{\Psi} & X \\ {\scriptstyle \pi_\Delta}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D}_r & \xrightarrow{\tilde{f}} & \mathbb{C} \end{array}$$

Proof. Branched disk Δ is endowed with a meromorphic 1-form $\Psi^*\omega$. Projection π_Δ is a primitive of $\Psi^*\omega$. Besides, since the image of Ψ is disjoint from the poles of f, function $f \circ \Psi$ is holomorphic on Δ.

Every zero of ω in the image of Ψ is disjoint from $\mathcal{PPL}(\omega, f)$. It follows that in any small enough open subset of Δ, $f \circ \Psi$ factorizes through π_Δ (see Definition 1.3). We deduce by analytic continuation that $f \circ \Psi$ is constant on fibers of π_Δ. Consequently, there exists a holomorphic function $\tilde{f}$ such that $f \circ \Psi = \tilde{f} \circ \pi_\Delta$. □

The critical radius can be characterized analytically as a radius of convergence. In particular, provided that the principal polar locus is nonempty, every point has a finite critical radius.

Lemma 2.8. *We consider a compact connected Riemann surface X endowed with a nonzero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. Any point z that is not a pole of ω and does not belong to $\mathcal{PPL}(\omega, f)$ has a finite critical radius r_z. Besides, given a locally isometric immersion Ψ of a branched disk (Δ, π_Δ) of radius r_{z_0} centered in z_0 into X, $\tilde{f} = f \circ \Psi \circ \pi_\Delta^{-1}$ does not extend as a holomorphic function to the boundary circle $\partial\mathbb{D}_{r_{z_0}}$.*

Proof. ????????????????????????

For any $r < r_{z_0}$, we consider the locally isometric immersion Ψ_r of a branched disk $(\Delta_r, \pi_{\Delta_r})$. If $\tilde{f} = f \circ \Psi_r \circ \pi_\Delta^{-1}$ extends holomorphically to the boundary circle $\partial\mathbb{D}_r$, then there is a uniform $\epsilon > 0$ such that for any $z \in \partial\mathbb{D}_r$, $\tilde{f}$ extends holomorphically to a disk of center z and radius ϵ.

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Let $r_{z_0} > 0$ be the critical radius of z_0 ($r_{z_0} < +\infty$, see Lemma 2.8).

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Provided the principal polar locus is nonempty, every point has a finite critical radius.

Lemma 2.9. *For a nonzero meromorphic 1-form ω and a meromorphic function f defined on a compact connected Riemann surface X such that $\mathcal{PPL}(\omega, f) \neq \emptyset$, any point $z \in X$ that is not a pole of ω satisfies $r_z < +\infty$.*

Proof. We assume by contradiction that for some $z_0 \in X^*$, we have $r_{z_0} = +\infty$. Since X is connected and $\mathcal{PPL}(\omega, f) \neq \emptyset$, there is a path γ between z_0 and some point $z_1 \in \mathcal{PPL}(\omega, f)$.

Following Lemma 2.5 and Definition 2.6, path γ is contained in the image of some locally isometric immersion Ψ of a branched disk (Δ, π_Δ) centered at z_0. Lemma 2.7 then implies that $f \circ \Psi$ is holomorphic and constant on fibers of π_Δ. Therefore, z_1 cannot belong to $\mathcal{PPL}(\omega, f)$ and we get a contradiction. □

The critical radius is a radius of convergence.

Lemma 2.10. *For any point z_0 that is not a pole of ω and does not belong to $\mathcal{PPL}(\omega, f)$, we have $\nu_z \geq 1$.*

Proof. Let $r_{z_0} > 0$ be the critical radius of z_0 ($r_{z_0} < +\infty$, see Lemma 2.8). If $\tilde{f} = f \circ \Psi \circ \pi_\Delta^{-1}$ extends holomorphically to the boundary circle $\partial \mathbb{D}_{r_{z_0}}$, then there is a uniform $\epsilon > 0$ such that for any $z \in \partial \mathbb{D}_{r_{z_0}}$, f extends holomorphically to a disk of center z and radius ϵ.

Translation surface (X, ω) is in particular a complete metric space (see Lemma 2.3). Therefore, any point of the closure of the image of immersion Ψ is contained in the image of the immersion of a branched disk of radius ϵ. It follows that for any radius r smaller than $r_{z_0} + \epsilon$, the image of a locally isometric immersion centered at z_0 of a branched disk of radius r is disjoint from $\mathcal{PPL}(\omega, f)$. This is a contradiction. Therefore, there is at least one point in $\partial \mathbb{D}_{r_{z_0}}$, where $\tilde{f} = f \circ \Psi \circ \pi_\Delta^{-1}$ does not extend holomorphically. □

2.4. **Voronoi diagrams.** Translation surface (X, ω) is stratified according to the values of a *Voronoi index* ν defined in terms of immersions of branched disks.

Definition 2.11. We consider a compact connected Riemann surface X endowed with a nonzero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. For any point z of critical radius r_z that is not a pole of ω and does not belong to $\mathcal{PPL}(\omega, f)$, we define the *Voronoi index* ν_z of z in the following way.

Let Ψ be a locally isometric immersion of a branched disk (Δ, π_Δ) of radius r_z centered in z into X. The *Voronoi index* $\nu(z)$ of z is the number of points of the boundary circle $\partial \mathbb{D}_{r_z}$ where $\tilde{f} = f \circ \Psi \circ \pi_\Delta^{-1}$ does not extend as a holomorphic function.

Assuming that the principal polar locus is nonempty, it follows from Lemma 2.8 that every point of the surface satisfies $\nu \geq 1$. The Voronoi index decomposes to underlying surface into:

- Voronoi cells (where $\nu = 1$);
- Voronoi edges (where $\nu = 2$);
- Voronoi vertices (where $\nu \geq 3$).

Definition 2.12. We consider a compact Riemann surface X endowed with a nonzero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. The *Voronoi diagram* $\mathcal{V}_{\omega,f}$ of pair (ω, f) is the union of the points z of X such that $\nu_z \geq 2$.

Voronoi edges are straight segments with respect to the singular flat metric induced by differential ω.

Proposition 2.13. *We consider a compact Riemann surface X endowed with a nonzero meromorphic 1-form ω and a meromorphic function f such that $\mathcal{PPL}(\omega, f) \neq \emptyset$. Then $\mathcal{V}_{\omega,f}$ is a union of geodesics of singular flat metric $|\omega|$.*

Proof. ???????????????????? For any point $z_0 \in \mathcal{V}_{\omega,f}$, there is a locally isometric immersion Ψ centered at z_0 of a branched disk (Δ, π_Δ) whose radius is the critical radius of z_0 (see Lemma 2.5).

We first consider the case $\nu_{z_0} = 2$. Let α, β be the two points of $\partial\mathbb{D}_{r_{z_0}}$ where $f \circ \Psi$ does not extend holomorphically. Then, then there is $\epsilon > 0$ such that

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2.5. **Cauchy measure of a Voronoi diagram.**

Using central angles at points of $\mathcal{PPL}(\omega, f)$, we define the so-called *Cauchy measure* $\mu_{\omega,f}$ of Voronoi diagram $\mathcal{V}_{\omega,f}$.

Definition 2.14. ???The Cauchy measure of each segment is the central angle of the incident poles. Since there are finitely many edges (and the measure of each of them is at most π), this measure is finite.???

Remark 2.15. Cauchy measure $\mu_{\omega,f}$ is invariant under scaling $\omega \mapsto \lambda\omega$ (with $\lambda \in \mathbb{C}^*$).

3. Zero-free regions

??? Two cases depending whether $\nu_z = 1$ because of a regular pole of f or a zero of ω. In the first case, the classical proof works (with Cauchy formula).

3.1. **Disk Lemma.**

Lemma 3.1. *In the open centered disk $\mathbb{D}_r$ of radius $r > 1$, we consider a meromorphic function f with a pole in 1 and no other pole in $\mathbb{D}_r$.*

For any ϵ satisfying $\epsilon < \frac{r-1}{2}$ there exists $M > 0$ such that for any $n \geq M$, no zero of $f^{(n)}$ belongs to the open centered disk $\mathbb{D}_\epsilon$ of radius ϵ.

Proof. There are $\lambda \in \mathbb{C}^*$ and $m \in \mathbb{N}^*$ such that function g defined by $g(z) = f(z) - \frac{\lambda}{(z-1)^m}$ is holomorphic on $\mathbb{D}_r$. We consider some radius t satisfying $1 < t < r$ and denote by G the maximal value of $g(z)$ when $|z| = t$.

For some $\epsilon > 0$, $n \in \mathbb{N}$ and $z \in \mathbb{D}_\epsilon$, we apply Cauchy's integral formula to $g^{(n)}(z)$ and obtain $g^{(n)}(z) = \frac{n!}{2i\pi} \int\limits_{\theta=0}^{2\pi} \frac{g(te^{i\theta})}{(te^{i\theta}-z)^{n+1}}$. It follows that $|g^{(n)}(z)| \leq \frac{n!G}{(t-\epsilon)^{n+1}}$.

Since $f^{(n)}(z) = g^{(n)}(z) + \frac{\lambda(-1)^n n!}{(z-1)^{m+n}}$, we deduce that $\frac{|f^{(n)}(z)|}{n!} \geq \frac{|\lambda|}{(1+\epsilon)^{m+n}} - \frac{G}{(t-\epsilon)^{n+1}}$. If $\epsilon < \frac{t-1}{2}$, there exists a constant M such that $f^n(z) \neq 0$ for any $z \in \mathbb{D}_\epsilon$. □

Lemma 3.2. *Let f be a meromorphic function in an open disk $\mathbb{D}_r$ of radius R centered at 0, where $0 < R \leq \infty$. Suppose that f has at least 2 poles in $\mathbb{D}_r$. Then, for any z on the Voronoi diagram determined by the poles of f and $\epsilon > 0$, there exists a positive integer ℓ_0 such that $B(z, \epsilon)$ contains a zero of $f^{(\ell)}$ for any positive integer $\ell_0 > \ell$.*

Proof. Let z_0 be such that $\partial B(z_0, r)$ contains two poles ξ_1 and ξ_2 of f and has no other pole in the interior. Then, in this disk we may write f as

$$f(z) = \frac{c_1}{(\xi_1 - z)^{d_1}} + \frac{c_2}{(\xi_2 - z)^{d_2}} + \phi(z)$$

where ϕ is analytic in the open disk $B(z_0, r)$, $c_1, c_2 \in \mathbb{C}$, and d_1, d_2 are the orders of ξ_1, ξ_2, respectively. We have

$$\begin{aligned} \frac{1}{n!} f^{(n)}(z) &= \frac{c_1 (d_1)_n}{n!(\xi_1 - z)^{d_1+n}} + \frac{c_2 (d_2)_n}{n!(\xi_2 - z)^{d_2+n}} + \frac{\phi^{(n)}}{n!} \\ &= \frac{c_1 (d_1)_n (\xi_2 - z)^{d_2+n} + c_2 (d_2)_n (\xi_1 - z)^{d_1+n} + \phi^{(n)}(z)(\xi_1 - z)^{d_1+n}(\xi_2 - z)^{d_2+n}}{n!(\xi_1 - z)^{d_1+n}(\xi_2 - z)^{d_2+n}} \end{aligned} \tag{3.1}$$

Here, $(a)_n$ is the Pochhammer symbol denoting the ascending product of n consecutive integers, starting from a.

We will now try to apply Rouche's theorem.

Consider the polynomial

$$G_n(z) = c_1 \frac{(d_1)_n}{n!}(\xi_2 - z)^{d_2+n} + c_2 \frac{(d_2)_n}{n!}(\xi_1 - z)^{d_1+n}.$$

The generating function of $G_n(z)$ is $\frac{c_1(\xi_2-z)^{d_2}}{(1-\xi_2+z)^{d_1}} + \frac{c_2(\xi_1-z)^{d_1}}{(1-\xi_1+z)^{d_2}}$.

So, $G_n(z)$ satisfies

$$G_n(z) = \alpha_1(z;n)(\xi_1 - z)^n + \alpha_2(z;n)(\xi_2 - z)^n$$

Then, by BKW theorem, z belongs to the final set of $G_n(z)$.

Claim: On $\partial B(z_0, \epsilon)$, we have

$$|G_n(z)| > \frac{1}{n!}|\phi^n(z)(\xi_1 - z)^{d_1+n}(\xi_2 - z)^{d_2+n}|$$

for n large.

In particular, $B(z, \epsilon)$ contains a zero of $\frac{1}{n!}f^n(z)$ for n large.

□

Corollary 3.3. *For any meromorphic function f on $\mathbb{C}$, the accumulation set of the zeros of $f^{(n)}$ is contained in the edges of the Voronoi diagram determined by the poles of f.*

Proof. For any point z contained in the interior of the Voronoi cell of some pole p of f, there is disk centered in z containing p and no other pole of f. Lemma 3.1 proves the existence a small disk centered in 0 containing no zero of $f^{(n)}$ provided n is large enough. It follows that the accumulation set of the zeros of iterated derivatives of f is disjoint from the interiors of Voronoi cells. □

4. Asymptotic distribution on Voronoi edges

Three cases depending whether $\nu(z) = 2$ because of two, one or zero regular poles of f.

5. Examples and applications

5.1. **Rational 1-forms.** Let $X = \mathbb{P}^1$ and $\omega(z) = R(z)dz$, where $R(z)$ is a rational function. where z is an affine coordinate of $\mathbb{P}^1$. Then the operator T_ω in the affine coordinate z is given by

$$T_\omega := \frac{1}{R(z)}\frac{d}{dz}.$$

In the previous sections we discussed the asymptotic zero distributions of $T_\omega^n(f)$ which we illustrate below in several special cases.

5.2. **Monomial linear operator applied to a function with one simple pole.** For any $\ell \in \mathbb{Z}$, set $\omega_\ell = z^{-\ell}dz$ and $T_\ell = z^\ell \frac{d}{dz}$. Let us calculate $T_\ell^n\left(-\frac{1}{z+a}\right)$, where $a \neq 0$.

Lemma 5.1. *For $\ell \geq 1$, one has $T_\ell^n\left(-\frac{1}{z+a}\right) = \frac{z^{(\ell-1)n+1}U_n(z,a)}{(z+a)^{n+1}}$. Here $U_n(z,a)$ is a binary form of degree $n-1$ satisfying the recurrence relation*

$$U_{n+1}(z,a) = ((\ell-2)nz + ((\ell-1)n+1)a)U_n(z,a) + z(z+a)\frac{\partial U_n(z,a)}{\partial z},$$

with the initial condition $U_1(z,a) = 1$.

IT SEEMS THAT THE FORMULA WORKS EVEN FOR NEGATIVE ℓ!

Proof. We use induction on n. For $n = 1$, one has $z^\ell \frac{d}{dz}\left(-\frac{1}{z+a}\right) = \frac{z^\ell}{(z+a)^2}$ which gives the base of induction. Now assuming that the result holds for indices up to n let us apply $\mathcal{D}_4$ to the same function $(n+1)$ times. One has

$$T_\ell^{n+1}\left(-\frac{1}{z+a}\right) = T_\ell\left(T_\ell^n\left(-\frac{1}{z+a}\right)\right) = T_\ell\left(\frac{z^{(\ell-1)n+1}U_n(z,a)}{(z+a)^{n+1}}\right).$$

Further,

$$T_\ell\left(\frac{z^{(\ell-1)n+1}U_n(z,a)}{(z+a)^{n+1}}\right) = z^{(\ell-1)(n+1)+1}\frac{(((\ell-1)n+1)U_n + U_n')(z+a)^{n+1} - (n+1)(z+a)^n zU_n}{(z+a)^{2n+2}}$$

$$= z^{(\ell-1)(n+1)+1}\frac{(((\ell-1)n+1)U_n + zU_n')(z+a) - (n+1)zU_n}{(z+a)^{n+2}}$$

$$= z^{(\ell-1)(n+1)+1}\frac{((\ell-1)nz + ((\ell-1)n+1)a)U_n + z(z+a)U_n'}{(z+a)^{n+2}},$$

where $U_n' = \frac{\partial}{\partial z}U_n$. Thus $U_{n+1}(z,a) = ((\ell-2)nz+((\ell-1)n+1)a)U_n+z(z+a)U_n'$. □

An illustration of the above root distribution can be found in Fig. 2.

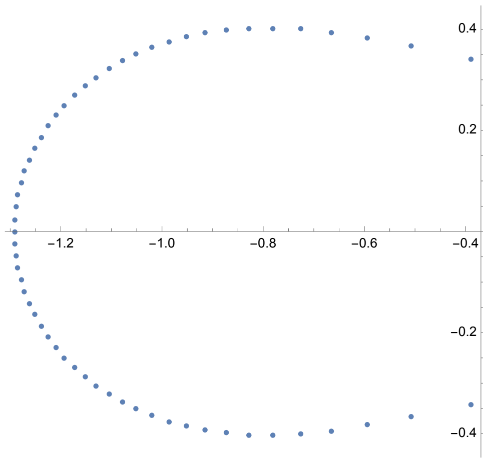

FIGURE 2. Roots of $U_n(z,a)$ for T_4^{65} and $a = 1$.

Guillaume's receipt for the formula of the respective lemniscate!

5.3. $T_2 = z^2\dfrac{d}{dz}$ **applied to an arbitrary rational function.** The simplest situation occurs for $\omega_2 = z^{-2}dz$, i.e. $T_2 = z^2\frac{d}{dz}$. In this case the coordinate for the flat metric induced by ω_2 is simply given by a new variable $y = -\dfrac{1}{z} \Leftrightarrow yz = -1$. In other words,

$$\frac{d}{dy} = z^2\frac{d}{dz} \Leftrightarrow \frac{dy}{dz} = \frac{1}{z^2}.$$

Consider the operator $T_2 = z^2\dfrac{d}{dz}$ and a rational function $f(z) = \sum_{i=1}^k \frac{\alpha_i}{z-z_i}, \quad \alpha_i \neq 0$. The limit set of $\mathcal{D}_2^n(R(z))$ is obtained as follows:

Theorem 5.2. *In the above notation, take* $y = -\dfrac{1}{z}$, *and let* $S^* = \left\{\dfrac{1}{z_1}, \dots, \dfrac{1}{z_k}\right\} \subset \mathbb{C}_y$ *be the set of the inverses to the poles of* $R(z)$ *in the* y*-plane. Take the Voronoi diagram* V^* *of* S^* *in the* y*-plane and consider its minus inverse in the* z*-plane. This is the limit set.*

Proof. With respect to the variable $y = -\frac{1}{z}$ the operator $\mathcal{D}_2$ coincides with $\frac{d}{dy}$. Thus

$$\mathcal{D}_2^n\left(\sum_{i=1}^{k}\frac{\alpha_i}{z-z_i}\right) = \frac{d^n}{dy^n}\left(\sum_{i=1}^{k}\frac{\alpha_i}{(-\frac{1}{y}-z_i)}\right) = -\frac{d^n}{dy^n}\left(\sum_{i=1}^{k}\frac{\alpha_i y}{1+z_i y}\right).$$

The set of poles for $\left(\sum_{i=1}^{k}\frac{\alpha_i y}{1+z_i y}\right)$ is given by $-S^* = \left\{-\frac{1}{z_1},\dots,-\frac{1}{z_k}\right\} \subset \mathbb{C}_y$. By the original Pólya's shire theorem the limiting root-counting measure of $\frac{d^n}{dy^n}\left(\sum_{i=1}^{k}\frac{\alpha_i y}{1+z_i y}\right)$ when $n \to \infty$ is supported on the Voronoi diagram of $-S^* = \left\{-\frac{1}{z_1},\dots,-\frac{1}{z_k}\right\}$. Since the map $z = -\frac{1}{y}$ is a biholomorphism between $\mathbb{C}P^1_z$ and $\mathbb{C}P^1_y$ the result follows. □

5.4. **Examples in genus** 1. Sangsan's section!

6. Outlook

1. Relevance for other fields: Lee-Yang zeros
Stationary phase approximation
Quantized potential theory

2. Since any rational function in 1 variable has more or less explicit primitive one can obtain rather explicit formulas for the accumulation set in case of genus 0.
3. The original shire theorem deals with $\mathbb{C}$ which is open and meromorphic functions on it. In particular, the main result of [10] shows that for an entire function of the form $R(z)EU(z)$ with polynomial U a certain part of the total mass of the limiting root-counting measure will be placed at ∞. In the present paper we only consider compact Riemann surface X, but appropriate modifications of our results should work for open X as well.

6.1. **The global geometry of translation surfaces.** An essential feature of the theory of translation surfaces is that the same objects have a complex-analytic side (a Riemann surface endowed with a holomorphic differential) and a geometric side (a polygon with pairs of sides identified by translations). Although these two descriptions are theoretically equivalent, going from one side to another is a delicate question in practice.

In a translation surface obtained by the gluing of a family of triangles, the lengths and the slopes of the edges are respectively the module and the argument of periods of the differential corresponding to these relative homology classes. However, starting with a complex structure (defined by a Fuchsian group for example) and an explicit holomorphic differential (in terms of modular forms), it is a difficult problem to determine which relative homology classes are represented by simple geodesic segments (in order to construct a triangulation).

In the current state of the art, the standard approach to obtain a geometric presentation of a translation surface is to discretize the circle of directions and integrate the differential equation corresponding to the differential form to find saddle connections.

If we can call the latter approach "classical", Theorem 1.5 suggests a "quantum" way from the complex-analytic data to the flat picture. For a given translation surface (X,ω), we consider a meromorphic function f with poles located at the zeros of ω. As $k \to +\infty$, zeros of $T^k_\omega f$ accumulate on the Voronoi diagram defined with respect to the zeros of ω. After an adequate number of iterations, the relative homology classes of the Delaunay triangulation (dual to the Voronoi tessellation) are characterized with a low error rate. EXAMPLE FIGURE GENUS ONE

6.2. **Fuchsian meromorphic connections.** Generalization to a even broader settings can be made as follows. Let X be a compact Riemann surface with a Fuchsian meromorphic connection ∇ on a line bundle $\mathcal{L}$. We can investigate the limit set of a global meromorphic section of $\mathcal{L}$ under iteration of ∇.

Fuchsian meromorphic connections induce complex affine structures (see [13]) providing local coordinates where ∇ is conjugated with $\frac{d}{dz}$. We still have a meaningful notion of affine disk immersion so Voronoi diagrams can be defined. Besides, the definition of Cauchy measures in terms of angles is suitable for a generalization to a complex affine settings (see Section 2.5). Nevertheless, an important difference with the current settings is that in most cases, meromorphic connections fail to be geodesically complete, as in the case of the Hopf torus $\mathbb{C}^*/\langle z \mapsto 2z \rangle$.

A family of Fuchsian meromorphic connection can already be handled with the methods of the current paper. A k-differential ω is a global meromorphic section of the k^{th} tensor power $K_X^{\otimes k}$ of the canonical bundle. In local coordinates, it is a complex analytic object of the form $h(z)dz^k$ where h is a meromorphic function. A k^{th} root $\omega^{1/k}$ of ω can be thought as a global meromorphic section of a line bundle twisted by some character χ valued in the complex multiplicative group $(\mathbb{C}^*, \times)$. Operator $T : f \mapsto \frac{df}{\omega^{1/k}}$ acting on the space of global sections of a suitable line bundle coincides with a Fuchsian meromorphic connection.

For a compact Riemann surface X endowed with a k-differential ω, the canonical k-cover (see [2] for details) is the smallest ramified cover $\pi : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ such that ω the k^{th} power of a globally defined meromorphic 1-form. This way, the limit set associated with operator T and some meromorphic section f of a line bundle $\mathcal{L}$ is the quotient of the limit set associated with an operator $T_{\tilde{\omega}^{1/k}}$ and a section of $\pi^*\mathcal{L}$ defined on a surface of higher genus $\tilde{X}$. The latter is obtained using Theorem 1.5.

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