TRAN’S CONJECTURE IS TRUE

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Abstract. In this note, we settle a conjecture of Khang Tran [6] claiming that for an arbitrary pair of polynomials \( A(z) \) and \( B(z) \), all zeros of every polynomial in the sequence \( \{P_n(z)\}_{n=1}^{\infty} \) satisfying the three-term recurrence relation of length \( k \)

\[ P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k} = 0 \]

with the standard initial conditions \( P_0(z) = 1, P_{-1}(z) = \cdots = P_{k-1}(z) = 0 \)

lie on the real (semi)-algebraic curve \( C \subset \mathbb{C} \) given by

\[ \Re \left( \frac{B_k(z)}{A(z)} \right) = 0 \quad \text{and} \quad 0 \leq (-1)^k \Re \left( \frac{B_k(z)}{A(z)} \right) \leq \frac{k}{(k-1)^{k-1}}. \]

1. Introduction

Linear recurrence relations with various types of coefficients have been studied for more than a century and appear in different contexts throughout the whole body of mathematics. In the present paper we discuss linear recurrences with fixed polynomial coefficients, settle a conjecture of Khang Tran [6] from 2014 and put it into a more general context.

The main object of our study is polynomial sequences \( \{P_n(z)\} \) which satisfy finite linear recurrence relations of the form

\[ P_n(z) + Q_1(z)P_{n-1}(z) + Q_2(z)P_{n-2}(z) + \cdots + Q_k(z)P_{n-k}(z) = 0 \quad (1.1) \]

and some initial conditions of the form

\[ P_0(z) = p_0(z), P_{-1}(z) = p_{-1}(z), \ldots, P_{-k+1}(z) = p_{-k+1}(z) \]

for some \( k \)-tuple \( In = (p_0(z), p_{-1}(z), \ldots, p_{-k+1}(z)) \) of initial polynomials.

A major result in the area of asymptotic behavior of roots of such \( \{P_n(z)\} \) was found in [1, 2]. It states that independently of the initial conditions the sequence \( \{\mu_n\} \) of the root-counting measures of \( \{P_n(z)\} \) converges in the weak sense to the measure \( \mu_{\mathcal{Q}} \) supported on \( \Gamma_{\mathcal{Q}} \), where \( \mathcal{Q} = (Q_1(z), Q_2(z), \ldots, Q_k(z)) \) and \( \Gamma_{\mathcal{Q}} \) is defined as follows. Consider the symbol equation of (1.1)

\[ t^k + Q_1(z)t^{k-1} + \cdots + Q_k(z) = 0. \quad (1.2) \]

For a given \( z \in \mathbb{C} \), let \( \tau_1(z) \geq \tau_2(z) \geq \cdots \geq \tau_k(z) \) be the \( k \)-tuple of absolute values of all \( k \) (not necessarily distinct) roots of (1.2) in the non-increasing order. Now define \( \Gamma_{\mathcal{Q}} \) as:

\[ \Gamma_{\mathcal{Q}} \subset \mathbb{C} \colon \{ z \in \mathbb{C} \mid \tau_1(z) = \tau_2(z) \}. \quad (1.3) \]

Definition 1. A pair \( (\mathcal{Q}, In) \) where \( \mathcal{Q} \) and \( In \) are as above is called exact if all roots of every polynomial \( P_n \), \( n = 1, 2, \ldots \) in (1.1) belong to \( \Gamma_{\mathcal{Q}} \).

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Obviously, exact pairs are rare since a small perturbation of $Jn$ typically destroys the exactness. It is unclear if for a given $\mathcal{Q}$ there exist some initial $k$-tuple $In$ which makes the pair $(\mathcal{Q}, In)$ exact. Typically, exact pairs occur when all polynomials in the sequence $\{P_n(z)\}$ are real-rooted, comp. [3, 5].

However in [6] K. Tran has found an interesting (conjectural) class of exact pairs of another sort and partially proved its exactness. Namely, Conjecture 6 of [6] claims the following.

**Conjecture A.** For an arbitrary pair of polynomials $A(z)$ and $B(z)$, all zeros of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length $k$

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k} = 0 \quad (1.4)$$

with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \cdots = P_{k-1}(z) = 0$ which do satisfy $A(z) \neq 0$ lie on the real (semi)-algebraic curve $\mathcal{C} \subset \mathbb{C}$ given by

$$\Re \frac{B^k(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq (-1)^k \Re \frac{B^k(z)}{A(z)} \leq \frac{k^k}{(k-1)^{k-1}}. \quad (1.5)$$

Moreover, these roots become dense in $\mathcal{C}$ when $n \to \infty$.

One can check that in this specific case, the latter curve $\mathcal{C}$ given by (1.5) is exactly the Beraha-Kahane-Weiss curve $\Gamma_{17}$. In [6] Conjecture A was proven for $k = 2, 3, 4$. In [7] Conjecture A was proven for arbitrary $k$, but only for polynomials $P_n$ with sufficiently large $n$. Several other aspects of this problem are discussed in [4, 8, 9]. The purpose if this note is settle Conjecture A in complete generality.

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### 2. Proofs

In this section we show that Conjecture A is true.

**Lemma 1.** For any $k \geq 2$, and $n = kf + i$, $i = 0, \ldots, k-1$,

$$P_{kf+i} = (-B)^i T^{(i)}_\ell(B^k, A), \quad (2.1)$$

where $T^{(i)}_\ell(u, v)$ is a binary form of degree $\ell$.

**Proof.** Obtained by induction using iteration of (1.4) with the standard initial conditions. \hfill $\square$

**Example 1.** For $k = 2$, $T^{(0)}_0 = T^{(1)}_0 = 1$, $T^{(0)}_1(u, v) = u - v$, $T^{(1)}_1(u, v) = (u - 2v)$, $T^{(0)}_2(u, v) = u^2 - 3uv + v^2$, $T^{(1)}_2(u, v) = u^2 - 4uv + 3v^2$.

**Example 2.** For $k = 3$, $T^{(0)}_0 = T^{(1)}_0 = T^{(2)}_0 = 1$, $T^{(0)}_1(u, v) = -u - v$, $T^{(1)}_1(u, v) = (-u - 2v)$, $T^{(2)}_1(u, v) = (-u - 3v)$, $T^{(0)}_2(u, v) = u^2 + 4uv + v^2$, $T^{(1)}_2(u, v) = u^2 + 3uv - v^2$, $T^{(2)}_2(u, v) = u^2 + 4uv + 2v^2$.

(CHECK CALCULATIONS!)
Lemma 2. For any $k \geq 2$, the $k$-tuple of sequences $\{T^{(i)}_\ell\}_{\ell=0}^\infty$, $i = 0, \ldots, k-1$ satisfy the recurrence relations:

$$
\begin{align*}
T^{(0)}_i &= (-1)^k u T^{(k-1)}_{i-1} - v T^{(0)}_{i-1}, \\
T^{(1)}_i &= T^{(0)}_i - v T^{(1)}_{i-1}, \\
T^{(2)}_i &= T^{(1)}_{i-1} - v T^{(2)}_{i-1}, \\
&\vdots \\
T^{(k-1)}_i &= T^{(k-2)}_{i-1} - v T^{(k-1)}_{i-1}.
\end{align*}
$$

(2.2)

Example 3. For $k = 2$, one has $T^{(0)}_0 = T^{(1)}_0 = 1$ and

$$
\begin{align*}
T^{(0)}_0 &= u T^{(1)}_0 - v T^{(0)}_0, \\
T^{(1)}_0 &= T^{(0)}_0 - v T^{(1)}_0.
\end{align*}
$$

Example 4. For $k = 3$, one has $T^{(0)}_0 = T^{(1)}_0 = T^{(2)}_0 = 1$ and

$$
\begin{align*}
T^{(0)}_0 &= -u T^{(1)}_0 - v T^{(0)}_0, \\
T^{(1)}_0 &= T^{(0)}_0 - v T^{(1)}_0, \\
T^{(2)}_0 &= T^{(1)}_0 - v T^{(2)}_0.
\end{align*}
$$

Proposition 1. For any $k \geq 2$, every $T^{(i)}_\ell(u, 1)$ is a real-rooted polynomial with integer coefficients. Moreover, for every fixed $i$ and any positive integer $\ell$, the roots of $T^{(i)}_{\ell-1}(u, 1)$ and $T^{(i)}_{\ell}(u, 1)$ are interlacing.

Proof. BLA □

CHECK with Andrei Martinez whether it is an AT-system (algebraic Tschebychev). Connection to multi-orthogonal polys.

Proof of Conjecture A. Equation (2.1) implies that if $n = k\ell + i$, then every root of $P_n(z) = P_{k\ell+i}(z)$ either is a root of $B(z)$ or is a root of $T^{(i)}_\ell(B(z), A(z))$. Since $T^{(i)}_\ell(u, v)$ is a binary form of degree $\ell$ such that $T^{(i)}_\ell(u, 1)$ is a real-rooted univariate polynomial with integer coefficients, then the value of the quotient $\frac{R^i_B(z)}{A(z)}$ has to be real. This is exactly the first condition in (1.5). Using the interlacing property and the fact that all roots of $P_n(z)$ belong to $C$ for sufficiently large $n$, we obtain that Conjecture A is valid. □

3. Final remarks

1. Find other examples/classes of exact pairs $(\mathbb{Q}, In)$. In particular, if one finds other multiorthogonal polynomials related to finite recurrence relations, do they lead to an exact situation?

References


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