

On existence of quasi-Strebel structures for meromorphic k -differentials

BORIS SHAPIRO and Guillaume TAHAR

Abstract. In this paper, motivated by the classical notion of a Strebel quadratic differential on a compact Riemann surface without boundary we introduce the notion of a quasi-Strebel structure for a meromorphic differential of an arbitrary order. It turns out that every differential of even order $k \geq 4$ satisfying certain natural conditions at its singular points admits such a structure. The case of differentials of odd order is quite different and our existence result involves some arithmetic conditions. We discuss the set of quasi-Strebel structures associated to a given differential and introduce the subclass of positive k -differentials. Finally, we provide a family of examples of positive rational differentials and explain their connection with the classical Heine-Stieltjes theory of linear differential equations with polynomial coefficients.

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1. Introduction

Definition 1. A (meromorphic) differential Ψ of order $k \geq 2$ on a compact orientable Riemann surface Y without boundary is a (meromorphic) section of the k -th tensor power $(T_{\mathbb{C}}^*Y)^{\otimes k}$ of the holomorphic cotangent bundle $T_{\mathbb{C}}^*Y$. The zeros and the poles of Ψ constitute the set Cr_{Ψ} of *critical points* (alias, singularities) of Ψ .

In what follows, we will use the convention that the order of a zero is a positive integer while the order of a pole is a negative integer. For a differential Ψ of order k given locally by $f(z)dz^k$ in a neighbourhood of a non-critical point, we define k locally distinct direction fields called *horizontal* which are given by the condition that $f(z)dz^k$ is positive. Integral curves of these direction fields are called (horizontal) *trajectories*. Any two consecutive direction fields differ by the angle $\frac{2\pi}{k}$ and for even $k = 2\ell$, they form ℓ locally distinct line fields. However, for a general differential of order $k > 2$, it is impossible to globally distinguish these k direction fields because every two of them are obtained by the analytic continuation of each other. Multiplication

of Ψ by the scalar factor $e^{\frac{\pi i}{k}}$, $i = \sqrt{-1}$ does not change these direction fields and their trajectories.

Extracting the k -th root, one can globally interpret an arbitrary differential of order k as a multi-valued meromorphic abelian differential on Y . Outside of the poles and the zeros of Ψ , it is locally representable as the k -th power of a holomorphic abelian differential ω . Integration of ω in a neighborhood of a point z_0 gives local coordinates whose transition maps are of the type $z \mapsto e^{i\theta}z + c$ where $\theta \in \{0, \frac{2\pi}{k} \dots, (k-1)\frac{2\pi}{k}\}$. In such a way, the Riemann surface Y punctured at its singularities is locally isometric to the complex plane \mathbb{C} endowed with the flat structure induced by dz . In other words, we obtain a flat metric on $Y \setminus Cr_\Psi$ induced by the integration of ω where the distance between two points p_1 and p_2 is $|\int_{p_1}^{p_2} \omega|$, see [3, 23]. (In particular, dz induces the usual Euclidean metric on \mathbb{C} .) The surface $Y \setminus Cr_\Psi$ with the latter flat metric will be called the *flat model* of Ψ .

This flat metric extends from $Y \setminus Cr_\Psi$ to all zeros and poles of order $\ell > -k$; such singularities are called *conical of order ℓ* . Any conical singularity of order ℓ has a neighborhood on Y isometric to a neighborhood of the apex of a flat cone with the total angle $\frac{(\ell+k)2\pi}{k}$. Analogously, a pole of order $\ell \leq -k$ corresponds to the point at ∞ of a flat cone with the total angle $-\frac{(\ell+k)2\pi}{k}$, see Proposition A. By convention, flat cones of angle 0 (i.e., poles of order $-k$) are half-infinite cylinders.

In general, differentials of order $k > 2$ are much less studied than their quadratic counterpart. Nevertheless, they sometimes appear in the literature, see, e.g., [3, 4, 8, 9, 13, 16, 18, 21, 23] and, as we already mentioned, are closely related to the theory of locally flat surfaces which is used in the study of the moduli spaces of curves and dynamics of interval exchange maps, see, e.g., [24]. Moreover, the theory of translation surfaces (which correspond to the abelian differentials) provides new important insights into the dynamics of billiards by means of algebraic geometry and renormalization theory.

Moduli spaces of pairs (Y, Ψ) consisting of a compact Riemann surface Y and a k -differential on it are naturally stratified by the loci where the differentials have the same set of multiplicities of their singular points. For example, in the case of genus zero, a meromorphic differential of order k can be globally represented as $\Psi(z) = R(z)dz^k$, where $R(z)$ is a rational function. So, up to a non-vanishing constant factor, Ψ is completely determined by the multiplicities and the positions of its singularities. In particular, the strata of k -differentials can be identified with certain configuration spaces of divisors on $\mathbb{C}P^1$ and are connected, see [7].

A natural system of coordinates on the latter strata is given by the periods of $\sqrt[k]{\Psi}$ with respect to the relative homology of Y punctured at all poles of order smaller than or equal to $-k$ taken with respect to the set of all conical singularities (i.e., all

zeros together with all poles of order exceeding $-k$). In the genus zero case, projective automorphisms allow us to assign positions of three singularities to $(0, 1, \infty)$ and the strata of k -differentials with m singularities on the sphere are complex-analytic orbifolds of dimension of $m - 2$. More details can be found in [3].

Local classification of singularities of differentials of order $k > 2$ under the action of the group of local biholomorphisms is quite similar to that of quadratic differentials and can be found in Proposition A below.

The main motivation of this paper is as follows. Recall that a quadratic differential on a compact Riemann surface Y is called *Strebel* if almost all its horizontal trajectories are closed. Such a phenomenon can never happen for a k -differential Ψ of order $k \geq 3$ unless it is a power of a 1-form or a quadratic differential, see Corollary 1 below. Thus for $k \geq 3$, in order to define something similar to a Strebel differential one should instead of using only smooth closed horizontal trajectories of Ψ (which might exist in some domains of Y) allow a certain type of closed broken trajectories and try to cover Y with those. However, to avoid trivialities, one should impose substantial restrictions on the set where broken trajectories can switch from one smooth piece to another. An attempt to develop an appropriate notion is carried out below by introducing quasi-Strebel structures, see Definition 10.

The purpose of this paper is to develop elements of a general theory of k -differentials which as much as possible resembles that of quadratic differentials and to point out the connection of a certain class of k -differentials with the classical Heine–Stieltjes theory. It seems very plausible that special types of k -differentials should naturally appear in the asymptotic analysis of linear ODE of high order, compare, e.g., [1]. In what follows, theorems, conjectures, etc., labeled by letters, are borrowed from the existing literature, while those labeled by numbers are hopefully new.

Our main result (Theorem 1) claims the existence of a quasi-Strebel structure for every meromorphic k -differential with admissible singularities if k is even. A similar statement for odd k , currently requires an additional (and very restrictive) hypothesis of rationality of the periods of a differential. On the other hand, we strongly believe that for k odd, there are arithmetic obstructions to the existence of a quasi-Strebel structure.

The paper is organized as follows. Section 2 contains main definitions and results related to k -differentials and quasi-Strebel structures. Section 3 introduces a special class of positive differentials and provides a short explanation of how such differentials appear in the Heine–Stieltjes theory. Finally, in Section 4 we present a number of relevant open questions.

2. Basic notions and the main theorem

2.1. Trivium on k -differentials. The next definitions are straightforward generalizations of those for quadratic differentials.

Definition 2. The *canonical length element* on Y associated with a k -differential Ψ given locally as $\Psi = f(z)dz^k$ is defined by

$$|dw| = |f(z)|^{\frac{1}{k}}|dz|.$$

(Notice that the associated area element on Y equals $|f(z)|^{\frac{2}{k}}dxdy$ where $z = x + iy$. Both $|dw|$ and the area element are globally well-defined.)

Definition 3. (a) The *norm* of a k -differential $\Psi = f(z)dz^k$ is defined as

$$\iint_{\Gamma} |f(z)|^{\frac{2}{k}}dxdy.$$

(Again $|f(z)|^{\frac{2}{k}}dxdy$ is a globally well-defined 2-form on Y .)

(b) If $\Psi = f(z)dz^k$ is meromorphic, then its singular point p is called *finite or conical* if p has a neighbourhood with a finite associated area.

Remark 1. Finite critical points of Ψ are exactly its zeros and poles of order greater than $-k$ (i.e., conical singularities) and a meromorphic k -differential Ψ has a finite norm if and only if all its critical points are finite.

Definition 4 (cp. Definition 20.1 in [22]). A trajectory of Ψ is called *critical* if it starts or ends at a conical singularity.

Definition 5. The *distinguished* or *natural* parameter W associated with a k -differential $\Psi = f(z)dz^k$ is defined by

$$W = \int \sqrt[k]{f(z)}dz$$

for some branch of the k -th root.

Remark 2. Obviously, a sufficiently small neighbourhood of any regular point p is homeomorphically mapped by a branch of W onto an open set in the W -plane. Notice that $dW = \sqrt[k]{f(z)}dz$ which implies $(dW)^k = f(z)dz^k$.

A standard basic result about k -differentials which can be found in several sources is as follows.

Lemma A. *The Euler characteristic of $(T_{\mathbb{C}}^*Y)^{\otimes k}$ equals $\chi(Y)k$, where $\chi(Y)$ is the Euler characteristic of the underlying curve Y . Therefore, the difference between the number of poles and zeros (counted with multiplicity) of a meromorphic differential Ψ of order k on Y equals $\chi(Y)k$. In particular, the number of poles minus the number of zeros of any rational k -differential Ψ equals $2k$.*

Next, following closely § 6 of Chapter 3 in [22], we provide normal forms for a k -differential near its critical points. (These results can be found in Proposition 3.1 of [3] and were independently obtained about 10 years ago by the first author (unpublished).)

Proposition A. *Given a k -differential Ψ on a sufficiently small disk V centered at 0, denote by m the order $\text{ord}_0\Psi$. Then there exists a conformal map $\xi : (\Delta_R, 0) \rightarrow (V, 0)$ defined on a disk of sufficiently small radius R , and a number $r \in \mathbb{C}$ such that*

$$\xi^*(\Psi) = \begin{cases} z^m dz^k & \text{if } m > -k \text{ or } k \nmid m, \\ \frac{r}{z^k} dz^k & \text{if } m = -k, \\ (z^{m/k} + \frac{s}{z})^k dz^k & \text{if } m < -k \text{ and } k \mid m, \end{cases}$$

where in the last case $s \in \mathbb{C}$ and $s^k = r$. The germ of ξ is unique up to multiplication by a $(m + k)$ -th root of unity when $m > -k$ or $k \nmid m$, and up to multiplication by a non-zero constant if $m = -k$.

The following statement describes possible behavior of non-critical global trajectories, see [23, Proposition 5.5].

Lemma B. *Non-critical trajectories of a k -differential Ψ are of the following 3 types:*

- (i) *closed trajectories (with or without selfintersections);*
- (ii) *trajectories (with or without selfintersections) one or both ends of which approach a pole of order smaller than or equal to $-k$;*
- (iii) *trajectories (with or without selfintersections) which are dense in some polygonal domains in the flat model of Ψ .*

Observe that the analytic continuation of a branch of $\sqrt[k]{\Psi}$ of a k -differential Ψ along some loop on Y may result in a different branch. The ratio between the values of these two branches is a k -th root of unity called the *holonomy* of the loop. This ratio describes the failure of the identification of the circle of directions in the parallel transport along the loop. Observe that in particular, the holonomy of a loop around a singularity of order n depends on the congruence class of n modulo k . To formulate our next result we need the following notion.

Definition 6. In the above notation, the *holonomy group* of Y is the subgroup of $G_k \subset U(1)$ generated by the holonomies of all possible loops on Y where G_k denotes

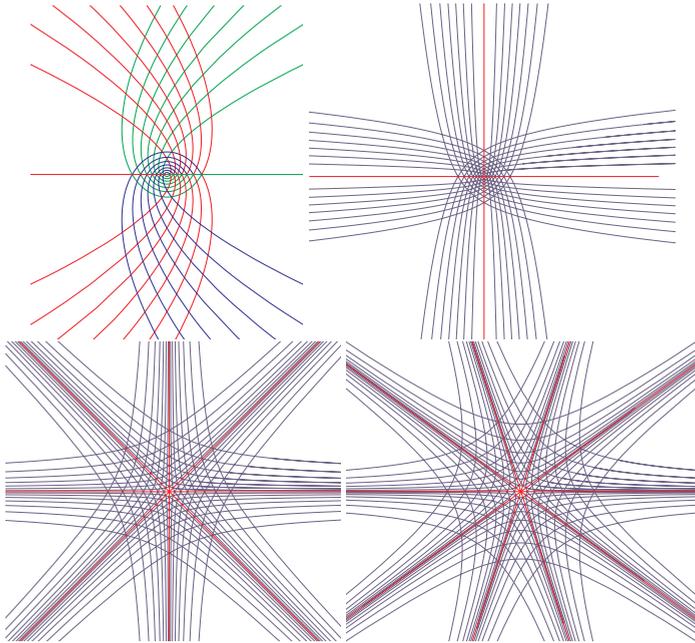


FIGURE 1
Local phase portraits of a cubic differential near a simple and double poles (top row) and simple and double zeros (bottom row).

the group of k -th roots of unity. Given a domain $\Omega \subseteq Y$, we define the holonomy of Ω as the subgroup of G_k generated by the holonomies of loops contained in Ω .

By a *density domain* we denote the closure of a trajectory in case (iii) of Lemma B without selfintersections. The following claim seems to be new.

Proposition 1. *The holonomy of a density domain is either trivial, i.e., consists either of $\{1\}$ or it is equal to $\{\pm 1\}$ in which case k is even.*

Remark 3. In the case when the holonomy group of a density domain is trivial the k -differential Ψ restricted to the density domain can be represented as the k -th power of a 1-form. In the case when the holonomy group of a density domain is $\{\pm 1\}$ k is even and Ψ restricted to the density domain can be represented as the $\frac{k}{2}$ -th power of a quadratic differential.

Proof. Consider a non-selfintersecting global horizontal trajectory γ which is not periodic. Its closure A is a subset of the surface Y foliated by horizontal trajectories parallel to γ . Since γ is non-selfintersecting these other trajectories are non-selfintersecting

either. Therefore, A is a polygonal domain on the flat model of Y whose boundary is a union of segments of horizontal trajectories parallel to γ . Thus A is a flat polygon with some of its sides identified.

Notice that such presentation is not unique. Nevertheless, the angle between every side of a polygon and the foliation by the trajectory lines is well-defined. Therefore, the holonomy of any simple loop that crosses exactly one side exactly once is either 1 or -1 . Such loops generate the fundamental group of A which finishes the proof. ■

Corollary 1. *If a meromorphic k -differential Ψ on a compact Riemann surface Y with $k \geq 3$ has a family of smooth periodic trajectories covering Y almost everywhere then there either exists a 1-form ω or a quadratic differential q such that either $\Psi = \omega^k$ or $\Psi = q^{\frac{k}{2}}$.*

Proof. If there is a family of smooth closed trajectories covering Y almost everywhere, then the direction field of Ψ tangent to this family is well-defined on the whole surface Y . This implies that the holonomy of Ψ along any closed loop on Y is equal to ± 1 . Assume first that the holonomy group of Ψ is trivial. Then there exists a 1-form ω on Y such that $\Psi = \omega^k$. (Indeed, in this situation every branch of $\sqrt[k]{\Psi}$ is a well-defined 1-form on Y). In the remaining case the holonomy group of Ψ is $\{\pm 1\}$ which implies that k branches split into $\frac{k}{2}$ pairs where the two branches in each pair are analytic continuations of each other. Moreover, the two branches in each pair coincide with the two branches of \sqrt{q} for some quadratic differential q . Additionally, $\Psi = q^{\frac{k}{2}}$ independently of the choice of a pair. ■

Definition 7. For a k -differential $\Psi = f(z)dz^k$, we call by its *dual* the differential $\Psi^* = i \cdot f(z)dz^k$, $i = \sqrt{-1}$. Obviously, Ψ^* is globally well-defined and attains purely imaginary values whenever Ψ attains real values and vice versa. Additionally, $(\Psi^*)^* = -\Psi$.

2.2. Introducing quasi-Strebel structures. The following definitions are inspired by the approach to Strebel quadratic differentials developed in [2] and by our previous results obtained in [5, 12, 17, 20].

Definition 8. Given a k -differential Ψ on Y , by its *finite broken trajectory* we mean a continuous curve on Y consisting of finitely many segments of trajectories of Ψ . A broken trajectory which is a non-selfintersecting closed curve on Y is called *closed*. Points where a trajectory is broken are called *switching points*. Usual periodic trajectories (which might exist for k -differentials, but can not cover almost all Y , see Corollary 1) is a particular case of closed broken trajectories.

Definition 9. Given a k -differential Ψ , we say that a continuous and piecewise smooth function $\Phi : Y \setminus Cr_\Psi \rightarrow \mathbb{R}$ is a *prelevel function* of Ψ if

- (i) Φ is non-constant on any open subset of Y ;
- (ii) all connected components of its level curves are broken trajectories of Ψ .

We say that a prelevel function Φ is a *level function* of Ψ if its *switching set* $S_{\Psi, \Phi}$ (i.e., the set of all its switching points) consists of finitely many piecewise smooth compact curves on Y .

We say that two level functions Φ_1 and Φ_2 of Ψ are *level-equivalent* if every connected component of any level curve of Φ_1 is a connected component of some level curve of Φ_2 and conversely.

Definition 10. Given a meromorphic k -differential Ψ defined on a compact Riemann surface Y and a level function $\Phi : Y \setminus Cr_\Psi \rightarrow \mathbb{R}$, we say that Φ defines a *quasi-Strebel structure* for Ψ (or that a pair (Ψ, Φ) is quasi-Strebel) if

- (i) Φ is a piecewise linear continuous function consisting of finitely many (local) linear functions in the charts of the flat model of Y ;
- (ii) at each point where Ψ is smooth its differential coincides with one of the k branches of $\text{Im}(\sqrt[k]{\Psi})$;
- (iii) almost all connected components of almost all level curves of Φ are closed broken trajectories of Ψ and their canonical lengths are bounded from above.

Level functions appearing in Definition 10 are called *piecewise standard* or *ps-level functions* for short. Obviously the switching set of any ps-level function Φ coincides with the subset of Y where Φ is not smooth and it consists of a finite collection of straight segments on the flat model. More exactly, the following result holds.

Lemma 1. *The switching set $S_{\Psi, \Phi}$ of a ps-level function Φ of a quasi-Strebel structure (Ψ, Φ) consists of finitely many segments of trajectories of Ψ for odd k , and trajectories of Ψ together with trajectories of its dual Ψ^* for even k .*

Proof. Indeed consider a generic point p of $S_{\Psi, \Phi}$. By the properties of the differential of Φ , in the flat model of Ψ , p lies on the line defined by the equation $l_1(W) = l_2(W)$, where W is a local canonical coordinate centered at p and $l_1(W) := \text{Im}(\zeta_1 W)$ and $l_2(W) := \text{Im}(\zeta_2 W)$ are the linear functions with $\zeta_1^k = \zeta_2^k = 1$. Any such line goes in the direction determined by the bisector of two k -th roots of unity. Such bisectors coincide (up to \pm -sign) with k -th roots of unity for odd k . For k even, bisectors either coincide (up to \pm -sign) with k -th roots of unity or with k -th roots of -1 . These are exactly the directions tangent to the trajectories of either Ψ or Ψ^* at the point p . ■

Remark 4. Recall that given a k -differential Ψ , its two ps-level functions Φ_1 and Φ_2 are *level-equivalent* if they have coinciding connected components of their level sets. Since we are only interested in the behavior of the family of closed broken trajectories obtained as connected components of the level curves of Φ_1 (resp. Φ_2) we can identify the quasi-Strebel structures (Ψ, Φ_1) and (Ψ, Φ_2) . Thus denoting by $[\Phi]$ the class of all ps-level functions equivalent to Φ , from now on we will say that a *quasi-Strebel structure* is a pair $(\Psi, [\Phi])$.

Definition 11. Given a meromorphic k -differential Ψ as above, we say that it has *admissible singularities* if

- (i) Ψ has no poles of order smaller than $-k$;
- (ii) at every pole of order $-k$ of Ψ the residue should be equal to $i^k \cdot a$ where $a \in \mathbb{R}$.

The motivation for the above definition is as follows.

Lemma 2. *If Ψ is a k -differential admitting a quasi-Strebel structure $(\Psi, [\Phi])$, then it has admissible singularities.*

Proof. Indeed, if Ψ has a pole of order smaller than or equal to $-k$, there exists a neighborhood V of this pole disjoint from the switching set $S_{\Psi, \Phi}$. If one of the level sets of Φ inside V is a periodic trajectory, then the pole has to be of order $-k$ with the residue of the form $i^k \cdot a$ where $a \in \mathbb{R}$. Otherwise there exists a level set whose intersection with V connects the boundary of V with the pole and therefore is of infinite length – contradiction. ■

Definition 12. Given a ps-level function Φ for a differential Ψ , we say that a point p in the switching set Sw_{Φ} is its *node* if an arbitrary small neighborhood of p in Sw_{Φ} is not a straight segment without boundary. (In other words, a node is a vertex of Sw_{Φ} considered as a graph embedded in Y .) In the above notation, connected components of the level curves of Φ containing nodes will be called *special*.

One type of nodes is described by the following lemma.

Lemma 3. *For any quasi-Strebel structure (Ψ, Φ) , each vertex of valency one of Sw_{Φ} (which is considered as an embedded graph on Y) is a singular point of Ψ whose order for k is odd, is not divisible by k and for k even, not divisible by $\frac{k}{2}$. Moreover, every singular point of Ψ whose order is not an integer multiple of $\frac{k}{2}$ is a node of Sw_{Φ} .*

(Obviously, vertices of valency one are nodes of Sw_{Φ} .)

Proof. Around a vertex of valency one of the switching set, the level curves are formed by some segments of smooth trajectories (possibly none) and one broken trajectory

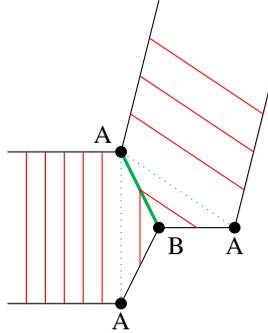


FIGURE 2

Cubic differential with one zero of order 2 (marked by A), one pole of order -2 (marked by B) and two poles of order -3 (at the ends of two cylinders). The thick segment connecting A and B is the switching set of the quasi-Strebel structure whose broken trajectories are shown by the parallel lines. Finally, the dashed lines are the special broken trajectories separating the cylinders.

with a switching point of angle θ , $0 < \theta < \pi$. Connecting these segments of trajectories by half circles, we get a loop around the vertex whose holonomy does not belong to $\{\pm 1\}$. Therefore, the vertex is a conical singularity whose total angle is not a multiple of π . Therefore its order is neither divisible by k nor by $\frac{k}{2}$ for k even.

Now if a conical singularity does not belong to Sw_Φ , then the level curves around it are formed only by some segments of smooth trajectories. Connecting as above these segments of trajectories by half circles, we get a loop around the vertex whose holonomy belongs to $\{\pm 1\}$. Thus the order of the conical singularity is an integer multiple of $\frac{k}{2}$. ■

Remark 5. Notice that for $k > 2$, there usually exist nodes which are not the singular points of Ψ , cf. Figure 6. We will call them *secondary singularities*. (These nodes are similar to additional turning points appearing in the asymptotic theory of linear ODE of order exceeding 2, see [1].)

2.3. Main theorem.

Theorem 1. (i) For even $k > 2$, every k -differential Ψ with admissible singularities has a quasi-Strebel structure.

(ii) For odd $k > 2$, any k -differential Ψ with admissible singularities such that, up to a common factor, the period of $\sqrt[k]{\Psi}$ along every path connecting any two conical singularities of Ψ belongs to $\mathbb{Q}[e^{\frac{2i\pi}{k}}]$ has a quasi-Strebel structure.

Remark 6. For k odd, differentials satisfying condition (ii) are dense in the space of all differentials with admissible singularities.

To prove Theorem 1, we will need the following preliminary statement.

Proposition 2. *For any $k \geq 3$ and any k -differential Ψ with admissible singularities, there exists a decomposition of the flat model of Ψ into triangles, trapezoids and infinite cylinders such that*

- (a) *each singularity of Ψ is one of its vertices;*
- (b) *each boundary edge of the decomposition is a segment of a horizontal trajectory.*

Proof. To start with, observe that if Ψ has a pole of order $-k$, then by Lemma 2 its residue is such that the corresponding infinite cylinder is bounded by a union of some critical trajectories and therefore it can be cut off from Y along this union. After cutting off all such cylinders, we get a new surface \tilde{Y} with finite area whose boundary consists of horizontal trajectories. (Recall that there exist only finitely many critical trajectories in Y since Ψ has a finite number of singularities.) Consider all the remaining critical trajectories satisfying the condition that they are nowhere dense in \tilde{Y} and remove their union from \tilde{Y} . All the remaining critical trajectories are dense and some of them are self-intersecting while others are not. Let us additionally cut \tilde{Y} along a certain part of every remaining critical trajectory having self-intersections. Namely, we remove a fragment of such trajectory starting at the conical singularity and ending at its first self-intersection. (If such a trajectory both starts and ends at conical singularities we do the latter procedure for both of its endpoints). Denote by CUT the constructed cut of Y which includes every curve we have removed so far.

Denote by γ one of the remaining critical trajectories. Then γ is dense in some domain of Y and non-selfintersecting. At this point, there are two cases: either γ hits CUT or it is dense in some domain $D \subseteq (Y \setminus CUT)$. In the first case, similarly to the above, we add to CUT the fragment of γ between the conical singularity and its first intersection with CUT .

Let us show by contradiction that the second case is impossible. Take some critical trajectory γ which is dense in some domain $D' \subseteq (Y \setminus CUT)$. Then any other critical trajectory which starts in D' must be dense in some subdomain $D'' \subseteq D'$.

The trajectory γ returns arbitrarily close to its starting point A (which is a singular point of Ψ). Therefore, there is an angular sector at A contained in D' and whose angle is at least π , see Figure 3. Since $k > 2$, there is still another non-selfintersecting critical trajectory α in D' starting at A transversally to γ . (Observe that this argument does not work for quadratic differentials which explains why quadratic differentials are not always quasi-Strebel.) Notice that α should also be dense in some $D'' \subseteq D'$ (because

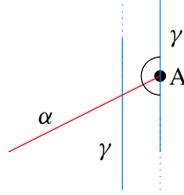


FIGURE 3
The local picture at a critical point lying in a density domain

the geometry of D' is that of a quadratic differential, see Proposition 1). Notice that the case $D'' = D'$ is impossible since α and γ are transverse. Taking α instead of γ we can iterate this step. However we get a contradiction after a finite number of steps since there is no infinite sequence of density domains strictly included in each other.

In this way, the connected components of the complement to the constructed cut CUT have corners with angles smaller than or equal to π . Indeed, at each singularity the angle between any two neighboring critical trajectories equals $2\pi/k < \pi$. Moreover, for any regular point where (at least) two segments of CUT intersect, there are either (at least) four angles smaller than π or there are three angles one of which is π . By the Gauss-Bonnet theorem, a flat surface such that every corner of its boundary has angle smaller than π is either a convex polygon (if at least one corner has an angle strictly smaller than π) or a cylinder (if every angle of every corner is π or, equivalently, if there are no corners). Cylinders can be cut out along some horizontal directions in order to get a parallelogram. Now, each convex polygon can be cut into a finite number of trapezoids and triangles by taking lines parallel to one side of the polygon. ■

Proof of Theorem 1. To settle Claim (i), we use Proposition 2 to decompose the flat surface Y of Ψ into infinite cylinders, triangles and trapezoids. Let us show that each of these tiles can be covered by a family of broken trajectories. Indeed, each infinite cylinder is covered by a family of smooth horizontal trajectories. Each triangle can be filled by the family of broken trajectories whose switching set consists of three segments of bisectors meeting at the incenter, i.e., the center of the inscribed circle, see the right part of Figure 4. Finally, any trapezoid can be decomposed into a symmetric trapezoid and a triangle. Symmetric trapezoids can be covered by closed broken trajectories in a standard way shown in the left part of Figure 4.

To prove Claim (ii), we can without loss of generality assume that all the periods of $\sqrt[k]{\Psi}$ lie in $\mathbb{Q}[\zeta]$ where ζ is a k -th root of unity. Firstly, define the subset $\mathcal{L} \subset Y$ consisting of all points p such that a period of Ψ over a path connecting p of one of the conical singularities is in $\mathbb{Q}[\zeta]$. (Observe that by our assumptions if one period

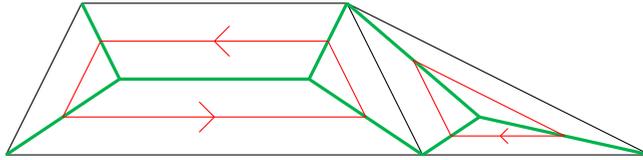


FIGURE 4
Tiling a general trapezoid by a symmetric trapezoid and a triangle and covering both tiles by closed broken trajectories

satisfies this property, then all of them satisfy it as well.) We want to show that every vertex created in a decomposition from Proposition 2 belongs to \mathcal{L} .

Indeed, choose two points A and C in \mathcal{L} . Let B be a point in the intersection of the horizontal trajectories passing through A and C resp. We want to show that B also belongs to \mathcal{L} . By choosing the appropriate k -th root we can assume that $\int_A^B \sqrt[k]{\Psi} \in \mathbb{R}$ and $\int_B^C \sqrt[k]{\Psi} \in \zeta \cdot \mathbb{R}$. Since $\mathbb{Q}[\zeta]$ considered as a vector space over \mathbb{Q} is the direct sum of $\mathbb{Q}[\zeta + \bar{\zeta}] \subset \mathbb{R}$ and $\zeta \cdot \mathbb{Q}[\zeta + \bar{\zeta}] \subset \zeta \cdot \mathbb{R}$, the decomposition of the number $\int_A^C \sqrt[k]{\Psi}$ into such summands is exactly $\int_A^C \sqrt[k]{\Psi} = \int_A^B \sqrt[k]{\Psi} + \int_B^C \sqrt[k]{\Psi}$. Thus $\int_A^B \sqrt[k]{\Psi}$ belongs to $\mathbb{Q}[\zeta + \bar{\zeta}] \subset \mathbb{Q}[\zeta]$ implying that $B \in \mathcal{L}$.

Therefore the decomposition provided by Proposition 2 consists of triangles and trapezoids whose vertices belong to \mathcal{L} . In Theorem 4 of [15] R. Kenyon proved that a convex polygon in \mathbb{C} is tileable with triangles whose angles are multiples of $\frac{\pi}{k}$ if and only if up to a common factor, its vertices lie in $\mathbb{Q}[e^{\frac{2i\pi}{k}}] \subset \mathbb{C}$. So, \tilde{Y} can be tiled with triangles whose sides are horizontal trajectories. Additionally, every such triangle can be filled by a family of broken trajectories with the switching set being the union of three bisectors, see the right part of Figure 4. ■

Remark 7. The result [15] of R. Kenyon quoted above states, in particular, that a convex polygon with sides in the directions of the k -th roots of unity cannot be tiled by triangles with sides in the same set of directions if the lengths of the sides of the polygon do not belong to the field $\mathbb{Q}[\cos(2\pi/k)]$. Because of this result we expect to find arithmetic obstructions to the existence of a quasi-Strebel structure for k -differentials with admissible singularities in the case of odd k .

3. Positive k -differentials and Heine–Stieltjes theory

The material of this section is somewhat similar to our treatment of gradient and positive gradient quadratic differentials presented in [2].

To connect quasi-Strebel structures with potential theory, let us recall some basic facts from complex analysis on Riemann surfaces, see, e.g., [11]. Let Y be a Riemann surface (open or closed) and h be a real- or complex-valued smooth function on Y .

Definition 13. The *Levy form* of h (with respect to a local coordinate z) is a $(1, 1)$ -form given by

$$(3.1) \quad \mu_h := 2i \frac{\partial^2 h}{\partial z \partial \bar{z}} dz \wedge d\bar{z}.$$

In terms of the real and imaginary parts (x, y) of z , μ_h is given by

$$\mu_h = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) dx \wedge dy = \Delta h dx \wedge dy.$$

If h is a smooth real-valued function, then μ_h is a signed measure on Y with a smooth density. In potential theory h is usually referred to as the (*logarithmic*) *potential* of the measure μ_h , see, e.g., [11, Ch. 3].

Notice that (3.1) makes sense for an arbitrary complex-valued distribution h on Y if one interprets μ_h as a 2-current on Y , i.e., a linear functional on the space of smooth compactly supported functions on Y , see, e.g., [10]. Such a current is necessarily exact since the inclusion of smooth forms into currents induces the (co)homology isomorphism. Notice that if Y is compact and connected, then the exactness of μ_h is equivalent to the vanishing of the integral of μ_h over Y .

Now observe that by definition, any ps-level function Φ for a k -differential Ψ is piecewise-harmonic and continuous in the complement to the set of poles of order $-k$ where it tends to $\pm\infty$ depending on the residue of Ψ at such a pole, see Definition 11. Thus Φ defines a distribution on the Riemann surface Y and its Levy form μ_Φ is a measure supported on the union of the switching set $S_{\Psi, \Phi}$ and the set of poles of order $-k$.

The final notion motivated by our original study of the Heine–Stieltjes theory is as follows.

Definition 14. A k -differential Ψ with at least one pole of order $-k$, is called *positive* if there exists a ps-level function Φ whose Levy form μ_Φ is a real measure the negative part of which is supported only on the set of poles of order $-k$ of Ψ .

Let us now present some families of rational k -differentials which are positive in the above sense. (In these examples the first author initially encountered quasi-Strebel structures and positive differentials of order exceeding 2, see [5, 12, 17, 20].)

Theorem A. *Given an arbitrary monic polynomial $Q_k(z) = z^k + a_1 z^{k-1} + \dots + a_k$ of degree k , consider the rational k -differential $\Psi = \frac{i^k dz^k}{Q_k(z)}$, $i = \sqrt{-1}$. Then Ψ is a positive differential.*

Theorem A is a special case of Corollary 2 below. Further, given a monic polynomial $Q(z)$ of degree ℓ and a positive integer $k \geq 2$, take the differential operator

$$(3.2) \quad T = Q(z) \frac{d^k}{dz^k}.$$

Assuming that $\ell \geq k$, consider the related Heine–Stieltjes problem which is as follows. For a non-negative integer n , find all pairs of monic polynomials $(V(z), S(z))$ such that $\deg V = \ell - k$ and $\deg S = n$ which solve the equation

$$(3.3) \quad TS_n(z) - (n)_k V(z)S(z) = Q(z) \frac{d^k S}{dz^k} - (n)_k V(z)S(z) = 0,$$

where $(n)_k := n(n - 1) \dots (n - k + 1)$. (V is classically known as the *Van-Vleck* polynomial and S as the *Stieltjes* polynomial for the equation (3.3).)

The following results can be found in [19], Theorem 4.

Theorem B. *In the above notation, for every $n \geq k$, there exist exactly $\binom{n+\ell-k}{\ell-k}$ pairs $(V(z), S(z))$ counting multiplicities which solve (3.3). Moreover, all roots of every $V(z)$ and every $S(z)$ are located in the convex hull $\text{Conv}(Q) \subset \mathbb{C}$ of the roots of $Q(z)$.*

Given a non-negative integer n , denote by $\{V_{i,n}\}$, $i = 1, \dots, \binom{n+\ell-k}{\ell-k}$ the set of all Van Vleck polynomials whose Stieltjes polynomials have degree n . In the above notation, assume that $\{V_{i,n}\}_{n=k}^\infty$ is a sequence of Van Vleck polynomials converging to some monic polynomial $\tilde{V}(z)$. (Recall that all $V_{i,n}(z)$ have degree $\ell - k$ and their roots lie in $\text{Conv}(Q)$ which implies that there exist plenty of converging sequences $\{V_{i,n}\}_{n=N}^\infty$.) Denote by $S_{i,n}$ the Stieltjes polynomial corresponding to $V_{i,n}$. (Here $\deg S_{i,n} = n$.)

Definition 15. The *root-counting measure* μ_P of a univariate polynomial $P = \prod_{j=1}^d (x - x_j)$ is, by definition, given by

$$\mu_d := \frac{1}{d} \sum_{j=1}^d \delta(x - x_j),$$

where $\delta(x - u)$ is the unit point mass located at u .

Theorem C. *In the above notation, the sequence of root-counting measure $\{\mu_{i,n}\}$ of $\{S_{i,n}\}_{n=k}^\infty$ converges to the probability measure $\mu := \mu_{\tilde{V},Q}$ satisfying the condition that*

$$\mathfrak{C}_\mu^k(z) = \frac{\tilde{V}(z)}{Q(z)}$$

a.e. in the complex plane \mathbb{C} , where $\mathfrak{C}_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z-\xi}$ is the Cauchy transform of μ .

Corollary 2. *The k -th differential $\Psi := \frac{i^k \cdot \tilde{V}(z)}{Q(z)} dz^k$ is positive. The corresponding ps -level function Φ is given by the logarithmic potential $u_\mu(z)$ of the probability measure μ where*

$$u_\mu(z) := \int_{\mathbb{C}} \log |z - \zeta| d\mu(\zeta).$$

Sketch (cf. the final section of [12] and § 4 of [6]). Recall that

$$\mathcal{E}_\mu(z) = \frac{\partial u_\mu(z)}{\partial z}; \quad \mu = \frac{1}{\pi} \frac{\partial \bar{\mathcal{E}}_\mu(z)}{\partial \bar{z}}$$

where the derivatives are understood in the distributional sense which implies that μ is the Levy form of $u(z)$. The logarithmic potential $u_\mu(z)$ is continuous outside the set of point masses of μ and harmonic outside its support. One can directly check that Theorem C can be reinterpreted as the statement that the level curves of the logarithmic potential $u_\mu(z)$ of the measure μ constructed in this theorem consist of segments of horizontal trajectories of the differential $\Psi(z) = \frac{i^k \tilde{V}(z)}{Q(z)} dz^k$. Indeed, the gradient of $u_\mu(z)$ equals $\bar{\mathcal{E}}_\mu(z)$ where bar stands for the usual complex conjugation. Thus the tangent direction $\delta \in \mathbb{C}$ to the level curve of $u_\mu(z)$ at a point p is given by $\kappa i \bar{\mathcal{E}}_\mu(p)$ for some real κ . Thus it will satisfy the condition $\frac{i^k \tilde{V}(p)}{Q(p)} \delta^k > 0$. Additionally, $u_\mu(z)$ approaches $\log |z|$ when $|z| \rightarrow \infty$ which implies that almost all connected components of its level curves are closed non-selfintersecting curves in \mathbb{C} . The measure μ is positive in \mathbb{C} and its extension to $\mathbb{C}P^1 \supset \mathbb{C}$ is the exact 2-current with the point mass -1 at ∞ . Observe that ∞ is the pole of order $-k$ for the k -differential $\Psi(z) = \frac{i^k \tilde{V}(z)}{Q(z)} dz^k$ with the residue $-i^k$. Finally, the support of μ consists of a finite number of curve segments globally forming an embedded forest in \mathbb{C} where each curve is either a segment of a horizontal trajectory of $\Psi(z)$ or a segment of a trajectory of $\Psi^*(z) = \frac{i^{k+1} \tilde{V}(z)}{Q(z)} dz^k$, see more details in [12]. ■

Concrete illustrations of the switching sets for $u(z)$ of such cubic differentials with $\deg \tilde{V} = 1$ and $\deg Q = 4$ are given in Figure 5, see more details and explanations in Figures 1 and 2 of [12].

The special case when $\deg Q = k$ is called *exactly solvable* and is considered in more detail in [17] and [5]. These papers contain substantial information about the switching sets of $u(z)$ for such differentials of order k which are topologically planar trees in \mathbb{C} with vertices of valency one coinciding with all roots of Q and special angles between the edges, see an illustration in Figure 6.

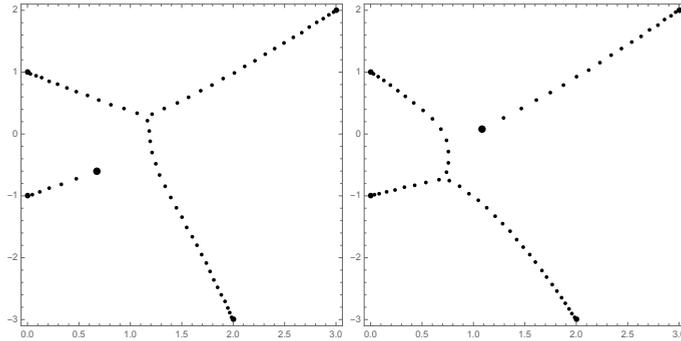


FIGURE 5

The switching sets for the logarithmic potential $u_\mu(z)$ in case of the cubic differentials $\Psi(z) = \frac{i^3 \cdot (z-\alpha) dz^3}{(z+i)(z-i)(z-2+3i)(z-3-2i)}$ with $\alpha \simeq 0.7 - 0.6i$ (left) and $\alpha \simeq 1.1 + 0.1i$ (right). (Possible choices of α are described in [12]).

4. Outlook

1. Recall that by the famous Jenkins Basic Structure Theorem [14, Theorem 3.5, pp. 38–39], for any quadratic differential Ψ given on a compact Riemann surface Y , up to a few exceptions, the set $Y \setminus (K_\Psi \cup Cr_\Psi)$ consists of a finite number of the so-called circle, ring, strip and end domains. Here K_Ψ is the union of all critical trajectories. (For the detailed definitions and information we refer to loc. cit). The names circle, ring and strip domain are reflecting the shapes of their images under the analytic continuation of the mapping given by the canonical coordinate; an end domain (also referred to as half-plane domain) is mapped by the canonical coordinate onto the half-plane.

Problem 1. Is there an analog of the Jenkins Basic Structure Theorem for higher order differentials?

Although due to self-intersections of trajectories of Ψ the existence of only finitely many types of possible domains for $k > 2$ seems problematic, one still wants (at least crudely) to distinguish their possible global behaviour.

2. The next question is related to Part (ii) of Theorem 1.

Problem 2. Find necessary and sufficient conditions for a k -differential Ψ of odd order k to admit a quasi-Strebel structure?

Hopefully it is at least possible to find some arithmetic obstructions to the existence of a quasi-Strebel structure for odd k .

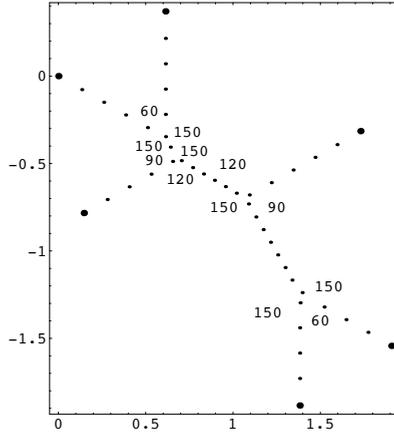


FIGURE 6

The switching set of $u(z)$ for the rational sextic differential $\Psi(z) = \frac{-dz^6}{z(z+i)(z-1-i)(z-2+2i)(z-2+3i)(z-3+2i)}$ shown in a chart of the corresponding flat surface. (The numbers are the angles between the neighbouring edges).

3. Typically a k -differential Ψ with admissible singularities allows for an abundance of non-equivalent quasi-Strebel structures in the sense that collections of their closed broken trajectories are distinct. To streamline the situation, one can introduce a natural partial order on quasi-Strebel structures of a given Ψ . Observe that the complement in Y to the union of all special connected components of level curves and poles of order $-k$ is a disjoint union of finitely many topological cylinders. Each of them is covered by closed broken trajectories of the same type. We will call such topological cylinders *packs of broken trajectories*.

In Figure 7 we present a simple example of two nonequivalent ps-level functions (quasi-Strebel structures) for the same quartic differential. Observe additionally that on the right picture in Figure 7, the switching set consists of three connected components one of which is non-contractible. On the left picture of Figure 7, the switching set does not contain the zero of order 12.

Given two ps-level functions Φ_1 and Φ_2 on Y , we say that Φ_1 gives a *coarser subdivision of Y* than Φ_2 if the closure of every pack of broken trajectories of Φ_1 is the union of the closures of some packs of broken trajectories of Φ_2 . Notation, $\Phi_1 \succeq \Phi_2$. In this situation we will also say that Φ_2 provides a *finer subdivision of Y* than Φ_1 . We say that a quasi-Strebel structure (Ψ, Φ) is *maximal* if there is no coarser ps-level function, i.e., no $\tilde{\Phi}$ such that $\tilde{\Phi} \succ \Phi$ exists.

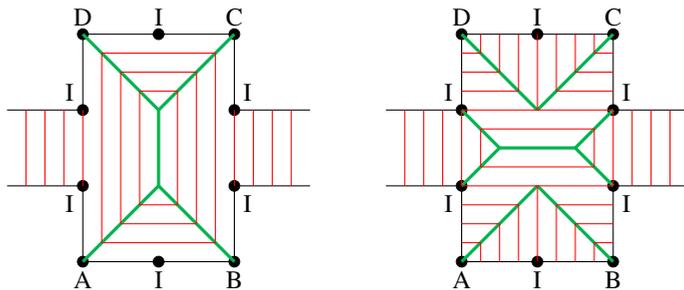


FIGURE 7
 A quartic differential with zero of order 12 (marked by I), four poles of order -3 (marked by A, B, C and D) and two poles of order -4 together with two nonequivalent ps-level functions. (The broken trajectories are shown by the thin lines while the switching set is shown by the thick ones.)

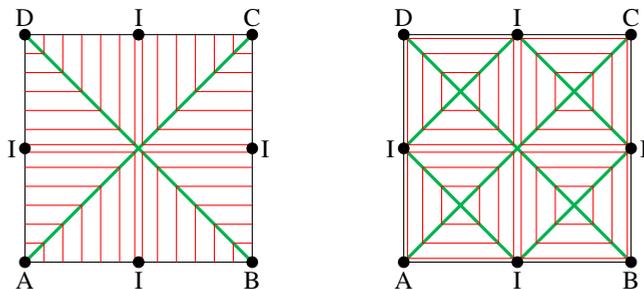


FIGURE 8
 A quartic differential with a zero of order 4 (marked by I) and four poles of order -3 (marked by A, B, C and D) together with two nonequivalent ps-level functions

Problem 3. Given a k -differential Ψ with admissible singularities, describe all its maximal quasi-Strebel structures.

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Boris SHAPIRO, Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden; *e-mail*: shapiro@math.su.se

Guillaume TAHAR, Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001 Israel; *e-mail*: tahar.guillaume@weizmann.ac.il