INTRODUCING BIZONOTOPAL ALGEBRAS FOR UNDIRECTED GRAPHS

ANATOL KIRILLOV, GLEB NENASHEV, BORIS SHAPIRO, AND ARKADY VAINTROB

Abstract. About two decades ago three types of zonotopal algebras (external, central, and internal) have been introduced for an arbitrary undirected graph G. They contain an abundance of information about G encoded in its Tutte polynomial. Below we introduce their bizonotopal analogs in which we double each edge of G. The new algebras are monomial and have intricate combinatorial properties. In particular, in the external and central cases, the Hilbert series of these algebras satisfy a new version of deletion-contraction property in which the contracted edge becomes a loop while in the internal case, the HiIbert series satisfy a more complicated 4-term relation.

1. Introduction

In what follows, we introduce three new algebras associated to an undirected graph G with possible loops and multiple edges which we denote by $\mathfrak{BZ}^e_G$, $\mathfrak{BZ}^c_G$, and $\mathfrak{BZ}^i_G$ and call *external, central, and internal bizonotopal* algebras of G respectively. They generalize the previously known external, central, and internal zonotopal algebras of G introduced in [ArPo, HoRo]. Furthermore, for a given graph G, we define a sequence of r-bizonotopal algebras where r is any integer such that $-r$ is smaller or equal the minimum degree of the vertices of G. In addition to the above 3 algebras, r-bizonotopal algebras split into a sequence of *superexternal* and a finite number of *subinternal* algebras. The basic idea behind the bizonotopal algebras is similar to that of the zonotopal algebras, but each undirected edge of G is substituted by a pair of directed edges with opposite orientations.

The resulting algebras are monomial and have intriguing combinatorial properties. They as well as their Hilbert series contain a lot of information about the underlying graph G. Quite unexpectedly, in the external and central case their Hilbert series satisfy a natural "deletion-contraction" property very similar to the classical one satisfied by the famous Tutte polynomials. In the interior case, the Hilbert series satisfy an intriguing 4-term relation involving 4 different graphs. Moreover, one can introduce a new multivariate polynomial associated to G which generalizes the multivariate Tutte polynomial of G and satisfies the new "deletion-contraction" property, see [KNSV].

The structure of the paper is as follows. In § 2, we introduce and describe the major properties of the external algebras. In § 3.1 we show that Hilbert series of external, central, and superexternal algebras satisfy one and the same "deletion-contraction" property, but with different initial conditions. In § 3.2 we describe a 4-term relation satisfied by the Hilbert series of (sub)internal algebras. In § 4 we prove a number of additional results valid in the central and in the interior cases.

Date: April 17, 2024.

2020 *Mathematics Subject Classification.* Primary 05A15; 05B35; 13B25; 16S32; 41A15 Secondary 05C31; 41A63; 47F05.

Key words and phrases. (bi)zonotopal algebras, monomial algebras, Hilbert series, deletion-contraction property.

Finally, in § 5 we provide examples of the Hilbert series of the latter algebras for complete graphs.

Acknowledgements. The third author wants to acknowledge the financial support of his research provided by the Swedish Research Council grant 2021-04900. He is sincerely grateful to Beijing Institute for Mathematical Sciences and Applications (BIMSA) for the hospitality in Fall 2023 where this research project has been initiated.

2. External bizonotopal algebra

The following definition was suggested by the first author about a decade ago, see p. 136 of [Ki].

Let $G = (V, E)$ be an undirected graph with the vertex set V and the edge set E. Label the vertices in V as $\{v_1, \dots, v_n\}$ where $n = |V|$. Now consider the double edge set $DE = E \cup \tilde{E}$ where for each edge $e \in E$, we introduce another edge $\tilde{e} \in \tilde{E}$ connecting the same pair of vertices. Choosing some field $\mathbb{K}$ consider the *external double edge algebra*

$$\mathcal{DE}^e_G = \frac{\mathbb{K}[DE]}{< x_e^2, x_{\tilde{e}}^2, x_e x_{\tilde{e}} >}$$

which is the quotient of the polynomial algebra generated by the variables x_e and $x_{\tilde{e}}$ corresponding to all edges in DE by the ideal generated by the squares of all edges in DE together with the relations $x_e x_{\tilde{e}} = 0$ for all edges in E.

To each vertex $v_\ell \in V$, $\ell = 1, \dots, n$, we associate the linear form

$$y_\ell = \sum_{i<\ell, e=(i,\ell)\in E} x_e + \sum_{j>\ell, \tilde{e}=(j,\ell)\in \tilde{E}} x_{\tilde{e}} = \sum_{k\neq \ell} x_{(k,\ell)} \tag{2.1}$$

in $\mathcal{DE}^e_G$.

Finally, for the graph $G = (V, E)$ with labelled vertices $(v_1, \dots, v_n)$, define its *external bizonotopal algebra* as the subalgebra $\mathfrak{B}\mathfrak{Z}^e_G \subset \mathcal{DE}^e_G$ generated by the linear forms $y_1, \dots, y_n$. Obviously, $\mathfrak{B}\mathfrak{Z}^e_G$ is a graded algebra which a priori depends on the choice of vertex labelling.

Remark 2.1. The last formula in (2.1) implies that $\mathfrak{B}\mathfrak{Z}^e_G$ is independent of a vertex labelling.

Remark 2.2. The main motivation for the consideration of bizonotopal algebras was an attempt to extend several known results valid for simple graphs (i.e. having no loops or multiple edges) by Orlik-Solomon and Postnikov-Shapiro-Shapiro.

The next result describes $\mathfrak{B}\mathfrak{Z}^e_G$ in terms of generators and relations.

Definition 2.3. A sequence of non-negative integers $(a_1, a_2, \dots, a_n)$ is called a *partial score vector* of G if there is a subset $E' \subset E$ of edges and their orientations such that outgoing degrees of vertices $\{v_1, \dots, v_n\}$ are given by $(a_1, a_2, \dots, a_n)$ respectively. We define the *G-parking function polytope* (denoted by $\mathcal{P}_G$) as the convex hull of all partial score vectors of G.

Remark 2.4. Observe that the classical parking function polytope (corresponding to a complex graph) and several its generalizations different from the above one have been studied in a number of recent papers, see e.g. [HLVM, AmWa] and references therein.

Lemma 2.5. *For any undirected graph $G = (V, E)$ on n vertices, one has that*
(i) the algebra $\mathfrak{B}\mathfrak{Z}^e_G$ is monomial;
(ii) $y_1^{a_1} y_2^{a_2} \dots y_n^{a_n} \neq 0$ in $\mathfrak{B}\mathfrak{Z}^e_G$ if and only if $(a_1, a_2, \dots, a_n)$ is a partial score vector.

Proof. We start with the following crucial observation. Given an edge $e = (i, j)$ with $i < j$, the variable x_e occurs only in the linear form y_j while the variable $x_{\tilde{e}}$ occurs only in the linear form y_i.

Therefore any monomial $\mathfrak{m}$ in the variables $x_e, x_{\tilde{e}}$ appears only in the expansion of the unique monomial in the variables $y_1, \dots, y_n$. Therefore $\mathfrak{BZ}_G^e$ is a monomial algebra. Furthermore, $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \neq 0$ if and only if there exists a subset of edges $E' \subset E$ and its orientation such that the outgoing degrees of the vertices $\{v_1, \dots, v_n\}$ are given by $(a_1, a_2, \dots, a_n)$ respectively. □

Lemma 2.5 shows that $\mathfrak{BZ}_G^e$ is the monomial algebra. Our next goal is to find its set of relations.

Notation 2.6. For any undirected graph $G = (V, E)$ and any subset $V_I = (v_{i_1}, v_{i_2}, \dots, v_{i_\ell})$ of its vertices, denote by κ_I the total number of edges of G either one or both vertices of which belong to V_I. Further, associate to V_I the set of monomials $\mathfrak{M}_I^e$ in the variables $v_{i_1}, v_{i_2}, \dots, v_{i_\ell}$ of the form

$$v_{i_1}^{k_1} v_{i_2}^{k_2} \dots v_{i_\ell}^{k_\ell} \text{ with } k_1 + k_2 + \dots + k_\ell = \kappa_I + 1.$$

Theorem 2.7. *For any undirected graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$, one has*

$$\mathfrak{BZ}_G^e \simeq \frac{\mathbb{K}[v_1, \dots, v_n]}{< \cup_{I \subseteq \{v_1, \dots, v_n\}} \mathfrak{M}_I^e >}.$$

Here the denominator is the monomial ideal generated by the union of all monomials appearing in $\mathfrak{M}_I^e$ where I runs over all non-empty subsets of V.

Proof. We first show that for any subset $V_I = (v_{i_1}, v_{i_2}, \dots, v_{i_\ell})$ of vertices of G and any multiindex $(k_1, k_2, \dots, k_\ell)$ satisfying the equality $k_1 + k_2 + \dots + k_\ell = \kappa_I + 1$, one has

$$y_{i_1}^{k_1} y_{i_2}^{k_2} \dots y_{i_\ell}^{k_\ell} = 0. \tag{2.2}$$

Here $y_1, y_2, \dots, y_n$ are given by (2.1). Indeed, if we expand $y_{i_1}^{k_1} y_{i_2}^{k_2} \dots y_{i_\ell}^{k_\ell}$ in the edge variables x_e and $x_{\tilde{e}}$ only edges at least one end of which belongs to V_I will be involved. Since in each monomial of the expansion the total number of such edges is κ_I, then for some $e \in E$, either x_e^2, $x_{\tilde{e}}^2$ or $x_e x_{\tilde{e}}$ will appear in every such monomial. Thus (2.2) follows.

In order to prove the converse, we use Lemma 2.5. It remains to show that if $v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} \neq 0$, then there is a subset $E' \subseteq E$ and its orientation such that the outgoing degrees of $(v_1, v_2, \dots, v_n)$ are $a_1, a_2, \dots, a_n$ respectively. Consider the bipartite graph $\mathcal{B}$ with two sets of vertices $B_1 = \{p_{1,1}, \dots, p_{1,a_1}, p_{2,1}, \dots, p_{2,a_2}, p_{3,1}, \dots, p_{n,a_n}\}$ and $B_2 = E$. Its set of edges is given by the pairs $(p_{i,k}, e)$ for all $v_i \in e$. Note that the condition $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \neq 0$ is equivalent to the assumptions of Hall's marriage theorem. Therefore there exists a perfect matching M in $\mathcal{B}$.

Let us now construct a subset of edges E' and its orientation. If $(p_{i,k}, e) \in M$, then we orient e away from the vertex i. It is easy to check that the corresponding score vector is $(a_1, a_2, \dots, a_n)$. □

Theorem 2.8. *The Hilbert series*

$$h_G^e(t) := \sum_{k \geq 0} \dim(\mathfrak{BZ}_G^e)^{(k)} \cdot t^k$$

has the following properties:

(1) *it is a polynomial of degree $|E|$ where $|E|$ is the total number of edges in G;*

(2) $\dim(\mathfrak{BZ}_G^e)^{(|E|)}$ *equals the number of spanning forests of G;*

(3) $\dim(\mathfrak{BZ}_G^e)^{(\ell)}, \ell = 1, 2, \dots, |E|$ *equals the number of partial score vectors of G with the sum of coordinates equal to ℓ;*

(4) *the total dimension* $\dim(\mathfrak{B}\mathfrak{Z}^e_G)$ *equals the number of integer points in the G-parking function polytope.*

Proof of (1)-(3) Theorem 2.8. Items (1) and (3) immediately follow from Lemma 2.5. Since the number of score vectors for any graph is equal to the number of spanning forests (see [KlWi]), then item (2) holds. □

Proposition 2.9. *For a graph G and* $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, *we have that* $(a_1, a_2, \dots, a_n) \in \mathcal{P}_G$ *if and only if* $(a_1, a_2, \dots, a_n)$ *is a partial score vector.*

Proof of Proposition 2.9 and item (4) of Theorem 2.8. By Theorem 2.7 we know that $(a_1, a_2, \dots, a_n)$ is a partial score vector if and only if

$$x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \notin \langle \cup_{I \subseteq \{v_1, \dots, v_n\}} \mathfrak{M}_I \rangle.$$

Hence, $(a_1, a_2, \dots, a_n)$ is a partial score vectors if and only if $\sum_{i \in I} a_i \leq \kappa_I$ for all $I \subset V$. Therefore, the set of partial score vectors is exactly the set of integer points in the polytope described by the latter inequalities. □

Let us also describe all the vertices of the polytope $\mathcal{P}_G$.

Theorem 2.10. *The integer vector* $(a_1, a_2, \dots, a_n)$ *is a vertex of* $\mathcal{P}_G$ *if and only there is a pair* $\pi \in S_n, k \leq n$ *such that*

$$a_{\pi_i} = \begin{cases} \text{the number of edges between } v_{\pi_i} \text{ and } \{v_{\pi_1}, \dots, v_{\pi_{i-1}}\}, & \text{if } i > k. \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.11. *For an arbitrary undirected graph G, the set function* κ_X *is submodular.*

Proof. We need to show that $\kappa_I + \kappa_J \geq \kappa_{I \cap J} + \kappa_{I \cup J}$. Remember that κ_X counts edges incident to X. Let us count the appearances of the edge (a, b) in the left-hand and in the right-hand sides of this inequality.

If a or b belongs to $I \cap J$, then we count this edge twice in the left-hand and twice in the right-hand sides. If $a, b \notin I \cap J$ and (a, b) is incident to at least one set, then we count it at least once in the left-hand and exactly once in the right-hand sides.

Since the appearance of each edge is a submodular function which implies that κ is also submodular. □

Proof of Theorem 2.10. We use induction on the number of vertices of G. For the empty graph everything is clear. To carry out the step of induction, let $\bar{a} = (a_1, a_2, \dots, a_n)$ be a vertex of $\mathcal{P}_G$. We have two possible cases presented below.

(i) For every $\emptyset \neq I \subset V$, we have $\sum_{i \in I} a_i < \kappa_I$. Note that if $a_{k_1}, a_{k_2} > 0$, then $\bar{a} + e_{k_1} - e_{k_2}, \bar{a} - e_{k_1} + e_{k_2} \in \mathcal{P}_G$. Since $\bar{a}$ is a vertex, then there is k such that $a_i = 0$ for $i \neq k$. Hence $\bar{a}$ is a linear combination of 0 and $deg(v_k)e_k$. Hence $\bar{a} = 0$, because $\bar{a}$ is a vertex with the condition $a_k < deg(v_k)$.

(ii) There exists $\emptyset \neq I \subset V$ such that $\sum_{i \in I} a_i = \kappa_I$. Let I be a minimal (by inclusion) such set. If $I = \{v_k\}$, then we consider $\pi_n = k$ and $G' = G - v_k$. Note that $(a_1, a_2, \dots a_{k-1}, a_{k+1}, \dots a_n)$ is a vertex $\mathcal{P}_{G'}$. Therefore by induction, we can construct a permutation $\pi_1, \pi_2, \dots \pi_{k-1}, \pi_{k+1}, \dots \pi_n$ of $\{1, 2, \dots, k-1, k+1, \dots n\}$.

It remains to consider the case $|I| > 1$. Take $k_1, k_2 \in I$. Since I is a minimal by inclusion set, $a_{k_1} > 0$ and $a_{k_2} > 0$. Since κ is a submodular function on subsets of vertices, there is no $J \subset V$ such that $\sum_{j \in J} a_j = \kappa_J$ and exactly one vertex of k_1, k_2 belongs to J (otherwise we can consider $I \subset J$, which is smaller than I).

Hence both $\bar{a} + e_{k_1} - e_{k_2}$ and $\bar{a} - e_{k_1} + e_{k_2}$ belong to $\mathcal{P}_G$. Let us check the claim for $\bar{a}' = \bar{a} + e_{k_1} - e_{k_2}$. Indeed, for any $J \subset V$, we have

$$\sum_{j \in J} a'_j = \sum_{j \in J} a_j + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J}.$$

If $\{k_1, k_2\} \cap J = \{k_1, k_2\}$ or $\{\emptyset\}$, then

$$\sum_{j \in J} a_j + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J} = \sum_{j \in J} a_j \leq \kappa_J.$$

If $\{k_1, k_2\} \cap J = \{k_1\}$ or $\{k_2\}$, then

$$\sum_{j \in J} a_j + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J} < \kappa_J + \mathbb{1}_{k_1 \in J} - \mathbb{1}_{k_2 \in J} \leq \kappa_J + 1.$$

Hence, $\bar{a} + e_{k_1} - e_{k_2} \in \mathcal{P}(G)$ and similarly, $\bar{a} - e_{k_1} + e_{k_2} \in \mathcal{P}(G)$. Therefore $\bar{a}$ is not a vertex, contradiction. $\square$

Note that for the majority of graphs G, it seems difficult to do the exact count of the number of vertices of the G-parking function polytope $\mathcal{P}_G$. However we have the following natural upper bound.

Proposition 2.12. *Given a graph G with n vertices, the number of vertices of $\mathcal{P}_G$ is at most $[(e-1)n!]$ with equality only for the complete graph K_n.*

Proof. Fixing $k \leq n$, it is easy to see that the vector from Theorem 2.10 depends only on $\pi_{k+1}, \dots, \pi_n$. Moreover if there is an edge between π_j and π_{j+1}, then we can interchange them. Hence, the number of vertices for k is at most $(n-k)!\binom{n}{n-k} = \frac{n!}{k!}$ with equality only for the complete graph.

Then the total number of vertices of $\mathcal{P}_G$ is at most

$$\sum_{k=1}^{n} \frac{n!}{k!} = n! \sum_{k=1}^{n} \frac{1}{k!} = n!(e-1) - n! \sum_{i=n+1}^{\infty} \frac{1}{i!} = [(e-1)n!]$$

with equality only for the complete graph. $\square$

Theorem 2.13. *For any two graphs G_1 and G_2 without isolated vertices, their algebras $\mathfrak{BZ}^e_{G_1}$ and $\mathfrak{BZ}^e_{G_2}$ are isomorphic if and only if G_1 and G_2 are isomorphic.*

Proof. Obviously for two isomorphic graphs, their algebras are also isomorphic. Let us prove the converse.

Given two graphs G_1 and G_2 such that $\mathfrak{BZ}^e_{G_1} \simeq \mathfrak{BZ}^e_{G_2}$ and $u \in \mathfrak{BZ}^e_{G_1}$, define $deg(u)$ as the smallest k such that $u^{k+1} = 0$. Note that, for non-zero $u \in (\mathfrak{BZ}^e_{G_1})^{(1)}$, $deg(u)$ is exactly the number of edges appearing in u. We know that $dim(\mathfrak{BZ}^e_{G_1})$ is equal to the number of vertices.

Consider a basis $u_1, u_2, \dots, u_n$ of $(\mathfrak{BZ}^e_{G_1})^{(1)}$ such that $deg(u_1) + deg(u_2) + \dots + deg(u_n)$ is the smallest possible. We have

$$deg(u_1) + deg(u_2) + \dots + deg(u_n) = 2 * |E(G_1)| - \#\{e : \ e \text{ is a loop}\},$$

since each non-loop edge appears at least in two different u_i and u_j and we have equality for the basis corresponding to the vertices.

Let $y_1, \dots, y_n \in \mathfrak{BZ}^e_{G_1}$ correspond to the vertices of G_1. Since $y_1, \dots, y_n$ and $u_1, \dots, u_n$ are two bases of $\mathfrak{BZ}^e_{G_1}$, there is a permutation $\pi \in S_n$ such that $u_i = c_{i,1}y_1 + \dots + c_{i,n}y_n$ has a non-zero coefficient c_{i,π_i}. Clearly $deg(u_i) \geq deg(y_{\pi_i})$, hence $deg(u_i) = deg(y_{\pi_i})$. Therefore the support of u_i has only edges corresponding to the edges incident to v_{π_i}. Hence, the number of edges between v_{π_i} and v_{π_j} is equal to $deg(u_i) + deg(u_j) - deg(u_i + \lambda u_j)$, where $\lambda \in \mathbb{K}$ is generic. Since we know how many edges are incident to π_i and how many edges exist between π_i and π_j for each

j, we can find the number of loops for the vertex π_i. In this way we can reconstruct G_1 from $\mathfrak{BZ}^e_{G_1}$ (Note that u_i is not necessary $c_{i,\pi_i} v_{\pi_i}$). Similarly we can reconstruct G_2 from $\mathfrak{BZ}^e_{G_2}$. Since $\mathfrak{BZ}^e_{G_2}$ and $\mathfrak{BZ}^e_{G_2}$ are isomorphic the underlying graphs G_1 and G_2 are isomorphic as well. □

3. r-bizonotopal algebras and their properties

Notation 3.1. Consider any undirected graph $G = (V; E)$. Choose an integer r such that $-r$ is smaller or equal than the minimal degree of vertices in G. (Obviously, if $r \geq 0$ the latter condition is automatically satisfied).

As above, for any subset $V_I = (v_{i_1}, v_{i_2}, \dots, v_{i_\ell})$ of its vertices, denote by κ_I the total number of edges of G at least one of vertices of which belong to V_I. Associate to V_I the set of monomials $\mathfrak{M}^r_I$ in the variables $v_{i_1}, v_{i_2}, \dots, v_{i_\ell}$ of the form

$$v_{i_1}^{k_1} v_{i_2}^{k_2} \dots v_{i_\ell}^{k_\ell} \text{ with } k_1 + k_2 + \cdots + k_\ell = \kappa_I + r.$$

In the above notation, define the *r-bizonotopal algebra* as

$$\mathfrak{BZ}^r_G := \frac{\mathbb{K}[v_1, \dots, v_n]}{< \cup_{I \subseteq \{v_1, \dots, v_n\}} \mathfrak{M}^r_I >}.$$

Denote by $h^r_G(t)$ the Hilbert series of $\mathfrak{BZ}^r_G$.

Case $r = 1$ corresponds to the above external bizonotopal algebra. Algebras corresponding to $r > 1$ are called *superexternal*. In analogy with the zonotopal algebras, we call the case $r = 0$ *central* and discuss it in more details in § 4.1. Case $r = -1$ called *internal* makes sense if the underlying graph has no isolated vertices and will be discussed in § 4.2. If the minimal degree of vertices in G equals $\theta \geq 2$, we can define $\mathfrak{BZ}^r_G$ for $-\theta \leq r < -1$. Such algebras are called *subinternal*.

The next result extends Theorem 2.13 and is analogous to the main result of [NeSh].

Theorem 3.2. *Given two graphs G_1, G_2 and $r > 1$, the algebras $\mathfrak{BZ}^r_{G_1}$ and $\mathfrak{BZ}^r_{G_2}$ are isomorphic if and only if the graphs G_1 and G_2 are isomorphic.*

Proof. Almost identical with that of Theorem 2.13. □

Remark 3.3. An analog of Theorem 2.13 for the central case is discussed in section § 4.1.

3.1. "Deletion-contraction" property.

Definition 3.4. For a given undirected multigraph G and its edge e (different from a loop), define
(i) the deletion $G - e$ as the graph G with the edge e deleted;
(ii) the contraction G/e as the graph G in which the end vertices of e are glued together and the edge e is transformed into a loop at this common vertex.

Notice that our definition of contraction differs from the standard one used for Tutte polynomials! The above operations are essential for the Hilbert series of external, central, and superexternal algebras.

Theorem 3.5. *For any undirected multigraph $G = (V, E)$, any non-negative integer r, and an edge e of G different from a loop, one has the "deletion-contraction" property:*

$$h^r_G(t) = h^r_{G/e}(t) + t \cdot h^r_{G-e}(t).$$

Proof. Assume that an edge e connects the vertices v_p and v_q. We want to describe all non-vanishing monomials in the monomial algebras $\mathfrak{BZ}^r_G$, $\mathfrak{BZ}^r_{G-e}$, and $\mathfrak{BZ}^r_{G/e}$. Let us fix the degrees d_i of all $v_i \in V \setminus \{v_p, v_q\}$. We are interested only in monomials $m = \prod_{v_i \in V\setminus\{v_p,v_q\}} v_i^{d_i}$ which do not vanish in $\mathfrak{BZ}^r_G$, because if such a monomial vanishes in $\mathfrak{BZ}^r_G$ then it also vanishes in both $\mathfrak{BZ}^r_{G-e}$ and $\mathfrak{BZ}^r_{G/e}$.

Define the three numbers:

- $a := min_{I\subset[n]\setminus\{p,q\}} \kappa_{I\cup\{p\}} + r - 1 - \sum_{i\in I} d_i$;
- $b := min_{I\subset[n]\setminus\{p,q\}} \kappa_{I\cup\{q\}} + r - 1 - \sum_{i\in I} d_i$;
- $c := min_{I\subset[n]\setminus\{p,q\}} \kappa_{I\cup\{p+q\}} + r - 1 - \sum_{i\in I} d_i$.

Recall that $n = |V|$.

From the definition of r-bizonotopal algebras we immediately get that the monomial

$$\widetilde{m} = m v_p^{d_p} v_q^{d_q}$$

does not vanish in $\mathfrak{BZ}^r_G$ if and only if $d_p \le a$, $d_q \le b$, and $d_p + d_q \le c$. Similarly $\widetilde{m}$ does not vanish in $\mathfrak{BZ}^r_{G\setminus e}$ if and only if $d_p \le a-1$, $d_q \le b-1$, and $d_p + d_q \le c-1$, because all the corresponding κ's are by 1 smaller than the original ones. For the third algebra $\mathfrak{BZ}^r_{G/e}$, $\widetilde{m}$ does not vanish if and only if $d_{pq} \le c$, where v_{pq} corresponds to the new vertex obtain by glueing v_p and v_q.

It remains to show that for any $0 \le c' \le c$, the number of integer solutions of the system $0 \le d_p \le a$, $0 \le d_q \le b$, and $d_p + d_q = c'$ is equal the number of solution of the system $0 \le d_p \le a-1$, $0 \le d_q \le b-1$, and $d_p + d_q = c'-1$ increased by 1.

Consider $J_1, J_2 \subset [n] \setminus \{p,q\}$ such that $a = \kappa_{J_1\cup\{p\}} + r - 1 - \sum_{i\in J_1} d_i$ and $b = \kappa_{J_2\cup\{q\}} + r - 1 - \sum_{i\in J_2} d_i$. We have

$$c' \le c \le \kappa_{J_1\cup J_2\cup\{p,q\}} + r - 1 - \sum_{i\in J_1\cup J_2} d_i = \kappa_{J_1\cup J_2\cup\{p,q\}} + r - 1 - \sum_{i\in J_1} d_i - \sum_{i\in J_2} d_i + \sum_{i\in J_1\cap J_2} d_i =$$

$$= \kappa_{J_1\cup J_2\cup\{p,q\}} + r - 1 - (\kappa_{J_1\cup\{p\}} + r - 1 - a) - (\kappa_{J_2\cup\{q\}} + r - 1 - b) + \sum_{i\in J_1\cap J_2} d_i =$$

$$= a + b + \kappa_{J_1\cup J_2\cup\{p,q\}} - \kappa_{J_1\cup\{p\}} - \kappa_{J_2\cup\{q\}} + \sum_{i\in J_1\cap J_2} d_i - (r-1).$$

Note that if $J_1 \cap J_2 \ne \emptyset$, then $\sum_{i\in J_1\cap J_2} d_i - (r-1) \le \kappa_{J_1\cap J_2}$. If $J_1 \cap J_2 = \emptyset$, then $\sum_{i\in J_1\cap J_2} d_i - (r-1) = -(r-1) = \kappa_\emptyset - (r-1) \le \kappa_\emptyset + 1$. Hence,

$$\begin{aligned} c' \le c \le a + b + \kappa_{J_1\cup J_2\cup\{p,q\}} - \kappa_{J_1\cup\{p\}} - \kappa_{J_2\cup\{q\}} + \kappa_{J_1\cap J_2} + 1 = \\ = a + b + \kappa'_{J_1\cup J_2\cup\{p,q\}} - \kappa'_{J_1\cup\{p\}} - \kappa'_{J_2\cup\{q\}} + \kappa'_{J_1\cap J_2} \le a + b, \end{aligned}$$

where κ' is the corresponding function for $G\setminus e$, which is submodular by Lemma 2.11.

We are ready to count the number of solutions of the system $0 \le d_p \le a$, $0 \le d_q \le b$, and $d_p + d_q = c'$. This number equals to the number of solutions of $0 \le x \le a$, $0 \le c' - x \le b$, hence $max(0, c'-b) \le x \le min(a, c')$. The latter number is $min(a, c') - max(0, c'-b) + 1$ (there is always a solution because $c' \le a + b$). Similarly, the number of solutions of the system $0 \le d_p \le a-1$, $0 \le d_q \le b-1$, and $d_p + d_q = c'-1$ is equal to

$$min(a-1, c'-1) - max(0, (c'-1)-(b-1)) + 1 = min(a, c') - max(0, c'-b)$$

which is one less. This observation concludes our proof. $\square$

Theorem 3.5 has the following consequence for the external and the central cases ($r = 1$ and $r = 0$ resp.)

Theorem 3.6. *For any undirected multigraph G and $r = 0$ or $r = 1$, one has:*
(a) the "deletion-contraction" property:

$$h^r_G(t) = h^r_{G/e}(t) + t \cdot h^r_{G-e}(t),$$

where the edge e of G is not a loop;
(b) the multiplicativity property:

$$h^r_{G_1 \sqcup G_2}(t) = h^r_{G_1}(t) \cdot h^r_{G_2}(t)$$

with the initial conditions:

$$h^r_{B_n}(t) = 1 + t + \cdots + t^{n+r-1} = \frac{1 - t^{n+r}}{1 - t}, \quad n \geq 0,$$

where B_n is the wedge of n loops.

Proof. The first part we already proved for all $r \geq 0$. It is easy to see that the central and the external algebras enjoy the multiplicative property as well. □

3.2. **4-term relation.** In the internal and the subinternal cases, the Hilbert series do not satisfy the above "deletion-contraction" property. However, they do satisfy the following intriguing 4-term relation which we use later to inductively calculate them.

Theorem 3.7. *Given a graph G, take $r \leq 0$ such that $\min_{v \in V} \deg(v) + r \geq 0$. Let $v_p, v_q, v_s \in V(G)$ be fixed distinct vertices such that $deg(v_s) \geq -r - 1$. Then*

$$h^r_{G_1}(t) - h^r_{G_2}(t) = t(h^r_{G_3}(t) - h^r_{G_4}(t)),$$

where $G_1 = G + (v_p, v_s) + (v_p, v_q)$, $G_2 = G + (v_q, v_s) + (v_p, v_q)$, $G_3 = G + (v_p, v_s)$, and $G_4 = G + (v_q, v_s)$.

Proof. The degrees of all vertices in G_i are at least $-r$ implying that all 4 ideals are well-defined and therefore the algebras are well-defined as well.

Let us fix the degrees d_i of all $v_i \in V \setminus \{v_p, v_q\}$. We are interested only in the monomials $m = \prod_{v_i \in V \setminus \{v_p, v_q\}} v_i^{d_i}$ which do not vanish in $\mathfrak{B}\mathfrak{Z}^r_{G_i}$, for some $i = \{1, 2, 3, 4\}$. It is easy to see that if m does not vanish in G_i, then it does not vanish in all 4 graphs. For the rest of the proof let us assume that m does not vanish in all 4 algebras.

Since the 4 graphs are very similar there are integers a, b, c such that

- $m v_p^{d_p} v_q^{d_q}$ does not vanish in G_1 if and only if
$$d_p + d_q \leq c + 1,\ d_p \leq a + 2 \text{ and } d_q \leq b + 1;$$
- $m v_p^{d_p} v_q^{d_q}$ does not vanish in G_2 if and only if
$$d_p + d_q \leq c + 1,\ d_p \leq a + 1 \text{ and } d_q \leq b + 2;$$
- $m v_p^{d_p} v_q^{d_q}$ does not vanish in G_1 if and only if
$$d_p + d_q \leq c,\ d_p \leq a + 1 \text{ and } d_q \leq b;$$
- $m v_p^{d_p} v_q^{d_q}$ does not vanish in G_2 if and only if
$$d_p + d_q \leq c,\ d_p \leq a \text{ and } d_q \leq b + 1.$$

We want to compare the differences $h^r_{G_1}(t) - h^r_{G_2}(t)$ and $t(h^r_{G_3}(t) - h^r_{G_4}(t))$ both of which which count some monomials. We can fix degrees of monomials and then it remains to prove that

$$\#(3.1) - \#(3.2) = \#(3.3) - \#(3.4),$$

where # is the number of non-negative integer solutions of the respective inequality

$$d_p + d_q = c' + 1,\ d_p \leq a + 2 \text{ and } d_q \leq b + 1; \tag{3.1}$$

$$d_p + d_q = c' + 1,\ d_p \leq a + 1 \text{ and } d_q \leq b + 2; \tag{3.2}$$

$$d_p + d_q = c',\ d_p \leq a + 1 \text{ and } d_q \leq b; \tag{3.3}$$

$$d_p + d_q = c',\ d_p \leq a \text{ and } d_q \leq b + 1. \tag{3.4}$$

Note that inequalities (3.1) and (3.2) have many common solutions. Similarly, (3.3) and (3.4) also have many common solutions. Hence, it is enough to show that

$$\#(3.6) - \#(3.7) = \#(3.8) - \#(3.9) \tag{3.5}$$

for the inequalities

$$d_p + d_q = c' + 1,\ d_p = a + 2 \text{ and } d_q \leq b + 1; \tag{3.6}$$

$$d_p + d_q = c' + 1,\ d_p \leq a + 1 \text{ and } d_q = b + 2; \tag{3.7}$$

$$d_p + d_q = c',\ d_p = a + 1 \text{ and } d_q \leq b; \tag{3.8}$$

$$d_p + d_q = c',\ d_p \leq a \text{ and } d_q = b + 1. \tag{3.9}$$

The difference $\#(3.6) - \#(3.8)$ counts the number of solutions of

$$d_p + d_q = c' + 1,\ d_p = a + 2 \text{ and } d_q = b + 1,$$

which is either one or zero. Similarly, the difference $\#(3.7) - \#(3.9)$ counts the number of solutions of

$$d_p + d_q = c' + 1,\ d_p = a + 1 \text{ and } d_q = b + 2,$$

which is either one or zero as well. Furthermore, these two differences are either both one or zero. Therefore, (3.5) holds which finishes the proof. $\square$

Using the 4-term relation, one can inductively calculate the Hilbert series of (sub)internal algebras for all graphs, but to do this we need to study the boundary cases in which the 4-term relation does not apply. They are given by the following simple lemmas and a proposition whose proofs we leave to a reader.

Lemma 3.8. *Given a graph G and $r \leq 0$ such that $-r$ equals the minimal degree of G, one has $h^r_G(t) \equiv 0$.*

Lemma 3.9. *Given a graph G, a negative integer r, and a vertex v_u such that $\deg(v_u) = -r+1$, one has $h^r_G(t) = h^r_{G'}(t)$, where G' is obtained from G by removing the vertex v_u and making all edges connecting $V(G) \setminus v_u$ and v_u into loops in G'.*

Lemma 3.10. *Given a graph G with a vertex v which has exactly one non-loop edge, one has*

$$h^r_G = (1 + t + t^2 + \ldots + t^{deg(v)-r-1})h^r_{G'},$$

where G' is obtained from G by removing the vertex v and adding an extra loop at the vertex connected to v by the latter edge.

Finally, we need to compute the Hilbert series of the (sub)interior algebras for the graphs with one and two vertices.

Proposition 3.11. *Given $r \leq 0$ and the graph G having a single vertex and $a > -r$ loops, one has that $h^r_G(t) = 1 + t + t^2 + \ldots + t^{a+r-1}$.*

Given $r \leq 0$ and the graph G with two vertices u, v having a loops at u, b loops at v, and c edges connecting u and v with $a + c, b + c > -r$, one has

$$h^r_G(t) = (1 + t + t^2 + \ldots + t^{a+c+r-1})(1 + t + t^2 + \ldots + t^{b+c+r-1})$$

truncated at the degree $t^{a+b+c-r}$.

Now we are ready to present a procedure calculating the Hilbert series of (sub)internal algebras of graphs using the above 4-term relation. Let us consider the lexicographic partial order of all graphs according to the triple $(|V|, |E|, min_{v\in V}deg(v))$. We want to calculate the Hilbert series of a given graph G under the assumption that we already know them for all graphs which are smaller than G in the latter partial order.

Algorithm calculating the Hilbert series of (sub)internal algebras

- If $min_{v\in V}deg(v) = -r$ or $-r+1$, see Lemma 3.8 and Lemma 3.9;
- If G is disconnected, then its Hilbert series is the product of Hilbert series of algebras corresponding to each connected component;
- If G is connected, $min_{v\in V}deg(v) \geq -r+2$ and $V(G) \leq 2$, see Proposition 3.11;
- If G is connected, $min_{v\in V}deg(v) \geq -r+2$ and $V(G) \geq 3$, then choose its vertex v_p of the smallest degree. We have two cases:
  Case 1°: v_p has (at least) two distinct neighbors v_q, v_s. Set $G' = G - (v_p, v_s) - (v_p, v_q)$, $G_1 = G$, $G_2 = G' + (v_q, v_s) + (v_p, v_q)$, $G_3 = G' + (v_p, v_s)$, and $G_4 = G' + (v_q, v_s)$. By Theorem 3.7
  $$h^r_G(t) = h^r_{G_2}(t) + t(h^r_{G_3}(t) - h^r_{G_4}(t)).$$
  Additionally, the smallest degree in G_2 is smaller than in G and G_3, G_4 have less edges; their minimal degrees are also smaller than in G.
  Case 2°: v_p has only one neighbor v_q. If there is exactly one edge between v_p and v_q we can use Lemma 3.10 .
  Assume that there are at least two edges (v_p, v_q). Since $V(G) \geq 3$ and G is connected, there is vertex $v_{p'} \neq v_p, v_q$, which is neighbor of v_q. Set $G' = G - (v_p, v_q) - (v_{p'}, v_q)$, $G_1 = G' + (v_{p'}, v_q) + (v_{p'}, v_p)$, $G_2 = G$, $G_3 = G' + (v_{p'}, v_p)$, and $G_4 = G' + (v_q, v_p)$. By Theorem 3.7,
  $$h^r_G(t) = h^r_{G_1}(t) + t(h^r_{G_3}(t) - h^r_{G_4}(t)).$$
  The graphs G_3, G_4 have less edges and G_1 has the same number of edges while the degree of vertex v_p remains the same. Then either the minimal degree in G_1 is smaller than that in G or we can calculate $h^r_{G_1}(t)$ using Case 2°. Therefore we can compute $h^r_{G_1}(t) + t(h^r_{G_3}(t) - h^r_{G_4}(t))$ in all situations and therefore obtain $h^r_G(t)$.

4. Additional results for central and internal bizonotopal algebras

4.1. **Central case.** We present more information about the central bizonotopal algebra $\mathfrak{BZ}^c_G := \mathfrak{BZ}^0_G$. We start with the definition of $\mathfrak{BZ}^c_G$ as a subalgebra similar to that of $\mathfrak{BZ}^e_G$. Let us define the *central double edge algebra* $\mathcal{DE}^c_G$ as

$$\mathcal{DE}^c_G = \frac{\mathbb{K}[DE]}{< x_e^2, x_{\tilde{e}}^2, \cup_{I\subset\{V(G)\}}\mathcal{X}_I >}.$$

Here $\mathcal{X}_I$ is the set of monomials $m = \prod x_e$ such that if edge e has both ends in I, then either x_e or $x_{\tilde{e}}$ belongs to m and if it has exactly one end in I, then we choose an ingoing edge to I. Note that degrees of all monomials in X_I equal to κ_I and the total number of such monomials is 2^μ where μ is the number of edges between vertices in I.

Theorem 4.1. *For any undirected graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, $\mathfrak{BZ}^c_G$ is isomorphic to the subalgebra of $\mathcal{DE}^c_G$ generated by the linear forms*

$$y_\ell = \sum_{i<\ell, e=(i,\ell)\in E} x_e + \sum_{j>\ell, \tilde{e}=(j,\ell)\in \tilde{E}} x_{\tilde{e}}.$$

Proof. Arguing as in Part (i) of Lemma 2.5, we get that the subalgebra of $\mathcal{DE}^c_G$ generated by $y_1, \dots, y_n$ is monomial. We need to check that $y_1^{a_1} \cdots y_n^{a_n} = 0$ in $\mathcal{DE}^c_G$ if and only if $v_1^{a_1} \cdots v_n^{a_n} = 0$ in $\mathfrak{BZ}^c_G$.

Clearly for any $I = \{i_1, \dots, i_\ell\}$, we have that for all $k_1 + k_2 + \dots + k_\ell = \kappa_I$,

$$y_{i_1}^{k_1} y_{i_2}^{k_2} \dots y_{i_\ell}^{k_\ell} = 0 \text{ in } \mathcal{DE}^c_G$$

which are exactly the set of all relations in the algebra $\mathfrak{BZ}^c_G$.

It remains to prove the converse. If $y_1^{a_1} \cdots y_n^{a_n} = 0$ in $\mathcal{DE}^e_G$, then by Theorem 2.7, we know that the similar holds for $v_1, \dots, v_n$. Further, assume that $y_1^{a_1} \cdots y_n^{a_n} = 0$ in $\mathcal{DE}^c_G$, but not in $\mathcal{DE}^e_G$. Hence, there is a monomial m in $y_1^{a_1} \cdots y_n^{a_n}$ in expression through $x_e, x_{\tilde{e}}, e \in E$ that vanishes in $\mathcal{DE}^c_G$, but not in $\mathcal{DE}^e_G$. Therefore it is divisible by some $m' \in \mathcal{X}_I$ for some I. We immediately get $\sum_{i \in I} a_i \geq \kappa_I$ and, hence, $v_1^{a_1} \cdots v_n^{a_n} = 0$ in $\mathfrak{BZ}^c_G$. □

Theorem 4.2. *For a connected graph G, the Hilbert series*

$$h^c_G(t) := \sum_{k \geq 0} \dim(\mathfrak{BZ}^c_G)^{(k)} \cdot t^k$$

has the following properties:

(1) *it is a polynomial of degree $|E| - 1$ where $|E|$ is the total number of edges in G;*

(2) $\dim(\mathfrak{BZ}^e_G)^{(|E|-1)}$ *equals the number of spanning trees of G.*

Proof. The first part is trivial. The second part follows from Theorem 3.6, because the number of trees satisfies both deletion-contraction and the multiplicative properties. □

4.2. **Internal case.** By definition, internal bizonotopal algebra $\mathfrak{BZ}^i_G$ equals $\mathfrak{BZ}^r_G$ for $r = -1$ with the additional restriction that G has no isolated vertices. We define $\mathcal{DE}^i_G$ in a similar way to $\mathcal{DE}^c_G$, namely

$$\mathcal{DE}^i_G = \frac{\mathbb{K}[DE]}{< x_e^2, x_{\tilde{e}}^2, \cup_{I \subset \{V(G)\}} \mathcal{X}_I^- >}.$$

Here $\mathcal{X}_I^-$ is the set of monomials m such that there is x_e such that $x_e m \in \mathcal{X}_I$. Clearly the degrees of all monomials in $\mathcal{X}_I^-$ are equal to $\kappa_I - 1$.

Theorem 4.3. *For any undirected graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ having no isolated vertices, $\mathfrak{BZ}^i_G$ is isomorphic to the subalgebra of $\mathcal{DE}^i_G$ generated by the linear forms*

$$y_\ell = \sum_{i<\ell, e=(i,\ell) \in E} x_e + \sum_{j>\ell, \tilde{e}=(j,\ell) \in \tilde{E}} x_{\tilde{e}}.$$

The proof copies that of Theorem 4.1.

Theorem 4.4. *For the complete graph K_n on $n \geq 4$ vertices,*

$$\deg \mathfrak{BZ}^i_{K_n} = \binom{n}{2} - 2, \quad \max\dim \mathfrak{BZ}^i_{K_n} = \binom{n-2}{2} n^{n-4},$$

where max dim *stands for the dimension of the algebra in its top degree.*

Proof. Since we have $\binom{n}{2}$ edges, any monomial $v_1^{a_1} v_2^{a_2} \dots v_n^{a_n}$ of degree $\binom{n}{2} - 1$ vanishes in $\mathfrak{BZ}^i_{K_n}$. Hence, it is enough to show that the dimension of $(\binom{n}{2} - 2)$-nd graded component is $\binom{n-2}{2} n^{n-4}$. This calculation will be done after introducing several combinatorial notions and Lemma 4.5. □

Let us introduce two sets X and Y of vectors with integer coordinates. The element $(b_1, b_2, \dots, b_n) \in \mathbb{Z}^n$ belongs to X if and only if the following conditions hold:

- $\sum_{i\in[n]} b_i = \binom{n}{2} - 2$;
- $\sum_{i\in I} b_i \le \kappa_I - 2 = \binom{|I|}{2} + |I|(n-|I|) - 2$, for all I.

The element $(b'_1, b'_2, \dots, b'_n) \in \mathbb{Z}^n$ belongs to Y if and only if the following conditions hold:

- $\sum_{i\in[n]} b'_i = \binom{n}{2} - 1$;
- $\sum_{i\in I} b'_i \le \kappa_I - 1 = \binom{|I|}{2} + |I|(n-|I|) - 1$, for all I.

By the definition of the internal bizonotopal algebra the dimension of $(\binom{n}{2} - 2)$-nd graded component of this algebra is equal to $|X|$ which we count below. We also know that $|Y|$ is equal to the number of trees in G, see e.g. Theorem 4.2. Hence, $|Y| = n^{n-2}$. Since if $(b_1, \dots, b_{n-1}, b_n) \in X$, then $(b_1, \dots, b_{n-1}, b_n+1) \in Y$, we have

$$|X| = |Y| - \#\{(b'_1, b'_2, \dots, b'_n) \in Y : \ b'_n = 0\} -$$

$$-\#\{(b'_1, b'_2, \dots, b'_n) \in Y : \ b'_n > 0 \text{ and } \sum_{j\in J} b'_j = \kappa_J - 1 \text{ for some } J \subset [n-1]\}$$

$$= |Y| - \#\{(b'_1, b'_2, \dots, b'_n) \in Y : \ \sum_{j\in J} b'_j = \kappa_J - 1 \text{ for some } J \subset [n-1]\}.$$

Denote by Z the set appearing in the latter part of the previous formula, i.e. define

$$Z := \#\{(b'_1, b'_2, \dots, b'_n) \in Y : \text{ and } \sum_{j\in J} b'_j = \kappa_J - 1 \text{ for some } J \subset [n-1]\}.$$

Note that for any element $(b'_1, b'_2, \dots, b'_n) \in Z$, there is unique maximal $J \subset [n-1]$ such that $\sum_{j\in J} b'_j = \kappa_J - 1$. Indeed assume by contradiction, that there exist two sets $J_1, J_2 \subset [n-1]$ such that $\sum_{j\in J_1} b'_j = \kappa_{J_1} - 1$ and $\sum_{j\in J_2} b'_j = \kappa_{J_2} - 1$. We have

$$\sum_{j\in J_1\cup J_2} b'_j = \sum_{j\in J_1} b'_j + \sum_{j\in J_2} b'_j - \sum_{j\in J_1\cap J_2} b'_j \ge \kappa_{J_1} - 1 + \kappa_{J_2} - 1 - (\kappa_{J_1\cap J_2} - 1) \ge \kappa_{J_1\cup J_2} - 1.$$

If the intersection $J_1 \cap J_2$ is empty then

$$\sum_{j\in J_1\cup J_2} b'_j = \sum_{j\in J_1} b'_j + \sum_{j\in J_2} b'_j = \kappa_{J_1} - 1 + \kappa_{J_2} - 1 \ge \kappa_{J_1\cup J_2} - 1.$$

We get that the union $J_1 \cup J_2$ also satisfies our property, i.e., neither J_1 nor J_2 are maximal by inclusion contradicting our assumption.

Therefore $Z = \sqcup_{I\subset[n-1]} Z_I$, where

$$Z_I := \{(b'_1, b'_2, \dots, b'_n) \in Z : \sum_{j\in I} b'_I = \kappa_I - 1 \text{ and } \sum_{j\in J} b'_J < \kappa_J - 1\ \forall J \subset [n-1] \text{ s.t. } I \subsetneq J\}.$$

The above leads to the following count

$$|X| = n^{n-2} - \sum_{\emptyset\ne I\subset[n-1]} |Z_I|.$$

Lemma 4.5. *For any $J \subset [n-1]$, we have*

$$|Z_J| = |J|^{|J|-2}(n-|J|)^{(n-|J|)-2}.$$

Observe that the right-hand side equals the product of the number of spanning trees in $K_{|J|}$ and the number of spanning trees in $K_{[n]\setminus|J|}$.

Proof. Set $J = \{j_1 < j_2 < \ldots < j_\ell\}$ and $[n] \setminus J = \{i_1 < i_2 < \ldots < i_{n-\ell}\}$.

Consider the maps $\phi_1 : \mathbb{Z}^n \to \mathbb{Z}^\ell$ and $\phi_2 : \mathbb{Z}^n \to \mathbb{Z}^{n-\ell}$ defined by
$$\phi_1(b) := (b_{j_1} - n + \ell, b_{j_2} - n + \ell, \ldots, , b_{j_\ell} - n + \ell),$$
and
$$\phi_2(b) := (b_{i_1}, b_{i_2}, \ldots, b_{i_{n-\ell-1}}, b_{i_{n-\ell}} - 1).$$
We want to show that $b \in Z_{n,J}$ if and only if $\phi_1(b) \in Y_\ell$ and $\phi_2(b) \in Y_{n-\ell}$.

Assume $b \in Z_{n,J}$, then for any $I \subset [\ell]$, we have
$$\sum_{t\in I} \phi_1(b)_t = \sum_{t\in I} b_{j_t} - n + \ell = \sum_{t\in I}(b_{j_t} - n + \ell) = \sum t \in I b_{j_t} + |I|(-n + \ell) \leq$$
$$\kappa_I - 1 + |I|(-n+\ell) = \binom{|I|}{2} + |I|(n-|I|) + |I|(-n+\ell) - 1 = \binom{|I|}{2} + |I|(\ell - |I|) - 1.$$
Hence $\phi_1(b) \in Y_\ell$.

For any subset $I \subset [n - \ell - 1]$, we have
$$\sum_{t\in I} \phi_2(b)_t = \sum_{j\in\{i_t : t\in I\}} b_j = \sum_{j\in\{i_t, t\in I\}\sqcup J} b_j - \sum_{j\in J} b_j =$$
$$= \sum_{j\in\{i_t,t\in I\}\sqcup J} b_j - (\kappa_J - 1) < (\kappa_{\{i_t,t\in I\}\sqcup J} - 1) - (\kappa_J - 1) = \binom{|I|}{2} + |I|(n - \ell - |I|)$$
and for all $I \subset [n - \ell]$ s.t. $(n - \ell) \in I$, we have
$$\sum_{t\in I} \phi_2(b)_t = \sum_{j\in\{i_t : t\in I\}} b_j - 1 = \sum_{j\in\{i_t,t\in I\}\sqcup J} b_j - \sum_{j\in J} b_j - 1 =$$
$$= \sum_{j\in\{i_t,t\in I\}\sqcup J} b_j - \kappa_J \leq \kappa_{\{i_t,t\in I\}\sqcup J} - \kappa_J - 1 = \binom{|I|}{2} + |I|(n - \ell - |I|) - 1.$$
Hence, $\phi_2(b) \in Y_{n-\ell}$.

Let us prove the converse. Assume that $\phi_1(b) \in Y_\ell$ and $\phi_2(b) \in Y_{n-\ell}$. We need to show that for any $I \subset [n]$, one has $\sum_{i\in I} b_i \leq \binom{|I|}{2} + |I|(n - |I|) - 1$. We will split the situation into several subcases.

Case 1: $I \cap J = \emptyset$. Since $\phi_2(b) \in Y_{n-\ell}$, we have
$$\sum_{i\in I} b_i \leq \binom{|I|}{2} + |I|(n-\ell-|I|) - 1 + 1 = \binom{|I|}{2} + |I|(n-|I|) - n\ell \leq \binom{|I|}{2} + |I|(n-|I|) - 1.$$
Case 2: $I \cap ([n] \setminus J) = \emptyset$. Since $\phi_1(b) \in Y_\ell$, we have
$$\sum_{i\in I} b_i = \sum_{i\in I}(b_i - n + \ell) + |I|(n-\ell) \leq \binom{|I|}{2} + |I|(\ell-|I|) - 1 + |I|(n-\ell) = \binom{|I|}{2} + |I|(n-|I|) - 1.$$
Case 3: $I \cap J \neq \emptyset$, $I \cap ([n] \setminus J) \neq \emptyset$. We have
$$\sum_{i\in I} b_i = \sum_{i\in I\cap J} b_i + \sum_{i\in I\cap([n]\setminus J)} b_i \leq \binom{|I\cap J|}{2} + |I \cap J|(n - |I \cap J|) - 1+$$
$$+\binom{|I \cap ([n] \setminus J)|}{2} + |I \cap ([n] \setminus J)|(n - \ell - |I \cap ([n] \setminus J)|) - 1 + 1 =$$
$$= \binom{|I|}{2} + |I|(n - |I|) - 1.$$
We also need to check that for $I = J$, we have the non-strict inequality, i.e., $\sum_{i\in J} b_i \leq \binom{|J|}{2} + |J|(n - |J|) - 1$. It follows from the second case.

It remains to check that for all I s.t. $J \subsetneq I \subset [n-1]$, we have the strict inequality. Indeed this happens in the third case and we do not use b_n, so we have same inequality, but without "+1".

We get that $b \in Z_J$ if and only if $\phi_1(b) \in Y_\ell$ and $\phi_1(b) \in Y_{n-\ell}$. Since $\phi_1 \otimes \phi_2 : \mathbb{Z}^n \to \mathbb{Z}^n$ is a bijection, the number of elements of Y_ℓ is the number of trees in K_ℓ and the number of elements of $Y_{n-\ell}$ is the number of trees in $K_{n-\ell}$. Therefore $|Z_J|$ is equal to $|J|^{|J|-2}(n-|J|)^{(n-|J|)-2}$. □

Proving the second part of Theorem 4.4. Since Z_J is equal to the number of trees in K_J times the number of trees in $K_{[n]\setminus J}$, we have $\sum_{\emptyset \neq I \subset [n-1]} |Z_I|$ is equal to the number of forests in K_n with exactly $n-2$ edges (two connected components). This number is equal to $n^{n-4}\frac{(n-1)(n+6)}{2}$, see OEIS A083483. We get

$$|X| = n^{n-2} - n^{n-4}\frac{(n-1)(n+6)}{2} = n^{n-4}\frac{(n-2)(n-3)}{2}.$$

□

Theorem 4.6. *The Hilbert series of the internal bizonotopal algebra of any* 3*-regular graph on* $n \geq 4$ *vertices equals* $(1+t)^n$.

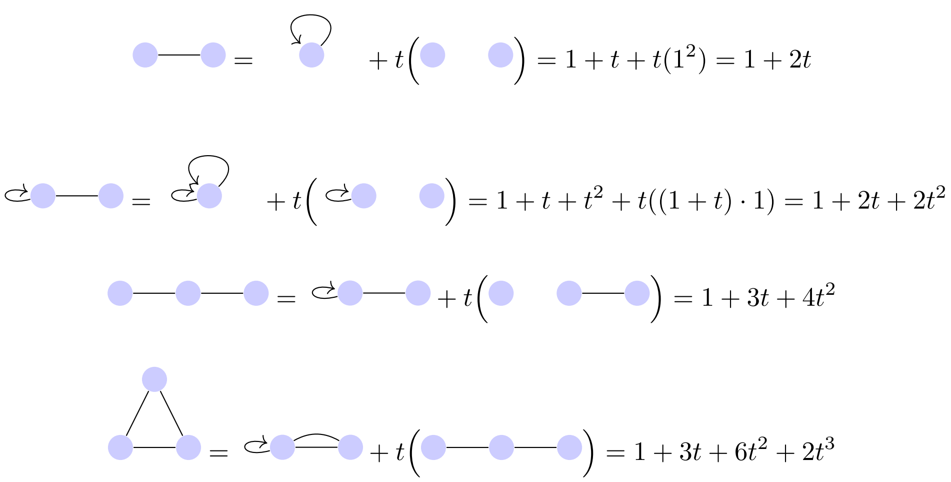
The Hilbert series of the internal bizonotopal algebra of any 4*-regular* 4*-edge-connected graph on* $n \geq 4$ *vertices equals* $(1+t+t^2)^n - nt^{2n-1} - t^{2n}$.

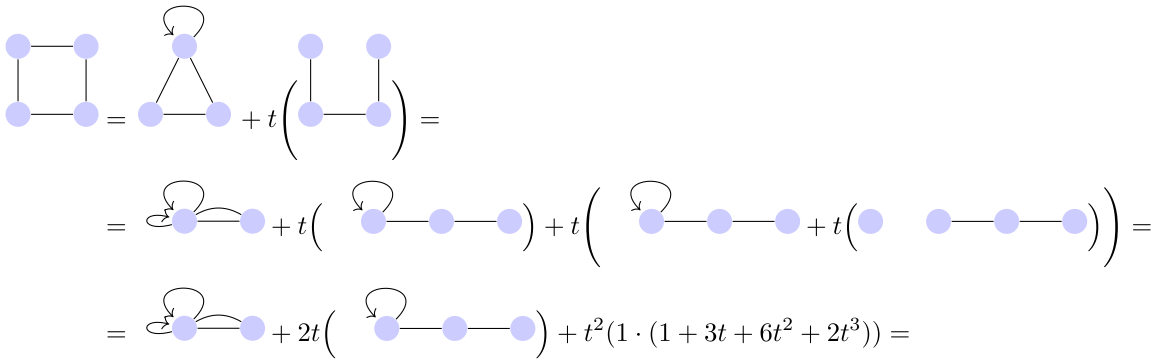
Proof. Take a 3-regular graph G on $n \geq 4$ vertices. Since $\kappa_{\{i\}} = 3$, we have $v_i^2 = 0$ in $\mathfrak{B}\mathfrak{Z}_G^i$. It is also easy to see that $v_1 v_2 \cdots v_n$ does not vanish in $\mathfrak{B}\mathfrak{Z}_G^i$, because for any subset of vertices, $\kappa_I - 1 \geq \frac{3}{2}|I| - 1 \geq |I| + 1$. Hence, we do not have square-free relations. Therefore $h_G^i(t) = (1+t)^n$.

Now take a 4-regular connected graph G on n vertices. Since $\kappa_{\{i\}} = 4$, we have $v_i^3 = 0$ in $\mathfrak{B}\mathfrak{Z}_G^i$. We also have $v_1^2 v_2^2 \cdots v_{i-1}^2 v_i v_{i+1}^2 \cdots v_n^2 = 0$ in $\mathfrak{B}\mathfrak{Z}_G^i$, because $\kappa_{[n]} = 2n$. For any i, the product $v_1^2 v_2^2 \cdots v_{i-1}^2 v_{i+1}^2 \cdots v_n^2$ does not vanish in $\mathfrak{B}\mathfrak{Z}_G^i$, because for any proper subset of vertices $\kappa_I - 1 \geq \frac{4}{2}(|I| - 4) + 4 - 1 = 2|I| + 1$. Therefore $h_G^i(t) = (1+t+t^2)^n - nt^{2n-1} - t^{2n}$. □

5. Appendix. Numerical results.

5.1. **Examples of calculation of Hilbert series and dimensions for bizonotopal algebras via recurrence relation.**





$$\ldots = 1 + 4t + 10t^2 + 14t^3 + 15t^4$$

5.2. Hilbert series and dimensions for bizonotopal algebras of complete graphs obtained using computer algebra.

External algebras (K_2–K_9)

1, 2;

1, 3, 6, 7;

1, 4, 10, 20, 31, 40, 38;

1, 5, 15, 35, 70, 121, 185, 255, 310, 335, 291;

1, 6, 21, 56, 126, 252, 456, 756, 1161, 1666, 2232, 2796, 3281, 3546, 3516, 2932;

1, 7, 28, 84, 210, 462, 924, 1709, 2954, 4809, 7420, 10906, 15309, 20559, 26454, 32655, 38591, 43589, 46984, 47649, 45150, 36961;

1, 8, 36, 120, 330, 792, 1716, 3432, 6427, 11376, 19160, 30864, 47748, 71184, 102524, 142920, 193117, 253240, 322596, 399344, 480390, 561472, 637400, 701296, 746089, 765640, 748532, 691720, 561948;

1, 9, 45, 165, 495, 1287, 3003, 6435, 12870, 24301, 43677, 75177, 124485, 199035, 308187, 463287, 677520, 965493, 1342513, 1823553, 2421927, 3147723, 4005819, 4993839, 6100350, 7303545, 8570601, 9855829, 11101599, 12241305, 13203705, 13902291, 14254524, 14195199, 13575951, 12369033, 10026505.

Dimensions: $3, 17, 144, 1623, 22804, 383415, 7501422, 167341283$ resp.

Central algebras (K_2–K_{10})

1;

1, 3, 3;

1, 4, 10, 16, 19, 16;

1, 5, 15, 35, 65, 101, 135, 155, 155, 125;

1, 6, 21, 56, 126, 246, 426, 666, 951, 1246, 1506, 1686, 1731, 1626, 1296;

1, 7, 28, 84, 210, 462, 917, 1667, 2807, 4417, 6538, 9142, 12117, 15267, 18327, 20958, 22827, 23667, 23107, 21112, 16807;

1, 8, 36, 120, 330, 792, 1716, 3424, 6371, 11152, 18488, 29184, 44052, 63792, 88852, 119288, 154645, 193880, 235292, 276592, 315078, 347880, 371820, 384112, 382817, 364232, 328392, 262144;

1, 9, 45, 165, 495, 1287, 3003, 6435, 12861, 24229, 43353, 74097, 121515, 191907, 292743, 432399, 619677, 863109, 1170073, 1545777, 1992195, 2506983, 3082599, 3705795, 4357593, 5013801, 5645313, 6219649, 6703245, 7064073, 7267815, 7285959, 7100739, 6660495, 5966613, 4782969;

1, 10, 55, 220, 715, 2002, 5005, 11440, 24310, 48610, 92278, 167410, 291730, 490270, 797170, 1257454, 1928575, 2881450, 4200670, 5983570, 8337880, 11377750, 15218050, 19966990, 25717165, 32535466, 40452550, 49452730, 59465230, 70357750, 81931942, 93922750, 106002685, 117791350, 128869900, 138786766, 147077890, 153294730, 157034680, 157852210, 155381665, 149411470, 139011220, 124170310, 100000000.

Dimensions: $1, 7, 66, 792, 11590, 200469, 90759016, 2301604074$ resp.

Internal algebras (K_2–K_{10}):

0;

1;

1, 4, 6, 4, 1;

1, 5, 15, 30, 45, 51, 45, 30, 15;

1, 6, 21, 56, 120, 216, 336, 456, 546, 580, 546, 456, 336, 216;

1, 7, 28, 84, 210, 455, 875, 1520, 2415, 3535, 4795, 6055, 7140, 7875, 8135, 7875, 7140, 6055, 4795, 3430;

1, 8, 36, 120, 330, 792, 1708, 3368, 6147, 10480, 16808, 25488, 36688, 50288, 65808, 82384, 98813, 113688, 125588, 133288, 135954, 133288, 125588, 113688, 98533, 81488, 61440;

1, 9, 45, 165, 495, 1287, 3003, 6426, 12789, 23905, 42273, 71127, 114387, 176463, 261891, 374808, 518301, 693693, 899857, 1132677, 1384803, 1645791, 1902663, 2140866, 2345553, 2503053, 2602341, 2636263, 2602341, 2502423, 2342907, 2134062, 1881243, 1596861, 1240029;

1, 10, 55, 220, 715, 2002, 5005, 11440, 24300, 48520, 91828, 165760, 286780, 477400, 767140, 1193104, 1799920, 2638800, 3765520, 5237200, 7107880, 9423040, 12213400, 15488560, 19231180, 23392456, 27889620, 32606080, 37394620, 42083800, 46487332, 50415760, 53689450, 56151700, 57679450, 58192656, 57660550, 56103820, 53573530, 50159560, 45988330, 41027500, 35399710, 28000000.

Dimensions: $1, 16, 237, 3892, 72425, 1521810, 35794801, 933875704$ resp.

References

[AmWa] A. Amanbayeva and D. Wang, The convex hull of parking functions of length n, Enumer. Comb. Appl. 2 (2022), no. 2, Paper No. S2R10, 10.

[ArPo] F. Ardila, A. Postnikov, Combinatorics and geometry of power ideals. Trans. Amer. Math. Soc.362(2010), no.8, 4357–4384.

[Ar] V. I. Arnold, Remarks on eigenvalues and eigenvectors of Hermitian matrices, Berry phase, adiabatic connections and quantum Hall effect, Selecta Math. 1 (1995), 1–19.

[BZ] J. Bell and J. Zhang, An isomorphism lemma for graded rings, Proceedings of the American Mathematical Society 145 (2017), 989–994.

[CGW] C.R. Coullard, J.G. del Greco and D.K. Wagner, Representations of bicircular matroids, Discr. Appl. Math. 32 (1991), 223–240.

[Do] M. Doob, An interrelation between line graphs, eigenvalues, and matroids, J. Combin. Theory Ser. B 15 (1973), 40–50.

[Dw] T.A. Dowling, A class of geometric lattices based on finite groups, J. Combin. Theory, Ser. B 14 (1973), 61–86.

[GM] M. Gondran and N. Minoux, Graphs and Algorithms, Wiley, 1984.

[Gr] J. Gross, Voltage graphs, Discr. Math. 9 (1974), 239–246.

[HLVM] M. Hanada, J. Lentrfer, and A. R. Vindas-Meléndez, Generalized parking function polytopes, Arxiv: 2212.06885.

[HoRo] O. Holtz, A. Ron, Zonotopal algebra, Adv. Math. 227 (2011), no. 2, 847–894.

[Ki] A. Kirillov, On some quadratic algebras II, in preparation.

[KNSV] A. Kirillov, G. Nenashev, B. Shapiro, A. Vaintrob, Introducing loopy polynomial of an undirected graph, in preparation.

[KlWi] D. J. Kleitman and K. J. Winston, Combinatorica 1 (1981), 49–54.

[Ma] L. R. Matthews, Bicircular matroids, Quart. J. Math. Oxford Ser. (2) 28 (1977), 213–228.

[Ne] G. Nenashev, Classification of external zonotopal algebras, Electron. J. Combin. 26 (2019), Pap. 1.32, 10 pp.

[NeSh] G. Nenashev, B. Shapiro, "K-theoretic" analogs of Postnikov-Shapiro algebra distinguishes graphs, Journal of Combinatorial Theory, Series A 148 (2017) 316–332.

[Neu] N. Neudauer, Graph representations of a bicircular matroid, Discr. Appl. Math. 118 (2002), 249–262.

[Ox] J. G. Oxley, Matroid theory, Oxford University Press, 2nd Edigion, 2011.

[PS] A. Postnikov and B. Shapiro, Trees, parking functions, syzygies, and deformations of monomials ideals, Trans. Amer. Math. Soc. 356 (8) (2004), 3109–3142.

[PSS] A. Postnikov, B. Shapiro and M. Shapiro, Algebras of curvature forms on homogeneous manifolds. Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., 1999, 227–235.

[SS] B. Shapiro and M. Shapiro, On ring generated by Chern 2-forms on SL_n/B, C. R. Acad. Sci. Paris Sér. I Math. 326(1) (1998), 75–80.

[ShVa] B. Shapiro, A. Vaintrob, On algebras and matroids associated to undirected graphs, arXiv:2005.06160.

[SOSG] R. Shull, J. Orlin, A. Shuchat and M. Gardner, The structure of bases in bicircular matroids, Discr. Appl. Math. 23 (1989), 267–283.

[SP] J.M.S. Simões-Pereira, On matroids on edge sets of graphs with connected subgraphs as circuits II, Discr. Math. 12 (1975), 55–78.

[St] R. Stanley, Problem 12191, Problems and Solutions, Amer. Math. Monthly 127 (2020), no. 6, 563–571.

[StPi] R. Stanley and J. Pitman, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, Discrete Comput. Geom. 27 (2002), no. 4, 603–634.

[Wa1] D.G. Wagner, The Algebra of flows in graphs, Adv. Appl. Math. 21 (1998), 644–684.

[Wa2] D.G. Wagner, Algebras related to matroids represented in characteristic zero, European J. Combin. 20 (1999), 701–711.

[Wa3] D.K. Wagner, Equivalent factor matroids of graphs, Combinatorica 8 (1988), 373–377.

[Za1] T. Zaslavsky, Signed graphs, Discr. Appl. Math. 4 (1982), 47–74.

[Za2] T. Zaslavsky, Biased graphs I: Bias, balance, and gains, J. Combin. Theory Ser. B 47 (1989), 32–52.

Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Huairou District, Beijing, China
Email address: kirillov@bimsa.cn

Department of Mathematics and Computer Science, St. Petersburg State University, St. Petersburg, 199178, Russia
Email address: glebnen@gmail.com

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: shapiro@math.su.se

Department of Mathematics, University of Oregon, Eugene, OR 97403. USA
Email address: vaintrob@uoregon.edu