O'K

DISCONJUGATE LINEAR ORDINARY DIFFERENTIAL EQUATIONS, FLAG VARIETIES AND CLASSIFICATION OF SYMPLECTIC LEAVES IN SEVERAL KAC-MOODY AND GELFAND-DIKII ALGEBRAS.

DOCTORAL DISSERTATION
BY
BORIS SHAPIRO

To be publicly discussed in the
Lecture Hall of the institute of Physics, Vanadisvägen 9, Stockholm,
Friday June 1, 1990, at 13.00,
for the degree of of Doctor of Philosophy

Department of Mathematics
University of Stockholm
Box 6701
S-113 85 Stockholm

Stockholm 1990
ISBN 91-7146-830-7

ABSTRACT

In the first paper of this thesis there is systematically studied the relation between the oscillatory properties of linear differential equations and behaviour of the induced left-invariant flow on the space of complete real flags. We estimate the ratio of the number of zeros for sequential Wronskians of two arbitrary fundamental solutions of any linear ordinary differential equation and thus obtain generalized Sturmian separation theorem for equations of arbitrary order. We show that the singularities of the boundary of the domain containing disconjugate equations are diffeomorphic to the generic sections of the so called trains (hypersurfaces consisting of all Schubert cells of positive codimension in the space of complete real flags).

In the second paper we show that the symplectic leaves in Kac-Moody algebras are enumerated by the monodromy operator and connected components in the space of curves on corresponding Lie groups with given monodromy operator and completely calculate these invariants in the cases of $\widehat{GL_3^+}$ and $\widehat{SO_n}$.

In the third and fourth papers we enumerate symplectic leaves of the Zamolodchikov lgebra and obtain the estimation of the number of leaves corresponding to the identity nonodromy for equations of the high order. In the last two papers we study Euler char-cteristics and its complex analogues of the (above mentioned) trains and their links and rove the coincidence of the sums of Betti numbers for arrangements of real trains and eir complexifications in the space $\mathbf{PT^*P^n}$.

DISCONJUGATE LINEAR ORDINARY DIFFERENTIAL EQUATIONS, FLAG VARIETIES AND CLASSIFICATION OF SYMPLECTIC LEAVES IN SEVERAL KAC-MOODY AND GELFAND-DIKII ALGEBRAS.

Boris Shapiro

MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
1990

Department of Mathematics
University of Stockholm
Box 6701
S-113 85 Stockholm

ABSTRACT

In the first paper of this thesis there is systematically studied the relation between the oscillatory properties of linear differential equations and behaviour of the induced left-invariant flow on the space of complete real flags. We estimate the ratio of the number of zeros for sequential Wronskians of two arbitrary fundamental solutions of any linear ordinary differential equation and thus obtain generalized Sturmian separation theorem for equations of arbitrary order. We show that the singularities of the boundary of the domain containing disconjugate equations are diffeomorphic to the generic sections of the so called trains (hypersurfaces consisting of all Schubert cells of positive codimension in the space of complete real flags).

In the second paper we show that the symplectic leaves in Kac-Moody algebras are enumerated by the monodromy operator and connected components in the space of curves on corresponding Lie groups with given monodromy operator and completely calculate these invariants in the cases of $\widehat{GL_3^+}$ and $\widehat{SO_n}$.

In the third and fourth papers we enumerate symplectic leaves of the Zamolodchikov algebra and obtain the estimation of the number of leaves corresponding to the identity monodromy for equations of the high order. In the last two papers we study Euler characteristics and its complex analogues of the (above mentioned) trains and their links and prove the coincidence of the sums of Betti numbers for arrangements of real trains and their complexifications in the space $\mathbf{PT^*P}^n$.

ISBN 91-7146-830-7
Akademitryck AB, Edsbruk 1990

INTRODUCTION

The results of this thesis are published in the following 6 papers:

[1]. B. Z. Shapiro, *Spaces of linear ordinary differential equations and flag varieties*, Izvesti Akad.Nauk USSR **54** 1 (1990), p. 25-37.
[2]. B. Z. Shapiro, *Classification of symplectic leaves in Kac-Moody algebras* $\widehat{\mathbf{GL}_3}$ *and* $\widehat{\mathbf{SO}_n}$ Funkt. Analyz i ego Prilozh. (to appear).
[3]. B. Z. Shapiro, *Discrete invariants of symplectic leaves for Zamolodchikov algebra an nondegenerate curves on* $\mathbf{S}^2$*, preprint, Moscow State University.*
[4]. B. Z. Shapiro, *On the* π_0 *of the space of closed nondegenerate curves on* $\mathbf{S}^n$, Bull. c Amer. Math. Soc. (to appear).
[5]. B. Z. Shapiro and M. Z. Shapiro, *The M-property of flag manifolds*, Madison centre c Mathematics publ. (to appear).
[6]. B. Z. Shapiro and A. D. Vainshtein, *Euler characteristics for links of Schubert cells i the space of complete flags*, in "The theory of singularities and its applications (serie Advances in Soviet Mathematics)," AMS publications, Providence RI, 1990.

A major portion of the study of the qualitative nature of solutions of differential equation may be traced to the famous 1836 paper of Sturm ([S]) dealing with oscillation, separatioı and comparison theorems for linear ordinary homogeneous second order differential equa tions. The associated work of Liouville introduced a type of boundary problem known as ; "Sturm-Liouville" problem involving an introduction to the study of asymptotic behavio of solutions of the linear second order differential equations by the use of integral equations

In the subsequent years the significance of the calculus of variations for such boundar problems was emphasized by Bliss and Morse. In particular, Morse ([M]) showed tha variational principles provided an appropriate environment for the extension of the self adjoint differential systems of the classical Sturmian theory.

The study of the action of the following group of transformations (in the modern terms the loop group of the trivial bundle $\mathbf{R}^1 \times \mathbf{S}^1$):

$$\begin{cases} z = z(x) \\ y(z) = m(x)y(x) \end{cases}$$

on the space of differential third order equations was started by Kummer in 1834 ([K]).

Further progress in this area (in the 19-th century) is connected with the expansion of Kummer's arguments to the case of linear ordinary differential equations of high order. Kummer's followers (in particular Laguerre, Brioschi, Halphen, Forsyth, Lie and Appell) studied high order equations in connection with the so called equivalence problem. Lie and Wilczynski proved that the above transformation is the most general change of variables preserving linearity and order of equations.

In the case of the second order equations the study of their normal forms under the action of the above group which for Hill equations coincides with the nontrivial extension of the group of diffeomorphisms of the circle (sometimes called the Virasoro group) was undertaken in different contexts (as orbits of the coadjoint representation of the Virasoro groups or projective structures on $\mathbf{S}^1$, etc.) by Kuiper, Lazutkin and Pankratova, Kirillov, Segal ([**Ku**,L-P,Ki,Se]).

For linear equations of high orders local normal forms were considered by Neuman ([**Ne**]). Supersymmetric generalizations of these problems (first investigated by Kirillov) were considered in [**Le**].

Qualitative theory of high order equations is hard apparently because there is no natural group giving satisfactory global classification of such equations. Still, the space of all linear ordinary differential equations of the given order has the structure of a Poisson manifold and one can calculate invariants of its symplectic leaves. A step in this direction is performed in this thesis. But this approach practically completely ignores global qualitative behavior of equations.

Recall that the linear ordinary n-th order differential equation (l.o.d.e.) given on the time interval I is called **disconjugate** if its arbitrary solution has on I at most $n-1$ zeros counted with multiplicities and **conjugate** otherwise. This property was studied by Vallee-Poussin, Markov, Polia, Hartman, Levin, Coppel, Reid, Sherman and others ([**P**,H,L,C,R,Sh]). Disconjugate equations form an important qualitative class. One can show that their behavior is similar in the following sense.

To an arbitrary solution ϕ of l.o.d.e. given on $I=[0,1]$ assign the set of multiplicities $K_\phi=(k_1,\dots,k_{l_\phi})$ of its sequential zeros. For two disconjugate equations u and w there exists a homogeneous diffeomorphism D of the spaces of their solutions preserving sets of multiplicities, i.e. $K_\phi=K_{D\phi}$. The study of this equivalence on the space of l.o.d.e. is a very promising problem.

Several positive results of the qualitative nature for the third and fourth order equations were obtained by Azbelev, Kondratiev, Levin ([**Az**,Ko,L]) and some others and there were constructed a lot of counterexamples. The simpliest of them (and at the time rather surprising) is Sansone' result of 1948 ([**Sa**]) on the third order l.o.d.e. all whose solutions are both sides oscillatory. It uses representation of a third order l.o.d.e. as a curve in $\mathbf{R}^3$ (whose coordinates are fundamental solutions of l.o.d.e.). On Fig.1 a 'prolonged cycloid' ν goes infinitely many times around the equator of the unit sphere $\mathbf{S}^2$ if time goes from $-\infty$ to $+\infty$. This curve may serve as a fundamental solution of l.o.d.e. and each linear hyperplane intersects ν infinitely many times.

More successful was the development of the qualitative theory in the context of Hamiltonian systems. Morse obtained the formula for the index of the extremal solution in positive definite variational problem with given boundary conditions in terms of the number of conjugate points and some residual term called 'the order of concavity'. Generalized

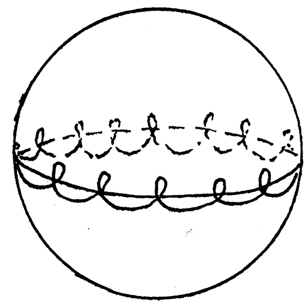


Fig.1.

Sturmian theory for linear Hamiltonian systems was developed by Lidskii, Bott, Edwards Duistermaat, Klingenberg and Arnold ([**Li**,B,E,D,A]). The main notion of this theory is the (induced by the system) flow on the space of Lagrangian Grassmann manifold (the space of all Lagrangian planes in $\mathbf{R}^{2n}$) and intersection of the orbits of this flow with Lagrangian trains (hypersurfaces of all Lagrangian planes nontransversal to the given one).

Methods of Hamiltonian Sturmian theory prompt the idea to consider instead of different individual solutions of a given l.o.d.e. its different fundamental solutions and this leads to some interesting qualitative results listed below.

While studying properties of l.o.d.e. we use Schubert stratification of the space of complete flags. Properties of these stratifications are of the independent interest and have many applications.

Now, we pass to detailed description of the results.

Paper 1.

Consider a linear ordinary n-th order differential equation (l.o.d.e.) given on the interval $I=[0,1]$:

$$L_n[x]=x^{(n)}+a_1(t)x^{(n-1)}+\dots+a_n(t)x=0 \qquad (1)$$

where $a_i(t)\in C^\infty[I]$.

DEFINITION. Let $\mathbf{F}_n$ be the variety of all complete flags in the linear space $\mathbf{V}$ of solutions of l.o.d.e. (1). **A flag curve fc** of l.o.d.e. (1) is a map $\mathbf{fc}:\mathbf{I}\to\mathbf{F}_n$ sending each moment $t\in\mathbf{I}$ onto a complete flag in $\mathbf{V}$ whose i-dimensional subspace consists of all solutions which have zero of multiplicity $\geq n-i$ at the moment t.

Recall that with each complete flag $\mathbf{f}$ there is related its Schubert cell decomposition $Sch_{\mathbf{f}}$ of the space $\mathbf{F}_n$ whose cells consists of all flags having with the subspaces of the given flag $\mathbf{f}$ given set of dimensions of intersections.

DEFINITION. A set of complete flags in $\mathbf{V}$ is called **generic** if any subset of Schubert cells belonging to different flags intersect transversally. Two complete flags are called **transversal** if they form a generic set (i.e. their subspaces are transversal) and **nontransversal** otherwise. The set $\mathbf{Tn_f}$ of all complete flags nontransversal to the given flag $\mathbf{f}$ is called the **train of f**.

Note that $\mathbf{Tn_f}$ consists of all positive-codimensional cells of $Sch_{\mathbf{f}}$. $\mathbf{Tn_f}$ is reducible and consists of $n-1$ (where n is dimension of the space) irreducible components $\mathbf{Tn}^i_{\mathbf{f}}$ consisting of all flags whose $(n-i)$-dimensional subspace is nontransversal to the i-dimensional subspace of $\mathbf{f}$.

THEOREM A. *The following three conditions are equivalent:*

(1) *equation (1) is conjugate on $I = [0,1]$;*
(2) *there exists a moment $t \in (0,1]$ such that a flag $\mathbf{fc}_t$ from the flag curve $\mathbf{fc}$ of equation (1) is nontransversal to $\mathbf{fc}_0$;*
(3) *the flag curve of equation (1) intersects the train of arbitrary flag.*

THEOREM B. *The sum of local multiplicities of intersection of the flag curve $\mathbf{fc}$ with $\mathbf{Tn}^i_g$ of arbitrary flag $\mathbf{g}$ (equal to the sum of dimensions of intersections of the i-dimensional subspace of the flag curve with the $(n-i)$-dimensional subspace of the flag $\mathbf{g}$) for a disconjugate equation (1) does not exceed $i(n-i)$.*

COROLLARY C (GENERALIZED STURM SEPARATION THEOREM).
If the sum of local multiplicities of the flag curve $\mathbf{fc}$ of some equation (1) with $\mathbf{Tn}^i_{\mathbf{g}}$ of some flag $\mathbf{g}$ exceeds $i(n-i)$ on some interval $\mathbf{I}$ then on this interval the equation is conjugate and thus intersects the train of arbitrary flag at least once.

COROLLARY D. *If $\sharp_1$ and $\sharp_2$ are positive sum of multiplicities of intersection of the flag curve of equation (1) with trains of two arbitrary flags $\mathbf{g}_1$ and $\mathbf{g}_2$ then*

$$\frac{6}{n^3-n+6} < \frac{\sharp_1}{\sharp_2} < \frac{n^3-n+6}{6}$$

Paper 2.

The space of operators $\{\frac{d}{dx} + A(x), A \in C^\infty(S^1, \mathcal{G})\}$, where $\mathcal{G}$ is a reductive Lie algebra is naturally identified with the dual space $\hat{\mathcal{G}}^*$ to the Kac-Moody algebra $\hat{\mathcal{G}}$ (which is the nontrivial 1-dimensional central extension of the loop algebra $\tilde{\mathcal{G}} = C^\infty(S^1, \mathcal{G})$, see [R-ST]). Under this identification the coadjoint action of $P \in \tilde{G}$ in $\hat{\mathcal{G}}$ (for $a \neq 0$) coincides with the gauge action on differential operators. Thus gauge classification of differential operators is equivalent to classification of the orbits of the coadjoint action in Kac-Moody algebras which are maximal nondegenerate submanifolds of the linear Poisson structure (i.e. symplectic leaves of Poisson-Lie bracket sometimes called Berezin-Kirillov bracket). At the same time the description of the classes of the first order matrix linear differential equations $\{(\frac{d}{dx} + A(x))\Psi = 0, \quad A \in C^\infty(S^1, \mathcal{G})\}$ of the above type with respect to gauge equivalence:

$\Psi \to P\Psi$ (or $A \mapsto P^{-1}P' + P^{-1}AP$), where $P \in \tilde{G} = C^\infty(S^1, \mathbf{G})$ is a well known problem of analysis. Denote by $\tilde{G}_\circ = C^\infty_\circ(S^1, \mathbf{G})$ the connected component of $\tilde{G}$ containing the trivial map onto the unit matrix (the components of $\tilde{G}$ are enumerated by the elements of $\pi_1(\mathbf{G})$).

FLOQUETS' THEOREM, SEE ([H]). *The only invariant of the matrix equation on the circle w.r.t. the action of gauge group $\tilde{G}$ is the conjugacy class of its monodromy operator in $\mathbf{G}$; w.r.t. the gauge transformations from $\tilde{G}_\circ$ the only invariant is the conjugacy class of its monodromy operator in the universal covering $\mathbf{G}_\circ$ of the group $\mathbf{G}$.*

THEOREM 2. *The conjugacy class of any element $M \in \mathbf{GL}_3^+(R)$ with the Jordan normal form (J.n.f.) different from the following one:*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

where $\lambda, \nu < 0$ ($\lambda = \nu$ is admissible) has two inverse images in the universal covering $\mathbf{GL}_3^\circ$ under the natural projection $\mathbf{GL}_3^\circ \to \mathbf{GL}_3^+$; conjugacy class of the operator with the above Jordan normal form has a connected inverse image in $\mathbf{GL}_3^0$.

THEOREM 2′. *The symplectic leaves of the linear Poisson structure on the Kac-Moody algebras related to $\mathbf{GL}_3^+$ and $\mathbf{SL}_3$ are enumerated by the real parameter $a \neq 0$, J.n.f. of operators and $\mathbf{Z}/2\mathbf{Z}$-invariant for any J.n.f. except one from Theorem 2.*

THEOREM 3. *The conjugacy class of any operator $M \in \mathbf{SO}_n$ has connected inverse image in $\mathbf{Spin}_n$ if and only if its spectrum contains both 1 and -1. Otherwise the inverse image consists of 2 components.*

Paper 3.

THEOREM A. *The number of connected components in the space $\mathcal{RS}^2(f_1, f_2)$ of right-oriented curves on $\mathbf{S}^2$ with given initial and final flags (f_1, f_2) equals 3 if there exists a (nonstrictly) disconjugate curve $\in \mathcal{RS}^2(f_1, f_2)$ and 2 otherwise.*

An oriented flag on $\mathbf{S}^2$ consists of a point p and an oriented circle C. One of the two open halfcircles into which C is divided by the points $p, \bar{p}$, where $\bar{p}$ is the point antipodal to p is called positive (negative) if a small positive (negative) push of the point p belongs to it and is denoted by C_p^+ (C_p^-). Given orientations of $\mathbf{S}^2$ and of the 'big circle ' C we can define upper- and lower hemispheres H_C^+ and H_C^- of $\mathbf{S}^2 \setminus C$ (the end of the right-oriented vector complementing the right-oriented pair of vectors in the plane containing $\mathbf{S}^2$ must intersect the upper hemisphere).

THEOREM B. *Flags $f_1 = (p_1, C_1)$ $f_2 = (p_2, C_2)$ can be connected by a right-oriented disconjugate curve if they form one of the following configurations:*

(1) *p_2 belongs to the upper hemisphere $H^+_{C_1}$ and C^+_{2,p_2} intersects with C^-_{1,p_1};*
(2) *p_2 belongs to the upper hemisphere $H^+_{C_1}$ and C^+_{2,p_2} passes through p_1;*
(3) *p_2 belongs to C^-_{1,p_1} and p_1 belongs to $H^+_{C_2}$;*
(4) *p_1 coincides with p_2 and tangent vectors to C^+_{2,p_2} and C^+_{1,p_1} defines the orientation of $\mathbf{S}^2$ opposite to the given one;*

(5) C_1 coincides with C_2 and p_2 belongs to C^-_{1,p_1};
(6) the flags f_1 and f_2 coincide.

Denote the space of all right-oriented conjugate curves given on $I = [0,1]$ with the fixed initial flag f by $CN(f)$ and the map sending each curve to its final flag by $\pi : CN(f) \to \mathbf{FO}_3$.

THEOREM C. *The map $\pi : CN(f) \to \mathbf{FO}_3$ satisfies the covering homotopy property.*

COROLLARY. *The space of the third order l.o.d.e. with all 1-periodic solutions consists of two disconnected parts one of which is contractible and the other is homotopically equivalent to the space of all closed paths on $\mathbf{FO}_3 \cong \mathbf{SO}_3$.*

The second part of the paper is devoted to the calculation of discrete invariant of symplectic leaves of Zamolodchikov algebra. Notice that the operators belonging to $\mathbf{GL}_3^+$ have one of the following 10 real Jordan normal forms (J.n.f.):

$$\text{a)}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \quad \text{b)}\begin{pmatrix} -a^2 & 0 & 0 \\ 0 & -b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \quad \text{c)}\begin{pmatrix} \lambda\cos\alpha & \lambda\sin\alpha & 0 \\ -\lambda\sin\alpha & \lambda\cos\alpha & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

$$\text{d)}\begin{pmatrix} a^2 & 1 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \quad \text{e)}\begin{pmatrix} -a^2 & 1 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \quad \text{f)}\begin{pmatrix} a^2 & 1 & 0 \\ 0 & a^2 & 1 \\ 0 & 0 & a^2 \end{pmatrix}$$

$$\text{g)}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \quad \text{h)}\begin{pmatrix} -a^2 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \quad \text{i)}\begin{pmatrix} a^2 & 1 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}$$

$$\text{j)}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}$$

THEOREM D. *The symplectic leaves of Zamolodchikov algebra are enumerated by the J.n.f. of monodromy operator and the invariant from $\mathbf{Z}/2\mathbf{Z}$ for the types b), g), h), i) and the invariant from $\mathbf{Z}/3\mathbf{Z}$ in the rest of the cases.*

Paper 4. We formulate here some new conjectures connected with the main result of paper 4.

CONJECTURE 1. *The space of the n-th order l.o.d.e. given on $I = [0,1]$ with the given monodromy matrix (Wronsky matrix $W(1)$ for the fundamental solution $\phi_0, \dots, \phi_{n-1}$ satisfying the relation $\phi_i^j(0) = \delta_i^j \quad i,j = \overline{0, n-1}$) consists of 3 connected components if there exists a disconjugate l.o.d.e. with given monodromy matrix and 2 otherwise.*

CONJECTURE 2. *The map $\phi : CD_n \to \mathbf{GL}_n^+$ from the space of l.o.d.e. given on $I = [0,1]$ sending each equation onto its monodromy matrix has the covering homotopy property.*

COROLLARY OF CONJECTURE 2. *The space PD_n of all l.o.d.e. whose solutions are periodic:*
(1) *is homotopically equivalent to the space of all closed curves on $\mathbf{SO}_n$ passing through a given point for even n;*

(2) *consists of two parts one of which is contractible and consists of (nonstrictly) disconjugate equations and the other one is homotopically equivalent to the space of all closed curves on $\mathbf{SO}_n$ passing through the given point for odd n.*

The result of the paper is the following estimation.

THEOREM A. *Card $\pi_0(PD_n) \geq 3$ for odd n.*

Paper 5.
A real algebraic variety $X^{\mathbf{R}}$ (the set of real points of $X^{\mathbf{C}}$) is called an M-manifold if $\sum b_i(X^{\mathbf{R}}) = \sum b_i(X^{\mathbf{C}})$. We shall also say in this case that $X^{\mathbf{R}}$ has the M-property.

DEFINITION: Two (incomplete in the general case) flags in $\mathbf{P}^n$ are called **transversal** if the intersection of any pair of their subspaces is of the minimal possible dimension.

Let $\mathbf{PT}^*\mathbf{P}^n$ denote the manifold of all flags in $\mathbf{P}^n$ consisting of a hyperplane and a distinguished point in it. (Notice that two flags belonging to $\mathbf{PT}^*\mathbf{P}^n$ are transversal if it holds for both that the distinguished point of one does not belong to the hyperplane of the another).

Hereinafter the term **'the set of flags in general position'** means that it belongs to some open dense domain in the space of all sets.

THEOREM A. *The locus of all flags from $\mathbf{PT}^*\mathbf{P}^n$ which are transversal to each flag from a given set 'in general position' has the M-property.*

THEOREM B. *There exists an open set of 4-tuples of real lines in $\mathbf{P}^3$ (flags in $\mathbf{F}^4$) such that the corresponding locus of all lines (flags) transversal to all lines (flags) from the given set does not possess the M-property.*

THEOREM 5.1. *For a set of flags $\hat{\mathbf{f}} = \{\mathbf{f}_1, \dots, \mathbf{f}_k\}$ on $\mathbf{RP}^2$ 'in general position' the variety $\mathcal{M}_{\hat{\mathbf{f}}}^{\mathbf{R}}$ of all flags transversal to all flags belonging to $\hat{\mathbf{f}}$ is homeomorphic to the disjoint union of $k^3 - k + k$ three-dimensional cells.*

THEOREM 5.2. *For a set of flags $\hat{\mathbf{f}} = \{\mathbf{f}_1, \dots, \mathbf{f}_k\}$ on $\mathbf{CP}^2$ 'in general position' the variety $\mathcal{M}_{\hat{\mathbf{f}}}^{\mathbf{C}}$ of all flags transversal to all flags belonging to $\hat{\mathbf{f}}$ has no torsion in the homology with coefficients in $\mathbf{Z}$ and its Betti numbers are as follows $b_0 = 1$, $b_1 = 2(k-1)$, $b_2 = 2(k-1)^2$, $b_3 = (k-1)^3$, $b_i = 0$ for $i \geq 4$.*

Paper 6.
Let c_σ be a cell from the decomposition Sch_f of the space of real or complex complete flags $\mathbf{F}_n$ (k denotes the ground field) corresponding to the permutation σ and B a sufficiently small $n(n-1)/2$-dimensional (over the ground field $\mathbf{k}$) ball with the origin at some point of c_σ.

DEFINITION. The manifold $A_\sigma = B \setminus \mathbf{Tn}_f$ will be called the link of the cell c_σ . By χ_σ we denote the Euler characteristics of A_σ :

$$\chi_\sigma = \sum_k (-1)^k \dim H^k(A_\sigma).$$

In the complex case let us introduce also the numbers

$$\chi_\sigma^{pq} = \sum_k (-1)^k \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^k(A_\sigma)$$

where Gr^W and Gr_F are the associated graded objects of the weight and the Hodge filtrations, respectively.

The n-dimensional $\mathbf{k}$-torus $T_n = (\mathbf{k} \setminus 0)^n$ acts on the manifold A_σ . In the coordinate representation given in §2 this action can be described as expansions and contractions of basis vectors. The orbits of this action have the following convenient description.

Consider the mapping sending each flag from A_σ to its line. This line determines an n-dimensional vector of 0's and 1's whose i-th coordinate equals 0 if the line lies in the subspace spanned by $e_{\sigma(1)}, \dots, e_{\sigma(i-1)}, e_{\sigma(i+1)}, \dots, e_{\sigma(n)}$ and 1 otherwise. The transversality of all flags from A_σ to the given pair of flags (see Lemma 2.3) implies that two coordinates (coinciding if the hyperplanes of these two flags coincide) of this vector must equal 1. Finally clear that two flags determine the same 0-1-vector if and only if they both belong to the same orbit. Therefore the orbits are enumerated by 0-1-vectors having 1's at two prescribed places (possibly coinciding):

$$A_\sigma = \bigcup_{w \in W_\sigma} O_{w\sigma} \tag{1}$$

where

$$W_\sigma = \{w = (w_1, \dots, w_n) \in \{0,1\}^n : w_1 = w_{\sigma^{-1}(n)} = 1\},$$

and $O_{w\sigma}$ is the orbit in A_σ corresponding to the vector w. The orbit $O_{w\sigma}$ is diffeomorphic to the product of several copies of $\mathbf{k}^*$ by the manifold A_π for a certain permutation $\pi \in \mathbf{S}_{n-1}$.

Let $\sigma \in \mathbf{S}_n$, $w \in W_\sigma$. To each pair (σ, w) assign a permutation $\pi(\sigma, w) \in \mathbf{S}_{n-1}$ in the following way.

Given an arbitrary sequence $I = \{i_1, \dots, i_k\}$ denote by $R(I)$ the sequence obtained by the following process: put

$$r_1 = i_1, \qquad r_l = \max\{r_{l-1}, i_l\}, \quad 1 < l \le k,$$

and delete all elements except the first one from each group of consecutive equal elements of the sequence $\{r_1, \dots, r_k\}$. Now let $I(w)$ be the ordered sequence of the numbers i such that $w_i = 1$. Put $J_\sigma(w) = \sigma^{-1}R(\sigma I(w))$; then a relation $1 = j_1 < j_2 < \dots < j_m = \sigma^{-1}(n)$ is obviously valid, m being the number of elements in $J_\sigma(w)$. Define $\pi(\sigma, w)$ by the formulas

$$\begin{aligned} \pi(\sigma, w)(i) &= \sigma(i+1) \quad \text{if } i \ne j_l - 1, & 2 \le l \le m, \\ \pi(\sigma, w)(j_l - 1) &= \sigma(j_{l-1}), & 2 \le l \le m. \end{aligned} \tag{2}$$

3.2. LEMMA. *The manifold $O_{w\sigma}$ is diffeomorphic to the direct product of $A_{\pi(\sigma,w)}$ by the variety $(\mathbf{k}^*)^{n(w)-1}$, where $n(w)$ is the number of nonzero entries in w.*

5.1. THEOREM. *In the real case*

$$\chi_\sigma = \sum_{w \in W_\sigma} (-1)^{n-n(w)} 2^{n(w)-1} \chi_{\pi(\sigma,w)},$$

where $n(w)$, as before, is the number of unitary entries in w.

Suppose X is an arbitrary complex manifold; denote by

$$P_X(t) = \sum_i \chi^{ii}(X) t^i,$$

$$h_k^{ij}(X) = \dim \mathrm{Gr}_F^i \mathrm{Gr}_{i+j}^W H^k(X).$$

5.3. THEOREM. *Put $P_\sigma(t) \equiv P_{A_\sigma}(t)$, then*

$$P_\sigma(t) = \sum_{w \in W_\sigma} t^{n-n(w)} (1-t)^{n(w)-1} P_{\pi(\sigma,w)}(t), \tag{8}$$

$$\chi_\sigma^{ij} = 0 \quad \text{for} \quad i \ne j, \tag{9}$$

where n(w) is the same that in Theorem 5.1.

5.4. COROLLARY. *In the complex case $\chi_\sigma = 0$.*

5.6. THEOREM. *For any $\sigma \in W_\sigma$*

$$\deg P_\sigma = d_\sigma, \qquad \chi_\sigma^{d_\sigma d_\sigma} = (-1)^{d_\sigma}, \tag{12}$$

$$\chi_\sigma^{ii} = (-1)^{d_\sigma} \chi_\sigma^{d_\sigma - i, d_\sigma - i}, \quad 0 \le i \le d_\sigma. \tag{13}$$

REFERENCES

[A] V. I. Arnold, *Sturmian theorems and symplectic geometry*, Funct. Analyz i ego Prilozh **19** 4 (1985), 1-10.

[Az] N. V. Azbelev and Z. B. Tsaljuk, *On the problem of zero distribution for solution of the third order linear ordinary differential equations*, Math. Sbornik **5** 5 (1960) 475-486.

[B] R. Bott, *On the iteration of closed geodesics and the Sturm intersection theory*, Comm Pure and Appl. Math. **9** 2 (1956), 171-206.

[C] W. A. Coppel, "Disconjugacy," Lect. Notes in Math., 1971.

[D] J. J. Duistermaat, *On the Morse index in variational calculus*, Adv. in Math. **21** 2 (1976), 173-195.

[E] H. M. Edwards, *A generalized Sturm theorem*, Ann. of Math. **80** 1 (1964), 22-57.

[H] P. Hartman, "Ordinary differential equations," John Willey and Sons, 1960.

[K] E. E. Kummer, *De generali quadam aequatione differentiali tertii ordinis.*, Progr. Evang. Konigl. Stadtgymnasiums, Liegnitz (1834).

[Ki] A. A. Kirillov, *Infinite dimensional Lie groups: their orbits, invariants and representations. Geometry of moments.*, in "Lect. Notes in Math.," 1982, pp. 101-123.

[Ko] V. A. Kondratiev, *Oscillatory properties of solutions of the equation $y^{(n)} + p(x)y = 0$*, Trudy MMO **10** (1961), 419-436.

[Ku] N. H. Kuiper, *Locally projective spaces of dimension one*, Mich. Math. jour. **2** 2 (1953), 95-97.

[L] A. Yu. Levin, *Disconjugacy of solutions of the equation* $x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p_n(t)x = 0$, Soviet Math. Surveys **24** 2 (1969), 43-96.
[Le] D.A.Leites, "Theory of supermanifolds," Petrozavodsk, KF AN SSSR, 1983.
[L-P] V. E. Lazutkin and T. F. Pankratova, *Normal forms and versal deformations for Hill equations*, Funct. Analyz i ego Prilozh. **9** 4 (1975).
[Li] V. B. Lidskii, *Oscillatory theorems for canonical systems of differential equations*, Doklady Akademii Nauk USSR **102** 5 (1955), 877-880.
[M] M. Morse, "The Calculus of Variations in the Large," AMS Coll. Publ., 1934.
[Ne] F. Neuman, *Global equivalence of linear differential equations in the real domain*, in "Lecture Notes in Mathematics," 1982.
[P] G. Polia, *On the mean value theorem corresponding to a given linear ordinary equation*, Trans. AMS **24** 4 (1924), 312-324.
[R] W. T. Reid, "Sturmian theory for ordinary differential equations," Appl. Math. Sci., 1980.
[R-ST]. A. G. Reiman, M. A. Semenov-Tyan-Shansky, *Loop algebras and nonlinear partial differential equations*, Doklady Akademii Nauk SSSR **251** (1980), 1310-1314.
[S] J. C. F. Sturm, *Memoire sur les equations differntielles lineares du second ordre*, J. Math. Pur. Appl. **1** (1836), 106-186.
[Sa] Sansone, *Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reele*, Revista Mat. Fis. Teor. Tucuman **6** (1948), 195-253.
[Sh] T. L. Sherman, *Conjugate points and simple zeros for ordinary linear differential equations*, Trans. AMS **146** 2 (1969), 397-411.

Linear differential equations and flag varieties

B. Z. Shapiro

§I. Introduction

Consider a linear ordinary n-th order differential equation (l.o.d.e.) given on the interva $I = [0,1]$:

$$L_n[x] = x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0 \tag{1}$$

where $a_i(t) \in C^\infty[I]$.

Main definition. Equation (1) is called **disconjugate** on **I** if its arbitrary nontrivial solution has on **I** less than n zeros counted with multiplicities and **conjugate** otherwise.

Definition Let $\mathbf{F}_n$ be the manifold of all complete flags in the linear space **V** of solutions of l.o.d.e. (1). The flag curve **fc** of l.o.d.e. (1) is the map $\mathbf{fc} : \mathbf{I} \to \mathbf{F}_n$ sending each moment $t \in \mathbf{I}$ onto a complete flag in **V** whose i-dimensional subspace consists of all solutions which have a zero of multiplicity $\geq n-i$ at the moment t.

Recall that with each complete flag **f** there is related its Schubert cell decomposition $Sch_{\mathbf{f}}$ of the space $\mathbf{F}_n$ whose cells consists of all flags having given set of dimensions of intersections with the subspaces of the given flag **f** .

Definition. The set of complete flags in **V** is called **generic** if any subset of Schubert cells belonging to different flags intersect transversally. Two complete flags are called **transversal** if they form a generic set (i.e. their subspaces are transversal) and **nontransversal** otherwise. The set $\mathbf{Tn}_{\mathbf{f}}$ of all complete flags nontransversal to the given flag **f** is called the **train of f**.

Notice that $\mathbf{Tn}_{\mathbf{f}}$ consists of all positive-codimensional cells of $Sch_{\mathbf{f}}$. The train $\mathbf{Tn}_{\mathbf{f}}$ is reducible and consists of $n-1$ (where n is dimension of the space) irreducible components $\mathbf{Tn}^i_{\mathbf{f}}$ consisting of all flags whose $(n-i)$-dimensional subspace is nontransversal to the i-dimensional subspace of **f**.

The results of the paper are:

THEOREM A. *The following three conditions are equivalent:*

(1) *equation (1) is conjugate on* $I = [0, 1]$;
(2) *there exists a moment* $t \in (0, 1]$ *such that flag* $\mathbf{fc}_t$ *from the flag curve* $\mathbf{fc}$ *of equation (1) is nontransversal to* $\mathbf{fc}_0$;
(3) *the flag curve of equation (1) intersects the train of arbitrary flag.*

THEOREM B. *The sum of local multiplicities of intersection of the flag curve* $\mathbf{fc}$ *with* $\mathbf{Tn}_g^i$ *of arbitrary flag* $\mathbf{g}$ *(equal to the sum of dimensions of intersections of* i*-dimensional subspace of the flag curve with* $(n-i)$*- dimensional subspace of flag* $\mathbf{g}$*) for a disconjugate equation (1) does not exceed* $i(n-i)$.

COROLLARY C (GENERALIZED STURM SEPARATION THEOREM).
If the sum of local multiplicities of the flag curve $\mathbf{fc}$ *of some equation (1) with* $\mathbf{Tn}_{\mathbf{g}}^i$ *of some flag* $\mathbf{g}$ *exceeds* $i(n-i)$ *on some interval* $\mathbf{I}$ *then on this interval the equation is conjugate and thus intersects the train of arbitrary flag at least once.*

COROLLARY D. *If* $\sharp_1$ *and* $\sharp_2$ *are positive sums of multiplicities of intersection of the flag curve of equation (1) with trains of two arbitrary flags* $\mathbf{g}_1$ *and* $\mathbf{g}_2$ *then*

$$\frac{6}{n^3-n+6} < \frac{\sharp_1}{\sharp_2} < \frac{n^3-n+6}{6}$$

Notice that by results of Kondratiev (see [K]) no separation theorem can be obtained in the terms of zeros of individual solutions of the high order l.o.d.e.

§2. CURVES AND TRAINS OF L.O.D.E.

DEFINITION. Let $\mathbf{P}(\mathbf{V}^*)$ be the projectivization (projective space associated with) of the space $\mathbf{V}$ of all solutions of equation (1). Projective curve of equation (1) is the map $\mathbf{p} : \mathbf{I} \to \mathbf{P}(\mathbf{V}^*)$ sending each moment $t \in \mathbf{I}$ onto the hyperplane of solutions having a zero at the moment t.

Another definition of the curve $\mathbf{p}$.

DEFINITION. The affine curve $\mathbf{a} : \mathbf{I} \to \mathbf{V}^*$ of l.o.d.e. (1) is defined by the relation

$$< \mathbf{a}(t), \phi >= \phi(t),$$

where ϕ is an arbitrary solution of (1).

For any t vector $\mathbf{a}$ annihilates the hyperplane of solutions of l.o.d.e. (1) vanishing at the moment t. Therefore the projectivization of the curve $\mathbf{a}$ coincides with the curve $\mathbf{p}$.

For arbitrary fundamental solutions $\phi_1, \dots, \phi_n$ of equation (1) components of the affine curve $\mathbf{a}(t)$ in the basis dual to the chosen fundamental solution coincide with $\phi_1(t), \dots, \phi_n(t)$.

DEFINITION. A point of the curve γ in the projective space $\mathbf{P}^n$ is called **nondegenerate** if in some affine chart containing this point we can choose affine coordinates such that coordinates of the germ of the curve itself will have the following expansion

$$(t + \dots, t^2 + \dots, \dots, t^n + \dots).$$

Projective curve it called **nondegenerate** if all its points are nondegenerate.

REMARK 1. Since Wronskian of arbitrary fundamental system of solutions of l.o.d.e. (1) does not vanish, its projective curve is nondegenerate.

REMARK 2. In each nondegenerate point the complete osculating flag is properly defined. It consists of osculating subspaces of all dimensions in the considered point.

REMARK 3. The above defined flag curve of l.o.d.e. consists of osculating flags to all points of the projective curve of l.o.d.e.

Let $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_{n-1})$ be a complete flag on $\mathbf{R}^n$ and $\mathbf{F}_n$ be the manifold of all complete flags in $\mathbf{R}^n$. In $\mathbf{F}_n$ define the set of $n-1$ circles $\{l_1, \dots, l_{n-1}\}$ passing through $\mathbf{f}$ and given by the relation

$$l_i = \{\mathbf{f}_1, \dots, \mathbf{f}_{i-1} \subset L_i \subset \mathbf{f}_{i+1} \subset \dots \subset \mathbf{f}_{n-1}\} \qquad i = \overline{1, n-1}$$

where L_i runs over the set of all i-dimensional subspaces satisfying the above inclusions.

Tangent lines to $l_1, \dots, l_{n-1}$ at the point $\mathbf{f}$ are linearly independent. Span with them the $(n-1)$-dimensional tangent subspace $\bar{C}_\mathbf{f} \in \mathbf{TF}_n$ and delete from $\bar{C}_\mathbf{f}$ all $(n-2)$-dimensional subspaces $C_{\mathbf{f},j}$, $j = \overline{1, n-1}$, where $C_{\mathbf{f},j}$ is spanned by all the tangent lines except the j-th one.

DEFINITION. The distribution

$$C_\mathbf{f} = \bar{C}_\mathbf{f} \setminus \cup_{j=1}^{n-1} C_{\mathbf{f},j}$$

is called the **Cartan distribution** on the space $\mathbf{F}_n$.

The immersed curve $\mathbf{f} : \mathbf{I} \to \mathbf{F}_n$ *is the flag curve of some equation (1) if and only if at any moment* $t \in \mathbf{I}$ *it is tangent to* $C_\mathbf{f}$.

PROOF: The flag curve of l.o.d.e.(1) is tangent to $\bar{C}_\mathbf{f}$ since infinitesimal motion of the i-dimensional subspace in the osculating flag to any projective curve belongs to its $(i+1)$-dimensional subspace. Since the curve is nondegenerate, the velocity vector of the motion of the i-dimensional subspace L_t does not belong to L_t. ∎

Recall that the train $\mathbf{Tn}_\mathbf{f}$ of a complete flag $\mathbf{f}$ is the stratified (with the Schubert cells) hypersurface; it consists of $(n-1)$ irreducible components $\mathbf{Tn}_\mathbf{f}^i$. There is the following bijection between the cells of Schubert decomposition in $\mathbf{F}_n$ and the elements of the permutation group $\mathbf{S}_n$ where $\mathbf{S}_n$ acts on the linear space $\mathbf{R}^n$ by the permutation of coordinates and each cell of the decomposition contains a unique element from the orbit of the given flag $\mathbf{f}$. Dimension of the cell corresponding to the given permutation $(i_1, \dots, i_n)$ equals the number of 'disorders' in the permutation (i.e. the number of the pairs (i_l, i_k) where $i_l > i_k$.

LEMMA. *The multiplicity of intersection of the germ* $\mathbf{fc} : [-\epsilon, \epsilon] \to \mathbf{F}_n$ *of the flag curve of some l.o.d.e. (1) with the train* $\mathbf{Tn}_\mathbf{g}$ *depends only on the cell of the Schubert decomposition* Sch_g *to which the intersection point* $\mathbf{fc}(0)$ *belongs and can be calculated as follows. Relate to the given flag* $\mathbf{f}$ *Schubert decompositions of all Grassmann varieties* $\mathbf{G}_{k,n}$, $k = \overline{1, n-1}$. *Then the multiplicity* $\sharp_k$ *of intersection of* $\mathbf{fc}$ *with* $\mathbf{Tn}_\mathbf{g}^k$ *equals the codimension of the cell of the decomposition to which the* k*-dimensional subspace at the moment of nontransversality belongs. For the permutation* $(i_1, \dots, i_n)$ *the multiplicity* $\sharp_k$ *is equal to*

$$\sharp_k = max(0, \Sigma_{m=n-k+1}^n) = (i - (n - m - 1))$$

PROOF: Let $\mathbf{a}$ be the germ of an affine curve of l.o.d.e. (1) (see definition above) such that its flag curve is nontransversal at $t = o$ to the complete flag $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_{n-1})$. The basis $e_1, \dots, e_n$ is called **adjusted** to $\mathbf{f}$ if for all i the space $\mathbf{f}_i$ is spanned by $e_1, \dots, e_i$. For any germ $\mathbf{a}$ of the affine curve of l.o.d.e. (1) there exists a unique adjusted to $\mathbf{f}$ basis such that coordinates of $\mathbf{a}$ have the following extension

$$\mathbf{a}_1(t) = \frac{t_{i_1}}{i_1!}, \dots, \mathbf{a}_n(t) = \frac{t_{i_n}}{i_n!},$$

where $(i_1, \dots, i_n)$ is the permutation corresponding to the cell of Schubert decomposition $Sch_{\mathbf{f}}$ to which $\mathbf{fc}(0)$ belongs. Consider the Wronsky matrix $W(t)$ for the fundamental solution $\mathbf{a}_1(t), \dots \mathbf{a}_n(t)$. The multiplicity of intersection of $\mathbf{fc}$ with $\mathbf{Tn_g}$ equals the multiplicity of intersection of $W(t)$ with $\nu^{-1}(\mathbf{Tn_g})$, where ν is the projection of the bundle $\mathbf{GL}_n \to \mathbf{F}_n$ mapping each nondegenerate matrix onto the complete flag whose i- dimensional subspace is spanned by the first i rows of the matrix. Expanding the elements of tho j-th row in the powers of t we get

$$(2) \qquad W_j(t) = \{\frac{t^{i_j}}{i_j!} + \dots, \frac{t^{i_j - 1}}{(i_j - 1)!} + \dots, \dots, 1 + \dots, 0 + \dots, 0 + \dots\},$$

The equation of the k-th component of $\nu^{-1}(\mathbf{Tn_g})$ in the considered basis is

$$\Delta_k = 0$$

where Δ_k is the minor formed by the last k rows and the first k columns of $W(t)$. We can define the necessary multiplicity restricting Δ_k to $W(t)$. Thus we must calculate the first nontrivial term in the expansion of the correspondent minor in the powers of t. The explicit expression (2) implies that for arbitrary k the multiplicity of the zero of Δ_k depends only on the permutation $(i_1, \dots, i_n)$ and can be calculated as follows. We must compare the transposed permutation $(i_n, \dots, i_1)$ with the identity permutation $(1, \dots, n)$ and calculate the sum of differences between the first k terms of $(i_n, \dots, i_1)$ and $(1, \dots, n)$. It equals $\Sigma_{m=n-k+1}^{n}(i_m - (n - m + 1))$. This value coincides with the square of corresponding Young diagram which equals the codimension of the considered Schubert cell in $\mathbf{G}_{k,n}$.

§3. (DIS)CONJUGACY CRITERION AND GENERALIZED SEPARATION THEOREM FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

LEMMA. *Equation (1) is disconjugate on the interval* $\mathbf{I}$ *if and only if arbitrary subspaces of arbitrary set of pairwise different flags from the flag curve of considered equation intersect transversally.*

PROOF: The following result is one of the classical disconjugacy criteria (see [L]). Equation (1) is disconjugate if and only if there exists a unique solution of arbitrary multipoint boundary value problem:

$$x^j(t_i) = x_{i,j}, \quad t_i \in \mathbf{I}, j = \overline{0, m_i}, \sum_{i=1,k}(m_i + 1) = n$$

This criterion is exactly equivalent to the statement of the lemma.

Theorem A will be proved by the following sequence of implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

The first step (2) $\Rightarrow$ (1). Let $\mathbf{fc}$ be the flag curve of equation (1) given on $\mathbf{I} = [0, 1]$. Suppose that some k-dimensional subspace of the flag $\mathbf{fc}(0)$ is nontransversal to the $(n-k)$-dimensional subspace of $\mathbf{fc}(\tau)$, $\tau \in (0, 1]$. If $\mathbf{fc}(0)$ and $\mathbf{fc}(\tau)$ are nontransversal then we can always choose such a pair. The sum of these subspaces belongs to some hyperplane in $\mathbf{V}^*$, where $\mathbf{V}$ is the linear space of solutions. This hyperplane defines some nontrivial solution of considered equation which has at the moment $t = 0$ zero of multiplicity $\geq n - k$ and at the moment τ zero of multiplicity $\geq k$. Thus, the considered equation is conjugate by the definition.

The second step (1) $\Rightarrow$ (3) will be proved by induction.

Case $n = 2$. The flag space $\mathbf{F}_2$ coincides with $\mathbf{S}^1$ and the train of any flag coincides with the flag itself. The flag curve of any equation moves along $\mathbf{F}_2$ with nonvanishing velocity. The existence of a solution with at least 2 zeros means that the flag curve passes through some point on $\mathbf{F}_2$ at least twice and thus covers the whole $\mathbf{F}_2$ intersecting trains of all flags.

Induction. Let $\mathbf{p} : [0, 1,] \to \mathbf{P}^{n-1}$ be the projective curve of the conjugate equation (see Introduction) and L the hyperplane with which $\mathbf{p}$ intersects at least n times counted with multiplicities and $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_{n-2})$ an arbitrary complete flag in $\mathbf{P}^{n-1}$. The multiplicity of intersection is continuous from below. So it suffices to consider the case when $\mathbf{f}$ is transversal to L. If $\mathbf{p}$ intersects $\mathbf{f}_{n-2}$ then this intersection point is the necessary moment of nontransversality. If $\mathbf{p}$ does not intersect $\mathbf{f}_{n-2}$ we can consider on the hyperplane $\mathbf{f}_{n-2} = \mathbf{P}^{n-2}$ the flag $\mathbf{F}_L$ obtained by intersection of the flag $\mathbf{f}$ with the hyperplane L. Moreover, we can project the curve $\mathbf{p}$ on $\mathbf{f}_{n-2}$ along the tangent lines, i.e. to map each point of $\mathbf{p}$ onto intersection of the tangent line to the curve $\mathbf{p}$ in the considered point with $\mathbf{f}_{n-2}$. If $\mathbf{p}$ is nondegenerate in $\mathbf{P}^{n-1}$ and $\mathbf{p}$ does not intersect $\mathbf{f}_{n-2}$ then $\mathbf{p_f}$ is a nondegenerate curve on $\mathbf{f}_{n-2} = \mathbf{P}^{n-2}$.

Now let us prove that $\mathbf{p_f}$ intersects the plane $L_{\mathbf{f}} = L \cap \mathbf{f}_{n-2}$ at least $n-1$ times counted with multiplicities. Let us begin with nonsimple zeros. If $\mathbf{p}$ intersects l at the moment t with some nontrivial multiplicity (which by definition equals the maximal dimension of the osculating subspace belonging to the hyperplane L) then $\mathbf{p_f}$ intersects L_f with the multiplicity reduced by 1 because of the projection along the line. Now on $\mathbf{p}$ we choose two neighboring geometrically different zeros $\mathbf{p}(t_i)$ and $\mathbf{p}(t_{i+1})$ and prove that on the interval (t_i, t_{i+1}) we can find the moment τ_i for which the tangent line to the point $p(\tau_i)$ intersects the plane $L_{\mathbf{f}}$ (the corresponding point is the intersection moment for the curve $p_{\mathbf{f}}$ (see Fig.1). Indeed, hyperplanes L and $\mathbf{f}_{n-2}$ separate the considered space $\mathbf{P}^{n-1}$ into two semispaces. The part of the curve p on the interval (t_i, t_{i+1}) lies in one of them. We can choose the projective chart in which the hyperplane L is horizontal. In this chart the point of $\mathbf{p}$ whose tangent line intersects $\mathbf{f}_{n-2}$ is the point in which the tangent line is horizontal. On the interval (t_i, t_{i+1}), consider the distance from the curve $\mathbf{p}$ to the plane L. This function necessarily has a maximum in the internal point since it grows on the both ends of the interval. At this point the tangent line is horizontal. ∎

The third step (3) $\Rightarrow$ (2) . Condition (3) is equivalent to the fact that the union of the trains of all points of the flag curve $\mathbf{fc}$ coincides with $\mathbf{F}_n$. Suppose that there is no moment t such that $\mathbf{fc}(0) \in \mathbf{Tn}_{\mathbf{fc}(t)}$. We shall show that there exists a flag $\mathbf{f} \in \mathbf{F}_n$ which does

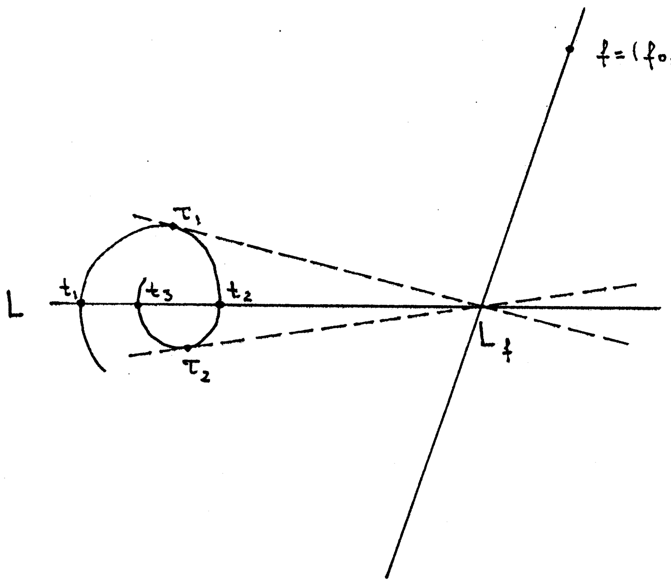


Fig.1

not belong to the union of trains taken over all points of **fc**. In our assumptions for each $\tau \in (0,1]$ there exists ϵ_τ such that the union of the trains of **fc** on the interval $[\tau, 1]$ does not intersect with the ϵ_τ-neighborhood of the flag **fc**(0). On the other hand, if we extend the germ of the flag curve of l.o.d.e. to a small negative time interval $(-\sigma, \sigma)$ then it remains disconjugate. So for sufficiently small negative moment $-\breve{\sigma}$ the flag **fc**$(-\breve{\sigma})$ which belongs to $(\epsilon - \tau)$ -neighborhood of **fc**(0) is transversal to all flags **fc**$(t)t \in [0,1]$. ∎

In order to prove theorem B we need the following proposition.

LEMMA (CF. [E-H]). *Suppose that* $\{\mathbf{fc}(t_1), \ldots, \mathbf{fc}(t_m)\}$ *is the set of flags from the flag curve of disconjugate equation and* $\{Sch_{\mathbf{f}_1}, \ldots, Sch_{\mathbf{f}_m}\}$ *is the set of correspondent Schubert decompositions. Then* $\{\mathbf{fc}(t_1), \ldots, \mathbf{fc}(t_m)\}$ *has the dimensional transversality property, i.e. the codimension of the intersection of an arbitrary set of cells* $C_1, \ldots, C_m$*, where* C_i *belongs to* Sch_i*, equals the maximum of* $\dim \mathbf{F}_n$ *and the sum of the codimensions.*

Now we pass to theorem B.

PROOF: Let $t_1, \ldots, t_m$ be different moments of nontransversality of the points of the flag curve **fc** of some disconjugate equation with some given flag **f** and $\sharp_{i,j}$ the multiplicity of the intersection of **fc** with the k-th component $\mathbf{Tn}_\mathbf{f}^k$ of the train $\mathbf{Tn}_\mathbf{f}$. Now let us show that $\sum_i \sharp_{i,k} \leq k(n-k)$. By the above lemma the set of flags $\{\mathbf{fc}(t_1), \ldots, \mathbf{fc}(t_m)\}$ is dimensional transversal, i.e. codimension of Schubert cells from their decompositions is equal to the sum of codimensions. Consider m Schubert decompositions of the Grassmann varieties $\mathbf{G}_{k,n}$ constructed for the flags $\mathbf{fc}(t_1), \ldots, \mathbf{fc}(t_m)$. By the above lemma, the k-dimensional plane of the flag **f** belongs to the union of the cells of codimension $\sharp_{n-k,i}$ of the i-th cell decomposition of $\mathbf{G}_{k,n}$. By the dimensional transversality the sum of the codimensions of the cells to whose intersection the k- dimensional plane of the flag **f** belongs can not exceed $\dim \mathbf{G}_{k,n} = k * (n-k)$. Consequently, $\sum_i \sharp_{i,k} \leq k * (n-k)$. In particular, we obtain once again that the number of the zeros of any solution does not exceed $n-1$.

Now let us prove corollary C.

PROOF: Indeed, if the sum of multiplicities of intersections of the flag curve of equation (1) with the k-th component of the train $\mathbf{Tn}_\mathbf{f}$ on some certain time interval exceeds $k(n-k)$ then equation is conjugate on this interval. So by theorem A on this interval there exists a moment of nontransversality to any flag. ∎

REFERENCES

[K]. V. A. Kondratiev, *Oscillatory properties of solutions of the equation* $y^{(n)}+p(x)y = 0$, Trudy Moskovskogo Matematicheskogo Obshestva **10** (1961), 419-436.

[L]. A. Ju. Levin, *Disconjugacy of solutions of equations* $x^{(n)}+p_1(t)x^{(n-1)}+\ldots+p_n(t)x = 0$, Soviet Math. Surveys **24** (1969), 43-96.

[E-H]. D. Eisenbud, J. Harris, *Divisors on general curves and cuspidal rational curves*, Inv.Math. **74** (1983), 374-418.

Classification of symplectic leaves in Kac-Moody algebras $\widehat{\mathbf{GL}_3}$ and $\widehat{\mathbf{SO}_n}$

B. Z. Shapiro

§I. Introduction

Recall (see [R-ST]) that the space of operators $\{a\frac{d}{dx} + A(x), A \in C^\infty(S^1, \mathcal{G})\}$, where $\mathcal{G}$ is a reductive Lie algebra is naturally identified with dual space $\hat{\mathcal{G}}^*$ to the Kac-Moody algebra $\hat{\mathcal{G}}$ (which is the 1-dimensional extension of the loop algebra $\tilde{\mathcal{G}} = C^\infty(S^1, \mathcal{G})$). Under this identification the coadjoint action $P \in \tilde{G}$ in $\hat{\mathcal{G}}^*$ coincides (for $a \neq 0$) with the gauge action on differential operators. Thus gauge classification of differential operators is equivalent to the classification of the orbits of coadjoint action in Kac-Moody algebras which are maximal nondegenerate submanifolds of the linear Poisson structure (i.e. the symplectic leaves of the Poisson-lie bracket, also known as Beresin-Kirillov bracket). At the same time the problem of description of the classes of the first order matrix linear differential equations $\{adx + A(x)\Psi = 0, \quad A \in C^\infty(S^1, \mathcal{G})\}$ of the above type with respect to gauge equivalence: $\Psi \to P\Psi$ (or $A \mapsto P^{-1}P' + P^{-1}AP$), where $P \in \tilde{G} = C^\infty(S^1, \mathbf{G})$ is a well known problem of analysis. Denote by $\tilde{G}_\circ = C^\infty_\circ(S^1, \mathbf{G})$ the connected component of $\tilde{G}$ containing the trivial map onto the identity matrix (components of $\tilde{G}$ are enumerated by the elements of $\pi_1(\mathbf{G})$).

Floquet's theorem, ([H]). *The only invariant of the matrix equation on the circle under the action of gauge group $\tilde{G}$ is the conjugacy class of its monodromy operator in* $\mathbf{G}$*; for gauge transformations from $\tilde{G}_\circ$ the only invariant is the conjugacy class of its monodromy operator in the universal covering* $\mathbf{G}_\circ$ *of the group* $\mathbf{G}$.

Proof: Indeed the multiplication of the fundamental solution $\Psi : \mathbf{R} \to \mathbf{G}$ by a periodic function changes the monodromy operator M of the considered equation only inside its conjugacy class. When $P \in \tilde{G}_\circ$ then we preserve also the homotopy type of the path on $\mathbf{G}$ given by the map Ψ on the period.

Classes of $\tilde{G}$-equivalence as well as the group $\tilde{G}$ itself can, generally, contain several connected components as long as for $\tilde{G}_\circ$-equivalence these classes are by the definition connected. Therefore, in classification problems the last equivalence is often more preferable.

In this paper we describe conjugacy classes in universal coverings $\mathbf{GL}_3^0, \mathbf{SL}_3^0$ and $\mathbf{Spin}_n$ of $\mathbf{GL}_3, \mathbf{SL}_3$ and $\mathbf{SO}_n$ respectively and formulate the hypothesis about $\mathbf{GL}_n$. Thus we obtain classification of symplectic leaves in the third order Kac-Moody algebras related to $\mathbf{GL}_3, \mathbf{SL}_3$ and arbitrary order algebras related to $\mathbf{SO}_n$.

Theorem 2. *The conjugacy class of any element $M \in \mathbf{GL}_3^+(R)$ with the Jordan normal form (J.n.f.) different from the following one:*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

where $\lambda, \nu < 0$ ($\lambda = \nu$ is admissible) has two inverse images in the universal covering $\mathbf{GL}_3^\circ$ under the natural projection $\mathbf{GL}_3^\circ \to \mathbf{GL}_3^+$; the conjugacy class of the operator with the above Jordan normal form has a connected inverse image in $\mathbf{GL}_3^0$.

Remark 1. Analogous theorem is valid for $\mathbf{SL}_3(R)$.

Theorem 2′. *Symplectic leaves of the linear Poisson structure on the Kac-Moody algebras related to $\mathbf{GL}_3^+$ and $\mathbf{SL}_3$ are enumerated by the real parameter $a \neq 0$, J.n.f. of operators and an invariant from $\mathbf{Z}/2\mathbf{Z}$ for any J.n.f. except the above one.*

Theorem 3. *The conjugacy class of any operator $M \in \mathbf{SO}_n$ has connected inverse image in $\mathbf{Spin}_n$ if and only if its spectrum contains both 1 and -1. In the opposite case the inverse image consists of 2 components.*

Proof of Theorem 2: It suffices to prove our statement in the $\mathbf{GL}_3^+$-case since $\mathbf{GL}_3^+ = \mathbf{SL}_3 \oplus$ {scalar matrices}, where the second summand is the center of the group.

At first we show that theorem 2 is equivalent to the following statement.

Let $C_M \in \mathbf{GL}_3^+(R)$ denote the conjugacy class of the operator M.

Proposition. *For operator M with J.n.f.:*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

where $\lambda, \nu < 0$, the embedding $C_M \hookrightarrow \mathbf{GL}_3^+$ induces epimorphism $\phi : \pi_1(C_M) \to \pi_1(\mathbf{GL}_3^+) = \mathbf{Z}/2\mathbf{Z}$. For other operators the induced homomorphism $\phi : \pi_1(C_M) \to 0 \in \pi_1(\mathbf{GL}_3^+)$ is trivial.

Equivalence of this proposition to Theorem 2 is obvious since $C_M \to \mathbf{GL}_3^+$ has connected inverse image in $\mathbf{GL}_3^0$ only in the case when there exists a closed path $\gamma \in C_M$ representing the generator of $\pi_1(\mathbf{GL}_3^+)$.

Our proof is based on the explicit consideration of all 10 real J.n. forms in $\mathbf{GL}_3^+$.

$$\text{a)}\begin{pmatrix} a^2 & 0 & 0\\ 0 & b^2 & 0\\ 0 & 0 & c^2\end{pmatrix}\quad \text{b)}\begin{pmatrix} -a^2 & 0 & 0\\ 0 & -b^2 & 0\\ 0 & 0 & c^2\end{pmatrix}\quad \text{c)}\begin{pmatrix} \lambda\cos\alpha & \lambda\sin\alpha & 0\\ -\lambda\sin\alpha & \lambda\cos\alpha & 0\\ 0 & 0 & c^2\end{pmatrix}$$

$$\text{d)}\begin{pmatrix} a^2 & 1 & 0\\ 0 & a^2 & 0\\ 0 & 0 & c^2\end{pmatrix}\quad \text{e)}\begin{pmatrix} -a^2 & 1 & 0\\ 0 & -a^2 & 0\\ 0 & 0 & c^2\end{pmatrix}\quad \text{f)}\begin{pmatrix} a^2 & 1 & 0\\ 0 & a^2 & 1\\ 0 & 0 & a^2\end{pmatrix}$$

$$\text{g)}\begin{pmatrix} a^2 & 0 & 0\\ 0 & a^2 & 0\\ 0 & 0 & c^2\end{pmatrix}\quad \text{h)}\begin{pmatrix} -a^2 & 0 & 0\\ 0 & -a^2 & 0\\ 0 & 0 & c^2\end{pmatrix}\quad \text{i)}\begin{pmatrix} a^2 & 1 & 0\\ 0 & a^2 & 0\\ 0 & 0 & a^2\end{pmatrix}$$

$$\text{j)}\begin{pmatrix} a^2 & 0 & 0\\ 0 & a^2 & 0\\ 0 & 0 & a^2\end{pmatrix}$$

LEMMA 1. *Suppose that operators M_0 and M_1 can be connected by a continuous path $M_t, t \in [0,1]$, where M_t for $t \in (0,1]$ has the same J.n. type as M_1 (i.e. belongs to the same class given above with probably another a, b, c, λ, α etc.). This means that M_0 is adjoint to M_1. Suppose also that $\phi(\pi_1(C_{M_0})) = 0$. Then $\phi(\pi_1(C_{M_1})) = 0$.*

PROOF: Let $\gamma = \{\theta(\tau), \tau \in [0, 2\pi]\} \in C_{M_1}$ be an arbitrary closed path in the conjugacy class of M_1. We will prove that it is contractible. By the definition of C_{M_1} for any path $\theta(\tau) \in C_{M_1}$ there exists a path $g(\tau) \in \mathbf{GL}_3^+, g(0) = e$ such that $\theta(\tau) = g^{-1}(\tau)M_1 g(\tau)$. The fact that the path $\theta(2\pi) = \theta(0)$ is closed means that $g(2\pi)$ belongs to the stabilizor $St(M_1)$ of the matrix M_1. The assumption that M_t for any $t \neq 0$ has the same type as M_1 means that their stabilizors coincide and one can choose a continuous path $g_t(2\pi)$ such that $g_t(2\pi) \in St_{M_t}$ for any $t \neq 0$. Then $g_{t_0}(\tau)$ for any t_0 defines the closed path

$$\theta_{t_0} = g_{t_0}^{-1}(\tau) M_{t_0} g_{t_0}$$

in the orbit C_{M_0} (the path is closed since $g_t(\pi) \in St_{M_t}$). Therefore 1-parameter family of closed paths $\theta_t(\tau)$ defines the homotopy of the path γ to some closed path γ' in C_{M_0}. The condition $\phi(\pi_1(C_{M_0})) = 0$ means that γ' is contractible on the group $\mathbf{GL}_3^+$. Thus γ is also contractible. ∎

COROLLARY: For operators M of the types c), f), i), j) and d), a), g) with positive eigenvalues arbitrary closed paths in C_M are contractible on $\mathbf{GL}_3^+$, i.e. $\phi(\pi_1(C_M)) = 0$.

PROOF: The scalar matrix λE belongs to the center and its orbit consists of one point so obviously $\phi(\pi_1(C_{\lambda E})) = 0$. The other mentioned matrices can be connected with a scalar matrix by the rather obvious paths within their types.

LEMMA 2. *Among the remaining types:*

$$\begin{pmatrix} \lambda & 0 & 0\\ 0 & \nu & 0\\ 0 & 0 & \mu\end{pmatrix},\quad \begin{pmatrix} \lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix},\quad \begin{pmatrix} \lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix}$$

with negative λ, ν the orbits of the first two types represent, in fact, nontrivial element of $\pi_1(\mathbf{GL}_3^+)$, i.e. $\phi(\pi_1(C_M)) \neq 0$.

PROOF: Indeed, the stabilizor of these types contains the following element:

$$\sigma = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Therefore the curve $g(\tau)$ such that $g(0) = e$, $g(2\pi) = \sigma$,

$$g(\tau) = \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\tau & \sin\tau\\ 0 & -\sin\tau & \cos\tau\end{pmatrix},$$

defines closed paths $\theta(\tau) = g^{-1}(\tau)Mg(\tau)$ in the corresponding orbits. These paths are noncontractible since they invert directions of the expanding eigenvector and preserve the invariant 2-dimensional contracting subspace. Remark that corresponding orbits are nonoriented. Now the necessary statement follows from the next proposition.

LEMMA 3. *For*

$$M = \begin{pmatrix} \lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$\phi(\pi_1(C_M)) = 0.$$

PROOF: We have the stabilizor of M consists of the matrices of the form

$$\begin{pmatrix} a & b & 0\\ 0 & a & 0\\ 0 & 0 & c\end{pmatrix}$$

where $c > 0$. Let $g(x)$ be the curve with $g(0) = e$, $g(2\pi) \in St_M$ defining the closed path θ in C_M. The family

$$M_t = \begin{pmatrix} \lambda & t & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix},\quad t \in [0,1]$$

defines the 2-chain $\theta_t(\tau) = g^{-1}(\tau)M_t g(\tau)$ since $g(2\pi) \in St_{M_1} = St_{M_\tau}$ for $\tau \neq 0$. However, the matrix

$$M_0 = \begin{pmatrix} \lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix}$$

under the action of the path $g(\tau)$ with $g(2\pi) \in St_{M_1}$ spans a contractible path. Indeed, the upper (2×2)-block of the stabilizor acts trivially on the scalar $(2 \times 2)-$ block of matrix M_0. We has just mentioned that the path generated by M_0 is noncontractible for the paths $g(\tau)$ where $g(2\pi)$ has a negative element c only.

This finishes the proof of Lemma 3 and Theorems 2 and 2′.

Now pass to the proof of Theorem 3.

PROOF: Recall ([M]) that the spectrum of any orthogonal operator M lies on the unit circle and it has the following real J.n.f.

$$\begin{pmatrix} -E_k & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & E_l & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \cos\phi_1 & \sin\phi_1 & \dots & 0 & 0 \\ 0 & 0 & -\sin\phi_1 & \cos\phi_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \cos\phi_n & \sin\phi_n \\ 0 & 0 & 0 & 0 & \dots & -\sin\phi_n & \cos\phi_n \end{pmatrix},$$

i.e. there are two (perhaps empty) invariant subspaces E_l and $-E_k$ on which M acts as the unit or minus unit matrix and the set of 2-planes on which M acts by rotations by the angle different from 0 and π. Let l and k denote the dimensions of E_l and $-E_k$ respectively. If k (or l) vanishes then we connect M with the minus unit and unit matrix, respectively. For example for $l = 0$ we use the following procedure: in each 2-plane with the rotation angle $> \pi$ we decrease it to 0 and in 2-planes with rotation angle in the interval $(\pi, 2\pi]$ we increase it to 2π. Thus, as earlier we have constructed a path in the class of matrices of the same type (and stabilizor) contracting an arbitrary path on C_M to the scalar matrix. If k and l are positive then as at the previous step we can define a path $g(\tau)$, where $g(0) = e$ and $g(2\pi)$ is the unit matrix except the block $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in the 2-subspace spanned by any vector from E_l and any vector from $-E_k$ changing orientation of the eigenvector and thus realizing the nontrivial π_1-element.

In the conclusion I would like to formulate the following hypothesis.

HYPOTHESIS. *The conjugacy class of a operator $M \in GL_n^+$ realizes its nontrivial π_1-element if and only if the spectrum of M contains two real eigenvalues of different signs with 1-dimensional Jordan blocks.*

REFERENCES

[R-ST]. A. G. Reiman, M. A. Semenov-Tyan-Shansky, *Loop algebras and nonlinear partial differential equations*, Doklady AN SSSR **251** (1980), 1310-1314.

[H]. P. Hartman, "Ordinary differential equations," John Wiley and Sons, 1961, p. 522.

[M]. A. I. Mal'tsev, "Foundations of linear algebra," Gostechizdat, 1948, p. 423.

Discrete invariants of symplectic leaves of Zamolodchikov algebra and nondegenerate curves on S^2

B. Z. Shapiro

Abstract. Zamolodchikov algebra (called also Gelfand -Dikii algebra or KdV- structure) is the algebra of coefficients of the third order linear ordinary differential equations with the Poisson bracket. In [K-O] one can find the classification of symplectic leaves of this bracket in the case of arbitrary order linear ordinary differential equations with periodic coefficients. The complete set of invariants of leaves consists of the pair {the conjugate class of monodromy operators and π_0 group of the space of differential equations with the given monodromy operator}. The space of the n-th order linear ordinary differential equations with given monodromy A retracts to the space of the so called nondegenerate curves on S^{n-1} with the boundary conditions determined by A. In this paper we completely calculate π_0 of the space of 3-rd order differential equations with given monodromy and obtain detailed information about the topology of the space of equations with unit monodromy (i.e. closed curves).

§I. INTRODUCTION

THE MAIN DEFINITION. A curve $\gamma : [0,1] \to \mathbf{RP}^n$ (or $\mathbf{S}^n$) is called **nondegenerate** if for any moment $t \in [0,1]$ at the point $\gamma(t)$ the complete osculating flag is properly defined. (Its i-dimensional subspace is spanned by $\gamma'(t), \dots, \gamma^{(i)}(t)$ which are properly determined in any affine chart containing the point $\gamma(t)$).

Instead of $\mathbf{RP}^n$ or $\mathbf{S}^n$ one can consider arbitrary n-dimensional manifold with the flat projective structure. Analogous notion in the Riemannian case was considered in [G].

With the given the n-th order linear ordinary differential equation (l.o.d.e.) $P\phi = 0$ one can associate with it the class Γ_p of $\mathbf{GL}_n$- equivalent curves in $\mathbf{R}^n$ such that coordinates $\gamma_p = (\phi_1, \dots, \phi_n)$, of any $\gamma_p \in \Gamma_p$ (in arbitrary basis) are an arbitrary fundamental solution of $P\phi = 0$.

The crucial property of γ_p is that $\forall t\ \gamma_p(t), \dots, \gamma^{(n-1)}(t)$ are linearly independent. Therefore the radial projection of γ_p on the standard embedded unit sphere is a nondegenerate curve on $\mathbf{S}^{n-1}$. Thus Γ_u retracts to the class of $\mathbf{GL}_n$-equivalent nondegenerate curves on $\mathbf{S}^{n-1}$.

DEFINITION. The **initial (final)** flag of a nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^{n-1}$ is a complete flag in $\mathbf{T}_{\gamma(0)}\mathbf{S}^{n-1}$ ($\mathbf{T}_{\gamma(0)}\mathbf{S}^{n-1}$) spanned by $\gamma'(0),\dots,\gamma^{(n-1)}(0)$ ($\gamma'(1),\dots,\gamma^{(n-1)}(1)$) respectively.

Fixing some basis $e_1,\dots,e_n$ in $\mathbf{R}^n$ we can choose for any equation $P\phi = 0$ the unique curve $\hat{\gamma}_p$ from Γ_p such that $\hat{\gamma}_p(0), \hat{\gamma}_p'(0), \dots, \hat{\gamma}_p^{(n-1)}(0)$ is the unit matrix and thus associate with $P\phi = 0$ its monodromy matrix $\mathcal{A}_p = (\gamma(1), \gamma'(1), \dots, \gamma^{(n-1)}(1))$.

The space $D_n(\mathcal{A})$ of all n-th order l.o.d.e. with given monodromy matrix retracts to the space all oriented nondegenerate curves on $\mathbf{S}^{n-1}$ with given initial and final flags (f_1, f_2) satisfying the relation $f_2 = \mathcal{A}(f_1)$.

DEFINITION. Fixing orientation on $\mathbf{S}^2$ we call a nondegenerate curve: $\gamma : [0,1] \to \mathbf{S}^2$ **right-oriented** if $\gamma'(t), \gamma''(t)$ defines the given orientation and **left-oriented** otherwise.

First results about the topology of the space of nondegenerate closed curves on $\mathbf{S}^2$ and $\mathbf{R}^3$ (i.e. of curves with coinciding initial and final flags) were obtained by J. Little [**L1**,L2].

PROPOSITION 1 [**L1**]. *The space $\mathcal{R}\mathbf{S}^2$ of all right-oriented curves on $\mathbf{S}^2$ consists of three connected components (see fig.1).*

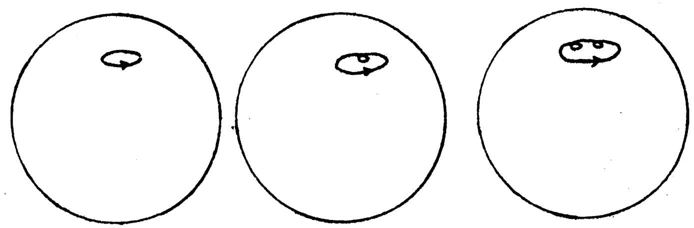


Fig.1

PROPOSITION 2 [**L2**]. *The space $\mathcal{R}\mathbf{R}^3$ of all right-oriented curves in $\mathbf{R}^3$ consists of 2 connected components.*

Earlier Whitney has shown that $\mathcal{R}\mathbf{R}^2$ consists of the countable number of connected components.

In the first part of the paper we generalize proposition 1 to the case of nondegenerate curves on $\mathbf{S}^2$ with arbitrary initial and final flags.

DEFINITION. A nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^2$ is called **strictly disconjugate** if the sum of local multiplicities (without signs) of its intersection with each 'big circle' on $\mathbf{S}^2$ does not exceed 3 and **conjugate** if it has with some 'big circle' at least three transversal intersections in the inner points.

Both properties are open and curves belonging to their boundary are called **nonstrictly disconjugate**.

The main result of the first part is:

THEOREM A. *The number of connected components in the space $\mathcal{R}\mathbf{S}^2(f_1, f_2)$ of right-oriented curves on $\mathbf{S}^2$ with given initial and final flags (f_1, f_2) equals 3 if there is a (non-strictly) disconjugate curve $\in \mathcal{R}\mathbf{S}^2(f_1, f_2)$ and 2 otherwise.*

An oriented flag on $\mathbf{S}^2$ consists of a point p and oriented circle C. The open segment on which C is divided by the pair $(p, \bar{p})$ (where $\bar{p}$ is the point antipodal to p) is called **positive (negative)** if it contains the small push in the positive (negative) direction of the point p and is denoted by C_p^+ (C_p^-). Fixing orientations of $\mathbf{S}^2$ and of the 'big circle ' C we can define upper and lower hemispheres H_C^+ and H_C^- of $\mathbf{S}^2 \setminus C$ (so that the vector complementing the right-oriented pair of vectors in the plane containing C to the right-oriented 3-tuple in $\mathbf{R}^3$ must intersect the upper hemisphere).

THEOREM B. *Flags $f_1 = (p_1, C_1)$ $f_2 = (p_2, C_2)$ can be connected by a right-oriented disconjugate curve if they form one of the following arrangements:*

(1) *p_2 belongs to the upper hemisphere $H_{C_1}^+$ and C_{2,p_2}^+ intersects with C_{1,p_1}^-;*
(2) *p_2 belongs to the upper hemisphere $H_{C_1}^+$ and C_{2,p_2}^+ passes through p_1;*
(3) *p_2 belongs to C_{1,p_1}^- and p_1 belongs to $H_{C_2}^+$;*
(4) *p_1 coincides with p_2; tangent vectors to $(C_{2,p_2}^+, C_{1,p_1}^+)$ defines the orientation of $\mathbf{S}^2$ opposite to the given one;*
(5) *C_1 coincides with C_2 and p_2 belongs to C_{1,p_1}^-;*
(6) *flags f_1 and f_2 coincide.*

Denote the space of all right-oriented conjugate curves given on $I = [0,1]$ with the fixed initial flag f by $CN(f)$ and the map taking each curve to its final flag by $\pi : CN(f) \to \mathbf{FO}_3$.

THEOREM C. *The map $\pi : CN(f) \to \mathbf{FO}_3$ satisfies the covering homotopy property.*

COROLLARY. *The space of the third order l.o.d.e. with all periodic solutions consists of two disconnected parts one of which is contractible and the other is homotopically equivalent to the space of all closed paths on $\mathbf{FO}_3 \cong \mathbf{SO}_3$ passing through the given point.*

The second part of the paper is devoted to the calculation of the discrete invariant for symplectic leaves of Zamolodchikov algebra. Recall that operators belonging to $\mathbf{GL}_3^+$ have one of the following 10 real Jordan normal forms (J.n.f.)

a) $\begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ b) $\begin{pmatrix} -a^2 & 0 & 0 \\ 0 & -b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ c) $\begin{pmatrix} \lambda\cos\alpha & \lambda\sin\alpha & 0 \\ -\lambda\sin\alpha & \lambda\cos\alpha & 0 \\ 0 & 0 & c^2 \end{pmatrix}$

d) $\begin{pmatrix} a^2 & 1 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ e) $\begin{pmatrix} -a^2 & 1 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ f) $\begin{pmatrix} a^2 & 1 & 0 \\ 0 & a^2 & 1 \\ 0 & 0 & a^2 \end{pmatrix}$

g) $\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ h) $\begin{pmatrix} -a^2 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ i) $\begin{pmatrix} a^2 & 1 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}$

j) $\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}$

THEOREM D. *Symplectic leaves in Zamolodchikov algebra are enumerated by the J.n.f. of monodromy operator and an invariant from* $\mathbf{Z}/2\mathbf{Z}$ *for the types b), g), h), i) and from* $\mathbf{Z}/3\mathbf{Z}$ *in the rest of the cases.*

PART I. NONDEGENERATE CURVES ON $\mathbf{S}^2$ WITH FIXED INITIAL AND FINAL FLAGS

§1. BASIC NOTIONS

As we have marked above the space $D(\mathcal{A})$ of n-th order l.o.d.e. (given on $I = [0,1]$) with given monodromy matrix $\mathcal{A}$ (i.e. those satisfying the condition $W(1) = \mathcal{A}$ where $W(t)$ is the Wronsky matrix such that $W(0) = E$) is contractable to the space of all oriented nondegenerate curves on $\mathbf{S}^{n-1}$ with the standard initial flag f_1 and fixed final flag f_2 determined by $\mathcal{A}$. So instead of $D_3(\mathcal{A})$ we will study the corresponding space of curves.

DEFINITION. Given a nondegenerate $\gamma : [0,1] \to \mathbf{S}^{n-1} \hookrightarrow \mathbf{R}^n$ and some basis one can define

(1) the matrix curve $\gamma_G : [0,1] \to \mathbf{GL}_n$ as

$$\begin{pmatrix} \gamma_1(t), & \dots, & \gamma_n(t) \\ \vdots & \vdots & \vdots \\ \gamma_1^{(n-1)}(t), & \dots, & \gamma_n^{(n-1)}(t) \end{pmatrix},$$

where γ_i is the i-th coordinate of γ (nondegeneracy provides that its determinant is nowhere vanishing);

(2) the flag curve $\gamma_f : [0,1] \to \mathbf{FO}_n$ consisting of osculating oriented flags to all points of γ, where $\mathbf{FO}_n$ is the space of all oriented flags in $\mathbf{R}^n$ (i-th space of osculating flag at the point $\gamma(t)$ is spanned by $\gamma(t), \gamma'(t), \dots, \gamma^{(i-1)}(t)$).

Space $\mathbf{FO}_n$ is equipped with the remarkable $n-1$-dimensional distribution (see [V-G]). Through each flag $f = (f_1, \dots, f_{n-1}) \in \mathbf{FO}_n$ pass $n-1$ oriented circles $\alpha_1(f), \dots, \alpha_{n-1}(f)$ where

$$\alpha_i = \{f_1 \subset \dots \subset f_{i-1} \subset L_i \subset f_{i+1} \subset \dots \subset f_{n-1}\}$$

and L_i is an arbitrary oriented i-dimensional linear subspace satisfying the above inclusions. Orientation of α_i is defined so that orientation of L_i and the unit vector of its velocity must define the proper orientation of f_{i+1}.

Cartan distribution $\mathbf{CFO}_n$ on $\mathbf{FO}_n$ is the distribution of $(\mathbf{R}^+)^{n-1}$ spanned by the positive half-tangents to α_i, $i = \overline{1, n-1}$.

One can easily see that the flag curve of any n-th order l.o.d.e. is everywhere tangent to $\mathbf{CFO}_n$.

Arbitrary Euclidean structure in $\mathbf{R}^n$ identifies the space $\mathbf{FO}_n$ of complete oriented flags with the group $\mathbf{SO}_n$ (we choose in each oriented i-dimensional subspace L_i the unique orthonormal vector v_i orthogonal to the oriented $(i-1)$-dimensional subspace L_{i-1} and such that (L_i, v_i) defines the given orientation of L_{i+1}).

Further we'll often use this isomorphism.

REMARK. The change of parameter on the nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^n$ causes reparameterization of the flag curve γ_f and multiplies matrix curve γ_g by a 1-parameter family of upper triangular matrices.

DEFINITION. Each parameterized nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^n$ defines in $\mathbf{R}^{n+1}$ the unique linear operator (called monodromy operator of γ) which takes the initial n-tuple $\gamma(0), \gamma'(0), \dots, \gamma^{(n)}(0)$ onto the final n-tuple $\gamma(1), \gamma'(1), \dots, \gamma^{(n)}(1)$.

DEFINITION. We say that the basis $e_1, \dots, e_{n+1}$ in $\mathbf{R}^{n+1}$ is adjusted to the oriented flag $f_1, \dots, f_{n+1}$ if $e_1, \dots, e_i$ span f_i with proper orientation.

REMARK. Reparameterization multiplies the matrix of monodromy operator by an upper triangular matrix with positive elements on the main diagonal in any basis adjusted to the initial osculating flag.

REMARK. Fixing any Euclidean structure in $\mathbf{R}^{n+1}$ we assign to each oriented (naturally parameterized) nondegenerate curve on $\mathbf{S}^n$ the unique orthogonal transformation mapping orthogonal $(n+1)$-tuple of vectors associated with its initial flag onto the orthogonal $(n+1)$-tuple of vectors associated with its final flag. In other words if we identify the initial flag with the unit matrix then the final flag will be identified with the unique orthogonal matrix determining the above transformation.

DEFINITION. Two complete oriented flags f_1, f_2 in $\mathbf{R}^{n+1}$ are called nontransversal if the intersection of at least one pair of their subspaces has the unproper dimension.

§2. DISCONJUGACY CRITERION AND TYPES OF NONSTRICTLY DISCONJUGATE CURVES

LEMMA. *If a nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^2$ is conjugate (i.e. there exists a 'big circle' intersecting it transversally in at least 3 inner points) then at least for one moment $t \in (0,1)$ the osculating flag $f(t)$ is nontransversal to $f(0)$.*

PROOF: Taking a conjugate curve $\gamma : [0,1] \to \mathbf{S}^2$ let us consider it as a family of curves γ_τ given on the increasing time interval $[0,\tau]$ $\tau \leq 1$. Obviously for sufficiently small τ the curve γ_τ is disconjugate. Then for some τ_0 the curve γ_{τ_0} becomes nonstrictly disconjugate and for $\tau_0 \leq \tau \leq 1$ the curve γ_τ is conjugate. We'll prove that initial and final flags of γ_{τ_0} (i.e. f_0 and f_{τ_0}) are nontransversal. Indeed the nonstrictly disconjugate curve γ has a big circle intersecting it with the sum of local multiplicities ≥ 3 and any of such circles doesn't intersect $\gamma \geq 3$ times transversally since in this case any small perturbation of the initial curve is also conjugate contradicting to nonstrict disconjugacy.

Hence we immediately obtain that the ' big circle' C conjugate to nonstrictly disconjugate γ can not intersect it in the inner point because its small push will intersect γ three times transversally. So in the considered case conjugate C must exist and pass only through the initial and final points of γ. Notice that by results of Arnold ([A]) the final point γ_{τ_0} of (nonstrictly) disconjugate curve γ_{τ_0} can't be antipodal to $\gamma(0)$. So we can work with planar curves. Below we show all the possible arrangements for initial and final flags. Other possibilities are easily rejected because one can find the small push of C with ≥ 3 transversal inner intersections.

So there are 5 different types of nonstrictly disconjugate curves on $\mathbf{S}^2$ (it will be proved in the next section).

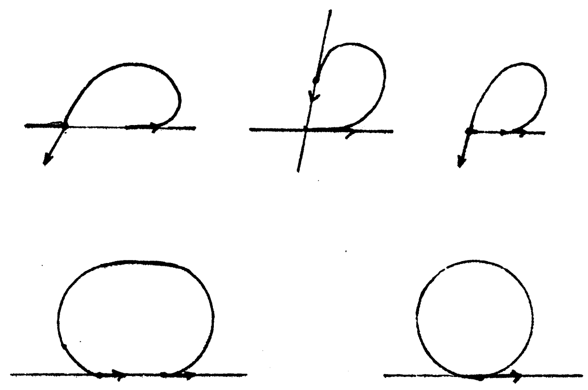


Fig.2

§3. CELLULAR DECOMPOSITION OF $\mathbf{FO}_3$, TOPOLOGY AND GEOMETRY OF TRAIN AND DISCONJUGACY DOMAIN

DEFINITION. The set $\mathbf{Tn}_f$ of all flags from $\mathbf{FO}_n$ nontransversal to the flag f is called the train of the flag f.

Consider $\mathbf{SL}_n$-action on the space $\mathbf{FO}_n$. If we fix arbitrary flag f and consider its stabilizor subgroup $St_f \subset \mathbf{SL}_n$ then it identifies with the subgroup T^+ of upper triangular (in any basis adjusted to f) matrices with positive elements on the main diagonal. One can easily see that the orbits of St_f-action on $\mathbf{FO}_n$ are cells (see [F]). If we fix an arbitrary basis $e_1, \dots, e_n$ in $\mathbf{R}^n$ then each cell contains the unique coordinate flag, i.e. flag whose subspaces coincide with oriented coordinate subspaces (we assume that orientation of the whole $\mathbf{R}^n$ is fixed).

These cells are enumerated by the elements of the group $\mathbf{Z}/2\mathbf{Z} \times \cdots \times \mathbf{Z}/2\mathbf{Z} \times S_n$ and their total number equals $2^{n-1} \times n!$.

Below we show the diagram of cellular decomposition for $\mathbf{SO}_3$, their adjointments and corresponding coordinate flags (drawn as flags in $\mathbf{TS}^2$). Adjointments of $\mathbf{FO}_n$-cells can be easily obtained from the classical Bruhat ordering (see [S]).

REMARK. By the definition $\mathbf{Tn}_f$ coincides with the union of all positive codimensional cells of the Schubert decomposition of $\mathbf{FO}_n$ associated with f (all cells except $2(n-1)$ full-dimensional ones).

Now we obtain the equation for $\mathbf{Tn}_f$. If we identify $\mathbf{FO}_n$ with $\mathbf{SO}_n$ then for any flag f represented by an orthogonal matrix (a_{ij}) then its train $\mathbf{Tn}_f$ is given by the equation

$$\Delta = \Delta_1(X) \dots \Delta_{n-1}(X) = 0$$

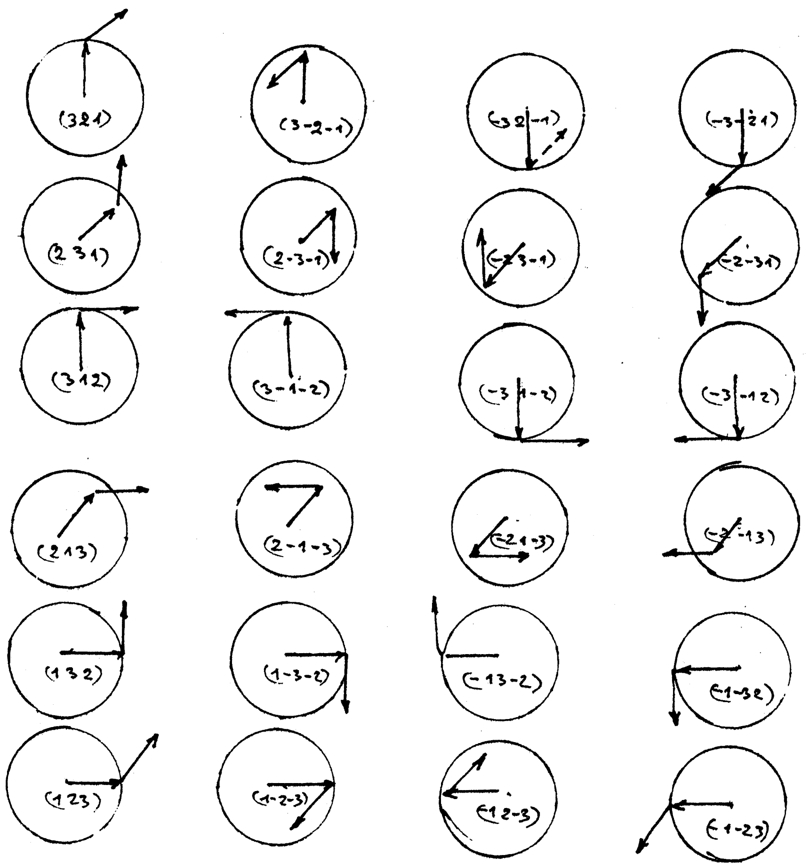


Fig.3. 24 cells in $\mathbf{FO}_3$

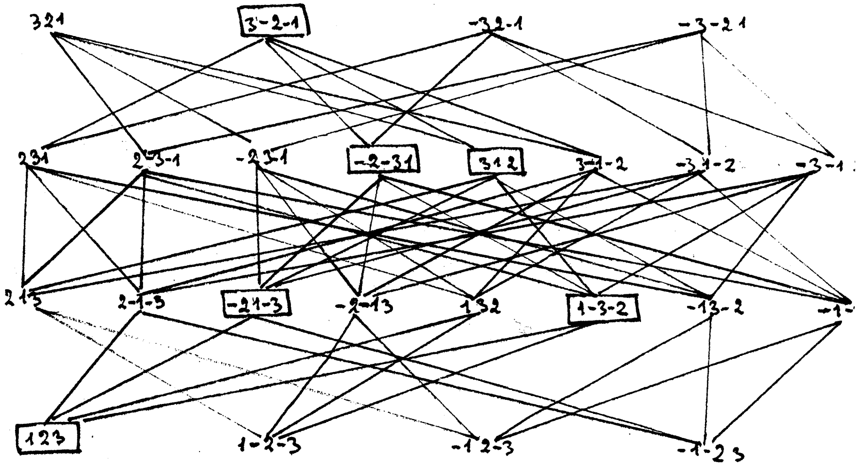


Fig.4. The adjointment of cells in $\mathbf{FO}_3$

where X denotes variable matrix $\in \mathbf{SO}_n$ and

$$\Delta_i(x): \left\| \begin{matrix} a_{11}, & a_{12}, & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}, & a_{i2}, & \dots & a_{in} \\ x_{11}, & x_{12}, & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-i,1}, & x_{n-i,2}, & \dots & x_{n-i,n} \end{matrix} \right\| = 0.$$

IMPORTANT EXAMPLE. If $f \in \mathbf{FO}_3$ is represented by the unit matrix then its train the equation $\Delta_1\Delta_2 = 0$, where Δ_i is the right main $(i \times i)$-minor of the orthogonal ma X.

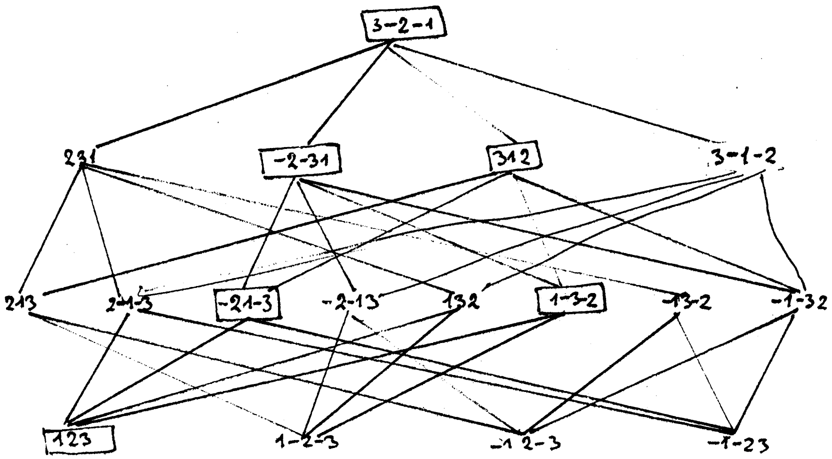


Fig.5. The closure of the open 3-dimensional cell (reachable strata are placed in boxes)

REMARK. For $\mathbf{FO}_n$ its $2(n-1)$ high-dimensional cells are given by inequalities

$$\pm\Delta_1 > 0,$$
$$\pm\Delta_2 > 0,$$
$$\vdots$$
$$\pm\Delta_{n-1} > 0.$$

Now we describe in details topology of $\mathbf{FO}_3$ and $\mathbf{Tn}_f$.

The space $\mathbf{FO}_3 = \mathbf{SO}_3$ is diffeomorphic to $\mathbf{RP}^3$; each of Δ_1 and Δ_2 is diffeomorphic to 2-torus which is separated by four circles consisting of eight 1-cells (1-skeleton of $\mathbf{Tn}_f$) into four 2-dimensional cells.

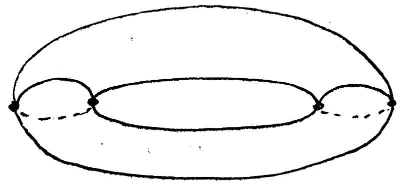


Fig.6

In the homogeneous coordinates (u, v) we have

$$\Delta_1 : |u|^2 = |v|^2$$
$$\Delta_2 : |u-v|^2 = |u+v|^2$$

Each of four 3-dimensional cells is bounded by four 2-dimensional cells which are divided into two pairs. Each pair belongs to its torus and 2-cells from each pair intersects with the other only by 4 vertices and thus form "the pillow". These pillows are glued to each other the way it's shown below

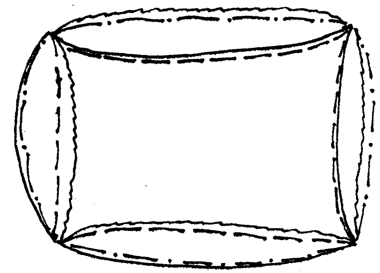


Fig.7

Good local illustration for $\mathbf{Tn}_f$ in the neighborhood of $f \in \mathbf{FO}_3$ is obtained as follows. The standard affine chart in $\mathbf{SO}_3$ is identified with the space of upper triangular matrices with the units on the main diagonal. Equation of the train of unitary matrix is $z(z - xy) = 0$, where $x \quad y \quad z$ are the following matrix entries

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

On the next figure one sees four 1-dimensional, eight 2-dimensional and four 3-dimensional cells (pairs of components connected by arrows glue through infinity).

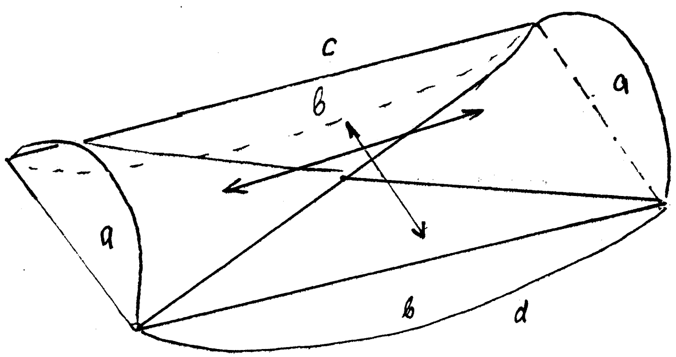


Fig.8

§4. Disconjugate domain of the flag

In section 1 we've mentioned that flag curve of nondegenerate curves are tangent to $(\mathbf{R}^+)^{n-1}$ of $\mathbf{SO}_n$-invariant Cartan distribution $\mathbf{CFO}_n$. Properties of trains are closely connected with the crucial properties of $\mathbf{CFO}_n$. For example if we consider the space of all germs of flag curves for l.o.d.e. starting at f then they fill the germ of domain called the local reachable domain. It coincides with one of the local components of $\mathbf{FO}_n \setminus \mathbf{Tn}_f$. On figure 8 the reachable domain for the flag 0 coincides with the component A and is given by the system of inequalities:

$$\begin{cases} z > 0, \\ z > xy, \\ x > 0, \\ y > 0 \end{cases}$$

By the lemma of section 2 we know that nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^2$ is disconjugate until the moment $\tau_0 \in (0,1]$ when $f_{\tau_0} \in \mathbf{Tn}_{f_0}$.

So all nondegenerate curves starting at f go into one connected component of $\mathbf{FO}_n \setminus \mathbf{Tn}_f$ and remains disconjugate until they reach its boundary. This component will be called the disconjugate domain of the flag f and denoted by D_f. Below we'll describe in details its boundary for the case of $\mathbf{FO}_3$.

If we ~~associate~~ identity f with the unit matrix then one can easily see that D_f is given by inequalities $\Delta_i > 0 \quad i = \overline{1,n}$ (see section 3). Now we consider the following question. What strata of D_f-boundary are reachable, i.e. their points are the final flags of nonstrictly disconjugate curves.

Lemma. *The generic point p of D_f-boundary is reachable if some (and therefore any) vector $v \in \mathbf{CFO}_n|_p$ goes outside the domain D_f and nonreachible otherwise. Stratum $S \subset \overline{D_f}$ of positive codimension is reachable by nonstrictly disconjugate curves if S can be represented as the intersection of the closures of reachable strata only.*

In the case of $\mathbf{FO}_3$ reachable strata are placed in boxes on fig.8. They are denoted by small letters and correspond to the signed permutations $(3-2-1), (-2-3\ 1), (3\ 1\ 2), (-2\ 1-3), (1-3-2), (1\ 2\ 3)$.

§5. On the π_0 of $\mathcal{RS}^2(f_1, f_2)$

Proof of theorem A (see introduction) is divided into the following two statements.

Lemma. *If f_1 and f_2 can be connected by a right oriented (nonstrictly) disconjugate curve then the subset $DS(f_1, f_2)$ of all disconjugate curves in $\mathcal{RS}^2(f_1, f_2)$ forms one connected component.*

Proof: We consider simultaneously all the five cases shown on fig.2. Initial and final flags (f_1, f_2) define three (probably coinciding) 'big circles', i.e. C_1, C_2 and $\overline{p_1, p_2}$ (which connects points p_1 and p_2). Curves from $DS(f_1, f_2)$ can't intersect C_1, C_2 and $\overline{p_1, p_2}$ in points different from p_1 and p_2.

So we have shown that curves from $DS(f_1, f_2)$ are simple convex arcs connecting p_1 and p_2 (see fig.9) and they obviously fill one connected component of $\mathcal{RS}^2(f_1, f_2)$.

Lemma. *For any pair (f_1, f_2) conjugate right oriented curves fill 2 connected components in $\mathcal{RS}^2(f_1, f_2)$.*

Proof: Slight modification of J.Little's arguments shows that arbitrary nondegenerate curve on $\mathbf{S}^2$ can be nondegenerately deformed with its ends fixed to some curve lying in some (probably closed) hemisphere. The situation on the hemisphere coincides with that on $\mathbf{R}^2$. According to Whitney the only invariant of connected components for nondegenerate curves on $\mathbf{R}^2$ with fixed initial and final flags (f_1, f_2) is the full rotation angle of its velocity. So if we prove that two arbitrary nondegenerate plane conjugate curves $\in \mathcal{RS}^2(f_1, f_2)$ whose difference of rotation angle is divisible by 4π are $\mathbf{S}^2$-nondegenerately homotopic the statement will be settled.

The following set of pictures shows how to increase the rotation angle by 4π. It resembles pulling of the spring.

So any planar "γ"-fragment can be nondegenerately deformed into "ω"- fragment. This procedure increases rotation angle by 4π. The proof completes by the remark that any conjugate curve can be deformed (with initial and final flags fixed) into a curve with "γ"-fragment.

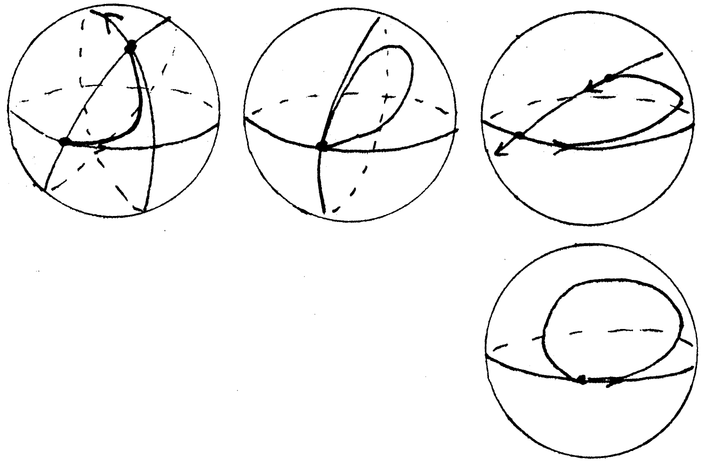


Fig.9

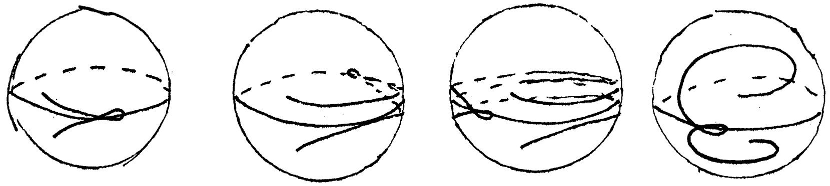


Fig.10

Indeed if C is the 'big circle' intersecting the conjugate curve $\gamma \geq 3$ times transversally then γ necessarily includes one of the following 2 fragments:

The first one is "γ" itself while the second is easily deformed into the first one.

§6. Covering homotopy for nondegenerate conjugate

curves on $\mathbf{S}^2$

Above we've shown that flag curves of nondegenerate curves on $\mathbf{S}^n$ are tangent to Cartan distribution $\mathbf{CFO}_n$ which is the left invariant distribution of $(\mathbf{R}^+)^{n-1}$ on $\mathbf{FO}_n = \mathbf{SO}_n$. We can consider the following general question. Having some nonholonomic distribution $\mathcal{F}$ of cones, i.e. distribution of cones which associated distribution of linear subspaces is nonholonomic, see [V-G]) on some manifold one can consider the space $\mathcal{REG}$ of all regular curves starting at some fixed tangent element and ending elsewhere and study the properties of the obvious map $\pi : \mathcal{REG} \to Tang$ sending regular curves onto the final tangent elements and find whether it satisfies the covering homotopy property. Recall that for linear nonholonomic distributions this is always valid by the results of S. Smale (see [S]). At the same time for the distribution of cones whose germs of reachable domains are different from the complete neighborhood of the initial points the situation at least locally is different, i.e. short curves can't satisfy the covering homotopy property (even for 1-parameter families) since we can choose deformations of the final tangent element pulling its attachment point outside the reachable domain and it can't be covered by the deformation of the curve. If this local situation is preserved globally (for example as for the distribution of parallel cones in $\mathbf{R}^n$ or negatively curved spaces) then nothing of the covering homotopy is left. Still in the case when the global reachable domain of any point coincides with the whole manifold and there exist closed contractible curves tangent to the distribution and passing through each tangent element (for 'positively curved' manifolds) then one can hope that for sufficiently large 'conjugate' curves the covering homotopy is valid. It will be very useful to turn these naive considerations into strict proofs for left-invariant distributions on the compact Lie groups and homogeneous spaces. Here we illustrate them on the example of the space of all conjugate curves on $\mathbf{S}^2$.

Proof of the covering homotopy property for the map $\pi : CN_f \to \mathbf{FO}_3$ which takes any curve (from the space CN_f of all conjugate curves starting at the fixed initial flag f) to its final flag derives from the following three lemmas.

Lemma 1, see [Sr]. *Let $w : I^n \to CN_f$ be an arbitrary continuous family of conjugate curves. Then there exists the continuous family $\phi : I^n \to (\mathbf{S}^2)^*$ of their conjugate 'big circles'.*

Definition. We call a conjugate curve on $\mathbf{S}^2$ the conjugate curve of the simplest type if its maximal sum of local multiplicities of intersections with any 'big circle' equals 3.

Proof: Since the first conjugate point on the conjugate curve depends continuously on parameters then one consider only conjugate curves close to the 5 types shown on Fig. 2. For which one can easily construct necessary families.

LEMMA 2. *Any continuous family of conjugate curves $w : I^n \to CN_f$ can be nondegenerately deformed (preserving their initial and final flags) to the following family $w_k : I^n \to CN_f$. Each curve from w_k coincides with the corresponding curve of w but starts with the k times passed little circle on $\mathbf{S}^2$.*

PROOF: According to Lemma 1 we can take simplest parts of all conjugate curves from w then deform each of them as it is shown on Fig. 10 and to obtain the necessary fragment at the beginning.

LEMMA 3. *For sufficiently big k the family w_k covers any deformation of the final flags on $\mathbf{FO}_3$.*

PROOF: Introduce on $\mathbf{FO}_3 \equiv \mathbf{SO}_3$ the Killing' metrics. We will cover any 1-parameter deformation of final flags in w_k (for sufficiently big k) Obviously the following procedure depends continuously on any compact space of parameters. Covering will be done as the iteration of two standard sequential steps. Firstly we choose some finite σ such that we can cover the σ-neighborhood of the final flag changing the last half of the circle by the appropriate halfs of the circle on the sphere which pass through $\bar{f}$ and the perturbed final flag $\tilde{f}$ (see Fig.11). Then we fulfil some standard smoothing procedure making the whole curve smooth. In order to cover the next σ-perturbation of $\tilde{f}$ we move as one fragment part of the curve starting with the flag g. Since $\mathbf{SO}_3$ acts on $\mathbf{FO}_3$ by isometries we have that the final flag moves on the distance not exceeding σ from its initial position and we can repeat the previous step.

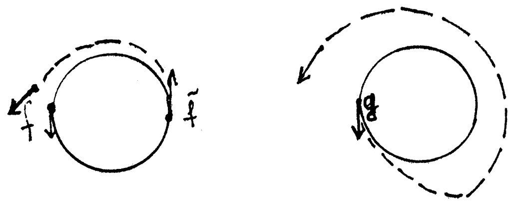


Fig.11

PART II. NONDEGENERATE CURVES ON $\mathbf{S}^2$ WITH GIVEN MONODROMY

This part is devoted to the calculation of π_0 of the space $D_3(A)$ of all third order linear ordinary differential equations with monodromy operator A which is the only discrete in-

variant of Gelfand-Dikii bracket. In the paper [SH2] we give the complete classificatio 3×3-matrix systems. This information will be useful for classification of ordinary equat For the space $D(A)$ we'll use interpretation quite different from those used in the first for equations with given monodromy matrix.

We fix some monodromy matrix $\mathcal{A}$ corresponding to the operator A consider its a on $\mathbf{FO}_3$ and look at all right-oriented curves satisfying the relation $f_2 = \mathcal{A}f_1$, whe and f_2 are as usual the initial and final flags of the curve. Among these curves those w correspond to the same differential equation are $\mathbf{SL}_3$-equivalent. So they are necess $St_{\mathcal{A}}$-equivalent, where $St_{\mathcal{A}}$ denotes the stabilizor subgroup of matrix $\mathcal{A}$.

Considering $\mathbf{FO}_3$ as the bundle over $\mathbf{S}^2$ with the fibre $\mathbf{S}^1$ (the set of all oriented on $\mathbf{S}^2$ passing through the given point) we have for each initial flag the problem alr considered in the first part and must observe how the situation changes when we ch the point of the base. For generic operators a fibre over the typical point is separated two parts (+arc and -arc) depending on whether the image $\mathcal{A}f$ of the given flag f bel to D_f or not. One of our aims is to describe connected components of +arcs and -ar the whole $\mathbf{FO}_3$. The main argument is the description of arc bifurcations when the point passes through invariant subspace of $\mathcal{A}$.

§1. CALCULATION OF $\pi_0(D(A))$

LEMMA 1. *$D(A)$ is homotopically equivalent to the factor of the space of all right-orie nondegenerate curves belonging to $\cup_{f\in\mathbf{FO}_3} \mathcal{R}\mathbf{S}^2(f,\mathcal{A}f)$ by $St(\mathcal{A})$, where $\mathcal{A}$ is some matrix representing operator A.*

PROOF: Indeed, since all curves corresponding to the same equation are $\mathbf{SL}_3$-equiva then (because we've fixed $\mathcal{A}$) they can differ only by $St(\mathcal{A})$-action. For arbitrary curve set of all its reparameterizations preserving initial and final flags is contractible. Th fore taking into account that the same radial projection on the unit sphere $\mathbf{S}^2$ have the curves differing by multiplication on arbitrary positive function we have the neces proposition.

DEFINITION. The set of all $x \in \mathbf{R}^n$ for which $x, \mathcal{A}x, \dots, \mathcal{A}^{n-1}x$ are linear depende called the **degeneration set** DgA of operator A.

REMARK. The set DgA is the union of all invariant subspaces of positive codimen and it is preserved by StA. In the complex case $\dim DgA$ equals the sum of sizes of biggest Jordan blocks with different eigenvalues.

LEMMA. *Intersection of DgA with $\mathbf{S}^2$ consists of*

(1) *of three 'big circles' in the case of J.n.forms a) and b);*
(2) *of one 'big circle' and a pair of antipodal points for J.n.f. c);*
(3) *of two circles (one of which is double for J.n.f. d) and e);*
(4) *of one circle for the J.n.f. f);*
(5) *and coincides with $\mathbf{S}^2$ otherwise.*

DEFINITION. A vector $x \notin DgA$ is called **positive** if the 3-tuple x, Ax, A^2x defines fixed orientation of $\mathbf{R}^3$ (which induce given orientation of $\mathbf{S}^2$) and **negative** otherwise.

DEFINITION. A flag $f \in \mathbf{FO}_3$ is called A-disconjugate if $Af \in Df$, i.e. there exist disconjugate curve in $\mathcal{R}\mathbf{S}^2(f, Af)$.

Any point which does not belong to DgA defines on the fiber $C_x = \pi^{-1}x$ four flags

f_α whose 'big circle' passes through p and Ap and the segment (p, Ap) is positively oriented;

f_β coinciding with f_α but with the opposite orientation of the 'big circle';

f_α^{-1} coinciding with $A^{-1}(f_\alpha)$;

f_β^{-1} coinciding with $A^{-1}(f_\beta)$.

All of them are different. Notice that if orientation of $\mathbf{S}^2$ is fixed then the fibre C_x has the induced orientation.

LEMMA. *If $x \notin DgA$ is a positive point then the +arc on C_x coincides with $(f_{\alpha^{-1}}, f_\alpha)$ and the -arc with $[f_\alpha, f_{\alpha^{-1}}]$;*

If $x \notin DgA$ is negative then the +arc on C_x coincides with $[f_\beta, f_{\beta^{-1}}]$ and the -arc with $(f_{\beta^{-1}}, f_\beta)$.

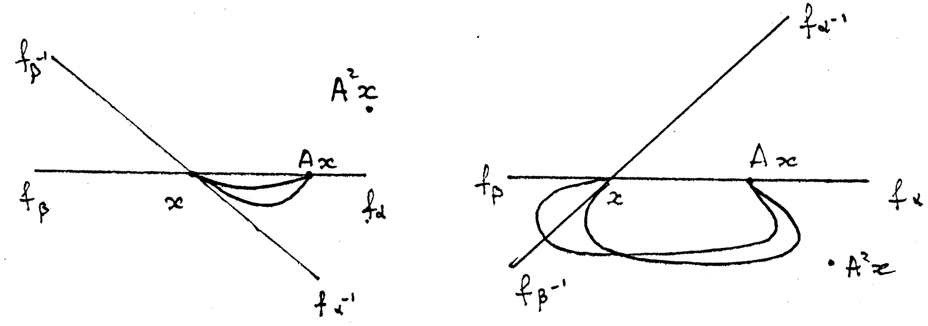


Fig 12. Arcs for positive and negative x

PROOF: According to results of §.4 the flag f_2 belongs to D_{f_1} if its point p_2 belongs to the upper hemisphere of the 'big circle' C_1 and the negative segment of f_2 intersects with the positive segment of f_1. In all our cases this condition can be easily verified. Moreover for positive x right-oriented curves vanish at the ends of the +arc since the final flag f_2 moves to the nonreachible strata of the completion of D_f while when x is negative then disconjugate curves become nonstrictly disconjugate at the ends of the +sarc. (f_2 moves to strata D and E).

Completion of the proof of Theorem D. Results of §2.1. and the covering homotopy property imply that the question about the number of connected components in the set of conjugate curves realizing the given monodromy operator reduces to the question whether the conjugacy class of operator A realize the nontrivial element of $\pi_1(\mathrm{GL}_n^+)$. So it remains to study what happens with disconjugate curves when we change the initial flag.

For generic operators , i.e. those whose DgA differs from the whole $\mathbf{S}^2$ the quest settled by the following proposition.

LEMMA.

If for $t \in [0,1)$ $x(t)$ is the curve of positive points on $\mathbf{S}^2$ such that $x(1) \in DgA$ the +arc vanishes, i.e. disconjugate curves on the positive components of $\mathbf{S}^2 \setminus DgA$ i separate connected component;

If for $t \in [0,1)$ $x(t)$ is negative and $x(1) \in DgA$ then the disconjugate curves dege to nonstrictly disconjugate and they are connected with the conjugate ones.

PROOF: We have to consider the following three cases

a) $x(1)$ is the generic point of the circle from DgA;

b) $x(1)$ is the eigenvector (i.e. the isolated point of DgA;

c) $x(1)$ is the generic point of the double circle.

In each of this cases the proposition is quite obvious.

COROLLARY. *The number of connected components filled by disconjugate curves the number of positive components in $\mathbf{S}^2 \setminus DgA$ up to the action of the stabilizor, equals 1 for J.n. forms a) - f).*

Cases g) - j) will be considered separately.

In the simplest case j) we work with closed nondegenerate curves and (by resu J.Little, see Introduction) there exists a separate disconjugate connected compone with nonstrictly disconjugate curves.

In the cases g) and i) for the point x lying outside the circle of eigenvectors (nonst disconjugate curves exist only for the unique direction of the tangent element and transform to the conjugate ones under the arbitrary small perturbation. To check can use convenient affine chart $\frac{x}{z}, \frac{y}{z}$ in which our linear transformations are of the $\begin{pmatrix} \frac{a^2}{c^2} & 0 \\ 0 & \frac{a^2}{c^2} \end{pmatrix}, \begin{pmatrix} -\frac{a^2}{c^2} & 0 \\ 0 & -\frac{a^2}{c^2} \end{pmatrix}$ and $\begin{pmatrix} 1 & \frac{1}{a^2} \\ 0 & 1 \end{pmatrix}$. So no new components appear.

In the remaining case h) disconjugate curves exist only if the angle between the r vector and the oriented flag line belongs to the interval $[0, \pi]$ in the above affine cha they fill two connected components in the upper and lower hemispheres which are ider by the action of stabilizor. ∎

REFERENCES

[A]. V. I. Arnold, *Ramified covering* $\mathbf{CP}^2 \to \mathbf{S}^4$, *hyperbolicity and projective topology*, Sib. Math. jo 5 (1988), 36-47.

[F]. D. B. Fuchs, *Classical manifolds*, in "Sovremennye problemy matematiki. Fundamental'nye n leniya," 1986, pp. 253-314.

[G]. M. Gromov, "Partial differential relations," Springer Verlag, 1986.

[L]. J. Little, *Nondegenerate homotopies of curves on the unit 2-sphere*, J. Diff. Geom. **4** (1970), 33

[K-O]. B. A. Khesin and V. Yu. Ovsienko, *Symplectic leaves of Gelfand-Dikii' brackets and homo classes of nondegenerate curves*, Funkt. Analyz i ego Priloz. **24** 1 (1990), 38-47.

[Sh]. B. Z. Shapiro Spaces of linear differential equations and flag varieties, Izvestia Akademii Nauk **54** 1 (1990), 25-37.

[Sr] T. L. Sherman, *Conjugate points and simple zeros for ordinary linear differential equations*, AMS **146** 2 (1969), 397-411.

[S]. R. Steinberg, "Lectures on Chevalley groups," Yale University, 1967.

[V-G]. A. M. Vershik and V. Ya. Gershkowitch, *Nonholonomic dynamical systems. Geometry of distributions and variational problems*, in "Sovremennye problemy matematiki. Fund. napravleniya.," pp. 5-85.

On the number of connected components in the space of closed nondegenerate curves on S^n

B. Z. Shapiro

THE MAIN DEFINITION. A parameterized curve $\gamma : \mathbf{I} \to \mathbf{R}^n$ is called **nondegenera** for any $t \in \mathbf{I}$ vectors $\gamma'(t), \dots, \gamma^{(n)}(t)$ are linearly independent. Analogously $\gamma : \mathbf{I} \to$ called **nondegenerate** if for any $t \in I$ the covariant derivatives $\gamma'(t), \dots, \gamma^{(n)}(t)$ spa tangent hyperplane to $\mathbf{S}^n$ at the point $\gamma(t)$ (compare with the notion of n-freedom in

Fixing an orientation in $\mathbf{R}^n$ or $\mathbf{S}^n$ we call a nondegenerate curve γ right-oriented orientations of $\gamma', \dots, \gamma^{(n)}$ coincide with the given one and left-oriented otherwise.

Nondegenerate curves on $\mathbf{S}^n$ are closely related with linear ordinary differential equ of $(n+1)$-th order. Such an equation defines two nondegenerate right-oriented curv $\mathbf{S}^n \subset V^{(n+1)*}$, where $V^{(n+1)*}$ is the $(n+1)$-dimensional vector space dual to the space solutions as follows. For each moment $t \in \mathbf{I}$ we choose the linear hyperplane in V^{n+1} solutions vanishing at t i.e. obtain a unique curve in the projective space $\mathbf{P}^n$. Raising it we obtain a pair of curves; both of them are right-oriented for an odd n and have op orientations for an even n (nondegeneracy follows from nonvanishing of Wronskian).

Nondegenerate curve $\gamma : [0,1] \to \mathbf{S}^n$ defines its monodromy operator $M \in \mathbf{GL}^+_{n+1}$ maps $\gamma(0), \gamma'(0), \dots, \gamma^{(n)}(0)$ to $\gamma(1), \gamma'(1), \dots, \gamma^{(n)}(1)$. On the other hand in the pape O] there is given a complete set of invariants for symplectic leaves of the second Gel Dikii bracket, namely its leaves are enumerated by pairs: a monodromy operator connected component of the space of right-oriented curves in the sphere with the monodromy operator.

In this paper we study the number of connected components for closed nondegen right-oriented curves (corresponding to the identity monodromy operator). The differ relation of nondegeneracy is also interesting in connection with the general theory principle (see [G]).

Let $\mathbf{NR}_n$ $(\mathbf{NS}_n)$ be the space of all nondegenerate closed right-oriented curv $\mathbf{R}^n(\mathbf{S}^n$ respectively).

The question we consider is how to calculate $\pi_o(\mathbf{NS}_n)$ and $\pi_o(\mathbf{NR}_n)$. For the first a similar question was studied in [F]. Notice that $\mathbf{NR}_2$ is homeomorphic to the in number of infinite-dimensional cells (this is probably due to Whitney). Later J.Little [L1,L2] studied $\mathbf{NS}_2$ and $\mathbf{NR}_3$ and proved the following W.Pohl' conjecture: $card(\pi_o(\mathbf{NS}_2)) = 3$ and $card(\pi_o(\mathbf{NR}_3)) = 2$. (The invariant which distinguishes closed nondegenerate curves is an element of π_1 of the image of the natural map $\nu : \mathbf{NR}_n \to \mathbf{SO}_n$, where $\nu(\gamma(t))$ equals the matrix obtained by orthogonalization of $\gamma'(t), \ldots, \gamma^{(n)}(t)$; and $s\nu : \mathbf{NS}_n \to \mathbf{SO}_{n+1}$, where $s\nu(\gamma(t))$ equals the matrix obtained by orthogonalization of $\gamma'(t), \ldots, \gamma^{(n)}(t)$ considered as vectors in $\mathbf{R}^{n+1}$ due to the standard embedding $\mathbf{S}^n \subset \mathbf{R}^{n+1}$.

U.Hamenstadt [H] continued the study of this question and formulated the following result (mentioned in [G]).

PROPOSITION. $\pi_o(\mathbf{NR}_k) = \pi_o(\mathbf{NS}_k)$ *and consists of 2 elements for any* $k > 2$.

There are several gaps in her proof and the proposition itself is wrong. Probably the true version is as follows:

CONJECTURE.

(1) $\pi_o(\mathbf{NR}_k) = \pi_o(\mathbf{NS}_k)$ *and consists of 2 elements for any odd* $k > 2$, *i.e. in this case the only invariant equals* π_1 *of the above mentioned map.*

(2) $\pi_o(\mathbf{NR}_k) = \pi_o(\mathbf{NS}_k)$ *and consists of 3 elements for any even* $k > 3$.

The only fact I am able to prove (and which contradicts Hamenstadts proposition) is:

THEOREM 1. $card(\pi_o(\mathbf{NS}_{2k})) \geq 3$

To prove this theorem we need several definitions.

DEFINITION 1. By a **linear subspace** of $\mathbf{S}^n$ we will understand any 'big circle' obtained as the intersection of $\mathbf{S}^n$ with any linear subspace in $\mathbf{R}^{n+1} \ni 0$ where $\mathbf{S}^n$ is standard embedded. Any k-tuple $(v_1, \ldots, v_k)$ of vectors tangent to $\mathbf{S}^n$ at some point p defines a linear subspace in $\mathbf{S}^n$ as its intersection with the subspace in $\mathbf{R}^{n+1}$ spanned by $p, v_1, \ldots, v_k$.

DEFINITION 2. Consider a closed nondegenerate curve γ on $\mathbf{S}^n$. The curve γ will be called **disconjugate** if the sum of local multiplicities of its intersection over all intersection points of the curve with any linear hyperplane (see above) is at most n and **conjugate** otherwise. (Local multiplicity is the degree of the restriction of the divisor on the curve, in the considered case it equals the maximal dimension of the osculating subspace to the curve contained in the hyperplane.)

REMARK 2. One can easily see that the local multiplicity of such intersection for a nondegenerate curve is at most n. It equals the maximal dimension of subspace osculating to the curve at this point and lying in the hyperplane.

DEFINITION 3. Denote by $\mathbf{S}^{n*}$ the sphere dual to the considered $\mathbf{S}^n$, i.e. the set of all oriented hyperplanes on $\mathbf{S}^n$. If γ is a nondegenerate curve on $\mathbf{S}^n$ then we can define the curve $\gamma^* \in \mathbf{S}^{n*}$ dual to γ *as the set of all oriented osculating hyperplanes tangent to* γ.

REMARK 3. Now we will also give the dual formulation of disconjugacy. For the curve γ^* dual to the disconjugate curve γ and an arbitrary point $x \in (\mathbf{S}^{n*})$ the sum of tangency orders over all hyperplanes passing through x and tangent to γ is at most n.

Notation. $\mathbf{ND}_{2k}$ $(\mathbf{NC}_{2k})$ will denote the space of all closed disconjugate (resp. conjugate) curves on $\mathbf{S}^{2k}$.

One can see that $\mathbf{ND}_{2k}$ is nonempty. For example, it contains the curve whose coordinates are $\sqrt{\frac{1}{k+1}}(1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots, \sin kt, \cos kt)$ (it lies on standard $\mathbf{S}^{2k}$).

Since $\mathbf{NC}_{2k}$ consist of at least 2 components differing by the element of π_1 of $\mathbf{GL}_{2k+1}$ (for example, iterating the above curve two or three times one obtains conjugate curves realizing different elements of π_1 of $\mathbf{GL}_n^+$), Theorem 1 follows from the next result.

LEMMA 1. $\mathbf{ND}_{2k}$ *is disconnected with* $\mathbf{NC}_{2k}$.

The proof is divided into 2 parts:

(1) $\mathbf{ND}_{2k}$ is open;

(2) $\mathbf{NC}_{2k}$ is open.

Instead of (1) we will prove the following more general fact. For any $m \geq n$ the set $\mathbf{NS}_n^m$ of all nondegenerate closed curves whose sum of intersection multiplicities with any hyperplane does not exceed m is open. Let $t_1, \ldots, t_p$ be pairwise different moments of intersection of γ with an arbitrary hyperplane $\mathbf{L}$ and $1 \leq k_1, \ldots, k_p \leq n$ be the set of corresponding local multiplicities ($\sum k_i \leq m$). Since γ is nondegenerate, then by the definition of multiplicity for each t_i there exist δ_i and ξ_i such that if $|\gamma - \bar{\gamma}|_{C^n} \leq \xi_i$ and the sum $\sharp$ of intersection multiplicities of $\bar{\gamma}$ with $\mathbf{L}$ if $\bar{\gamma}$ belongs to the δ_i-neighborhood of $\gamma(t_i)$ is at most k_i (the lower index $\mathbf{C}^n$ means that the distance is taken with respect to the metrics with n derivatives). Let U_i denote the δ_i-neighborhoods of the points $\gamma(t_i)$. Denote $\bar{\xi} = \min\{\xi_1, \ldots, \xi_p\}$ and let ρ be the distance between $\gamma \setminus \{U_1, \ldots, U_p\}$ and $\mathbf{L}$ in the ordinary metrics of $\mathbf{R}^n$. Finally, take $\xi = \min\{\bar{\xi}, \rho/2\}$; we see that if $|\gamma - \bar{\gamma}|_{C^n} < \xi/2$ then for any hyperplane $\mathbf{L}$ $\sharp(\mathbf{L} \cap \bar{\gamma}) \leq m$.

The idea of the proof of (2) is as follows. A hyperplane $\mathbf{L}$ is called **conjugate** relative to a conjugate curve $\gamma \subset \mathbf{S}^n$ if the sum of local intersection multiplicities of $\mathbf{L}$ with γ exceeds n. If a curve is conjugate, i.e. there exists a conjugate hyperplane then (by the result of Sherman ([S]) there exists a hyperplane with $\geq 2k$ transversal intersections. Consequently, for arbitrary sufficiently small deformation $\bar{\gamma}$ of γ the same hyperplane also intersects transversally γ at least 2k times. Thus $\bar{\gamma}$ is also conjugate and $\mathbf{NC}_{2k}$ is open.

I will give a selfcontained geometrical proof of Sherman's result whose main idea is due to V.I.Arnold.

DEFINITION 4. A point $x \in \mathbf{S}^n$ belonging to the small neighborhood of the germ of the nondegenerate curve $\gamma : [-\varepsilon, \varepsilon] \to \mathbf{S}^n$ is called **hyperbolic** if there exists n different hyperplanes passing through x and tangent to γ. The germ of the whole set of hyperbolic points is called the germ of the hyperbolic domain $\mathbf{H}$.

REMARK 4. One can easily show that $\mathbf{H}_\gamma$ is diffeomorphic to the product of the interval by the germ of the swallow's tail's pyramid (this pyramid is the subset of all n-tuples $(a_1, \ldots, a_n)$ for which the polynomial $x^{n+1} + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ has n simple real zeros (see [AVG]).

LEMMA 2. *An arbitrary hyperplane* $\mathbf{L} \subset \mathbf{S}^n$ *intersecting the germ of a nondegenerate curve* $\gamma : [-\varepsilon, \varepsilon] \to \mathbf{S}^n$ *(with any possible multiplicity* $\sharp$*) intersects the germ of its hyperbolic domain* $\mathbf{H}_\gamma$.

PROOF: The first case. If $\sharp$ is odd then ends of γ belong to the different 'semispaces' (local connected components in $\mathbf{S}^n \in \mathbf{L}$). Hence there exists a C^1-close to γ germ which entirely belongs to $\mathbf{H}_\gamma$ and also intersects $\mathbf{L}$.

The second case. If $\sharp$ is even then γ lies in one of the 'semispaces' and we need additional arguments.

DEFINITION 3. **The collar** Γ of the germ γ of a nondegenerate curve $\gamma : [-\varepsilon, \varepsilon] \to \mathbf{S}^n$ is the germ of 2-dimensional surface formed by the positive semitangents to γ. (The semitangent is called **positive** if it contains the tangent vector). **The collar** δ**-boundary** is the curve γ_δ of the ends of the tangent vectors whose lengths equal to δ.

LEMMA 3. *If* $\sharp(\gamma \cap \mathbf{L})$ *is even then half of the collar* δ*-boundary corresponding to the negative time interval* $[-\varepsilon, 0)$ *and* γ *itself lie in the different 'semispaces'.*

PROOF: We can always choose on $\mathbf{S}^n$ the appropriate affine coordinates $x_1, \dots, x_n$ with respect to which γ is parameterized as follows

$$\gamma_1 = t + \dots, \gamma_2 = t^2/2! + \dots, \gamma_{2k} = t^{2k}/2k! + \dots, \gamma_n = t^n/n! + \dots$$

Let the tangent subspace be spanned by the first $(2k-1)$ coordinates (since $\sharp$ is even), belong to $\mathbf{L}$ and the coordinate vector e_{2k} be transversal to $\mathbf{L}$. Then locally $\gamma_{2k}(t)$ is positive. The collar δ-boundary is given by the equation

$$\gamma_\delta = \gamma + \delta\gamma'$$

Hence 2k-th coordinate of γ_δ is equal to

$$\gamma_{\delta,2k} = \delta t^{2k-1}/(2k-1)! + t^{2k}/2k! + \dots$$

and for negative small $t \gamma_{\delta,2k}$ is negative. ▮

This lemma along with the observation that the collar belongs to the closure of $\mathbf{H}_\gamma$ implies that there also exists a germ of a curve lying entirely in $\mathbf{H}_\gamma$ whose ends belong to the different 'semispaces'. Lemma 2 is completely proved. ▮

Now we can give a geometrical proof of Sherman's result reducing it to Lemma 2. The essence of Arnold's lemma is as follows. If we have an $(n-1)$-dimensional linear system of hyperplanes on $\mathbf{S}^n$ containing a hyperplane which has the maximal possible order of tangency with a germ of nondegenerate curve then we can choose a hyperplane belonging to this system which has with the considered germ n transversal intersections. (The lemma itself has a dual formulation).

Now consider a closed conjugate curve $\gamma \subset \mathbf{S}^n$ and a conjugate hyperplane $\mathbf{L}$ (i.e. such that $\sharp(\gamma \cap \mathbf{L}) > n$). Let $(\gamma \cap \mathbf{L}) = (\gamma(t_o), \dots, \gamma(t_p))$ and $k_o, \dots, k_p$ be the corresponding p-tuple of multiplicities ($\sum_{i=1}^p k_i > n$). We additionally assume that k_o is the maximal possible intersection multiplicity at the point $\gamma(t_o)$ on the set of all conjugate hyperplanes.(In the dual space this means that we have a nondegenerate curve and a point for which there are p hyperplanes tangent to the curve with multiplicities $k_1, \dots, k_p$). In $(\mathbf{S}^{n*})$, consider the linear subspace $\mathbf{L}$ of the hyperplanes in $\mathbf{S}^n$, satisfying the following conditions:

$$\mathcal{L} = \{\mathbf{L} \subseteq \mathbf{S}^n \mid \text{hyperplane } \mathbf{L} \text{ such that}$$
$$\sharp(\mathbf{L} \cap \gamma)|_{t_1} = k_1, \dots, \sharp(\mathbf{L} \cap \gamma)|_{t_p} = k_p, \sum_i k_i = n - k_o + 1\},$$

where all the conditions are linearly independent. One can easily prove that such conditions can be chosen.

Then $\dim(\mathcal{L}) = k_o - 1$.

LEMMA 4. *If* $\mathcal{M}_t$ *is a linear subspace* $\subseteq \mathbf{S}^{n*}$ $(\dim \mathcal{M} = n - k_0)$ *passing through* $\gamma^*($
spanned by $(\gamma^*)'(t), \dots, (\gamma^*)^{(n-k_o)}(t)$ *then* $\dim(\mathcal{L} \cap M_t) = 0$.

PROOF: Assume that $dim(\mathcal{L} \cap M_t) > 0$. Then there exists a conjugate hyperplane
for which $\sharp(\gamma \cap \mathbf{L})|_{t_o} > k$. Contradiction. ▮

Now we finish the proof of Sherman's result. Projecting a point of the germ $\gamma(t)$ alo
on the subspace spanned by $\gamma^{(n-k_o+1)}(t)$ and $\mathbf{L}$ we get the situation of Arnold's lemm
are able to find a conjugate hyperplane with k transversal intersections in the neighbo
of $\gamma(t_o)$ and after this a hyperplane with $\geq n$ transversal intersections. ▮

REFERENCES

[G]. M. L. Gromov, "Partial differential relations," Springer-Verlag, 1986.

[L1]. J. Little, *Nondegenerate homotopies of curves on the unit 2-sphere*, J. Diff. Geom. **4** (1970), 3

[L2]. J. Little, *Third order nondegenerate homotopies of the space curves*, J. Diff. Geom. **5** (1971), 5

[H]. U. Hamenstadt, *Zur Theorie von Carno-Caratheodory Metriken und ihren Anwendungen*,
Math. Schr. **180** (1987), 1-64.

[S]. T. L. Sherman, *Conjugate points and simple zeros for ordinary linear differential equations*, T
the AMS **146** (1969), 397-411.

[AVG]. V. I. Arnold, A. N. Varchenko,S. M. Gusein-Zade, "Singularities of differential maps," Nauka,

[F]. W. Fenchel, *Uber Krummung und Windung geschlossener Raumkurven*, Math.Ann. **101**
238-252.

[K-O]. B. A. Khesin, V. Yu. Ovsienko, *Symplectic leaves of Gelfand-Dikii's brackets and homotopical of nondegenerate curves*, Funct. Analiz i ego Priloz. **24** (1990), 38-47.

The M-property of flag varieties

B. Z. Shapiro, M. Z. Shapiro

§1. Introduction

The well known Smith inequality implies that for an arbitrary real algebraic variety $X^{\mathbf{R}}$ we have $\sum b_i(X^{\mathbf{R}}) \leq \sum b_i(X^{\mathbf{C}})$, where b_i denotes the i-th Betti number with coefficients in $\mathbf{Z}/2\mathbf{Z}$ and $X^{\mathbf{C}}$ denotes the complexification of $X^{\mathbf{R}}$ (see for example [**CF**]).

In the particular case of a planar real algebraic curve this inequality is called Harnack's inequality and the planar curves for which Harnack's inequality is in fact the equality are called **M-curves**. M-curves were studied by several authors, see [**Ro**,**Ra**,**Hk**,**P**].

§1.1. The main definition

A real algebraic variety $X^{\mathbf{R}}$ (the set of real points of $X^{\mathbf{C}}$) is called an **M-manifold** if $\sum b_i(X^{\mathbf{R}}) = \sum b_i(X^{\mathbf{C}})$. (We shall also say that in this case $X^{\mathbf{R}}$ has the M-property.)

There are several articles by authors from the Leningrad and Gorky schools about M-surfaces (see [**H**,**V**]).

In the beginning of 70's it was found that several configuration spaces have the M-property. For example in the paper [**O-S**] by Orlik and Solomon it was proved that the complement of an arbitrary arrangement of real hyperplanes has the M-property (see also [**O-S**,**V-G**]).

In this paper we will consider other configuration spaces and establish the M-property for another series of manifolds. In this sense our work is the development of [**O-S**].

§1.2. The main results

Definition: Two (incomplete, generally) flags in $\mathbf{P}^n$ are called **transversal** if the intersection of any pair of their subspaces has the minimal possible dimension.

Let $\mathbf{PT^*P}^n$ denote the manifold of all flags in $\mathbf{P}^n$ consisting of a hyperplane and a distinguished point in it. (Notice that two flags belonging to $\mathbf{PT^*P}^n$ are transversal if it holds for both that a distinguished point of one does not belong to the hypersurface of the another).

Hereinafter the term **'the set of flags in general position'** means that it belongs to some open dense domain in the space of all sets.

Theorem A. *The locus of all flags from $\mathbf{PT^*P}^n$ which are transversal to each flag fr given set in general position has the M-property (see §2).*

Theorem B. *There exists an open set of 4-tuples of real lines in $\mathbf{P}^3$ (flags in $\mathbf{F}^4$) that the corresponding locus of all lines (flags) transversal to all lines (flags) from the set violates the M-property.*

The authors are very thankful to V. I. Arnold and A. N. Varchenko for their inter this work and useful discussions of the subject.

§2. The M-property of $\mathbf{PT^*P}^n$

This section contains the proof of theorem A.

2.1. Definition 1. The train $\mathbf{Tn_f}$ of a flag $\mathbf{f}$ is the locus of all flags nontransversa . If $\hat{\mathbf{f}}$ is an arbitrary set $\hat{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ of flags in $\mathbf{P}^n$ then we denote

$$\mathbf{Tn}_{\hat{\mathbf{f}}} = \bigcup_{\mathbf{f}_i \in \hat{\mathbf{f}}} \mathbf{Tn}_{\mathbf{f}_i}.$$

Definition 2. Let V_1, V_2 be two n-dimensional vector spaces. The subspace $L \subset V_1$ is called **decomposable** if $L = L_1 \oplus L_2$, where $L_1 \subset V_1, L_2 \subset V_2$. We denote by $V_1 \oplus V_2 \to V_1$; by $p_2 : V_1 \oplus V_2 \to V_2$ projections onto the first and the second sumr respectively. Let $\Phi : V_1 \times V_2 \to \mathbf{R}(\mathbf{C})$ be a pairing (possibly degenerate). With the p $\Phi : V_1 \times V_2 \to \mathbf{R}(\mathbf{C})$ we shall associate a quadratic form $\bar{\Phi}(V \oplus V) \to \mathbf{R}(\mathbf{C})$ such $\bar{\Phi}(v) = \Phi(p_1 v, p_2 v)$.

2.2. Lemma. *Let V_1, V_2 be n-dimensional linear spaces; $\Phi : V_1 \times V_2 \to \mathbf{R}$ a pairin $\bar{\Phi}(V_1 \oplus V_2) \to \mathbf{R}$ the associated quadratic form. If L is decomposable then the restrictio has zero signature (the number of positive squares minus the number of negative squ*

Proof: For any pairing Φ we can choose appropriate coordinates x_i on V_1 and y_i on that for any

$$v = (x_1, \ldots, x_n); \quad w = (y_1, \ldots, y_n)$$

$$\Phi(v, w) = \sum_{i=1}^{r} x_i y_i,$$

where $r = \operatorname{rank}\Phi \leq n$.

The signature of the quadratic form $\hat{\Phi} = x_1 y_1 + \cdots + x_r y_r$ equals zero. For any de posable vector space L the restriction $\Phi|_L$ is also a pairing and $\bar{\Phi}|_L = (\bar{\Phi|_L})$. ∎

2.3. Definition. Let $\hat{\mathbf{R}}^n(\hat{\mathbf{C}}^n)$ denote the one-point compactification of $\mathbf{R}^n$ (C an arbitrary real or complex-valued function on $\mathbf{R}^n(\mathbf{C}^n)$ and V the variety of solu $\{\Phi(x) = 0\}$.

The subvariety $\hat{V} = V \cup \infty$ in $\hat{\mathbf{R}}^n(\hat{\mathbf{C}}^n)$ is the one-point compactification of V.

2.4. The aim of this section is to study relative mod ∞ homology of the variety the nonhomogeneous quadratic form $\bar{\Phi}$ associated with a pairing Φ (see 2.1. and 2.2.) following lemma is obvious.

LEMMA. *The nonhomogeneous quadric function associated with a pairing $\Phi : V_1 \times V_2 \to$ **R**(**C**) can be transformed by the $Aff(V_1) \times Aff(V_2)$-action to one of the following:*

(*) $$A)\bar{\Phi} = l(x_i, y_i) \quad 1)\bar{\Phi} = l(x_i, y_i) + \sum_{2}^{m}(x_i \cdot y_i), m \leq n,$$

(**) $$B)\bar{\Phi} = 0 \quad 2)\bar{\Phi} = \sum_{1}^{m}(x_i \cdot y_i) - 1, \quad m \leq n,$$

(***) $$C)\bar{\Phi} = 1 \quad 3)\bar{\Phi} = \sum_{1}^{m}(x_i \cdot y_i), \quad m \leq n,$$

where $l(x_i, y_i)$ is a nonconstant linear function.

2.5. LEMMA. *The Poincare polynomials for the homology of the pair $(\hat{V}, mod\infty)$ are listed below:*

	A	B	C	1	2	3
R	t^{2n-1}	t^{2n}	0	t^{2n-1}	$t^{2n-1} + t^{2n-m}$	$t^{2n-1} + 2t$
C	t^{4n-2}	t^{4n}	0	t^{4n-2}	$t^{4n-2} + t^{4n-2m-1}$	$t^{4n-2} + t^{4n-2m} + t$

PROOF: We shall start with the $\mathbf{R}^{2n}$. The cases A), B), C), 1) are obvious, e.g.

1) $(\hat{V}, mod\infty)$ is homeomorphic to $(\mathbf{S}^{2n-1}$, mod point);

2) $(\hat{V}, mod\infty)$ is homeomorphic to the Thom space of a $(2n-m)$- dimensional vector bundle over $\mathbf{S}^{m-1}$.

3) The third case splits into two subcases:

3a) $m = n$;

3b) $m < n$

Case 3a). Since V is homogeneous $(\hat{V}, mod\infty)$ is homeomorphic to the suspension mod(point) over its intersection with the unit sphere.

This intersection is given by the system:

$$\begin{cases} \sum_{i=1}^{n} x_i^2 = 1/2 \\ \sum_{i=1}^{n} y_i^2 = 1/2 \end{cases}$$

Hence the intersection is homeomorphic to $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ and $(\hat{V}, mod\infty)$ is homeomorphic to the suspension mod(point) over $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$.

Case 3b). $(\hat{V}, mod\infty)$ is homeomorphic to the Thom space mod(point) of the 2(n-m)-dimensional bundle over $\mathbf{S}^{m-1} \times \mathbf{S}^{m-1}$ whose fiber over one point is glued by a cell.

Now let us pass to the $\hat{\mathbf{C}}_{2n}$.

1) $(\hat{V}, mod\infty)$ is homeomorphic to the sphere $\mathbf{S}^{4n-2}$ (mod point).

To study the other cases we must decompose the left sides of (**) and (***) into re imaginary parts:

$$\begin{cases} \sum((Rex_i)^2 + (Imy_i)^2 - (Imx_i)^2 - (Rey_i)^2) \\ 2\sum((Rex_i)(Imx_i) - (Rey_i)(Imy_i)) \end{cases}$$

Substituting $\bar{y}$ instead of y we obtain:

$$\begin{cases} \sum((Rex_i)^2 + (Imy_i)^2 - (Imx_i)^2 - (Rey_i)^2) \\ 2\sum((Rex_i)(Imx_i) + (Rey_i)(Imy_i)) \end{cases}$$

Thus in case 2) we see that $(\hat{V}, mod\infty)$ is homeomorphic to the Thom space (mod of the $(4n-2m)$-dimensional bundle over $\mathbf{S}^{2m-1}$.

Case 3 splits, as before, into two subcases.

3a). $(\hat{V}, mod\infty)$ is homeomorphic to the suspension mod (point) over the spheric $\mathbf{STS}^{2n-1}$ associated with $\mathbf{TS}^{2n-1}$.

3b). $(\hat{V}, mod\infty)$ is homeomorphic to the Thom space mod (point) of the 4(n dimensional bundle over suspension over $\mathbf{STS}^{2m-1}$ one fiber of which is glued by a

The proof of the remaining statements of the lemma is the calculation of the susp homology and the Thom space homology in terms of the homology of their base spa

2.6. Now we shall go ahead with the proof of Theorem A.

DEFINITION. A set of flags from $\mathbf{PT^*P}^n$ is called minimally degenerate if the spanned by the points of these flags intersects nontrivially with the intersection hyperplanes of the flags from the considered set.

DEFINITION. The set of flags from $\mathbf{PT^*P}^n$ is called **degenerate** if it contains a mir degenerate subset, and **nondegenerate** otherwise.

Now let us give the precise formulation of Theorem A.

THEOREM A. *The complement to the union of the trains for a nondegenerate set from $\mathbf{PT^*P}^n$ possesses the M-property.*

PROOF: Embed the space $\mathbf{PT^*P}^n$ into $\mathbf{P}^n \times \mathbf{P}^{n*}$ as a quadric given by the eq $\sum_{i=0}^{n} x_i y_i = 0$, where $x_0 : x_1 : \cdots : x_n;\ \ y_0 : y_1 : \cdots : y_n$ are homogeneous coor in $\mathbf{P}^n$ and $\mathbf{P}^{n*}$ respectively. We call a variety $L \subset P^n \times P^{n*}$ a 'projective subsp $L = L_1 \times L_2$ where $L_1 \subset P^n$ and $L_2 \subset P^{n*}$ are projective subspaces. If the codimen $L \subset P^n \times P^{n*}$ equals 1 then we call L a 'hyperplane'.

Having introduced these definitions we can notice that the train of any flag coincid the intersection of Q with the union of two 'hyperplanes'. The complement in $\mathbf{P}^n$ to the union of these 'hyperplanes' coincides with $\mathbf{A}^n \times \mathbf{A}^{n*}$ and thus the complen $\mathbf{PT^*P}^n$ to the train of some flag can be realized as an affine quadric Q_A in $\mathbf{A}^n \times \mathbf{A}^n$ by the equation $x_0 + y_0 + \sum_{i=2}^{n} x_i y_i = 0$.

The union (intersection) of the irreducible components of the trains for various the union (intersection) of Q with the corresponding 'hyperplanes'. Thus any intersec the irreducible components is the intersection of Q with some 'projective subspaces' pass from the projective spaces to the affine ones then the inverse image of the con 'projective subspace' is identified with the linear decomposable subspace since the restriction of the quadratic form $\sum x_i y_i$ onto this subspace has zero signature according to Lemma 2.4. Hence the equation of the irreducible components in any affine chart will be given by one of the equations (*), (**) or (***), i.e. any intersection of irreducible components mod ∞ is the M-manifold considered in Lemmas 2.4. - 2.5.

To finish the proof we will show that in the term E_1 of the relative Mayer-Vietoris spectral sequence (see [B]) all differentials are 0 for the union of trains for a nondegenerate set $\hat{\mathbf{f}} = \{\mathbf{f}_0, \dots, \mathbf{f}_m\}$ of flags from $\mathbf{PT^*P}^n$ degenerates .

Consider the space

$$(\cup \mathbf{Tn}_{\hat{\mathbf{f}}}, \ \bmod \mathbf{Tn}_{\mathbf{f}_0}) =$$
$$(\bigcup_{j=1,2; i=0,m} N_i^j, \ \bmod (N_0^1 \cup N_0^2)) =$$
$$(\bigcup_{j=1,2; i=0,m} \hat{N}_i^j, \ \bmod \infty)$$

where $N_i^{1,(2)}$ is the first (second) irreducible component of the i-th flag train. (The flags from the first component satisfy the condition that the hyperplane of the considered flag contains the point of the i-th flag; conversely, the points of flags belonging to the second component belong to on the hyperplane of the i-th flag.) Let $\hat{N}_i^j$ as before denote the one-point compactification of N_i^j . Hereafter we shall work in the chart for which $N_0^1 \cup N_0^2$ belong to the hyperplane at infinity. Under these assumptions 'hyperplanes' are hyperplanes and 'projective subspaces' are affine.

Let W denote the intersection

$$W = \bigcap_{s=1}^{k_0} (N_{l_s}^1 \cap (\cap_{p=1}^{k_1} N_{m_p}^l) \cap (\cap_{q=1}^{k_2} N_{n_q}^2),$$

where l_s, m_p, n_q are pairwise different.

Then:

(i) for $k_1 \neq 0$ or $k \neq 0$ W is defined by the type (*)equation;

(ii) for $k_0 \geq 2, k_1 = 0, k_2 = 0$ W is defined by the type (**)equation;

(iii) for $k_0 = 1, k_1 = 0, k_2 = 0$ W is defined by the type (***)equation.

This follows from the fact that the intersection of the irreducible components of the pairwise different flags is nonsingular. The structure of the term E_1 in the considered relative Mayer-Vietoris spectral sequence is shown on Fig.1, where asterisks mark the nontrivial places and arrows indicate the possible nontrivial differentials.

The proof of the triviality of the considered differentials is similar to the analogous proof for the complex case (see §4).

DEFINITION: Given quadric Q and the set $\mathcal{L} = \{L_1, \dots, L_k\}$ of the affine hyperplanes in $\mathbf{R}^n$ we will say that the set $\mathcal{L}$ is h-singular if there exists a subset of indices $i_1, \dots, i_h$ such that either the intersection $L_{i_1} \cap \dots \cap L_{i_h} \cap Q$ is singular or the set of hyperplanes $L_{i_1}, \dots, L_{i_h}$ is not in general position (i.e. there exists a nontransversal intersection); and h-nonsingular otherwise.

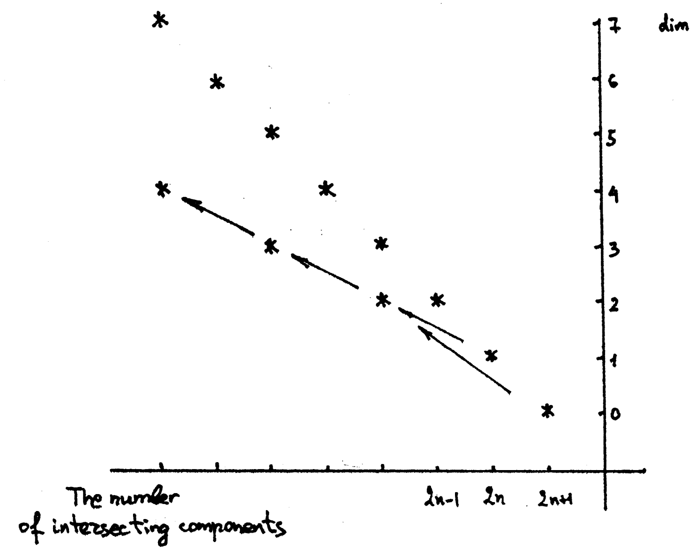


Fig.1. The structure of the term E_1

Let $\alpha = \{\alpha_j\} = \{i_1^j, \dots, i_h^j\}$ be the set of multiindices, where $1 \leq j \leq \nu$ and $\mathcal{L}_c$ $\bigcup_{j=1}^{\nu}(L_{i_1}^j \cap \dots \cap L_{i_h}^j)$. Any continuous deformation $\{L_1, \dots, L_k\}$ in the locus of all nonsingular sets can be extended to a continuous deformation of $\mathcal{L}_\alpha \cap Q$.

Hence, if we manage to prove that in the considered quadric we can nondegenera deform an arbitrary nonsingular set of hyperplanes in a neighborhood of infinity then triviality of all differentials d_t for $t \geq 1$. As shown on Fig.1. the 'suspicious' different exist for $t \leq 3$ and this will imply that they can 'strike' only in the middle- dimensic homology, i.e. the homology of the intersections defined by (**) and (***).

Let us start with (***). The hypersurface $Q = 0$ is a cone. (For the nondegenerate of flags this can happen only for the intersection with the train of no more than one i Consider the intersection of Q with an affine subspace L which in its turn is the intersec of several irreducible components of the other flag trains. In the case of the nondege ate flag set, L can not contain the cone's vertex (the origin). If we make a homoth transformation we obtain the necessary nondegenerate deformation.

Now consider (**). The quadric Q is given by the equation $\sum_{i=1}^n x_i y_i = 1$. Intersec with the hyperplane

$$\alpha_0 + \sum \alpha_i x_i = 0.$$

The conditions of the singularity of this intersection is as follows: There exist x_i^0, y_i^0 :

(1) $$\alpha_0 + \sum \alpha_i x_i^0 = 0$$

(2) $$\sum x_i^0 y_i^0 = 1$$

(3) $$\operatorname{rk}\begin{pmatrix} y_1^0 & \dots & y_n^0 & x_1^0 & \dots & x_n^0 \\ \alpha_1 & \dots & \alpha_n & 0 & \dots & 0 \end{pmatrix} = 1.$$

Conditions (2) and (3) contradict each other. Hence any intersection of the considered quadric with one hyperplane is nonsingular. Consequently, $d_1 \equiv 0$.

Analogously the intersection of the quadric with a subspace given by the system:

$$\begin{cases} \alpha_0^1 + \sum_{i=1}^n \alpha_i^1 x_i = 0 \\ \alpha_0^2 + \sum_{i=1}^n \alpha_i^2 x_i = 0 \end{cases}$$

is nonsingular if and only if the vectors $(\alpha_i)^1$ and $(\alpha_i)^2$ are linear independent. Similarly, the intersection of these hyperplanes with Q is nonsingular if the set of hyperplanes is nondegenerate and all the pairwise intersections are nonsingular. Consequently, $d_3 \equiv 0$.

Thus for $t \le 3$ we need only to define when the intersection $L_1 \cap L_2 \cap Q$ is singular, where $L_1 \cap L_2$ is given by:

$$\begin{cases} \alpha_0 + \sum_{i=1}^n \alpha_i x_i = 0 \\ \beta_0 + \sum_{i=1}^n \beta_i y_i = 0 \end{cases}$$

The conditions of degeneracy are as follows: There exist x_i^0, y_i^0 such that :

(1') $$\alpha_0 + \sum_{i=1}^n \alpha_i x_i^0 = 0$$

(2') $$\beta_0 + \sum_{i=1}^n \beta_i y_i^0 = 0$$

(3') $$\sum x_i^0 y_i^0 = 1$$

(4') $$\operatorname{rk}\begin{pmatrix} y_1^0 & \dots & y_n^0 & x_1^0 & \dots & x_n^0 \\ \alpha_1 & \dots & \alpha_n & 0 & \dots & 0 \\ 0 & \dots & 0 & \beta_1 & \dots & \beta_n \end{pmatrix} = 2$$

(4') implies that

$$y_1^0/\alpha_1 = \dots = y_n^0/\alpha_n = \alpha;$$

$$x_1^0/\beta_1 = \dots = x_n^0/\beta_n = \beta$$

(1) and (2) imply

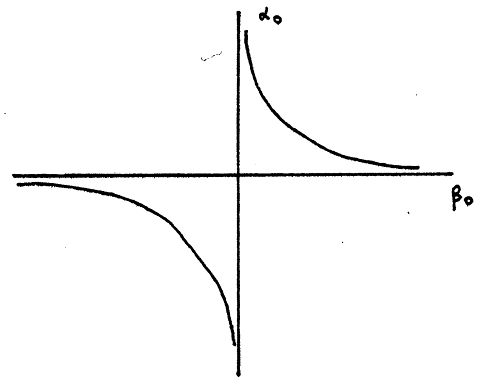
$$\alpha = -\alpha_0 / \sum \alpha_i \beta_i;$$

$$\beta = -\beta_0 / \sum \alpha_i \beta$$

and finally (3) gives the singularity condition:

$$\alpha_0 \beta_0 = \sum \alpha_i \beta_i$$

For fixed α_i, β_i let us depict on the (α_0, β_0)-plane the graph $\alpha_0 \beta_0 = const$ (Fig.2). Thi graph does not bound any compact connected component on the plane. Consequently an nondegenerate pair of hyperplanes can be nonsinqularity deformed in a neighborhood of th infinity.



Thus we see that a nondegenerate set of hyperplanes can be nonsingularly deformed in neighborhood of the infinity.

This finishes the proof of the degeneracy for the spectral sequence in the term E_1 in t] real case.

In the complex case the same arguments prove the degeneracy of the spectral sequen (see §4).

§3. Example of violation of the M-principle.

In this section we will consider the example of the flag variety $\mathcal{M}$ and the set of flags f which the complement to the union of their trains violates the M-property.

As before $\mathcal{M}$ denotes the flag space, $\mathbf{Tn}_f$ the flag train. Let $\hat{\mathbf{f}} = \{\mathbf{f}_0, \dots, \mathbf{f}_d\}$ be a set of flags,

$$\mathbf{Tn}_{\hat{\mathbf{f}}} = \bigcup_{i=0}^{d} \mathbf{Tn}_{\mathbf{f}_i}$$

and

$$\mathcal{M}_{\hat{\mathbf{f}}} = \mathcal{M} \setminus \mathbf{Tn}_{\hat{\mathbf{f}}}.$$

THEOREM B. *There exists an open nonempty set U of 4-tuples of lines in $\mathbf{RP}^3$ such that for each $\hat{\mathbf{f}} \in U$*

$$\sum b_i(\mathcal{M}^{\mathbf{C}}_{\hat{\mathbf{f}}}) > \sum b_i(\mathcal{M}^{\mathbf{R}}_{\hat{\mathbf{f}}})$$

where $\mathcal{M}^{\mathbf{C}}(\mathcal{M}^{\mathbf{R}})$ is the complement in the complex (real) Grassmann manifold $\mathbf{G}_{2,4}$ to the union of the trains of these lines.

PROOF: The group $\mathbf{PGL}_4$ acts transitively on 3-tuples of pairwise nonintersecting lines in $\mathbf{P}^3$. There exists a transformation that maps the three lines under consideration onto the surface of the standard hyperboloid of one sheet $x_0^2 = x_1^2 + x_2^2 + x_3^2$.

A hyperboloid of one sheet has two families of lines or rulings (denoted as family 1 and 2 respectively) such that the lines from one family do not intersect, while any pair of lines from different families has a one-point intersection. Since the three lines in consideration do not intersect, therefore they belong to one of these families (say, family 1). Consequently the intersection of their trains is one-dimensional and coincides with the other family of rulings (family 2) (see [R]).

The union of the points belonging to any of these families coincides with the whole hyperboloid. Each point of the hyperboloid belongs to on a unique line from the family 1. Given an arbitrary fourth line we thus obtain that the intersection of the trains of four lines is the set of lines from the family 2 passing through the points of intersection of the fourth line with the hyperboloid (see Fig.3).

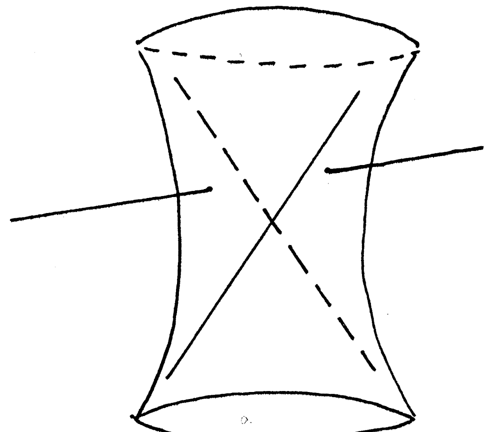
In the complex case for the set of 4-tuples of lines 'in general position' the number of the intersection points of 4-tuples of trains equals 2; in the real case there exists an open nonempty subset U in the space of 4-tuples of lines for which the intersection of their trains is empty.

The Smith inequality implies that $\dim E_1^C \geq \dim E_1^R$. Moreover, theorem C from §4 implies that for a set of lines 'in general position'

$$\dim H_*(\cup \mathbf{Tn}^{\mathbf{C}}_{\hat{\mathbf{f}}}, \mod \mathbf{Tn}^{\mathbf{C}}_{\mathbf{f}_0}) = \dim E_1^{\mathbf{C}} \geq$$
$$\dim E_1^{\mathbf{R}} \geq \dim H_*(\cup \mathbf{Tn}^{\mathbf{R}}_{\hat{\mathbf{f}}}, \mod \mathbf{Tn}^{\mathbf{R}}_{\mathbf{f}_0}).$$

Thus the complement to a union of trains possesses the M-property if at least $\dim E_1^{\mathbf{C}} = \dim E_1^{\mathbf{R}}$, i.e. M-property is fulfilled for the intersection of arbitrary subset of trains belonging to $\hat{\mathbf{f}}$. Since, as we've seen above for the 4-tuples of lines belonging to U

$$2 = H^*(\bigcap_{i=1}^{4} \mathbf{Tn}^{C}_{\mathbf{f}_i}) > H^*(\bigcap_{i=1}^{4} \mathbf{Tn}^{R}_{\mathbf{f}_i}) = 0.$$



Then $\dim E_1^{\mathbf{C}}(\dots) > \dim E_1^{\mathbf{R}}$ and the M-property is violated.

COROLLARY: There exists an open nonempty set $\mathcal{V}$ of 4-tuples of complete flags in $\mathbf{R}^4$ fo which $\sum b_i(\mathcal{M}^{\mathbf{C}}_{\hat{\mathbf{f}}}) > \sum b_i(\mathcal{M}^{\mathbf{R}}_{\hat{\mathbf{f}}})$ for each $\hat{\mathbf{f}} \in \mathcal{V}$.

PROOF: Consider the 4-tuples of flags containing the 2-dimensional subspaces belonging t the domain $\mathcal{U}$ from the previous theorem. For them the M-property is also violated, sinc in this case $\dim E_1^{\mathbf{C}} > \dim E_1^{\mathbf{R}}$.

§4. THE DEGENERACY OF THE MAYER-VIETORIS SPECTRAL SEQUENCE

Consider a cellular complex K which is represented as the union of cellular subcomplexe $K = \cup_i K_i$. Then there exists a spectral sequence converging to the homology of K (se [B]).

LEMMA. *Consider the following diagram:*

$$\begin{array}{ccc} E & \xrightarrow{\pi} & C \\ \downarrow{\scriptstyle s} & & \\ A & & \end{array}$$

where A, E, C are projective varieties, C is irreducible and s has the following property. Let E_c denote the inverse image $\pi^{-1}(c)$ for any $c \in C$ and A_c denote the image $s(E_c)$. Assume that s restricted to E_c is an embedding.

Then there exists an open dense set $U \in C$ such that for arbitrary c_1, c_2 from U, A_{C_1} is homotopic in A to A_{C_2}.

4.1. LEMMA. *For any $c_0 \in C$ there exists a nonempty open $U \subset C^d$ such that for any set $(c_1, \ldots, c_d) \in U$ the map $I_* : (\bigcup A_{c_j}, mod A_{c_0}) \to (A, mod A_{c_0})$ is homologically trivial (i.e. $I_* = 0$ on the homological level).*

PROOF: Since A_c is an algebraic variety then it is a neighborhood retract in A. Let Ω be its retracting neighborhood. There exists a nonempty open and hence dense locus $U \subset C$ such that for any $\{u_1, \ldots, u_d\}$, $\{v_1, \ldots, v_d\} \in U$ the union $\cup A_{u_j}$ is homotopic to $\cup A_{v_j}$ by the above Lemma. Choose $\{u_1, \ldots, u_d\}$ belonging to the ξ -neighborhood of the point $\{c_0, \ldots, c_0\} \cap U$.

Thus $\cup A_{u_j} \subset \Omega$. Retracting Ω to A_{c_0} we contract $\cup A_{u_j}$ to A_{c_0} . Thus the following composition:

$$\cup A_{v_j} \xrightarrow{\text{homotopy}} \cup A_{u_j} \xrightarrow{\text{retraction}} A_{c_0}$$

gives us a chain whose boundary coincides with $\cup A_{v_i} mod A_{c_0}$.

Now we are able to prove the following crucial result.

THEOREM C. *For any $c \in C$ there exists a nonempty subset $U \subset C^d$ such that for any $c_1, \ldots, c_d \in U$ the relative Mayer-Vietoris sequence for $x = \cup A_{c_j}$ mod A_{c_0} degenerates in the E_1-term.*

PROOF: The action of all differentials d_s for $s \geq 1$ is induced by the sequence of embeddings. Let $\{\tau_1, \ldots, \tau_k\} \subset \{c_1, \ldots, c_d\}$ be a subset. Then

$$A_{\tau_1} \cap A_{\tau_2} \cap \cdots \cap A_{\tau_k} \hookrightarrow$$
$$A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_m} \cap \cdots \cap A_{\tau_k};$$
$$(A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_p} \cap \cdots \cap A_{\tau_k}) \cup (A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_g} \cap \cdots \cap A_{\tau_k}) \hookrightarrow$$
$$A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_p} \cap \cdots \cap \hat{A}_{\tau_g} \cap \cdots \cap A_{\tau_k};$$
$$(A_{\tau_1} \cap \cdots \cap A_{\tau_{k-s}} \cap A_{\tau_{k-s+1}}) \cup \cdots \cup (A_{\tau_1} \cap \cdots \cap A_{\tau_{k-s}} \cap A_{\tau_k}) \hookrightarrow$$
$$A_{\tau_1} \cap \cdots \cap A_{\tau_{k-s}}$$

where ˆ means the omission of the corresponding term. On the homological level the maps induced by these embeddings are trivial by Lemma 4.1.

§5. EXAMPLE. CASE $\mathbf{PT^*P^2}$

In this section we shall give an explicit example of the theorem in the simplest case of flags on $\mathbf{RP}^2$. The main results are:

THEOREM 5.1. *For a set of flags $\hat{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ on $\mathbf{RP}^2$ 'in general position' the variety $\mathcal{M}^{\mathbf{R}}_{\hat{\mathbf{f}}}$ of all flags transversal to all flags belonging to $\hat{\mathbf{f}}$ is homeomorphic to the disjoint union of $k^3 - k + k$ three-dimensional cells.*

THEOREM 5.2. *For a set of flags $\hat{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ on $\mathbf{CP}^2$ 'in general position' the Z-homology of the variety $\mathcal{M}^{\mathbf{C}}_{\hat{\mathbf{f}}}$ of all flags transversal to all flags belonging to $\hat{\mathbf{f}}$ are torsion free and its Betti numbers are $b_0 = 1$, $b_1 = 2(k-1)$. $b_2 = 2(k-1)^2$, $b_3 = (k-1)^3$, $b_i = 0 \quad i \geq 4$*

Each flag in $\mathbf{RP}^2$ consists of a line and its point.

DEFINITION. **Bifurcation lines** are lines passing through points of different flags from $\hat{\mathbf{f}}$.

DEFINITION. The set $\hat{\mathbf{f}}$ of flags on $\mathbf{RP}^2$ is called a set **'in general position'** if:

a) flags are mutually transversal;

b) none of the 3-tuples of the flag lines intersect in one point;

c) none of the 3-tuples of the flag points belong to one line;

d) no bifurcation line passes through the intersection point of the flag lines.

Under these assumptions we prove the Theorem 5.1.

PROOF: Consider a set $\hat{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ of k flags 'in general position'. The lines of these flags divide $\mathbf{RP}^2$ into $N_1 = k(k-1)/2$ open polygons (see condition b). Let $\mathbf{M}$ denote one of these polygons and $\Omega(\mathbf{M})$ denote the locus of flags transversal to $\hat{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ whose points belong to $\mathbf{M}$. $\Omega(\mathbf{M})$ is a one-dimensional bundle the fiber of which is an interval and the base B identifies with the set of lines crossing $\mathbf{M}$ and not passing through the points of flags $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$. Let $l_1, \ldots, l_k$ denote the lines on $\mathbf{RP}^{2^*}$ dual to the points of flags $\hat{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$. By condition c) the lines $l_1, \ldots, l_k$ are 'in general position'.

DEFINITION. We call a polygon $\mathbf{M}^* \subset \mathbf{RP}^{2^*}$ **dual** to the polygon $\mathbf{M} \subset \mathbf{RP}^2$ if the points of $\mathbf{M}^*$ are dual to the lines on $\mathbf{RP}^2$ nonintersecting $\mathbf{M}$. If $\mathbf{M}$ is affine (i.e. contained in some affine chart) and convex then $\mathbf{M}^*$ is also affine and convex. If $\mathbf{M}$ is open then $\mathbf{M}^*$ is closed and vice versa. The base B of the bundle $\Omega(B)$ equals $\mathbf{RP}^{2^*} \setminus (\mathbf{M}^* \cup l_1 \cup \cdots \cup l_k)$. All the lines $l_1, \ldots, l_k$ intersect the closure of $\mathbf{M}^*$.

5.3. LEMMA. *B consists of $k + q(\mathbf{M})$ open two-dimensional cells, where $q(\mathbf{M})$ denotes the number of bifurcation lines intersecting with $\mathbf{M}$.*

COROLLARY. *$\Omega(\mathbf{M})$ consists of $k + q(\mathbf{M})$ open three-dimensional cells.*

The proof of the lemma is an immediate consequence of the following more general fact.

PROPOSITION. *Let $\mathcal{A}$ be a nonstrictly convex domain of $\mathbf{RP}^2$. Then the number of connected components on which $\mathcal{A}^- = \mathbf{RP}^2 \setminus \mathcal{A}$ is separated by k lines 'in general position' each of them intersecting $\mathcal{A}$ equals $k + r$, where r is the number of pairwise intersections belonging to $\mathcal{A}$. All the components on which $\mathcal{A}^-$ is separated are contractible.*

PROOF: For $k = 1$ the statement is true. Suppose that it proved for $k-1$ and δ is a segment (or point) of the intersection of $\mathcal{A}^-$ with the k -th line. Then the points of intersection of l_k with the other lines which belong to $\mathcal{A}^-$ divide $l_k \setminus \delta$ into $\nu + 1$ intervals, where ν is the number of these points. Each of these intervals is a simple path connecting two boundary points of the contractible (by inductive hypothesis) domain. Consequently, the inclusion of a new line l_k increases the number of domains on which $\mathcal{A}^-$ is divided by $\nu + 1$. Their contractability is obvious. ∎

The total number of connected components of $\mathcal{M}^R$ is equal to the sum of the number of connected components of $\Omega(M)$ over all the polygons M. Thus it equals $N_3 = kN_1 + N_2$, where N_2 is the total number of domains on which bifurcation lines are divided by the set of flag lines. By conditions a) and d) N_2 equals $k^2(k-1)/2$. Consequently $N_3 = k^3 - k^2 + k$. Theorem 5.1. is proved.

5.4. PROOF OF THEOREM 5.2:

DEFINITION. The **train** $\mathbf{Tn}(\mathbf{f})$ of a complete flag $\mathbf{f}$ is the set of all complete flags non-transversal to $\mathbf{f}$.

LEMMA. *The restriction of the train of a complete flag* $\mathbf{f}$ *given in* $\mathbf{R}^n(\mathbf{C}^n)$ *on the complement in* $\mathbf{F}_n$ *(where* $\mathbf{F}_n$ *is the manifold of all complete flags) to the train of some flag* $\mathbf{g}$ *transversal to* $\mathbf{f}$ *is diffeomorphic to the surface given by the equation:*

$$\Delta : \Delta_1 \dots \Delta_{n-1} = o, \tag{5}$$

where Δ_i *are the determinants of* $(i \times i)$*-minors formed by the first* i *rows and the last* i *columns of the upper triangular* $(n \times n)$*-matrix with the units on the main diagonal.*

PROOF: Let $e_1, \dots, e_n$ be a basis in $\mathbf{R}^n(\mathbf{C}^n)$. We construct two flags $\mathbf{f}_d$ and $\mathbf{f}_i$ such that the i-dimensional subspaces of $\mathbf{f}_d$ are spanned by $e_n, \dots, e_i$ and the i-dimensional subspaces of $\mathbf{f}_i$ are spanned by $e_1, \dots, e_i$. Since the group $\mathbf{GL}_n$ acts on the pairs of transversal flags transitively, then it suffices to consider only the pair $(\mathbf{f}_d, \mathbf{f}_i)$. Take the bundle $\pi_n : \mathbf{GL}_n \to \mathbf{F}_n$, where π_n maps each nondegenerate matrix to the complete flag whose i-dimensional subspace is spanned by the first i rows of the matrix. $\pi_n^{-1}(\mathbf{F}_n \setminus \mathbf{Tn}_{\mathbf{f}_i})$ consists of matrices whose principal minors do not vanish. This implies that we can construct a section in $\pi_n^{-1}(\mathbf{F}_n \setminus \mathbf{Tn}_{\mathbf{f}_i})$ choosing in each fiber an upper triangular matrix with the units on the main diagonal. Thus the group of upper triangular matrices with the units on the main diagonal is the natural affine chart for the cell $(\mathbf{F}_n \setminus \mathbf{Tn}_{\mathbf{f}_i})$. Under this identification the condition of nontransversality of k-dimensional subspace of the flag corresponding to some upper triangular matrix to the $(n-k)$-dimensional subspace of flag $\mathbf{f}$ coincides with the vanishing of the determinant of $((n-k) * (n-k))$-minor formed by its first rows and last columns. Lemma is proved.

REMARK. For $\mathbf{PT}^*\mathbf{P}^2$ the local equation of any train component equals :

$$z = 0; \quad z = xy \tag{6}$$

DEFINITION. Let $\hat{\mathbf{f}} = \{\mathbf{f}_1, \dots, \mathbf{f}_k\}$ be an arbitrary set of complete flags in $\mathbf{C}^n$. As before, $\widehat{\mathbf{Tn}}_{\hat{\mathbf{f}}}$ denotes the one-point compactification of

$$\mathbf{Tn}_{\hat{\mathbf{f}}} = \bigcup_{i=1}^{k-1} \mathbf{Tn}_{\mathbf{f}_i} \cap (\mathbf{F}_n \setminus \mathbf{Tn}_{\mathbf{f}_k})$$

and

$$\mathcal{M}_{\hat{\mathbf{f}}} = \mathbf{F}_n \setminus \bigcup_{i=1}^{k} \mathbf{Tn}_{\mathbf{f}_i}$$

LEMMA. $\tilde{H}_{i-1}(\mathcal{M}_{\hat{\mathbf{f}}}) = \tilde{H}^{n(n-1)-i}(\widehat{\mathbf{Tn}}_{\hat{\mathbf{f}}})$

PROOF: $\mathbf{Fn} \setminus \mathbf{Tn}_{\mathbf{f}_k}$ is diffeomorphic to the $n * (n-1)$-dimensional cell. The Alexander duality implies (see [**D**]):

$$\tilde{H}_{i-1}(\mathcal{M}_{\hat{\mathbf{f}}}) = H_c^{n(n-i)-i}(\bigcup_{i=1}^{k} \cap(\mathbf{F}_n \setminus \mathbf{Tn}_{\mathbf{f}_k})) = \tilde{H}_c^{n(n-1)-i}(\widehat{\mathbf{Tn}}_{\hat{\mathbf{f}}})$$

where $\tilde{H}_j$ and $\hat{H}_c^J$ denote the reduced homology and the compact support homology respectively. Consequently:

$$\tilde{H}_c^{n(n-1)-i}(\mathbf{Tn}_{\hat{\mathbf{f}}}) = \tilde{H}^{n(n-1)-i}(\widehat{\mathbf{Tn}}_{\hat{\mathbf{f}}}, \infty) = \tilde{H}^{n(n-1)-i}(\widehat{\mathbf{Tn}}_{\hat{\mathbf{f}}}, \infty),$$

where ∞, as before, denotes the compactifying point. To compute $\widetilde{H}^*(\widehat{\mathbf{Tn}}_{\hat{\mathbf{f}}})$ we use (according to the described machinery) the Mayer-Vietoris spectral sequence.

LEMMA. *Each of two irreducible components of* $\mathbf{Tn}_{\hat{\mathbf{f}}}$ *in* $\mathbf{C}^3$ *is biholomorphically equivalent to* $\mathbf{C}^2$. *These components intersect in a pair of complex lines, which in their turn intersect in a point.*

PROOF: See formula (6).

COROLLARY: The cohomology groups of a compactified train have no torsion and their nontrivial Betti numbers are equal to 1, 0, 1, 2, 2.

Proof. Use the Mayer-Vietoris exact sequence.

LEMMA. *The one-point compactification of the intersection* $\mathbf{Tn}_{\mathbf{f}_1} \cap \mathbf{Tn}_{\mathbf{f}_2}$ *is homeomorphic to the complex depicted on Fig.4. and is homotopically equivalent to the bouquet of 4 spheres* $\mathbf{S}^2$ *and 6 circles.*

PROOF: The geometrical description of $\mathbf{Tn}_{\mathbf{f}_1} \cap \mathbf{Tn}_{\mathbf{f}_2}$ before its compactification is as follows. It consists of 4 components A, B, C, D:

A corresponds to the case when the flag line passes through the points of $\mathbf{f}_1$ and $\mathbf{f}_2$;

B corresponds to the case when the flag line passes through the point of $\mathbf{f}_1$ and its point lies on the line of $\mathbf{f}_2$;

C, the flag point coincides with the intersection of $\mathbf{f}_1-$ and $\mathbf{f}_2$-lines;

D, the flag line passes trough the point of while its point lies on the $\mathbf{f}_1$ -line.

Components A and C are diffeomorphic to $\mathbf{S}^2\setminus$ {point}, while B and D are diffeomorphic to $\mathbf{S}^2\setminus$ {two points}.

The following pairs of components have one-point intersections (which are pairwise different): (A, B), (A, D), (B, C), (C, D). On Fig.4. there are shown noncompactified and compactified intersections $\mathbf{Tn}_{\mathbf{f}_1} \cap \mathbf{Tn}_{\mathbf{f}_2}$. From this figure one can easily obtain the necessary facts.

COROLLARY: The cohomology of $\mathbf{Tn}_{\mathbf{f}_1} \widehat{\cap} \mathbf{Tn}_{\mathbf{f}_2}$ is torsion-free and its nontrivial Betti numbers are 1, 6, 4.

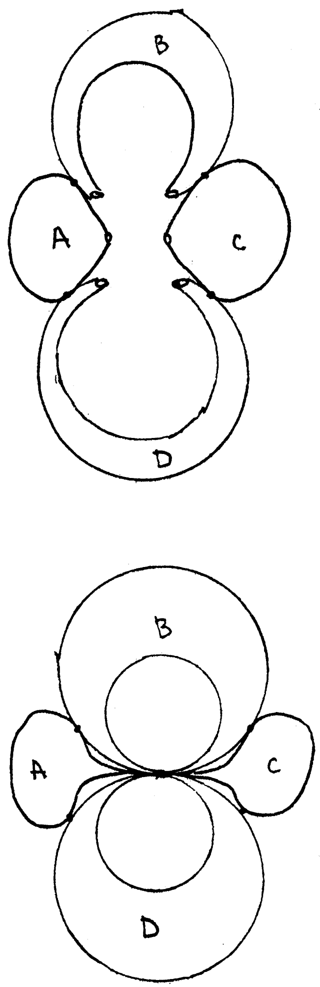


Fig.4. Non- and one-point compactified $\mathbf{Tn}_{f_1} \cap \mathbf{Tn}_{f_2}$

LEMMA. *Any 3-tuple of flags $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ on $\mathbf{CP}^2$ 'in general position' has 6 flags nontransversal*

$\underbrace{\mathbb{Z}\times\dots\times\mathbb{Z}}_{2(k-1)}$		
$\underbrace{\mathbb{Z}\times\dots\times\mathbb{Z}}_{2(k-1)}$		
$\underbrace{\mathbb{Z}\times\dots\times\mathbb{Z}}_{k-1}$	$\underbrace{\mathbb{Z}\times\dots\times\mathbb{Z}}_{4C_2^{k-1}}$	
	$\underbrace{\mathbb{Z}\times\dots\times\mathbb{Z}}_{6C_2^{k-1}}$	
		$\underbrace{\mathbb{Z}\times\dots\times\mathbb{Z}}_{6C_3^{k-1}}$

Fig.5 Structure of the term E_1

to each of them. The lines of three flags pass trough the pair of points of $\mathbf{f}_i$ and $\mathbf{f}_j$ wh their points lie on the line of $\mathbf{f}_k$; the lines of three others connect the intersection point $\mathbf{f}_i$ and $\mathbf{f}_j$ -lines with the point of $\mathbf{f}_k$, where (i, j, k) is an arbitrary permutation of 1, 2 a 3.

The term E_1 of the Mayer-Vietoris spectral sequence for the reduced cohomology of $\tilde{\mathbf{T}}$ is shown on Fig.5. The degeneracy results are proven in the previous part of this paper.

REFERENCES

[Ro]. V. A. Rochlin, *Complex topological characteristics of real algebraic curves,*, Russian math. surve **33** 5 (1978), 77-89.
[C-F]. P. E. Conner,E. E. Floyd, "Differentiable periodic maps,," Springer-Verlag, 1964.
[Ra]. V. Ragsdale, *On the arrangement of the real branches of plane algebraic curves*, Amer.J. of Math. (1906), 377-404.
[Hk]. A. Harnack, *Ueber die Vieltheiligkeit der ebenen algebraischen Curven*, Math.Ann. **10** (1876), 189-1
[P]. I. G. Petrovsky, *On the topology of real plane algebraic curves,*, Ann.Math. **39** (1938), 189-209.
[H] V. M. Harlamov, *Real algebraic surfaces*, Proceedings Inter. Cong. Math., Helsinki (1978), 421-4
[V]. O. Ja. Viro, *Construction of multicomponent real algebraic surfaces*, Soviet Math. Dokl. **20** (197 991-995.
[O-S]. P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Inv.Math. (1980), 167-189.

[V-G]. A .N. Varchenko, I. M. Gelfand, *On the Haviside' functions of the hyperplane arrangements*, Funct. Anal. i ego prilozh. **21** (1987), 1-18.
[B]. K. S. Brown, "Cohomology of groups," Springer-Verlag, 1982.
[R]. M. Reid, "Undergraduate Algebraic Geometry," London Math.Soc. student texts,.
[D]. A. Dold, "Lectures on algebraic topology," Springer-Verlag, 1972.

Euler characteristics for links of Schubert cells in the space of complete flags

B. Z. Shapiro and A. D. Vainshtein

§1. Introduction

Let $\mathbf{F}_n$ be the space of complete flags in $\mathbf{k}^n$ (where $\mathbf{k}$ is $\mathbf{R}$ or $\mathbf{C}$). With an arbitrary complete flag $f \in \mathbf{F}_n$ we associate the standard Schubert cell decomposition Sch_f of the space $\mathbf{F}_n$ whose cells are enumerated by elements from $\mathbf{S}_n$ while the dimension over $\mathbf{k}$ of such a cell equals the number of inversions in the corresponding permutation (see for example [**FF**] §5.4).

Definition. The train $\mathbf{Tn}_f$ of the flag $f \in \mathbf{F}_n$ is the union of all cells of Sch_f of positive codimension.

Let c_σ be the cell of the decomposition Sch_f corresponding to the permutation σ and B a sufficiently small $n(n-1)/2$-dimensional (over $\mathbf{k}$) ball with the origin at some point of c_σ.

Definition. The manifold $A_\sigma = B \setminus \mathbf{Tn}_f$ is called the link of the cell c_σ. By χ_σ we denote the Euler characteristic of A_σ :

$$\chi_\sigma = \sum_k (-1)^k \dim H^k(A_\sigma).$$

In the complex case we introduce also the numbers

$$\chi_\sigma^{pq} = \sum_k (-1)^k \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^k(A_\sigma)$$

where Gr^W and Gr_F are the associated graded objects of the weight and the Hodge filtrations, respectively.

In this paper we describe a construction which enables us to reduce the calculation of χ_σ and χ_σ^{pq} for $\mathbf{F}_n$ to similar calculations for $\mathbf{F}_{n-1}$ and give the results of the calculations in low dimensions. We also formulate a relation between χ_σ for the real case and χ_σ^{pq} for the complex one and establish certain properties of the latter numbers.

§2. Sylvester manifolds, flags transversal to a given pair of flags and links of Schubert cells

2.1. Let M be an arbitrary matrix over the ring of polynomials in d variables with the coefficients from the field $\mathbf{k}$.

DEFINITION. The Sylvester polynomial of the matrix M is the product of all its main minors; the Sylvester manifold of the matrix M is the complement in $\mathbf{k}^d$ to the set of zeros of the Sylvester polynomial of M.

Let σ be an arbitrary permutation from $\mathbf{S}_n$. Assign to permutation σ a following matrix M_σ over the ring of polynomials in $n(n-1)/2$ variables: take the upper triangular matrix with unit diagonal elements and independent variables x_{ij} above the diagonal and permutate its columns with the help of σ.

EXAMPLE. Let $\sigma = (2,3,1,4)$, then

$$M_\sigma = \begin{pmatrix} x_{13} & 1 & x_{12} & x_{14} \\ x_{23} & 0 & 1 & x_{24} \\ 1 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Sylvester polynomial of this matrix equals $x_{14}x_{34}(x_{12}x_{24} - x_{14})$.

2.2. Let $e_1, \dots, e_n$ be an arbitrary basis in $\mathbf{k}^n$.

DEFINITION. A complete flag is called **coordinate** if all its subspaces are spanned by the vectors of the basis and **standard coordinate** if each its i-dimensional subspace is spanned by $e_1, \dots, e_i$.

Coordinate flags are enumerated by permutations and each Schubert cell of the Schubert decomposition associated with a standard coordinate flag contains the unique coordinate flag while the permutations corresponding to these flag and cell coincide.

Given a permutation $\sigma = (i_1, i_2, \dots, i_n)$, we call the permutation $\bar\sigma = (i_n, \dots, i_2, i_1)$ transversal to σ. The coordinate flag $f_{\bar\sigma}$ corresponding to the permutation $\bar\sigma$ is the unique coordinate flag transversal to f_σ.

Let (f_1, f_2) be an arbitrary pair of flags in $\mathbf{F}_n$. Evidently, there exists a basis in $\mathbf{k}^n$ for which f_1 is the standard flag and f_2 is a coordinate flag. Each such basis can be obtained from another one via the multiplication by a nondegenerate lower triangular matrix. Thus a permutation $\sigma \in \mathbf{S}_n$ is assigned to each pair of flags from $\mathbf{F}_n$. The manifold of all flags transversal to a given pair of flags will be denoted by V_σ.

In what follows we shall often use the following fibration $\mathbf{GL}_n(\mathbf{k}) \to \mathbf{F}_n$. Fix some basis in $\mathbf{k}^n$ thus identifying $\mathbf{GL}_n(\mathbf{k})$ with the set of the nondegenerate $n \times n$-matrices and map each matrix onto the flag whose i-dimensional subspace is spanned by the first i rows of the matrix.

2.3. LEMMA. *The manifolds A_σ and $V_{\bar\sigma}$ are diffeomorphic to the Sylvester manifold of the matrix M_σ.*

PROOF: At first we prove the statement about $V_{\bar\sigma}$. The matrix M_σ defines the mapping of $\mathbf{k}^{n(n-1)/2}$ to $\mathbf{GL}_n(\mathbf{k})$. Let us ascertain now that the set of all flags transversal to $f_{\bar\sigma}$ can be identified with the image of this mapping. Suppose g is such a flag and $\bar\sigma = (i_1, \dots, i_n)$. Since the line of the flag g is transversal to the linear subspace spanned by $e_{i_1}, \dots, e_{i_{n-1}}$,

it contains a unique vector whose projection along $\{e_{i_1}, \dots, e_{i_{n-1}}\}$ coincides with the basis vector e_{i_n}; we place the coordinates of this vector in the first row of our matrix. Since the 2-plane of the flag g is transversal to the subspace spanned by $e_{i_1}, \dots, e_{i_{n-2}}$, it contains a unique vector whose projection along $\{e_{i_1}, \dots, e_{i_{n-2}}\}$ coincides with the basis vector $e_{i_{n-1}}$; we place the coordinates of this vector in the second row of the matrix, etc. At the end of this process we obtain a matrix belonging, obviously, to the image of the mapping determined by M_σ. Consider now the condition that the flag g represented by this matrix is transversal to the standard coordinate flag (represented by the unit matrix). The transversality of the $(n-i)$-dimensional subspace of the standard flag and of the i-dimensional subspace of g is equivalent to the nondegeneracy of the matrix constituted of the first $n-i$ rows of the unitary matrix and the first i rows of the matrix representing g, i.e. to the nonvanishing of the i-th main minor of the latter.

Now we prove the statement concerning A_σ. Consider the set of all flags belonging to the $n(n-1)/2$-dimensional ball with the origin at f_σ. If the radius of this ball is sufficiently small then all these flags are transversal to $f_{\bar\sigma}$. Therefore by previous arguments the set of these flags is identified with the image of a small ball under the mapping to $\mathbf{GL}_n(\mathbf{k})$ defined by the matrix M_σ. Just as before, the fact that a flag does not belong to the train of the initial (standard coordinate) flag is equivalent to the nonvanishing of the Sylvester polynomial of the matrix M_σ.

§3. Stratification of the manifold A_σ

3.1. The n-dimensional $\mathbf{k}$-torus $T_n = (\mathbf{k} \setminus 0)^n$ acts on the manifold A_σ. In the coordinate representation given in §2 this action can be described as expansions and contractions of basis vectors. The orbits of this action have the following convenient description.

Consider the mapping sending each flag from A_σ to its line. This line determines an n-dimensional vector of 0's and 1's whose i-th coordinate equals 0 if the line belongs t the subspace spanned by $e_{\sigma(1)}, \dots, e_{\sigma(i-1)}, e_{\sigma(i+1)}, \dots, e_{\sigma(n)}$ and 1 otherwise. The transversality of all flags from A_σ to the given pair of flags (see Lemma 2.3) implies that two coordinates (coinciding if the hyperplanes of these two flags coincide) of this vector must equal 1. Clear that two flags determine the same 0-1-vector if and only if they both belong to the same orbit. Therefore the orbits are enumerated by 0-1-vectors having 1's at two prescribed places (possibly coinciding):

$$A_\sigma = \bigcup_{w \in W_\sigma} O_{w\sigma} \tag{1}$$

where

$$W_\sigma = \{w = (w_1, \dots, w_n) \in \{0,1\}^n : w_1 = w_{\sigma^{-1}(n)} = 1\},$$

$O_{w\sigma}$ is the orbit in A_σ corresponding to the vector w.

We shall prove now that $O_{w\sigma}$ is diffeomorphic to the product of several copies of $\mathbf{k}^*$ by the manifold A_π for a certain permutation $\pi \in \mathbf{S}_{n-1}$.

Let $\sigma \in \mathbf{S}_n$, $w \in W_\sigma$. To each pair (σ, w) we assign a permutation $\pi(\sigma, w) \in \mathbf{S}_{n-1}$ in the following way.

Given an arbitrary sequence $I = \{i_1, \dots, i_k\}$ denote by $R(I)$ the sequence obtained by the following process: put

$$r_1 = i_1, \qquad r_l = \max\{r_{l-1}, i_l\}, \quad 1 < l \le k,$$

and delete from each group of consecutive equal elements of the sequence $\{r_1, \dots, r_k\}$ all elements except the first one. Now let $I(w)$ be the ordered sequence of the numbers i such that $w_i = 1$. Put $J_\sigma(w) = \sigma^{-1}R(\sigma I(w))$; then a relation $1 = j_1 < j_2 < \dots < j_m = \sigma^{-1}(n)$ is obviously valid, m being the number of elements in $J_\sigma(w)$. Define $\pi(\sigma, w)$ by the formulas

$$\begin{aligned} \pi(\sigma,w)(i) &= \sigma(i+1) \quad \text{if } i \ne j_l - 1, \quad 2 \le l \le m, \\ \pi(\sigma,w)(j_l - 1) &= \sigma(j_{l-1}), \qquad\qquad 2 \le l \le m. \end{aligned} \tag{2}$$

3.2. Lemma. *The manifold $O_{w\sigma}$ is diffeomorphic to the direct product of $A_{\pi(\sigma,w)}$ by $(\mathbf{k}^*)^{n(w)-1}$, where $n(w)$ is the number of nonzero entries in w.*

Proof: Let $A_{w\sigma}$ be the Sylvester manifold of the matrix $M_{w\sigma}$ obtained from M_σ via replacing its first row by the vector $w\sigma = (w_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(n)})$. Then $O_{w\sigma} \cong A_{w\sigma} \times (\mathbf{k}^*)^{n(w)-1}$. Let us prove that the Sylvester manifolds of $M_{w\sigma}$ and of $M_{\pi(\sigma,w)}$ are diffeomorphic. Indeed, put $\{r_1, \dots, r_m\} = R(\sigma I(w))$, $r_0 = 0$ and define the matrix L by the relations

$$l_{ii} = (-1)^{w_{\sigma(i)}}, \qquad 1 \le i \le n;$$

$$l_{ij} = \begin{cases} 1 & \text{if } \ w_{\sigma(j)} = 1,\ r_{k-1} \le j \le r_k,\ k \in \{1, \dots, m\}; \\ 0 & \text{otherwise;} \end{cases} \qquad i \ne j.$$

Since L is a nondegenerate lower triangular matrix, the Sylvester polynomials for $M_{w\sigma}$ and $M_{w\sigma}L$ coincide up to the factor ± 1. The first row of $M_{w\sigma}L$ contains the unique unity: in the last column. Hence, the Sylvester polynomial for this matrix is equal to that for its submatrix obtained by deleting of the first row and of the last column; however, by linear changes of variables the latter can be reduced to $M_{\pi(\sigma,w)}$, q.e.d.

3.3. Call vectors $w^1, w^2 \in W_\sigma$ equivalent ($w^1 \sim w^2$) if $\pi(\sigma, w^1) = \pi(\sigma, w^2)$. Relations (2) imply

$$w^1 \sim w^2 \quad \text{iff} \quad J_\sigma(w^1) = J_\sigma(w^2).$$

Denote by $\tilde{W}_\sigma$ the quotient set $W_\sigma/\sim$ and by $\tilde{W}_\sigma(w)$ the equivalence class containing w.

The set $\{1, \dots, n\}$ can be decomposed in the disjoint union of the three subsets N_0, N_1 and N_{01}. Here N_0 contains all j such that $w'_j = 0$ for all $w' \in \tilde{W}_\sigma(w)$, N_1 contains all j such that $w'_j = 1$ for all $w' \in \tilde{W}_\sigma(w)$, N_{01} contains all other j's. The definition of J_σ implies that

$$\begin{aligned} i \in N_1 \quad &\text{iff} \quad i \in J_\sigma(w); \\ i \in N_0 \quad &\text{iff} \quad \exists l : j_l < i < j_{l+1},\ \sigma(i) > \sigma(j_{l+1}). \end{aligned}$$

Define vectors $\underline{w}$ and $\overline{w}$ by the relations

$$\underline{w}_i = \begin{cases} 1 & \text{if } \ i \in N_1, \\ 0 & \text{otherwise,} \end{cases} \qquad \overline{w}_i = \begin{cases} 0 & \text{if } \ i \in N_0, \\ 1 & \text{otherwise,} \end{cases} \qquad 1 \le i \le n.$$

It is easy to see that $\tilde{W}_\sigma(w) = \{w' : \underline{w}_i \le w'_i \le \overline{w}_i,\ i = 1, \dots, n\}$.

For an arbitrary permutation τ denote by D_τ the set of all pairs (i, k) such that $i < k$ and $\tau(i) < \tau(k)$ and let d_τ be the cardinality of D_τ.

3.4. Lemma. *For arbitrary $\sigma \in \mathbf{S}_n$, $w \in W_\sigma$*

$$d_\sigma - d_{\pi(\sigma,w)} = n(\underline{w}) - 1 + 2(n - n(\overline{w})).$$

Proof: Let us define a mapping $\pi^* : D_\pi \to D_\sigma$ (since σ and w are fixed, we can and will omit dependence of π and of π^* on these parameters). To do this, decompose each of D_σ and D_π in 6 subsets.

Let $(i, k) \in D_\sigma$. Put

$$\begin{aligned} (i,k) \in D_\sigma^1 \quad &\text{iff} \quad i \notin N_1, k \notin N_1, \\ (i,k) \in D_\sigma^2 \quad &\text{iff} \quad i \in N_1, k \in N_{01}, \\ (i,k) \in D_\sigma^3 \quad &\text{iff} \quad i \in N_{01}, k \in N_1, \\ (i,k) \in D_\sigma^4 \quad &\text{iff} \quad i \in N_1, k \in N_0, \\ (i,k) \in D_\sigma^5 \quad &\text{iff} \quad i \in N_0, k \in N_1, \\ (i,k) \in D_\sigma^6 \quad &\text{iff} \quad i \in N_1, k \in N_1. \end{aligned}$$

The corresponding partition of D_π is defined in the following way: let $(i, k) \in D_\pi$, then

$$(i,k) \in D_\pi^l \quad \text{iff} \quad (i+1, k+1) \in D_\sigma^l, \quad 1 \le l \le 6.$$

For $(i,k) \in D_\pi^1$ put $\pi^*(i,k) = (i+1, k+1)$. We have $i+1 < k+1$ while $\sigma(i+1) = \pi(i) < \pi(k) = \sigma(k+1)$ by (2), hence $(i+1, k+1) \in D_\sigma^1$. Evidently, π^* defines a bijection of D_π^1 and D_σ^1.

For $(i,k) \in D_\pi^2$ we have $i + 1 = j_l$, $l > 1$ (recall that j_l is the l'th element in $J_\sigma(w)$). Put $\pi^*(i,k) = (j_{l-1}, k+1)$. Then $j_{l-1} < j_l = i+1 < k+1$ and by (2) $\sigma(j_{l-1}) = \pi(j_l - 1) < \pi(k) = \sigma(k+1)$, hence $(j_{l-1}, k+1) \in D_\sigma^2$. Since $(s,t) \in D_\sigma^2$ implies the existence of $s' \in N_1$ such that $s < s' < t$, we see that π^* defines a bijection of D_π^2 and D_σ^2.

For $(i,k) \in D_\pi^3$ we have $k+1 = j_l$, $l > 1$. Put $\pi^*(i,k) = (i+1, j_l)$. We have $i+1 < j_l$ while by (2) $\sigma(i+1) = \pi(i) < \pi(j_l - 1) = \sigma(j_{l-1}) < \sigma(j_l)$, hence $(i+1, j_l) \in D_\sigma^3$. Evidently, π^* defines a bijection of D_π^3 and D_σ^3.

For $(i,k) \in D_\pi^4$ we have $i+1 = j_l$, $l > 1$. Put $\pi^*(i,k) = (j_{l-1}, k+1)$. As in the case of D_π^2 we obtain $(j_{l-1}, k+1) \in D_\sigma^4$. Evidently, π^* is injective on D_π^4. Let $(s,t) \in D_\sigma^4$; from the definition of $J_\sigma(w)$ it follows that $(s,t) \in \pi^*(D_\pi^4)$ iff there exists s' such that $s < s' < t$. Since for each $t \in N_0$ there exists a unique $s \in N_1$ such that the pair (s,t) does not possess the above property, we see that the number of elements in D_σ^4 exceeds that in D_π^4 by card $N_0 = n - n(\overline{w})$.

For $(i,k) \in D_\pi^5$ we have $k+1 = j_l$, $l > 1$. Put $\pi^*(i,k) = (i+1, j_{l-1})$. By (2) $\sigma(i+1) = \pi(i) < \pi(j_l - 1) = \sigma(j_{l-1})$. Let us prove that $i + 1 < j_{l-1}$. Indeed, by the definition of N_0 the opposite inequality would imply $\sigma(J_{l-1}) < \sigma(i+1)$, i.e. $\pi(k) < \pi(i)$ which contradicts $(i,k) \in D_\pi^5$. Therefore $(i+1, j_{l-1}) \in D_\sigma^5$. Evidently, π^* is injective on D_π^5. Let $(s,t) \in D_\sigma^5$; from the definition of $J_\sigma(w)$ follows that $(s,t) \in \pi^*(D_\pi^5)$ iff $t \ne j_m$ (as before, m denotes the cardinality of $J_\sigma(w)$). Since for each $s \in N_0$ D_σ^5 contains the unique pair (s, j_m), we see that the number of elements in D_σ^5 exceeds that in D_π^5 by card $N_0 = n - n(\overline{w})$.

For $(i,k) \in D^6_\pi$ we have $i+1 = j_l$, $k+1 = j_r$, $1 < l < r$. Put $\pi^*(i,k) = (j_{l-1}, j_{r-1})$. We have $j_{l-1} < j_{r-1}$ while by (2) $\sigma(j_{l-1}) = \pi(j_l - 1) < \pi(j_r - 1) = \sigma(j_{r-1})$, hence $(j_{l-1}, j_{r-1}) \in D^6_\sigma$. Evidently, π^* is injective on D^6_π. Let $(s,t) \in D^6_\sigma$; from the definition of $J_\sigma(w)$ follows that $(s,t) \in \pi^*(D^6_\pi)$ iff $t \neq j_m$. Since D^6_σ contains the unique pair (s, j_m) for each $s \in N_1$, $s \neq j_m$, we see that the number of elements in D^6_σ exceeds that in D^6_π by card $n_1 = n(\underline{w}) - 1$.

Therefore

$$d_\sigma - d_\pi = \sum_{l=1}^{6} (\text{card}\, D^l_\sigma - \text{card}\, D^l_\pi) = n(\underline{w}) - 1 + 2(n - n(\overline{w})),$$

q.e.d.

§4. Euler characteristics of stratified manifolds for cohomology with compact supports

4.1. It is known (see e.g. [M]) that the theory of cohomology with compact supports is "the theory of a single space". It means that if $Y \subset X$ is a closed subset of a locally compact topological space X then for $U = X \setminus Y$ one has $H^i_c(X,Y) \equiv H^i_c(U)$ and the exact sequence of the pair can be written as

(3) $$\cdots \to H^k_c(U) \to H^k_c(X) \to H^k_c(Y) \to H^{k+1}_c(U) \to \cdots$$

thus implying

(4) $$\chi_c(X) = \chi_c(U) + \chi_c(Y).$$

The following proposition was mentioned by many authors (see e.g. [G] and especially [V]).

4.2. Lemma. *Suppose X is an n-dimensional manifold represented as a finite disjoint union of open manifolds X_i: $X = \cup_i X_i$. Then*

$$\chi_c(X) = \sum_i \chi_c(X_i).$$

Proof: Consider the filtration $X = X^n \supset X^{n-1} \supset X^{n-2} \supset \cdots \supset X^0 = \emptyset$, where X^l is the union of all X_i's whose dimension does not exceed l and apply (4) consequently to the pairs (X^n, X^{n-1}), (X^{n-1}, X^{n-2}) and so on.

It turns out that a similar proposition is true for χ^{pq}_c's of quasiprojective manifolds. (From now on in this section we shall follow mainly [D].)

4.3. Theorem. *Suppose X is a complex quasiprojective manifold represented as a finite disjoint union of quasiprojective manifolds X_i: $X = \cup_i X_i$. Then*

$$\chi^{pq}_c(X) = \sum_i \chi^{pq}_c(X_i).$$

The following property of the exact sequence (3) for quasiprojective manifolds immediately implies the Theorem.

4.4. Lemma. *Let $Y \subset X$ be a closed quasiprojective submanifold of a quasiprojective manifold X and $U = X \setminus Y$. Then the exact sequence (3) is an exact sequence of Hodge structures.*

Proof: Choose a compactification $\bar{X} \supset X$ and denote by X' the complement to X in $\bar{X}$, by $\bar{Y}$—the closure of Y in $\bar{X}$ and by Y'—the complement to Y in $\bar{Y}$. Then sequence (3) is reduced to

$$\cdots \to H^k(\bar{X}, X' \cup \bar{Y}) \to H^k(\bar{X}, X') \to H^k(\bar{Y}, Y') \to H^{k+1}(\bar{X}, X' \cup \bar{Y}) \to \cdots.$$

By the excision isomorphism, the third group is isomorphic to $H^k(X' \cup \bar{Y}, X')$, thus (3) is reduced to the exact sequence of the triple $(\bar{X}, X' \cup \bar{Y}, X')$. However, the exact sequence of a triple is an exact sequence of Hodge structures; to prove this it is sufficient to check that Hodge structures are respected by the connecting homomorphism. The latter fact is implied by the similar property of the exact sequence of a pair.

4.5. To prove Theorem 4.3 it suffices to consider the case $X = X_1 \cup X_2$. If both X_1 and X_2 are closed the Theorem is implied immediately by Lemma 4.4. In general, let $\bar{X}_1$ be the closure of X_1 in X and $C = \bar{X}_1 \cap X_2$. Then X_1 is open in $\bar{X}_1$ (since X_1 is quasiprojective), $\bar{X}_1$ is closed in X and C is closed in X_2. Applying Lemma 4.4 three times we obtain

$$\begin{aligned}\chi^{pq}_c(X) &= \chi^{pq}_c(\bar{X}_1) + \chi^{pq}_c(X_2 \setminus C) = \\ &= \chi^{pq}_c(X_1) + \chi^{pq}_c(C) + \chi^{pq}_c(X_2 \setminus C) = \\ &= \chi^{pq}_c(X_1) + \chi^{pq}_c(X_2),\end{aligned}$$

which completes the proof.

§5. Main results

5.1. Theorem. *In the real case*

$$\chi_\sigma = \sum_{w \in W_\sigma} (-1)^{n-n(w)} 2^{n(w)-1} \chi_{\pi(\sigma,w)},$$

where $n(w)$ as before is the number of unit entries in w.

Proof: From (1) and Lemma 4.2 it follows that

(5) $$\chi_c(A_\sigma) = \sum_{w \in W_\sigma} \chi_c(O_{w\sigma}).$$

Lemma 2.3 and the explicit description of the orbits $O_{w\sigma}$ (see section 3.1) imply $\dim O_{w\sigma} = \frac{n(n-1)}{2} - (n - n(w))$. Hence by Lemma 3.2.

$$\begin{aligned}\chi_c(O_{w\sigma}) &= \sum_k (-1)^k h^k_c(O_{w\sigma}) = \sum_k (-1)^k h^{n(n-1)/2-n+n(w)-k}(O_{w\sigma}) = \\ &= \sum_k (-1)^k h^0\left((\mathbf{R}^*)^{n(w)-1}\right) h^{n(n-1)/2-n+n(w)-k}(A_{\pi(\sigma,w)}) = \\ &= 2^{n(w)-1} \sum_k (-1)^k h^{k+1-n(w)}_c(A_{\pi(\sigma,w)}) = (-2)^{n(w)-1} \chi_c(A_{\pi(\sigma,w)}) = \\ &= (-1)^{(n-1)(n-2)/2} (-2)^{n(w)-1} \chi_{\pi(\sigma,w)}.\end{aligned}$$

From the above relation and formula (5) follows that

$$\chi_\sigma = (-1)^{n(n-1)/2}\chi_c(A_\sigma) = \sum_{w\in W_\sigma}(-1)^{n-n(w)}2^{n(w)-1}\chi_{\pi(\sigma,w)},$$

q.e.d.

5.2. To handle the complex case we need the following technical proposition. Suppose X is an arbitrary complex manifold; denote

$$P_X(t) = \sum_i \chi^{ii}(X)t^i,$$
$$h_k^{ij}(X) = \dim \mathrm{Gr}_F^i \mathrm{Gr}_{i+j}^W H^k(X).$$

LEMMA. *Let $X = U \times Y$ and the following assumptions are true:*
1) $\chi^{ij}(Y) = 0$ *for* $i \neq j$*;*
2) $h_k^{ij}(U) = 0$ *for* $i \neq j$.

Then

(6) $$P_X(t) = P_U(t)P_Y(t),$$
(7) $$\chi^{ij}(X) = 0 \quad \text{for} \quad i \neq j.$$

PROOF: Evidently, assumption (2) implies that

$$h_k^{ij}(X) = \sum_{p,q,l} h_l^{pq}(U)h_{k-l}^{i-p,j-q}(Y) = \sum_{p,l} h_l^{pp}(U)h_{k-l}^{i-p,j-p}(Y).$$

Hence

$$\chi^{ij}(X) = \sum_k (-1)^k h_k^{ij}(X) = \sum_{p,l}(-1)^l h_l^{pp}(U)\sum_k(-1)^{k-l}h_{k-l}^{i-p,j-p}(Y) =$$
$$= \sum_{p,l}(-1)^l h_l^{pp}(U)\chi^{i-p,j-p}(Y) = \sum_p \chi^{pp}(U)\chi^{i-p,j-p}(Y).$$

From this relation we see that (7) follows immediately from assumption (1). Now,

$$P_X(t) = \sum_i\sum_p \chi^{pp}(U)\chi^{i-p,i-p}(Y)t^i = \left(\sum_p \chi^{pp}(U)t^p\right)\left(\sum_j \chi^{jj}(Y)t^j\right) =$$
$$= P_U(t)P_Y(t),$$

which coincides with (6).

5.3. THEOREM. *Put $P_\sigma(t) \equiv P_{A_\sigma}(t)$, then*

(8) $$P_\sigma(t) = \sum_{w\in W_\sigma} t^{n-n(w)}(1-t)^{n(w)-1}P_{\pi(\sigma,w)}(t),$$
(9) $$\chi_\sigma^{ij} = 0 \quad \text{for} \quad i \neq j,$$

where $n(w)$ is the same that in Theorem 5.1.

PROOF: Denote $P_{w\sigma}(t) \equiv P_{O_{w\sigma}}(t)$. Lemmas 3.2 and 5.2 imply

(10) $$P_{w\sigma}(t) = P_{(\mathbf{C}^*)^{n(w)-1}}P_{\pi(\sigma,w)}(t) = (1-t)^{n(w)-1}P_{\pi(\sigma,w)}(t).$$

For an arbitrary complex manifold X define a polynomial

$$P_X^c(t) = \sum_i t^i \sum_k \dim \mathrm{Gr}_F^i \mathrm{Gr}_{2i}^W H_c^k(X)$$

and extend on this the abbreviated notions P_σ^c and $P_{w\sigma}^c$. Evidently,

(11) $$P_X^c(t) = t^d P_X(1/t),$$

where d is the complex dimension of X. According to Theorem 4.3 relation (1) implies

$$P_\sigma^c(t) = \sum_{w\in W_\sigma} P_{w\sigma}^c(t).$$

Together with (11) this implies

$$t^{n(n-1)/2}P_\sigma(1/t) = \sum_{w\in W_\sigma} t^{n(n-1)/2-n+n(w)}P_{w\sigma}(1/t).$$

Introducing (10) in the above relation and redenoting $\frac{1}{t}$ by t we obtain (8).

5.4. COROLLARY. *In the complex case $\chi_\sigma = 0$.*

PROOF: Evidently, $\chi_\sigma = \sum_{i,j}\chi_\sigma^{ij}$. Together with (8), (9) this implies $\chi_\sigma = \sum_i \chi_\sigma^{ii} = P_\sigma(1) = 0$.

5.5. A connection between χ_σ for the real case (ad hoc denote it by $\chi_\sigma^{\mathbf{R}}$) and χ_σ^{ii} is given by the following proposition.

COROLLARY. $\chi_\sigma^{\mathbf{R}} = \sum_i |\chi_\sigma^{ii}|$.

PROOF: Follows immediately from Theorem 5.1, formula (8) and the relation $\sum_i |\chi_\sigma^{ii}| = P_\sigma(-1)$.

5.6. Theorem. *For any $\sigma \in W_\sigma$*

$$\deg P_\sigma = d_\sigma, \qquad \chi_\sigma^{d_\sigma d_\sigma} = (-1)^{d_\sigma}, \tag{12}$$

$$\chi_\sigma^{ii} = (-1)^{d_\sigma} \chi_\sigma^{d_\sigma - i, d_\sigma - i}, \quad 0 \le i \le d_\sigma. \tag{13}$$

Proof: For an arbitrary $\tilde{w} \in \tilde{W}_\sigma$ put

$$Q_{\tilde{w}}(t) = \sum_{w \in \tilde{w}} t^{n-n(w)}(1-t)^{n(w)-1} P_{\pi(\sigma,w)}(t).$$

Then (8) can be rewritten as

$$P_\sigma(t) = \sum_{\tilde{w} \in \tilde{W}_\sigma} Q_{\tilde{w}}(t) \tag{14}$$

while

$$\begin{aligned} Q_{\tilde{w}}(t) &= \sum_{i=0}^{n(\overline{w})-n(\underline{w})} \binom{n(\overline{w}) - n(\underline{w})}{i} t^{n-n(\overline{w})+i}(1-t)^{n(\overline{w})-i-1} P_{\pi(\sigma,w)}(t) = \\ &= t^{n-n(\overline{w})}(1-t)^{n(\underline{w})-1} P_{\pi(\sigma,w)}(t) \end{aligned} \tag{15}$$

with w satisfying $\tilde{W}_\sigma(w) = \tilde{w}$.

Suppose that (12) is already proved for permutations from $\mathbf{S}_{n-1}$ (the basis of the induction is trivial). Then

$$\deg Q_{\tilde{w}} = n - n(\overline{w}) + n(\underline{w}) - 1 + d_{\pi(\sigma,w)}.$$

By Lemma 3.4 this relation implies $\deg Q_{\tilde{w}} = d_\sigma - n + n(\overline{w})$. This means that only one polynomial in the right hand side of (14) has degree d_σ (the corresponding w consists of all 1's), while the degrees of all other polynomials are strictly less then d_σ. Together with (15) this implies (12).

Now note that by (12) relation (13) is equivalent to

$$P_\sigma(t) = (-t)^{d_\sigma} P_\sigma(1/t).$$

Suppose that this relation is already proved for permutations from $\mathbf{S}_{n-1}$. From (15) and Lemma 3.4 we see that

$$Q_{\tilde{w}}(1/t) = (-t)^{-d_\sigma} Q_{\tilde{w}}(t).$$

Introducing this into (14) we obtain the required relation.

§6. Calculations for low dimensions

Theorems 5.1 and 5.3 enable us to calculate χ_σ and P_σ for any $\mathbf{S}_n$ consecutively by n. The results for $n = 1, 2, 3, 4$ are displayed below.

n	σ	χ_σ	P_σ
1	1	1	1
2	(1,2)	2	$1-t$
	(2,1)	1	1
3	(1,2,3)	6	$1-2t+2t^2-t^3$
	(1,3,2), (2,1,3)	4	$1-2t+t^2$
	(2,3,1), (3,1,2)	2	$1-t$
	(3,2,1)	1	1
4	(1,2,3,4)	20	$1-3t+4t^2-4t^3+4t^4-3t^5+t^6$
	(1,3,2,4)	18	$1-3t+5t^2-5t^3+3t^4-t^5$
	(1,2,4,3), (2,1,3,4)	16	$1-3t+4t^2-4t^3+3t^4-t^5$
	(2,1,4,3), (2,3,1,4), (3,1,2,4), (1,3,4,2), (1,4,2,3)	12	$1-3t+4t^2-3t^3+t^4$
	(3,2,1,4), (1,4,3,2), (2,4,1,3), (3,1,4,2)	8	$1-3t+3t^2-t^3$
	(2,3,4,1), (4,1,2,3)	6	$1-2t+2t^2-t^3$
	(3,4,1,2), (2,4,3,1), (3,2,4,1), (4,1,3,2), (4,2,1,3)	4	$1-2t+t^2$
	(4,2,3,1), (3,4,2,1), (4,3,1,2)	2	$1-t$
	(4,3,2,1)	1	1

Moreover, $\chi_{(1,2,3,4,5)} = 52$, $\chi_{(1,2,3,4,5,6)} = 104$.

The natural conjecture that for n fixed the maximal value of χ_σ is achieved at $\sigma = (1, 2, \ldots, n)$ fails. Indeed, already for $n = 5$ one has

$$\chi_{(1,3,2,4,5)} = \chi_{(1,2,4,3,5)} = 56, \chi_{(1,3,4,2,5)} = 60$$

(the latter value is maximal for $\mathbf{S}_5$).

It would be interesting to study the topology of A_σ. For $n = 1, 2, 3$ in the real case al A_σ are disconnected unions of cells. The same is apparently true for $n = 4$ thus addin one more number—52—to the list of the numbers of connected components of the set of al flags transversal to a given transversal pair of flags (see [A]). For $n > 4$ the topology of A_σ can be nontrivial.

Note that to each permutation σ from $\mathbf{S}_n$ one can assign the permutation $\hat{\sigma} = (\sigma, n+1)$ from $\mathbf{S}_{n+1}$. Evidently, $A_{\hat{\sigma}}$ is homotopically equivalent to A_σ (in fact $A_{\hat{\sigma}}$ is a cylinder over A_σ). Hence, on the set of all possible permutations one obtains the generalized Bruhat ordering, whose maximal element is "the inverse permutation of all positive integers". Does cohomology of the corresponding A_σ stabilize? If so, find the stable cohomology ring.

It would be also interesting to find the cohomology and the mixed Hodge structure of A_σ in the complex case. We managed to obtain the answer for $n \leq 4$. In all this cases the corresponding mixed Hodge structure is pure, namely, $h_i^{ii}(A_\sigma) = |\chi_\sigma^{ii}|$ while all the other h_k^{ij} vanish. It is tempting to prove that the same is true for all A_σ, but most likely it is a low dimensional effect. It is apparently easy to prove (decomposing A_σ in the disjoint union of quasiprojective manifolds each diffeomorphic to the product of the certain numbers of copies of $\mathbf{C}$'s and $\mathbf{C}^*$'s) that in the mixed Hodge structure of A_σ the h_k^{ij} always vanish for $i \neq j$. On the other hand, the analogs of A_σ for Grassmann manifolds turn out to be isomorphic to $\mathbf{GL}_n(\mathbf{C})$ while the mixed Hodge structure of these fails to be pure.

References

[A] V. I. Arnold, *Ten problems*, in "The theory of singularities and its applications (series Advances in Soviet Mathematics) (to appear)," AMS publications, Providence RI.

[D] A. H. Durfee, *Algebraic varieties which are a disjoint union of subvarieties*, in "Geometry & Topology. Manifolds, Varieties & Knots (C. McCrory and T. Schifrin, eds.)," Marcel Dekker, New-York & Boston, 1987, pp. 99–102.

[FF] A. T. Fomenko and D. B. Fuchs, "A Course in Homotopical Topology," Nauka, Moscow, 1989.

[G] V. A. Ginzburg, *Equivariant cohomology and Kaehlerian geometry*, Funct. Anal. i ego Prilozh. **21** 4 (1987), 19–34.

[M] W. S. Massey, "Homology and Cohomology Theory," Marcel Dekker, New-York & Boston, 1978.

[V] O. Y. Viro, *On integral calculus based on Euler characteristics*, in "Topology & Geometry—Rohlin Seminar (O. Y. Viro, ed.). Lecture Notes in Mathematics, vol.1346," Springer-Verlag, N.Y., 1988, pp. 32-46.