

Root-counting Measures of Jacobi Polynomials and Topological Types and Critical Geodesics of Related Quadratic Differentials

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To Mikael Passare, in memoriam

Abstract. Two main topics of this paper are asymptotic distributions of zeros of Jacobi polynomials and topology of critical trajectories of related quadratic differentials. First, we will discuss recent developments and some new results concerning the limit of the root-counting measures of these polynomials. In particular, we will show that the support of the limit measure sits on the critical trajectories of a quadratic differential of the form $Q(z) dz^2 = \frac{az^2+bz+c}{(z^2-1)^2} dz^2$. Then we will give a complete classification, in terms of complex parameters a , b , and c , of possible topological types of critical geodesics for the quadratic differential of this type.

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1. Introduction: From Jacobi polynomials to quadratic differentials

Two main themes of this work are asymptotic behavior of zeros of certain polynomials and topological properties of related quadratic differentials. The study of asymptotic root distributions of hypergeometric, Jacobi, and Laguerre polynomials with variable real parameters, which grow linearly with degree, became a rather hot topic in recent publications, which attracted attention of many authors [14], [15], [16], [17], [18], [22], [24], [25], [27]. In this paper, we survey some known results in this area and present some new results keeping focus on Jacobi polynomials.

Recall that the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ of degree n with complex parameters α, β is defined by

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k},$$

where $\binom{\gamma}{k} = \frac{\gamma(\gamma-1)\dots(\gamma-k+1)}{k!}$ with a non-negative integer k and an arbitrary complex number γ . Equivalently, $P_n^{(\alpha,\beta)}(z)$ can be defined by the well-known Rodrigues formula:

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!} (z-1)^{-\alpha} (z+1)^{-\beta} \left(\frac{d}{dz}\right)^n [(z-1)^{n+\alpha} (z+1)^{n+\beta}].$$

The following statement, which can be found, for instance, in [24, Proposition 2], gives an important characterization of Jacobi polynomials as solutions of second-order differential equation.

Proposition 1. *For arbitrary fixed complex numbers α and β , the differential equation*

$$(1-z^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)z)y' + \lambda y = 0$$

with a spectral parameter λ has a non-trivial polynomial solution of degree n if and only if $\lambda = n(n + \alpha + \beta + 1)$. This polynomial solution is unique (up to a constant factor) and coincides with $P_n^{(\alpha,\beta)}(z)$.

Working with root distributions of polynomials, it is convenient to use root-counting measures and their Cauchy transforms, which are defined as follows.

Definition 1. For a polynomial $p(z)$ of degree n with (not necessarily distinct) roots ξ_1, \dots, ξ_n , its *root-counting measure* μ_p is defined as

$$\mu_p = \frac{1}{n} \sum_{i=1}^n \delta_{\xi_i},$$

where δ_ξ is the Dirac measure supported at ξ .

Definition 2. Given a finite complex-valued Borel measure μ compactly supported in \mathbb{C} , its *Cauchy transform* \mathcal{C}_μ is defined as

$$\mathcal{C}_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi}. \tag{1.1}$$

and its logarithmic potential u_μ is defined as

$$u_\mu(z) = \int_{\mathbb{C}} \log |z - \xi| d\mu(\xi).$$

We note that the integral in (1.1) converges for all z , for which the Newtonian potential $U_{|\mu|}(z) = \int_{\mathbb{C}} \frac{d|\mu|(\xi)}{|\xi - z|}$ of μ is finite, see, e.g., [19, Ch. 2].

In case when $\mu = \mu_p$ is the root-counting measure of a polynomial $p(z)$, we will write \mathcal{C}_p instead of \mathcal{C}_{μ_p} . It follows from Definitions 1 and 2 that the Cauchy

transform $\mathcal{C}_p(z)$ of the root-counting measure of a monic polynomial $p(z)$ of degree n coincides with the normalized logarithmic derivative of $p(z)$; i.e.,

$$\mathcal{C}_p(z) = \frac{p'(z)}{np(z)} = \int_{\mathbb{C}} \frac{d\mu_p(\xi)}{z - \xi}, \tag{1.2}$$

and its logarithmic potential $u_p(z)$ is given by the formula:

$$u_p(z) = \frac{1}{n} \log |p(z)| = \int_{\mathbb{C}} \log |z - \xi| d\mu_p(\xi). \tag{1.3}$$

Let $\{p_n(z)\}$ be a sequence of Jacobi polynomials $p_n(z) = P_n^{(\alpha_n, \beta_n)}(z)$ and let $\{\mu_n\}$ be the corresponding sequence of their root-counting measures. The main question we are going to address in this paper is the following:

Problem 1. Assuming that the sequence $\{\mu_n\}$ weakly converges to a measure μ compactly supported in \mathbb{C} , what can be said about properties of the support of the measure μ and about its Cauchy transform \mathcal{C}_μ ?

Regarding the Cauchy transform \mathcal{C}_μ , our main result in this direction is the following theorem.

Theorem 1. *Suppose that a sequence $\{p_n(z)\}$ of Jacobi polynomials*

\Leftarrow disp

$$p_n(z) = P_n^{(\alpha_n, \beta_n)}(z)$$

satisfies conditions:

- (a) the limits $A = \lim_{n \rightarrow \infty} \frac{\alpha_n}{n}$ and $B = \lim_{n \rightarrow \infty} \frac{\beta_n}{n}$ exist, and $1 + A + B \neq 0$;
- (b) the sequence $\{\mu_n\}$ of the root-counting measures converges weakly to a probability measure μ , which is compactly supported in \mathbb{C} .

Then the Cauchy transform \mathcal{C}_μ of the limit measure μ satisfies almost everywhere in \mathbb{C} the quadratic equation:

$$(1 - z^2)\mathcal{C}_\mu^2 - ((A + B)z + A - B)\mathcal{C}_\mu + A + B + 1 = 0. \tag{1.4}$$

The proof of Theorem 1 given in Section 2 consists of several steps. Our arguments in Section 2 are similar to the arguments used in a number of earlier papers on root asymptotics of orthogonal polynomials.

Equation (1.4) of Theorem 1 implies that the support of the limit measure μ has a remarkable structure described by Theorem 2 below. And this is exactly the point where quadratic differentials, which are the second main theme of this paper, enter into the play.

Theorem 2. *In notation of Theorem 1, the support of μ consists of finitely many trajectories of the quadratic differential*

$$Q(z) dz^2 = - \frac{(A + B + 2)^2 z^2 + 2(A^2 - B^2)z + (A - B)^2 - 4(A + B + 1)}{(z - 1)^2(z + 1)^2} dz^2$$

and their end points.

Thus, to understand geometrical structure of the support of μ we have to study geometry of critical trajectories, or more generally critical geodesics of the quadratic differential $Q(z) dz^2$ of Theorem 1. We will consider a slightly more general family of quadratic differentials $Q(z; a, b, c) dz^2$ depending on three complex parameters $a, b, c \in \mathbb{C}$, $a \neq 0$, where

$$Q(z; a, b, c) dz^2 = \frac{az^2 + bz + c}{(z-1)^2(z+1)^2} dz^2. \quad (1.5)$$

It is well known that quadratic differentials appear in many areas of mathematics and mathematical physics such as moduli spaces of curves, univalent functions, asymptotic theory of linear ordinary differential equations, spectral theory of Schrödinger equations, orthogonal polynomials, etc. Postponing necessary definitions and basic properties of quadratic differentials till Section 3, we recall here that any meromorphic quadratic differential $Q(z) dz^2$ defines the so-called *Q-metric* and therefore it defines *Q-geodesics* in appropriate classes of curves. Motivated by the fact that the family of quadratic differentials (1.5) naturally appears in the study of the root asymptotics for sequences of Jacobi polynomials and is one of very few examples allowing detailed and explicit investigation in terms of its coefficients, we will consider the following two basic questions:

- 1) How many simple critical *Q*-geodesics may exist for a quadratic differential $Q(z) dz^2$ of the form (1.5)?
- 2) For given $a, b, c \in \mathbb{C}$, $a \neq 0$, describe topology of all simple critical *Q*-geodesics.

A complete description of topological structure of trajectories of quadratic differentials (1.5) which, in particular, answers questions 1) and 2), is given by lengthy Theorem 5 stated in Section 9.

The rest of the paper consists of two parts and is structured as follows. The first part, which is the area of expertise of the first author, includes Sections 2, 4, and 5. Section 2 contains the proof of Theorem 1 and related results. The material presented in Section 4 is mostly borrowed from a recent paper [12] of the first author. It contains some general results connecting signed measures, whose Cauchy transforms satisfy quadratic equations, and related quadratic differentials in \mathbb{C} . In particular, these results imply Theorem 2 as a special case. In Section 5, we formulate a number of general conjectures about the type of convergence of root-counting measures of polynomial solutions of a special class of linear differential equations with polynomial coefficients, which includes Riemann's differential equation.

Remaining sections constitute the second part, which is the area of expertise of the second author. In Section 3, we recall basic information about quadratic differentials, their critical trajectories and geodesics. This information is needed for presentation of our results in Sections 6–10. In Section 6, we describe possible domain configurations for the quadratic differentials (1.5). Then, in Section 7, we describe possible topological types of the structure of critical trajectories of

quadratic differentials of the form (1.5). Finally in Sections 8–10, we identify sets of parameters corresponding to each topological type. The latter allows us to answer some related questions.

We note here that our main proofs presented in Sections 6–10 are geometrical based on general facts of the theory of quadratic differentials. Thus, our methods can be easily adapted to study trajectory structure of many quadratic differentials other than quadratic differential (1.5).

Section 11 is our Figures Zoo, it contains many figures illustrating our results presented in Sections 6–10.

2. Proof of Theorem 1

To settle Theorem 1 we will need several auxiliary statements. Lemma 1 below can be found as Theorem 7.6 of [3] and apparently was originally proven by F. Riesz.

Lemma 1. *If a sequence $\{\mu_n\}$ of Borel probability measures in \mathbb{C} weakly converges to a probability measure μ with a compact support, then the sequence $\{\mathcal{C}_{\mu_n}(z)\}$ of its Cauchy transforms converges to $\mathcal{C}_\mu(z)$ in L^1_{loc} . Moreover there exists a subsequence of $\{\mathcal{C}_{\mu_n}(z)\}$ which converges to $\mathcal{C}_\mu(z)$ pointwise almost everywhere.*

The next result is recently obtained by the first author jointly with R. Bøgvad and D. Khavinson, see Theorem 1 of [13] and has an independent interest.

Proposition 2. *Let $\{p_m\}$ be any sequence of polynomials satisfying the following conditions:*

1. $n_m := \deg p_m \rightarrow \infty$ as $m \rightarrow \infty$,
2. *almost all roots of all p_m lie in a bounded convex open $\Omega \subset \mathbb{C}$ when $n \rightarrow \infty$. (More exactly, if In_m denotes the number of roots of p_m counted with multiplicities which are located in Ω , then $\lim_{m \rightarrow \infty} \frac{In_m}{n_m} = 1$), then for any $\epsilon > 0$,*

$$\lim_{m \rightarrow \infty} \frac{In'_m(\epsilon)}{n_m} = 1,$$

where $In'_m(\epsilon)$ is the number of roots of p'_m counted with multiplicities which are located inside $\Omega(\epsilon)$, the latter set being the ϵ -neighborhood of Ω in \mathbb{C} .

The next statement is a strengthening of Lemma 8 of [5] based on Proposition 2.

Lemma 2. *Let $\{p_m\}$ be any sequence of polynomials satisfying the following conditions:*

1. $n_m := \deg p_m \rightarrow \infty$ as $m \rightarrow \infty$,
2. *the sequence $\{\mu_m\}$ (resp. $\{\mu'_m\}$) of the root-counting measures of $\{p_m\}$ (resp. $\{p'_m\}$) weakly converges to compactly supported measures μ (resp μ').*
Then u and u' satisfy the inequality $u \geq u'$ with equality on the unbounded component of $\mathbb{C} \setminus \text{supp}(\mu)$. Here u (resp. u') is the logarithmic potential of the limiting measure μ (resp. μ').

Proof. Without loss of generality, we can assume that all p_m are monic. Let K be a compact convex set containing almost all the zeros of the sequences $\{p_m\}$ and $\{p'_m\}$, i.e., $\lim_{m \rightarrow \infty} \frac{In_m(K)}{n_m} = \lim_{m \rightarrow \infty} \frac{In'_m(K)}{n_m} = 1$. By (1.3) we have

$$u(z) = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log |p_m(z)|$$

and

$$u'(z) = \lim_{m \rightarrow \infty} \frac{1}{n_m - 1} \log \left| \frac{p'_m(z)}{n_m} \right| = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log \left| \frac{p'_m(z)}{n_m} \right|$$

with convergence in L^1_{loc} . Hence by (1.2),

$$u'(z) - u(z) = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log \left| \frac{p'_m(z)}{n_m p_m(z)} \right| = \lim_{m \rightarrow \infty} \frac{1}{n_m} \log \left| \int \frac{d\mu_m(\zeta)}{z - \zeta} \right|. \tag{2.1}$$

Now, if ϕ is a positive compactly supported test function, then

$$\begin{aligned} \int \phi(z)(u'(z) - u(z)) dA(z) &= \lim_{m \rightarrow \infty} \frac{1}{n_m} \int \phi(z) \log \left| \int \frac{d\mu_m(\zeta)}{z - \zeta} \right| dA(z) \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{n_m} \int \phi(z) \int \frac{d\mu_m(\zeta)}{|z - \zeta|} dA(z) \\ &= \lim_{m \rightarrow \infty} \frac{1}{n_m} \iint \frac{\phi(z) dA(z)}{|z - \zeta|} d\mu_m(\zeta) \end{aligned} \tag{2.2}$$

where dA denotes Lebesgue measure in the complex plane. Since $1/|z|$ is locally integrable, the function $\int \phi(z)|z - \zeta|^{-1} dA(z)$ is continuous, and hence bounded by a constant M for all z in K . Since asymptotically almost all zeros of $\{p_m\}$ belong to K , the last expression in (2.2) tends to 0 when $m \rightarrow \infty$. This proves that $u' \leq u$.

In the complement of $\text{supp } \mu$, u is harmonic and u' is subharmonic, hence $u' - u$ is a negative subharmonic function. Moreover, in the complement of $\text{supp } \mu$, $p'_m/(n_m p_m)$ converges to the Cauchy transform $\mathcal{C}(z)$ of μ a.e. in \mathbb{C} . Since $\mathcal{C}(z)$ is a nonconstant holomorphic function in the unbounded component of $\mathbb{C} \setminus \text{supp } \mu$, it follows from (2.1) that $u' - u \equiv 0$ there. \square

Notice that Lemma 2 implies the following interesting fact.

Corollary 1. *In notation of Lemma 2, if $\text{supp } \mu$ has Lebesgue area 0 and the complement $\mathbb{C} \setminus \text{supp } \mu$ is path-connected, then $\mu = \mu'$. In particular, in this case the whole sequence $\{\mu'_m\}$ weakly converges to μ .*

In general, however $\mu \neq \mu'$ as shown by a trivial example of the sequence $\{z^n - 1\}_{n=1}^\infty$. Also even if $\mu = \lim_{m \rightarrow \infty} \mu_n$ exists the limit $\lim_{m \rightarrow \infty} \mu'_n$ does not have to exist for the whole sequence. An example of this kind is the sequence $\{p_n(z)\}$ where $p_{2l}(z) = z^{2l} - 1$ and $p_{2l+1}(z) = z^{2l+1} - z$, $l = 1, 2, \dots$

Luckily, the latter phenomenon can never occur for sequences of Jacobi polynomials, see Proposition 3 below. (Apparently it cannot occur for a much more general class of polynomial sequences introduced in § 5.)

Lemma 3. *If the sequence $\{\mu_n\}$ of the root-counting measures of a sequence of Jacobi polynomials $\{p_n(z)\} = \{P_n^{(\alpha_n, \beta_n)}(z)\}$ weakly converges to a measure μ compactly supported in \mathbb{C} , and the sequence $\{\mu'_n\}$ of the root-counting measures of a sequence $\{p'_n(z)\}$ weakly converges to a measure μ' compactly supported in \mathbb{C} , then one of the following alternatives holds:*

- (i) *the sequences $\left\{\frac{\alpha_n + \beta_n}{n}\right\}$ and $\left\{\frac{\beta_n - \alpha_n}{n}\right\}$ (and, therefore, the sequences $\left\{\frac{\alpha_n}{n}\right\}$ and $\left\{\frac{\beta_n}{n}\right\}$) are bounded;*
- (ii) *the sequence $\left\{\frac{\alpha_n + \beta_n}{n}\right\}$ is unbounded and the sequence $\left\{\frac{\beta_n - \alpha_n}{n}\right\}$ is bounded, in which case $\{\mu_n\} \rightarrow \delta_0$ where δ_0 is the unit point mass at $z = 0$ (or, equivalently, $\mathcal{C}_{\delta_0}(z) = 1/z$);*
- (iii) *both sets $\left\{\frac{\alpha_n + \beta_n}{n}\right\}$ and $\left\{\frac{\beta_n - \alpha_n}{n}\right\}$ are unbounded, in which case, there exists at least one $\kappa \in \mathbb{C}$ and a subsequence $\{n_m\}$ such that $\lim_{m \rightarrow \infty} \frac{\beta_{n_m} - \alpha_{n_m}}{\alpha_{n_m} + \beta_{n_m}} = \kappa$ and $\{\mu_{n_m}\} \rightarrow \delta_\kappa$, where δ_κ is the unit point mass at $z = \kappa$ (or, equivalently, $\mathcal{C}_{\delta_\kappa}(z) = 1/(z - \kappa)$).*

Proof. Indeed, assume that the alternative (i) does not hold. Then there is a subsequence $\{n_m\}$ such that at least one of $\left|\frac{\alpha_{n_m} + \beta_{n_m}}{n_m}\right|$, $\left|\frac{\beta_{n_m} - \alpha_{n_m}}{n_m}\right|$ is unbounded along this subsequence. By our assumptions $\mu_n \rightarrow \mu$ and $\mu'_n \rightarrow \mu'$ weakly. Hence, by Lemma 1, there exists a subsequence of indices along which $\mathcal{C}_{\mu_n} := \frac{p'_n}{np_n}$ pointwise converges to \mathcal{C}_μ and $\mathcal{C}_{\mu'_n} := \frac{p''_n}{(n-1)p'_n}$ pointwise converges to $\mathcal{C}_{\mu'}$ a.e. in \mathbb{C} . Consider the sequence of differential equations satisfied by $\{p_n\}$ and divided termwise by $n(n-1)p_n$:

$$(1 - z^2) \frac{p''_n}{(n-1)p'_n} \cdot \frac{p'_n}{np_n} + \left(\frac{(\beta_n - \alpha_n) - (\alpha_n + \beta_n + 2)z}{n-1} \right) \frac{p'_n}{np_n} + \frac{n + \alpha_n + \beta_n + 1}{n-1} = 0. \tag{2.3}$$

If for a subsequence of indices, $\left|\frac{\beta_n - \alpha_n}{n}\right| \rightarrow \infty$ while $\left|\frac{\alpha_n + \beta_n}{n}\right|$ stays bounded, then the Cauchy transform \mathcal{C}_μ of the limiting (along this subsequence) measure μ must vanish identically in order for (2.3) to hold in the limit $n \rightarrow \infty$. But $\mathcal{C}_\mu \equiv 0$ is obviously impossible.

On the other hand, if for a subsequence of indices, $\left|\frac{\alpha_n + \beta_n}{n}\right| \rightarrow \infty$ while $\left|\frac{\beta_n - \alpha_n}{n}\right|$ stays bounded, then the limit of (2.3) when $n \rightarrow \infty$ coincides with $-z\mathcal{C}_\mu + 1 = 0 \Leftrightarrow \mathcal{C}_\mu = \frac{1}{z}$ implying $\mu = \delta_0$. Thus in Case (ii), the sequence $\{\mu_n\}$ converges to δ_0 .

Now assume, that on a subsequence of indices, both $\left|\frac{\alpha_n + \beta_n}{n}\right|$ and $\left|\frac{\beta_n - \alpha_n}{n}\right|$ tend to ∞ . Then dividing (2.3) by $\frac{\alpha_n + \beta_n}{n}$ and letting $n \rightarrow \infty$, we conclude that

the sequence $\left\{ \frac{\beta_n - \alpha_n}{\alpha_n + \beta_n} \right\}$ must be bounded. Therefore there exists its subsequence which converges to some $\kappa \in \mathbb{C}$. Taking the limit along this subsequence, we obtain

$$(z - \kappa)\mathcal{C}_\mu = 1.$$

This is true for all z , for which the Cauchy transform converges, i.e., almost everywhere outside the support of μ . Using the main results of [7, 8] claiming that the support of μ consists of piecewise smooth compact curves and/or isolated points together with the fact that \mathcal{C}_μ must have a discontinuity along every curve in its support, we conclude that the support of μ is the point $z = \kappa$. Thus in Case (iii), the sequence $\{\mu_{n_m}\}$ converges to δ_κ . \square

The next statement provides more information about Case (i) of Lemma 3.

Proposition 3. *Assume that the sequence $\{\mu_n\}$ of the root-counting measures for a sequence of Jacobi polynomials $\{p_n(z) = P_n^{(\alpha_n, \beta_n)}(z)\}$ weakly converges to a compactly supported measure μ in \mathbb{C} . Assume additionally that $\lim_{n \rightarrow \infty} \frac{\alpha_n}{n} = A$ and $\lim_{n \rightarrow \infty} \frac{\beta_n}{n} = B$ with $1 + A + B \neq 0$. Then, for any positive integer j , the sequence $\{\mu_n^{(j)}\}$ of the root-counting measures for the sequence $\{p_n^{(j)}(z)\}$ of the j th derivatives converges to the same measure μ .*

Proof. Observe that if an arbitrary polynomial sequence $\{p_m\}$ of increasing degrees has almost all roots in a convex bounded set $\Omega \subset \mathbb{C}$, then, by Proposition 2, almost all roots of $\{p'_m\}$ are in Ω_ϵ , for any $\epsilon > 0$. Therefore, if the sequence $\{\mu_m\}$ of the root-counting measures of $\{p_m\}$ weakly converges to a compactly supported measure μ , then there exists at least one weakly converging subsequence of $\{\mu'_m\}$. Additionally, by the Gauss–Lucas Theorem, the support of its limiting measure belongs to the (closure of the) convex hull of the support of μ . Thus the weak convergence of $\{\mu_m\}$ implies the existence of a weakly converging subsequence $\{\mu'_{n_m}\}$.

Proposition 3 is obvious in Cases (ii) and (iii) of Lemma 3. Let us concentrate on the remaining Case (i). Our assumptions imply that along a subsequence of the sequence $\left\{ \frac{p'_n}{np_n} \right\}$ of Cauchy transforms of polynomials p_n converges pointwise almost everywhere. We first show that the above sequence $\left\{ \frac{p'_n}{np_n} \right\}$ cannot converge to 0 on a set of positive measure.

Indeed, the differential equation satisfied by p_n after its division by $n(n - 1)p_n$ is given by (2.3). Since the sequences $\left\{ \frac{\alpha_n + \beta_n}{n} \right\}$ and $\left\{ \frac{\beta_n - \alpha_n}{n} \right\}$ converge and $1 + A + B \neq 0$, equation (2.3) shows that $\frac{p'_n}{np_n}$ cannot converge to 0 on a set of positive measure. Analogously, we see that $\frac{p''_n}{(n-1)p'_n}$ cannot converge to 0 on a set of positive measure either. Indeed, differentiating (2.3), we get that p'_n satisfies the equation

$$(1 - z^2)p'''_n + ((\beta_n - \alpha_n) - (\alpha_n + \beta_n + 4)z)p''_n + (n(n + \alpha_n + \beta_n + 1) + (\alpha_n + \beta_n + 2))p'_n = 0.$$

Using the same analysis as for p_n , we can conclude that the limit $\frac{p_n''}{n(n-1)p_n}$ along a subsequence exists pointwise and is non-vanishing almost everywhere.

Denote the logarithmic potentials of the root-counting measures associated to p_n and p'_n by u_n and u'_n respectively. Denote their limits by u and u' (where u' a priori is a limit only along some subsequence). With a slight abuse of notation, the following holds

$$|u - u'| = \lim_{n \rightarrow \infty} |u_n - u'_n| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{p_n''}{n(n-1)p_n} \right| = 0$$

due to the above claim about $\frac{p_n''}{n(n-1)p_n}$. But since $u \geq u'$ by Lemma 2, we see that $u = u'$ and, in particular u' exists as a limit over the whole sequence. Hence the asymptotic root-counting measures of $\{p_n\}$ and $\{p'_n\}$ actually coincide. Similar arguments apply to higher derivatives of the sequence $\{p_n\}$. \square

Proof of Theorem 1. The polynomial $p_n(z) = P_n^{(\alpha_n, \beta_n)}(z)$ satisfies equation (2.3). By Proposition 3 we know that, under the assumptions of Theorem 1, if $\left\{ \frac{p'_n}{np_n} \right\}$ converges to \mathcal{C}_μ a.e. in \mathbb{C} , then the sequence $\left\{ \frac{p_n''}{np_n'} \right\}$ also converges to the same \mathcal{C}_μ a.e. in \mathbb{C} . Therefore, the expression $\frac{p_n''}{n^2 p_n} = \frac{p_n'' p'_n}{n^2 p_n p'_n}$ converges to \mathcal{C}_μ^2 a.e. in \mathbb{C} . Thus \mathcal{C}_μ (which is well defined a.e. in \mathbb{C}) should satisfy the equation

$$(1 - z^2)\mathcal{C}_\mu^2 - ((A + B)z + A - B)\mathcal{C}_\mu + A + B + 1 = 0,$$

where $A = \lim_{n \rightarrow \infty} \frac{\alpha_n}{n}$ and $B = \lim_{n \rightarrow \infty} \frac{\beta_n}{n}$. \square

Remark 1. Apparently the condition that the sequences $\left\{ \frac{\alpha_n}{n} \right\}$ and $\left\{ \frac{\beta_n}{n} \right\}$ are bounded should be enough for the conclusion of Theorem 1. (The existence of the limits $\lim \frac{\alpha_n}{n}$ and $\lim \frac{\beta_n}{n}$ should follow automatically with some weak additional restriction.) Indeed, since the sequences $\left\{ \frac{\alpha_n}{n} \right\}$ and $\left\{ \frac{\beta_n}{n} \right\}$ are bounded, we can find at least one subsequence $\{n_m\}$ of indices along which both sequences of quotients converge. Assume that we have two possible distinct (pairs of) limits (A_1, B_1) and (A_2, B_2) along different subsequences. But then the same complex-analytic function $\mathcal{C}_\mu(z)$ should satisfy a.e. two different algebraic equations of the form (1.4) which is impossible at least for generic (A_1, B_1) and (A_2, B_2) .

3. Preliminaries on quadratic differentials

In this section, we recall some definitions and results of the theory of quadratic differentials on the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Most of these results remain true for quadratic differentials defined on any compact Riemann surface. But for the purposes of this paper, we will focus on results concerning the domain structure and properties of geodesics of quadratic differentials defined on $\overline{\mathbb{C}}$. For more information

on quadratic differentials in general, the interested reader may consult classical monographs of Jenkins [21] and Strebel [33] and papers [30] and [31].

A quadratic differential on a domain $D \subset \overline{\mathbb{C}}$ is a differential form $Q(z) dz^2$ with meromorphic $Q(z)$ and with conformal transformation rule

$$Q_1(\zeta) d\zeta^2 = Q(\varphi(z)) (\varphi'(z))^2 dz^2, \quad (3.1)$$

where $\zeta = \varphi(z)$ is a conformal map from D onto a domain $G \subset \overline{\mathbb{C}}$. Then zeros and poles of $Q(z)$ are critical points of $Q(z) dz^2$, in particular, zeros and simple poles are finite critical points of $Q(z) dz^2$. Below we will use the following notations. By H_p , C , and H we denote, respectively, the set of all poles, set of all finite critical points, and set of all infinite critical points of $Q(z) dz^2$. Also, we will use the following notations: $C' = \overline{\mathbb{C}} \setminus H$, $C'' = \overline{\mathbb{C}} \setminus H_p$, $C''' = \overline{\mathbb{C}} \setminus (C \cup H)$.

A trajectory (respectively, orthogonal trajectory) of $Q(z) dz^2$ is a closed analytic Jordan curve or maximal open analytic arc $\gamma \subset D$ such that

$$Q(z) dz^2 > 0 \quad \text{along } \gamma \quad (\text{respectively, } Q(z) dz^2 < 0 \quad \text{along } \gamma).$$

A trajectory γ is called *critical* if at least one of its end points is a finite critical point of $Q(z) dz^2$. By a closed critical trajectory we understand a critical trajectory together with its end points z_1 and z_2 (not necessarily distinct), assuming that these end points exist.

Let $\overline{\Phi}$ denote the closure of the set of points of all critical trajectories of $Q(z) dz^2$. Then, by Jenkins' Basic Structure Theorem [21, Theorem 3.5], the set $\overline{\mathbb{C}} \setminus \overline{\Phi}$ consists of a finite number of *circle, ring, strip and end domains*. The collection of all these domains together with so-called *density domains* constitute the so-called *domain configuration* of $Q(z) dz^2$. Here, we give definitions of circle domains and strip domains only; these two types will appear in our classification of possible domain configurations in Section 5. Figures 1–4 show several domain configurations with circle and strip domains. For the definitions of other domains, we refer to [21, Ch. 3].

We recall that a *circle domain* of $Q(z) dz^2$ is a simply connected domain D with the following properties:

- 1) D contains exactly one critical point z_0 , which is a second-order pole,
- 2) the domain $D \setminus \{z_0\}$ is swept out by trajectories of $Q(z) dz^2$ each of which is a Jordan curve separating z_0 from the boundary ∂D ,
- 3) ∂D contains at least one finite critical point.

Similarly, a *strip domain* of $Q(z) dz^2$ is a simply connected domain D with the following properties:

- 1) D contains no critical points of $Q(z) dz^2$,
- 2) ∂D contains exactly two boundary points z_1 and z_2 belonging to the set H (these boundary points may be situated at the same point of $\overline{\mathbb{C}}$),
- 3) the points z_1 and z_2 divide ∂D into two boundary arcs each of which contains at least one finite critical point,

4) D is swept out by trajectories of $Q(z) dz^2$ each of which is a Jordan arc connecting points z_1 and z_2 .

As we mentioned in the Introduction, every quadratic differential $Q(z) dz^2$ defines the so-called (singular) Q -metric with the differential element $|Q(z)|^{1/2} |dz|$. If γ is a rectifiable arc in D then its Q -length is defined by

$$|\gamma|_Q = \int_{\gamma} |Q(z)|^{1/2} |dz|.$$

According to their Q -lengths, trajectories of $Q(z) dz^2$ can be of two types. A trajectory γ is called *finite* if its Q -length is finite, otherwise γ is called *infinite*. In particular, a critical trajectory γ is finite if and only if it has two end points each of which is a finite critical point.

An important property of quadratic differentials is that transformation rule (8.1) respects trajectories and orthogonal trajectories and their Q -lengths, as well as it respects critical points together with their multiplicities and trajectory structure nearby.

Definition 3. A locally rectifiable (in the spherical metric) curve $\gamma \subset \mathbb{C}'$ is called a Q -geodesic if it is locally shortest in the Q -metric.

Next, given a quadratic differential $Q(z) dz^2$, we will discuss geodesics in homotopic classes. For any two points $z_1, z_2 \in \mathbb{C}'$, let $\mathcal{H}^J = \mathcal{H}^J(z_1, z_2)$ denote the set of all homotopic classes H of Jordan arcs $\gamma \subset \mathbb{C}'$ joining z_1 and z_2 . Here the letter J stands for “Jordan”. It is well known that there is a countable number of such homotopic classes. Thus, we may write $\mathcal{H}^J = \{H_k^J\}_{k=1}^{\infty}$.

Every class H_k^J can be extended to a larger class H_k by adding non-Jordan continuous curves γ joining z_1 and z_2 , each of which is homotopic on \mathbb{C}' to some curve $\gamma_0 \in H_k^J$ in the following sense.

There is a continuous function $\varphi(t, \tau)$ from the square $I^2 := [0, 1] \times [0, 1]$ to \mathbb{C}' such that

- 1) $\varphi(0, \tau) = z_1, \varphi(1, \tau) = z_2$ for all $0 \leq \tau \leq 1$,
- 2) $\gamma_0 = \{z = \varphi(t, 0) : 0 \leq t \leq 1\}$,
- 3) $\gamma = \gamma_1 = \{z = \varphi(t, 1) : 0 \leq t \leq 1\}$,
- 4) For every fixed $\tau, 0 < \tau < 1$, the curve $\gamma_{\tau} = \{z = \varphi(t, \tau) : 0 \leq t \leq 1\}$ is in the class H_k^J .

The following proposition is a special case of a well-known result about geodesics, see, e.g., [33, Theorem 18.2.1].

Proposition 4. For every k , there is a unique curve $\gamma' \in H_k$, called Q -geodesic in H_k , such that $|\gamma'|_Q < |\gamma|_Q$ for all $\gamma \in H_k, \gamma \neq \gamma'$. This geodesic is not necessarily a Jordan arc.

A Q -geodesic from z_1 to z_2 is called *simple* if $z_1 \neq z_2$ and γ is a Jordan arc on \mathbb{C}''' joining z_1 and z_2 . A Q -geodesic is called *critical* if both its end points belong to the set of finite critical points of $Q(z) dz^2$.

Proposition 5. *Let $Q(z) dz^2$ be a quadratic differential on $\overline{\mathbb{C}}$. Then for any two points $z_1, z_2 \in \mathbb{C}'$ and every continuous rectifiable curve γ on \mathbb{C}''' joining the points z_1 and z_2 there is a unique shortest curve γ_0 belonging to the homotopic class of γ .*

Furthermore, γ_0 is a geodesic in this class.

Definition 4. Let $z_0 \in \mathbb{C}'$. A geodesic ray from z_0 is a maximal simple rectifiable arc $\gamma : [0, 1) \rightarrow \mathbb{C}''' \cup \{z_0\}$ with $\gamma(0) = z_0$ such that for every $t, 0 < t < 1$, the arc $\gamma((0, 1))$ is a geodesic from z_0 to $z = \gamma(t)$.

Lemma 4. *Let D be a circle domain of $Q(z) dz^2$ centered at z_0 and let $\gamma_a : [0, 1) \rightarrow \mathbb{C}''' \cup \{a\}$ be a geodesic ray from $a \in \partial D$ such that $\gamma_a([0, t_0]) \subset \overline{D}$ for some $t_0 > 0$.*

Then either γ_a enters into D through the point a and then approaches to z_0 staying in D or γ_a is an arc of some critical trajectory $\gamma \subset \partial D$.

Lemma 5. *Let a be a second-order pole of $Q(z) dz^2$ and let Γ be the homotopic class of closed curves on \mathbb{C}'' separating a from $H_p \setminus \{a\}$. Then there is exactly one real $\theta_0, 0 \leq \theta_0 < 2\pi$, such that the quadratic differential $e^{i\theta_0} Q(z) dz^2$ has a circle domain, say D_0 , centered at a . Furthermore, the boundary ∂D_0 is the only critical Q -geodesic (non-Jordan in general) in the class Γ .*

In particular, Γ may contain at most one critical geodesic loop.

We will need some simple mapping properties of the canonical mapping related to the quadratic differential $Q(z) dz^2$, which is defined by

$$F(z) = \int_{z_0} \sqrt{Q(z)} dz$$

with some $z_0 \in \overline{\mathbb{C}}$ and some fixed branch of the radical. A simply connected domain D without critical points of $Q(z) dz^2$ is called a Q -rectangle if the boundary of D consists of two arcs of trajectories of $Q(z) dz^2$ separated by two arcs of orthogonal trajectories of this quadratic differential. As well a canonical mapping $F(z)$ maps any Q -rectangle conformally onto a geometrical rectangle in the plane with two sides parallel to the horizontal axis.

4. Cauchy transforms satisfying quadratic equations and quadratic differentials

Below we relate the question for which triples of polynomials (P, Q, R) the equation

$$P(z)\mathcal{C}^2 + Q(z)\mathcal{C} + R(z) = 0, \tag{4.1}$$

with $\deg P = n + 2, \deg Q \leq n + 1, \deg R \leq n$ admits a compactly supported signed measure μ whose Cauchy transform satisfies (4.1) almost everywhere in \mathbb{C} to a certain problem about rational quadratic differentials. We call such measure μ a *motherbody measure* for (4.1).

For a given quadratic differential Ψ on a compact surface \mathcal{R} , denote by $K_\Psi \subset \mathcal{R}$ the union of all its critical trajectories and critical points. (In general, K_Ψ can

be very complicated. In particular, it can be dense in some subdomains of \mathcal{R} .) We denote by $DK_\Psi \subseteq K_\Psi$ (the closure of) the set of finite critical trajectories of (4.2). (One can show that DK_Ψ is an imbedded (multi)graph in \mathcal{R} . Here by a *multigraph* on a surface we mean a graph with possibly multiple edges and loops.) Finally, denote by $DK_\Psi^0 \subseteq DK_\Psi$ the subgraph of DK_Ψ consisting of (the closure of) the set of finite critical trajectories whose both ends are zeros of Ψ .

A non-critical trajectory $\gamma_{z_0}(t)$ of a meromorphic Ψ is called *closed* if $\exists T > 0$ such that $\gamma_{z_0}(t+T) = \gamma_{z_0}(t)$ for all $t \in \mathbb{R}$. The least such T is called the *period* of γ_{z_0} . A quadratic differential Ψ on a compact Riemann surface \mathcal{R} without boundary is called *Strebel* if the set of its closed trajectories covers \mathcal{R} up to a set of Lebesgue measure zero.

Going back to Cauchy transforms, we formulate the following necessary condition of the existence of a motherbody measure for (4.1).

Proposition 6. *Assume that equation (4.1) admits a signed motherbody measure μ . Denote by $D(z) = Q^2(z) - 4P(z)R(z)$ the discriminant of equation (4.1). Then the following two conditions hold:*

- (i) *any connected smooth curve in the support of μ coincides with a horizontal trajectory of the quadratic differential*

$$\Theta = -\frac{D(z)}{P^2(z)}dz^2 = \frac{4P(z)R(z) - Q^2(z)}{P^2(z)}dz^2. \tag{4.2}$$

- (ii) *the support of μ includes all branching points of (4.1).*

Remark. Observe that if $P(z)$ and $Q(z)$ are coprime, the set of all branching points coincides with the set of all zeros of $D(z)$. In particular, in this case part (ii) of Proposition 6 implies that the set DK_Θ^0 for the differential Θ should contain all zeros of $D(z)$.

Remark. Proposition 6 applied to quadratic differential $Q(z)dz^2$ of Theorem 1 implies Theorem 2.

Proof. The fact that every curve in $\text{supp}(\mu)$ should coincide with some horizontal trajectory of (4.2) is well known and follows from the Plemelj–Sokhotsky’s formula. It is based on the local observation that if a real measure $\mu = \frac{1}{\pi} \frac{\partial \mathcal{C}}{\partial z}$ is supported on a smooth curve γ , then the tangent to γ at any point $z_0 \in \gamma$ should be perpendicular to $\overline{\mathcal{C}_1(z_0)} - \overline{\mathcal{C}_2(z_0)}$ where \mathcal{C}_1 and \mathcal{C}_2 are the one-sided limits of \mathcal{C} when $z \rightarrow z_0$, see, e.g., [5]. (Here $\bar{}$ stands for the usual complex conjugation.) Solutions of (4.1) are given by

$$\mathcal{C}_{1,2} = \frac{-Q(z) \pm \sqrt{Q^2(z) - 4P(z)R(z)}}{2P(z)},$$

their difference being

$$\mathcal{C}_1 - \mathcal{C}_2 = \frac{\sqrt{Q^2(z) - 4P(z)R(z)}}{P(z)}.$$

Since the tangent line to the support of the real motherbody measure μ satisfying (4.1) at its arbitrary smooth point z_0 , is orthogonal to $\mathcal{C}_1(z_0) - \mathcal{C}_2(z_0)$, it is exactly given by the condition $\frac{4P(z_0)R(z_0) - Q^2(z_0)}{P^2(z_0)} dz^2 > 0$. The latter condition defines the horizontal trajectory of Θ at z_0 .

Finally the observation that $\text{supp } \mu$ should contain all branching points of (4.1) follows immediately from the fact that \mathcal{C}_μ is a well-defined univalued function in $\mathbb{C} \setminus \text{supp } \mu$. □

In many special cases statements similar to Proposition 6 can be found in the literature, see, e.g., recent [1] and references therein.

Proposition 6 allows us, under mild nondegeneracy assumptions, to formulate necessary and sufficient conditions for the existence of a motherbody measure for (4.1) which however are difficult to verify. Namely, let $\Gamma \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ with affine coordinates (\mathcal{C}, z) be the algebraic curve given by (the projectivization of) equation (4.1). Γ has bidegree $(2, n+2)$ and is hyperelliptic. Let $\pi_z : \Gamma \rightarrow \mathbb{C}$ be the projection of Γ on the z -plane \mathbb{CP}^1 along the \mathcal{C} -coordinate. From (4.1) we observe that π_z induces a branched double covering of \mathbb{CP}^1 by Γ . If $P(z)$ and $Q(z)$ are coprime and if $\deg D(z) = 2n + 2$, the set of all branching points of $\pi_z : \Gamma \rightarrow \mathbb{CP}^1$ coincides with the set of all zeros of $D(z)$. (If $\deg D(z) < 2n + 2$, then ∞ is also a branching point of π_z of multiplicity $2n + 2 - \deg D(z)$.) We need the following lemma.

Lemma 6. *If $P(z)$ and $Q(z)$ are coprime, then at each pole of (4.1), i.e., at each zero of $P(z)$, only one of two branches of Γ goes to ∞ . Additionally the residue of this branch at this zero equals that of $-\frac{Q(z)}{P(z)}$.*

Proof. Indeed if $P(z)$ and $Q(z)$ are coprime, then no zero z_0 of $P(z)$ can be a branching point of (4.1) since $D(z_0) \neq 0$. Therefore only one of two branches of Γ goes to ∞ at z_0 . More exactly, the branch $\mathcal{C}_1 = \frac{-Q(z) + \sqrt{Q^2(z) - 4P(z)R(z)}}{2P(z)}$ attains a finite value at z_0 while the branch $\mathcal{C}_2 = \frac{-Q(z) - \sqrt{Q^2(z) - 4P(z)R(z)}}{2P(z)}$ goes to ∞ where we use the agreement that $\lim_{z \rightarrow z_0} \sqrt{Q^2 - 4P(z)R(z)} = Q(z_0)$. Now consider the residue of the branch \mathcal{C}_2 at z_0 . Since residues depend continuously on the coefficients $(P(z), Q(z), R(z))$ it suffices to consider only the case when z_0 is a simple zero of $P(z)$. Further if z_0 is a simple zero of $P(z)$, then

$$\text{Res}(\mathcal{C}_2, z_0) = \frac{-2Q(z_0)}{2P'(z_0)} = \text{Res}\left(-\frac{Q(z)}{P(z)}, z_0\right),$$

which completes the proof. □

By Proposition 6 (besides the obvious condition that (4.1) has a real branch near ∞ with the asymptotics $\frac{\alpha}{z}$ for some $\alpha \in \mathbb{R}$) the necessary condition for (4.1) to admit a motherbody measure is that the set DK_Θ^0 for the differential (4.2) contains all branching points of (4.1), i.e., all zeros of $D(z)$. Consider $\Gamma_{cut} := \Gamma \setminus \pi_z^{-1}(DK_\Theta^0)$. Since DK_Θ^0 contains all branching points of π_z , Γ_{cut} consists of some number of

open sheets, each projecting diffeomorphically on its image in $\mathbb{CP}^1 \setminus DK_{\Theta}^0$. (The number of sheets in Γ_{cut} equals to twice the number of connected components in $\mathbb{C} \setminus DK_{\Theta}^0$.) Observe that since we have chosen a real branch of (4.1) at infinity with the asymptotics $\frac{\alpha}{z}$, we have a marked point $p_{br} \in \Gamma$ over ∞ . If we additionally assume that $\deg D(z) = 2n + 2$, then ∞ is not a branching point of π_z and therefore $p_{br} \in \Gamma_{cut}$.

Lemma 7. *If $\deg D(z) = 2n + 2$, then any choice of a spanning (multi)subgraph $G \subset DK_{\Theta}^0$ with no isolated vertices induces the unique choice of the section S_G of Γ over $\mathbb{CP}^1 \setminus G$ which:*

- a) *contains p_{br} ;*
- b) *is discontinuous at any point of G ;* c) *is projected by π_z diffeomorphically onto $\mathbb{CP}^1 \setminus G$.*

Here by a spanning subgraph we mean a subgraph containing all the vertices of the ambient graph. By a section of Γ over $\mathbb{CP}^1 \setminus G$ we mean a choice of one of two possible values of Γ at each point in $\mathbb{CP}^1 \setminus G$. After these clarifications the proof is evident.

Observe that the section S_G might attain the value ∞ at some points, i.e., contain some poles of (4.1). Denote the set of poles of S_G by Poles_G . Now we can formulate our necessary and sufficient conditions.

Theorem 3. *Assume that the following conditions are valid:*

- (i) *equation (4.1) has a real branch near ∞ with the asymptotic behavior $\frac{\alpha}{z}$ for some $\alpha \in \mathbb{R}$;*
- (ii) *$P(z)$ and $Q(z)$ are coprime, and the discriminant $D(z) = Q^2(z) - 4P(z)R(z)$ of equation (4.1) has degree $2n + 2$;*
- (iii) *the set DK_{Θ}^0 for the quadratic differential Θ given by (4.2) contains all zeros of $D(z)$;*
- (iv) *Θ has no closed horizontal trajectories.*

Then (4.1) admits a real motherbody measure if and only if there exists a spanning (multi)subgraph $G \subseteq DK_{\Theta}^0$ with no isolated vertices, such that all poles in Poles_G are simple and all their residues are real, see notation above.

Proof. Indeed assume that (4.1) satisfying (ii) admits a real motherbody measure μ . Assumption (i) is obviously necessary for the existence of a real motherbody measure and the necessity of assumption (iii) follows from Proposition 6 if (ii) is satisfied. The support of μ consists of a finite number of curves and possibly a finite number of isolated points. Since each curve in the support of μ is a trajectory of Θ and Θ has no closed trajectories, then the whole support of μ consists of finite critical trajectories of Θ connecting its zeros, i.e., belongs to DK_{Θ}^0 . Moreover the support of μ should contain sufficiently many finite critical trajectories of Θ such that they include all the branching points of (4.1). By (ii) these are exactly all zeros of $D(z)$. Therefore the union of finite critical trajectories of Θ belonging to the support of μ is a spanning (multi)graph of DK_{Θ}^0 without isolated vertices. The

isolated points in the support of μ are necessarily the poles of (4.1). Observe that the Cauchy transform of any (complex-valued) measure can only have simple poles (as opposed to the Cauchy transform of a more general distribution). Since μ is real the residue of its Cauchy transform at each pole must be real as well. Therefore the existence of a real motherbody under the assumptions (i)–(iv) implies the existence of a spanning (multi)graph G with the above properties. The converse is also immediate. \square

Remark. Observe that if (i) is valid, then assumptions (ii) and (iv) are generically satisfied. Notice however that (iv) is violated in the special case when $Q(z)$ is absent. Additionally, if (iv) is satisfied, then the number of possible motherbody measures is finite. On the other hand, it is the assumption (iii) which imposes severe additional restrictions on admissible triples $(P(z), Q(z), R(z))$. At the moment the authors have no information about possible cardinalities of the sets Poles_G introduced above. Thus it is difficult to estimate the number of conditions required for (4.1) to admit a motherbody measure. Theorem 3 however leads to the following sufficient condition for the existence of a real motherbody measure for (4.1).

Corollary 2. *If, additionally to assumptions (i)–(iii) of Theorem 3, one assumes that all roots of $P(z)$ are simple and all residues of $\frac{Q(z)}{P(z)}$ are real, then (4.1) admits a real motherbody measure.*

Proof. Indeed if all roots of $P(z)$ are simple and all residues of $\frac{Q(z)}{P(z)}$ are real, then all poles of (4.1) are simple with real residues. In this case for any choice of a spanning (multi)subgraph G of DK_Θ^0 , there exists a real motherbody measure whose support coincides with G plus possibly some poles of (4.1). Observe that if all roots of $P(z)$ are simple and all residues of $\frac{Q(z)}{P(z)}$ are real one can omit assumption (iv). In case when Θ has no closed trajectories, then all possible real motherbody measures are in a bijective correspondence with all spanning (multi)subgraphs of DK_Θ^0 without isolated vertices. In the opposite case such measures are in a bijective correspondence with the unions of a spanning (multi)subgraph of DK_Θ^0 and an arbitrary (possibly empty) finite collection of closed trajectories. \square

5. Does weak convergence of Jacobi polynomials imply stronger forms of convergence?

Observe that, if one considers an arbitrary sequence $\{s_n(z)\}$, $n = 0, 1, \dots$ of monic univariate polynomials of increasing degrees, then even if the sequence $\{\theta_n\}$ of their root-counting measures weakly converges to some limiting probability measure Θ with compact support in \mathbb{C} , in general, it is not true that the roots of s_n stay on some finite distance from $\text{supp } \Theta$ for all n simultaneously. Similarly nothing can be said in general about the weak convergence of the sequence $\{\theta'_n\}$ of the root-counting measures of $\{s'_n(z)\}$. However we have already seen that the situation with sequences of Jacobi polynomials seems to be different, compare Proposition 3.

In the present appendix we formulate a general conjecture (and give some evidence of its validity) about sequences of Jacobi polynomials as well as sequences of more general polynomial solutions of a special class of linear differential equations which includes Riemann’s differential equation.

Consider a linear ordinary differential operator

$$\mathfrak{d}(z) = \sum_{i=1}^k Q_i(z) \frac{d^i}{dz^i} \tag{5.1}$$

with polynomial coefficients. We say that (5.1) is *exactly solvable* if a) $\deg Q_j \leq j$, for all $j = 1, \dots, k$; b) there exists at least one value j_0 such that $\deg Q_{j_0}(z) = j_0$. We say that an exactly solvable operator (5.1) is *non-degenerate* if $\deg Q_k = k$.

Observe that any exactly solvable operator $\mathfrak{d}(z)$ has a unique (up to a constant factor) eigenpolynomial of any sufficiently large degree, see, e.g., [5]. Fixing an arbitrary monic polynomial $Q_k(z)$ of degree k , consider the family \mathcal{F}_{Q_k} of all exactly solvable operators of the form (5.1) whose leading term is $Q_k(z) \frac{d^k}{dz^k}$. (\mathcal{F}_{Q_k} is a complex affine space of dimension $\binom{k+1}{2} - 1$.) Given a sequence $\{\mathfrak{d}_n(z)\}$ of exactly solvable operators from \mathcal{F}_{Q_k} of the form

$$\mathfrak{d}_n(z) = Q_k(z) \frac{d^k}{dz^k} + \sum_{i=1}^{k-1} Q_{i,n}(z) \frac{d^i}{dz^i},$$

we say that this sequence has a *moderate growth* if, for each $j = 1, \dots, k - 1$, the sequence of polynomials $\left\{ \frac{Q_{j,n}(z)}{n^{k-j}} \right\}$ has all bounded coefficients. (Recall that $\forall n$, $\deg Q_{j,n} \leq j$.)

Conjecture 1. *For any sequence $\{\mathfrak{d}_n(z)\}$ of exactly solvable operators of moderate growth, the union of all roots of all the eigenpolynomials of all $\mathfrak{d}_n(z)$ is bounded in \mathbb{C} .*

Now take a sequence $\{s_n(z)\}$, $\deg s_n = n$ of polynomial eigenfunctions of the sequence of operators $\mathfrak{d}_n(z) \in \mathcal{F}_{Q_k}$. (Observe that, in general, we have a different exactly solvable operator for each eigenpolynomial but with the same leading term.)

Conjecture 2. *In the above notation, assume that $\{\mathfrak{d}_n(z)\}$ is a sequence of exactly solvable operators of moderate growth and that $\{s_n(z)\}$ is the sequence of their eigenpolynomials (i.e., $s_n(z)$ is the eigenpolynomial of $\mathfrak{d}_n(z)$ of degree n) such that:*

- a) *the limits $\tilde{Q}_j(z) := \lim_{n \rightarrow \infty} \frac{1}{n^{k-j}} Q_{j,n}(z)$, $j = 1, \dots, k - 1$ exist;*
- b) *the sequence $\{\theta_n\}$ of the root-counting measures of $\{s_n(z)\}$ weakly converges to a compactly supported probability measure Θ in \mathbb{C} ,*

then

(i) the Cauchy transform \mathcal{C}_Θ of Θ satisfies a.e. in \mathbb{C} the algebraic equation

$$Q_k(z) \left(\frac{\mathcal{C}_\Theta}{\gamma}\right)^k + \sum_{j=1}^{k-1} \tilde{Q}_j(z) \left(\frac{\mathcal{C}_\Theta}{\gamma}\right)^j = 1, \tag{5.2}$$

where $\gamma = \lim_{n \rightarrow \infty} \frac{\sqrt[k]{\lambda_n}}{n}$, λ_n being the eigenvalue of $s_n(z)$.

(ii) for any positive $\epsilon > 0$, there exist n_ϵ such that, for $n \geq n_\epsilon$, all roots of all eigenpolynomials $s_n(z)$ are located within ϵ -neighborhood of $\text{supp } \Theta$, i.e., the weak convergence of $\theta_n \rightarrow \Theta$ implies a stronger form of this convergence.

Certain cases of Part (i) of the above Conjecture are settled in [5] and [9] and a version of Part (ii) is discussed in an unpublished preprint [11].

Now we present some partial confirmation of the above conjectures. Consider the family of linear differential operators of second order depending on parameter λ and given by

$$T_\lambda = Q_2(z) \frac{d^2}{dz^2} + (Q_1(z)\lambda + P_1(z)) \frac{d}{dz} + (\lambda^2 + p\lambda + q)Q_0, \tag{5.3}$$

where $Q_2(z)$ is a quadratic polynomial in z , $Q_1(z)$ and $P_1(z)$ are polynomials in z of degree at most 1, and Q_0 is a non-vanishing constant. (Observe that our use of parameter λ here is the same as of the parameter γ in the latter Conjecture.)

Denote $Q_i(z) = \sum_{j=0}^i q_{ji}z^j$, $i = 0, 1, 2$ and put $P_1 = p_{11}z + p_{01}$. The quadratic polynomial

$$q_{22} + q_{11}t + q_{00}t^2 \tag{5.4}$$

is called the *characteristic polynomial* of T_λ . Here $q_{22} \neq 0$ and $q_{00} = Q_0 \neq 0$.

Definition 5. We say that the family T_λ has a *generic type* if the roots of (5.4) have distinct arguments (and in particular 0 is not a root of (5.4) which is guaranteed by $q_{22} \neq 0$ together with $q_{00} \neq 0$), comp. [9].

Below we will denote the roots of characteristic polynomial (5.4) by α_1 and α_2 . Thus T_λ has a generic type if and only if $\arg \alpha_1 \neq \arg \alpha_2$.

Lemma 8. Equation (5.4) has two roots with the same arguments if and only if $q_{22}q_{00} = \rho q_{11}^2$, where $0 \leq \rho \leq \frac{1}{4}$.

Proof. Straightforward calculation, see Example 1 of [10]. □

Lemma 9. In the above notation, for a family T_λ of generic type, there exists a positive integer N such that, for any integer $n \geq N$, there exist two eigenvalues $\lambda_{1,n}$ and $\lambda_{2,n}$ such that the differential equation

$$T_\lambda(y) = 0 \tag{5.5}$$

has a polynomial solution of degree n . Moreover, $\lim_{n \rightarrow \infty} \frac{\lambda_{i,n}}{n} = \alpha_i$ where α_1, α_2 are the roots of the characteristic polynomial of T_λ .

Proof. Observe that for any $\lambda \in \mathbb{C}$, the operator T_λ acts on each linear space Pol_n of all polynomials of degree at most n , $n = 0, 1, 2, \dots$, and its matrix presentation $(c_{ij})_{i,j=0}^n$ in the standard monomial basis $(1, z, z^2, \dots, z^n)$ of Pol_n is an upper-triangular matrix with diagonal entries

$$c_{jj} = j(j - 1)q_{22} + jq_{11} + q + (jq_{11} + p)\lambda + q_{00}\lambda^2.$$

Therefore, for any given non-negative integer n , we have a (unique) polynomial solution of (5.5) of degree n if and only if $c_{nn} = 0$ but $c_{jj} \neq 0$ for $0 \leq j < n$. The asymptotic formula for $\lambda_{i,n}$ follows from the form of the equation $c_{nn} = 0$. The genericity assumption that the equations

$$n(n - 1)q_{22} + nq_{11} + q + (nq_{11} + p)\lambda + q_{00}\lambda^2 = 0$$

and

$$j(j - 1)q_{22} + jq_{11} + q + (jq_{11} + p)\lambda + q_{00}\lambda^2 = 0$$

should not have a common root, for $0 \leq j < n$ and n sufficiently large, is clearly satisfied if we assume that the characteristic equation does not have two roots with the same argument. \square

We can now prove the following stronger result.

Proposition 7. *For a general type family of differential operators T_λ of the form (5.3), all roots of all polynomial solutions of $T_\lambda(p) = 0$, $\lambda \in \mathbb{C}$ are located in some compact set $K \subset \mathbb{C}$.*

Proof. Since T_λ is assumed to be of general type, one gets $Q_0 \neq 0$. Therefore, without loss of generality we can assume that $Q_0 = 1$ in (5.5). Let $\{p_n\}$, $\deg(p_n) = n$ be a sequence of eigenpolynomials for (5.5), and assume that $\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \alpha$. (By Lemma 9, α equals either α_1 or α_2 .) Define $w_n = \frac{p'_n}{\lambda_n p_n}$ and notice that $p_n = e^{\lambda_n \int w_n dz}$. We then have

$$p'_n = \lambda_n w_n p_n; p''_n = (\lambda_n^2 w_n^2 + \lambda_n w'_n) p_n.$$

Substituting these expressions in (5.5), we obtain:

$$p_n(Q_2(z)(\lambda_n^2 w_n^2(z) + \lambda_n w'_n(z)) + \lambda_n^2 Q_1(z)w_n(z) + P_1(z)\lambda_n w_n(z) + \lambda_n^2 + p\lambda_n + q) = 0.$$

For each fixed n , near $z = \infty$ we can conclude that

$$Q_2(z)(\lambda_n^2 w_n^2(z) + \lambda_n w'_n(z)) + \lambda_n^2 Q_1(z)w_n(z) + P_1(z)\lambda_n w_n(z) + \lambda_n^2 + p\lambda_n + q = 0.$$

This relation defines a rational function w_n near infinity. We will show that the sequence $\{w_n\}$ converges uniformly to an analytic function w in a sufficiently small disc around ∞ . Moreover w does not vanish identically. Proposition 7 will immediately follow from this claim. Introducing $t = \frac{1}{z}$, one obtains

$$\tilde{Q}_2 \left(\left(\frac{w_n}{t} \right)^2 - \frac{1}{\lambda_n} w'_n \right) + \tilde{Q}_1 \left(\frac{w_n}{t} \right) + \frac{1}{\lambda_n} \tilde{P}_1 \left(\frac{w_n}{t} \right) + 1 + \frac{p}{\lambda_n} + \frac{q}{\lambda_n^2} = 0,$$

where $\tilde{Q}_2(t) := t^2 Q_2(1/t)$, $\tilde{Q}_1(t) := t Q_1(1/t)$ and $\tilde{P}_1(t) := t P_1(1/t)$. Expand $w_n = c_1 t + c_2 t^2 + \dots$ in a power series around ∞ , i.e., around $t = 0$. (By a slight

abuse of notation, we temporarily disregard the fact that the coefficients c_k depend on n until we make their proper estimate.) Set $(w_n/t)^2 = b_0 + b_1t + \dots$. Then

$$b_k = c_1c_{k+1} + c_2c_k + \dots + c_kc_2 + c_{k+1}c_1.$$

Finally, introduce $\epsilon_n = 1/\lambda_n$. Using these notations we obtain the following system of recurrence relations for the coefficients c_k :

$$\begin{aligned} q_{22}c_1^2 + (q_{11} - \epsilon_nq_{22} + \epsilon_np_{11})c_1 + 1 + \epsilon_np + \epsilon_n^2q &= 0, \\ q_{22}(b_1 - 2\epsilon_nc_2) + q_{12}(b_0 - \epsilon_nc_1) + (q_{11} + \epsilon_np_{11})c_2 + (q_{01} + \epsilon_np_{01})c_1 &= 0, \\ q_{22}(b_2 - 3\epsilon_nc_3) + q_{12}(b_1 - 2\epsilon_nc_2) + q_{02}(b_0 - \epsilon_nc_1) \\ + (q_{11} + \epsilon_np_{11})c_3 + (q_{01} + \epsilon_np_{01})c_2 &= 0, \end{aligned}$$

and, more generally,

$$\begin{aligned} q_{22}(b_k - (k + 1)\epsilon_nc_{k+1}) + q_{12}(b_{k-1} - k\epsilon_nc_k) + q_{02}(b_{k-2} - (k - 1)\epsilon_nc_{k-1}) \\ + (q_{11} + \epsilon_np_{11})c_{k+1} + (q_{01} + \epsilon_np_{01})c_k = 0 \quad \text{for } k \geq 2. \end{aligned}$$

Therefore, for any given n , we get 2 possible values for $c_1(n)$, which tend to the roots of $q_{22}t^2 + q_{11}t + 1 = 0$ as $n \rightarrow \infty$. Notice that $c_1(n) \rightarrow \frac{1}{\alpha}$ as $n \rightarrow \infty$. Choosing one of two possible values for c_1 , we uniquely determine the remaining coefficients (as rational functions of the previously calculated coefficients). Introducing $\tilde{b}_k = b_k - 2c_1c_{k+1}$, we can observe that \tilde{b}_k is independent of c_{k+1} and we obtain the following explicit formulas:

$$\begin{aligned} c_2 &= -\frac{q_{12}(c_1^2 - \epsilon_nc_1) + (q_{01} + \epsilon_np_{01})c_1}{(2c_1 - 2\epsilon_n)q_{22} + q_{11} + \epsilon_np_{11}}, \\ c_3 &= -\frac{q_{22}\tilde{b}_2 + q_{12}(b_1 - 2\epsilon_nc_2) + q_{02}(b_0 - \epsilon_nc_1) + (q_{01} + \epsilon_np_{01})c_2}{(2c_1 - 3\epsilon_n)q_{22} + q_{11} + \epsilon_np_{11}}, \end{aligned}$$

and more generally,

$$\begin{aligned} c_k &= -\frac{q_{22}\tilde{b}_{k-1} + q_{12}(b_{k-2} - (k - 1)\epsilon_nc_{k-1})}{(2c_1 - k\epsilon_n)q_{22} + q_{11} + \epsilon_np_{11}} \\ &\quad + \frac{q_{02}(b_{k-2} - (k - 3)\epsilon_nc_{k-3}) + (q_{01} + \epsilon_np_{01})c_{k-1}}{(2c_1 - k\epsilon_n)q_{22} + q_{11} + \epsilon_np_{11}}. \end{aligned}$$

We will now include the dependence of c_k on n and show that the coefficients $c_k(n)$ are majorated by the coefficients of a convergent power series independent of n . First we show that the denominators in these recurrence relations are bounded from below. Notice that under our assumption, the rational functions w_n exist and have a power series expansion near $z = \infty$ with coefficients given by the above recurrence relations. Therefore the denominators in these recurrences do not vanish. Notice also that $\epsilon_n \simeq \frac{c_1(n)}{n}$ asymptotically. For fixed k , it is therefore clear that the limits

$$\lim_{n \rightarrow \infty} (2c_1(n) - k\epsilon_n)q_{22} + q_{11} + \epsilon_np_{11} = \lim_{n \rightarrow \infty} 2c_1(n)q_{22} + q_{11}$$

vanish if and only if the characteristic polynomial (5.4) has a double root. We must however find a uniform bound for $c_k(n)$ valid for all k simultaneously. Indeed, there might exist a subsequence $I \subset \mathbb{N}$ of k_n such that

$$\lim_{n \in I; n \rightarrow \infty} (2c_1(n) - k_n \epsilon_n) q_{22} + q_{11} + \epsilon_n p_{11} = 0. \tag{5.6}$$

(1) But this implies, using the asymptotics of $c_1(n)$ and ϵ_n , the existence of a real number r such that $\frac{1-r}{\alpha} = -\frac{q_{22}}{2q_{11}}$ which is clearly impossible if the characteristic equation does not have two roots with the same argument. Thus we have established a positive lower bound for the absolute value of the denominators in the recurrence relations for the coefficients c_k . The latter circumstance gives us a possibility of majorizing the coefficients $c_k(n)$ independently of k and n . Namely, if there is a unbounded sequence $k_n \epsilon_n$, then we can factor it out from the rational functions in the recurrence. The existence of the sequence mentioned above follow from an elementary lemma stated below, which we leave without a proof. Thus, Proposition 7 is now settled. \square \Leftarrow (1)?

Lemma 10. *Consider a recurrence relation $c_{m+1} = P_m(c_1, \dots, c_m)$ where each P_m is a polynomial and assume that $d_{m+1} = Q_m(d_1, \dots, d_m)$ is a similar recurrence relation whose polynomials have all positive coefficients. If the polynomials under consideration satisfy the inequalities*

$$|P_m(z_1, \dots, z_m)| \leq Q_m(|z_1|, \dots, |z_m|),$$

then the power series $\sum c_i z^i$ is dominated by the series $\sum d_i z^i$ whenever $d_1 \geq |c_1|$.

6. Domain configurations of normalized quadratic differentials

Let $Q(z; a, b, c) dz^2$ be a quadratic differential of the form (1.5). Multiplying $Q(z; a, b, c) dz^2$ by a non-zero constant $A \in \mathbb{C}$, we rescale the corresponding Q -metric $|Q|^{1/2} |dz|$ by a positive constant $|A|^{1/2}$. Hence $A Q(z; a, b, c) dz^2$ has the same geodesics as the quadratic differential $Q(z; a, b, c) dz^2$ has. Obviously, multiplication does not affect the homotopic classes. Thus, while studying geodesics of the quadratic differential $Q(z; a, b, c) dz^2$, we may assume without loss of generality that it has the form

$$Q(z) dz^2 = -\frac{(z - p_1)(z - p_2)}{(z - 1)^2(z + 1)^2} dz^2. \tag{6.1}$$

In Sections 6–9, we will work with the generic case; i.e., we assume that

$$p_1 \neq \pm 1, \quad p_2 \neq \pm 1, \quad p_1 \neq p_2, \tag{6.2}$$

unless otherwise is mentioned. Some typical configurations in the limit (or non-generic) cases are shown in Figures 5a–5g. Expanding $Q(z)$ into Laurent series at $z = \infty$, we obtain

$$Q(z) = -\frac{1}{z^2} + \text{higher degrees of } z \quad \text{as } z \rightarrow \infty. \tag{6.3}$$

Since the leading coefficient in the series expansion (6.3) is real and negative it follows that $Q(z) dz^2$ has a circle domain D_∞ centered at $z = \infty$. The boundary $L_\infty = \partial D_\infty$ of D_∞ consists of a finite number of critical trajectories of the quadratic differential $Q(z) dz^2$ and therefore L_∞ contains at least one of the zeros p_1 and p_2 of $Q(z) dz^2$.

Next, we will discuss possible trajectory structures of $Q(z) dz^2$ on the complement $D_0 = \mathbb{C} \setminus \overline{D_\infty}$. As we have mentioned in Section 3, according to the Basic Structure Theorem, [21, Theorem 3.5], the domain configuration of a quadratic differential $Q(z) dz^2$ on $\overline{\mathbb{C}}$, which will be denoted by \mathcal{D}_Q , may include circle domains, ring domains, strip domains, end domains, and density domains. For the quadratic differential (6.1), by the Three Pole Theorem [21, Theorem 3.6], there are no density domains in its domain configuration \mathcal{D}_Q . In addition, since $Q(z) dz^2$ has only three poles of order two each, the domain configuration \mathcal{D}_Q does not contain end domains and may contain at most three circle domains centered at $z = \infty$, $z = -1$, and $z = 1$.

We note here that \mathcal{D}_Q may have strip domains (also called *bilaterals*) with vertices at the double poles $z = -1$ and $z = 1$ but \mathcal{D}_Q does not have ring domains. Indeed, if there were a ring domain $\widehat{D} \subset D_0$ with boundary components l_1 and l_2 then, by the Basic Structure Theorem, each component must contain a zero of $Q(z) dz^2$. In particular, $p_1 \neq p_2$ in this case. Suppose that l_1 contains a zero p_1 and that $p_1 \in L_\infty$. Then L_∞ contains a critical trajectory γ' , which has both its end points at p_1 . There is one more critical trajectory γ'' , which has one of its end points at p_1 . This trajectory γ'' is either lies on the boundary of the circle domain D_∞ or it lies on the boundary of the ring domain \widehat{D} . Therefore the second end point of γ'' must be at a zero of $Q(z) dz^2$. Since the only remaining zero is p_2 , which lies on the boundary component l_2 not intersecting l_1 , we obtain a contradiction with our assumption. The latter shows that \mathcal{D}_Q does not have ring domains.

Next, we will classify topological types of domain configurations according to the number of circle domains in \mathcal{D}_Q . The first digit in our further classifications stands for the section where this classification is introduced. The second and further digits will denote the case under consideration.

6.1. Assume first that \mathcal{D}_Q contains three circle domains $D_\infty \ni \infty$, $D_{-1} \ni -1$, and $D_1 \ni 1$. Then, of course, there are no strip domains in \mathcal{D}_Q . In this case, the domains D_∞, D_{-1}, D_1 constitute an extremal configuration of the Jenkins extremal problem for the weighted sum of reduced moduli with appropriate choice of positive weights $\alpha_\infty, \alpha_{-1}$, and α_1 ; see, for example, [33], [30], [31]. More precisely, the problem is to find all possible configurations realizing the following maximum:

$$\max (\alpha_\infty^2 m(B_\infty, \infty) + \alpha_{-1}^2 m(B_{-1}, -1) + \alpha_1^2 m(B_1, 1)) \quad (6.4)$$

over all triples of non-overlapping simply connected domains $B_\infty \ni \infty$, $B_{-1} \ni -1$, and $B_1 \ni 1$. Here, $m(B, z_0)$ stands for the reduced module of a simply connected domain B with respect to the point $z_0 \in B$; see [21, p.24].

Since the extremal configuration of problem (6.4) is unique it follows that the domains D_∞ , D_{-1} , and D_1 are symmetric with respect to the real axis. In particular, the zeros p_1 and p_2 are either both real or they are complex conjugates of each other. Of course, this symmetry property of zeros can be derived directly from the fact that the leading coefficient of the Laurent expansion of $Q(z)$ at each its pole is negative in the case under consideration. We have three essentially different possible positions for the zeros:

- (a) $-1 < p_2 < p_1 < 1$,
- (b) $1 < p_2 < p_1$ or $p_1 < p_2 < -1$,
- (c) $p_1 = \bar{p}_2 = p$, where $\Im p > 0$.

We note here that in the case when $-1 < p_2 < 1$ and, in addition, $p_1 > 1$ or $p_1 < -1$ the domain configuration \mathcal{D}_Q must contain a strip domain.

Case (a). The trajectory structure of $Q(z) dz^2$ corresponding to this case is shown in Figure 1a. There are three critical trajectories: γ_{-1} , which is on the boundary of D_{-1} and has both its end points at $z = p_2$; γ_1 , which is on the boundary of D_1 and has both its end points at $z = p_1$, and γ_0 , which is the segment $[p_2, p_1]$.

Case (b). An example of a domain configuration for the case $1 < p_2 < p_1$ is shown in Figure 1b. The boundary of D_1 consists of a single critical trajectory γ_1 having both end points at p_2 . The boundary of D_{-1} consists of critical trajectories γ_∞ , γ_1 , and γ_0 , which is the segment $[p_2, p_1]$. In the case $p_1 < p_2 < -1$, the domain configuration is similar.

Case (c). Since the domain configuration is symmetric, p_1 and p_2 both belong to the boundary of D_∞ . Furthermore, there are three critical trajectories: γ_{-1} , which joins p_1 and p_2 and intersects the real axis at some point $d_{-1} < -1$, γ_1 , which joins p_1 and p_2 and intersects the real axis at some point $d_1 > 1$, and γ^0 , which joins p_1 and p_2 and intersects the real axis at some point d_0 , $-1 < d_0 < 1$. In this case, $\gamma_1 \cup \gamma_0 \subset \partial D_1$, $\gamma_{-1} \cup \gamma_0 \subset \partial D_{-1}$. An example of a domain configuration of this type is shown in Figure 1c.

6.2. Next we consider the case when \mathcal{D}_Q has exactly two circle domains. Suppose that these domains are $D_\infty \ni \infty$ and $D_{-1} \ni -1$. In this case it is not difficult to see that L_∞ contains exactly one zero. Indeed, if $p_1, p_2 \in L_\infty$, then L_∞ must contain one or two critical trajectories joining p_1 and p_2 . Suppose that L_∞ contains one such trajectory, call it γ_0 . Since $p_1, p_2 \in L_\infty$ the boundary of D_∞ must contain a trajectory γ_1 , which has both its end points at p_1 and a trajectory γ_{-1} , which has both its end points at p_2 . Thus, $\gamma_1 \cup \{p_1\}$ and $\gamma_{-1} \cup \{p_2\}$ each surrounds a simply connected domain, which must contain a critical point of $Q(z) dz^2$. This implies that $z = -1$ and $z = 1$ are centers of circle domains of $Q(z) dz^2$, which is the case considered in part **6.1(a)**.

If L_∞ contains two critical trajectories joining p_1 and p_2 , then there are critical trajectories γ' having one of its end points at p_1 and γ'' having one of its end points at p_2 . If $\gamma' = \gamma''$, then $D_0 \setminus \gamma'$ consists of two simply connected domains, which in this case must be circle domains of $Q(z) dz^2$ as it is shown in Figure 1c.

If $\gamma' \neq \gamma''$, then each of these trajectories must have its second end point at one of the poles $z = -1$ or $z = 1$. Moreover, if γ' has an end point at $z = -1$ then γ'' must have its end point at $z = 1$. Thus, there is no second circle domain of $Q(z) dz^2$ in this case. Instead, there is one circle domain D_∞ and a strip domain, call it G_2 , as it shown in Figures 3a–3e.

Now, let p_1 be the only zero of $Q(z) dz^2$ lying on L_∞ . Then L_∞ consists of a single critical trajectory of $Q(z) dz^2$, call it γ_∞ , together with its end points, each of which is at p_1 . There is one more critical trajectory, call it γ_1^+ , that has one of its end points at p_1 . Then the second end point of γ_1^+ is either at the point p_2 or at the second-order pole at $z = 1$.

If γ_1^+ terminates at p_2 , then there is one more critical trajectory, call it γ_2 , having one of its end points at p_2 . Since D_{-1} is a circle domain and ∂D_{-1} contains at least one zero of $Q(z) dz^2$ it follows that γ_2 belongs to the boundary of D_{-1} . Since γ_2 lies on the boundary of D_{-1} it have to terminate at a finite critical point of $Q(z) dz^2$ and the only possibility for this is that γ_2 terminates at p_2 . In this case, γ_∞ , γ_1^+ , and γ_2 divide $\overline{\mathbb{C}}$ into three circle domains, the case which was already discussed in part 6.1(b).

Suppose that γ_1^+ joins the points $z = p_1$ and $z = 1$. Then \mathcal{D}_Q contains a strip domain G_1 . Since $z = 1$ is the only second-order pole of $Q(z) dz^2$, which has a non-negative non-zero leading coefficient, the strip domain G_1 has both its vertices at the point $z = 1$. Furthermore, one side of G_1 consists of two critical trajectories γ_∞ and γ_1^+ . Therefore there is a critical trajectory, call it γ_1^- of $Q(z) dz^2$ lying on ∂G_1 , which joins $z = 1$ and $z = p_2$. Now, the remaining possibility is that the boundary of D_{-1} consists of a single critical trajectory γ_{-1} , which has both its end points at p_2 . Then G_1 is the only strip domain in \mathcal{D}_Q and the second side of G_1 consists of the critical trajectories γ_1^- and γ_{-1} . Two examples of a domain configuration of this type, symmetric and non-symmetric, are shown in Figure 2a and Figure 2b.

6.3. Finally, we consider the case when D_∞ is the only circle domain of $Q(z) dz^2$. We consider two possibilities.

Case (a). Suppose that both zeros p_1 and p_2 belong to the boundary of D_∞ . As we have found in part 6.2 above, the domain configuration in this case consists of the circle domain D_∞ and the strip domain G_2 . The boundary of D_∞ consists of two critical trajectories γ_∞^+ and γ_∞^- and their end points, while the boundary of G_2 consists of the trajectories γ_∞^+ , γ_∞^- , γ_1 , and γ_{-1} and their end points, as it is shown in Figures 3a–3c.

Case (b). Suppose that the boundary L_∞ of D_∞ contains only one zero p_1 . Then there is a critical trajectory γ_∞ having both its end points at p_1 such that $L_\infty = \gamma_\infty \cup \{p_1\}$. Since p_1 is a simple zero of $Q(z) dz^2$ there is one more critical trajectory having one of its end points at p_1 . The second end point of this trajectory is either at the pole $z = 1$, or at the pole $z = -1$, or at the zero $z = p_2$. Depending on which of these possibilities is realized, this trajectory will be denoted by γ_1 , or γ_{-1} , or γ_0 , respectively. Thus, we have two essentially different subcases.

Case **(b1)**. Suppose that there is a critical trajectory γ_0 joining the zeros p_1 and p_2 . Then there are two critical trajectories, call them γ_1 and γ_{-1} , each of which has one of its end point at p_2 . We note that $\gamma_1 \neq \gamma_{-1}$. Indeed, if $\gamma_1 = \gamma_{-1}$, then the closed curve $\gamma_1 \cup \{p_2\}$ must enclose a bounded circle domain of $Q(z) dz^2$, which does not exist. Furthermore, γ_1 and γ_{-1} both cannot have their second end points at the same pole at $z = 1$ or $z = -1$. If this occurs then again γ_1 and γ_{-1} will enclose a simply connected domain having a single pole of order 2 on its boundary, which is not possible. The remaining possibility is that one of these critical trajectories, let assume that γ_1 , joins the zero $z = p_2$ and the pole at $z = 1$ while γ_{-1} joins $z = p_2$ and $z = -1$.

In this case the domain configuration \mathcal{D}_Q consists of the circle domain D_∞ and the strip domain G_2 ; see Figure 3d and Figure 3e. The boundary of G_2 consists of two sides, call them l_1 and l_2 . The side l_1 is the set of boundary points of G_2 traversed by the point z moving along γ_1 from $z = 1$ to $z = p_2$ and then along γ_{-1} from the point $z = p_2$ to $z = -1$. The side l_2 is the set of boundary points of G_2 traversed by the point z moving along γ_1 from $z = 1$ to $z = p_2$, then along γ_0 from $z = p_2$ to $z = p_1$, then along γ_∞ from $z = p_1$ to the same point $z = p_1$, then along γ_0 from $z = p_1$ to $z = p_2$, and finally along γ_{-1} from $z = p_2$ to $z = -1$.

Case **(b2)**. Suppose that there is a critical trajectory γ_1 joining the zero p_1 and the pole $z = 1$. Then there is a strip domain, call it G_1 , which has both its vertices at the pole $z = 1$ and has the critical trajectories γ_1 and γ_∞ on one of its sides, call it l_1^1 . More precisely, the side l_1^1 is the set of boundary points of G_1 traversed by the point z moving along γ_1 from $z = 1$ to $z = p_1$, then along γ_∞ from $z = p_1$ to the same point $z = p_1$, and then along γ_1 from $z = p_1$ to $z = 1$.

Let l_1^2 denote the second side of G_1 . Since a side of a strip domain always has a finite critical point it follows that l_1^2 contains two critical trajectories, call them γ_0^+ and γ_0^- , which join the pole $z = 1$ with zero $z = p_2$. There is one critical trajectory of $Q(z) dz^2$, call it γ_{-1} , which has one of its end points at $z = p_2$. Since $z = -1$ is a second-order pole, which is not the center of a circle domain, there should be at least one critical trajectory of $Q(z) dz^2$ approaching $z = -1$ at least in one direction. Since the end points of all critical trajectories, except γ_{-1} , are already identified and they are not at $z = -1$, the remaining possibility is that γ_{-1} has its second end point at $z = -1$. In this case there is one more strip domain, call it G_2 , which has vertices at the poles $z = 1$ and $z = -1$ and sides l_2^1 and l_2^2 . Two examples of configurations with one circle domain and two strip domains, symmetric and non-symmetric, are shown in Figure 4a and Figure 4b. Now we can identify all sides of G_1 and G_2 . The side l_1^2 is the set of boundary points of G_1 traversed by the point z moving along γ_0^+ from $z = 1$ to $z = p_2$ and then along γ_0^- from $z = p_2$ to $z = 1$. The side l_2^1 is the set of boundary points of G_2 traversed by the point z moving along γ_0^+ from $z = 1$ to $z = p_2$ and then along γ_{-1} from $z = p_2$ to $z = -1$. Finally, the side l_2^2 is the set of boundary points of G_2 traversed by the point z moving along γ_0^- from $z = 1$ to $z = p_2$ and then along γ_{-1} from $z = p_2$ to $z = -1$; see Figure 4a and Figure 4b.

Case **(b3)**. In the case when there is a critical trajectory joining the zero p_1 and the pole $z = -1$, the domain configuration is similar to one described above, we just have to switch the roles of the poles at $z = 1$ and $z = -1$.

Remark 2. We have described above all possible configurations in the generic case; i.e., under conditions (6.2). The remaining special cases can be obtained from the generic case as limit cases when $p_2 \rightarrow -1$, when $p_2 \rightarrow p_1$; etc. In the case $p_1 = p_2$, possible configurations are shown in Figures 5a–5c.

In the case when $p_2 = -1$, $p_1 \neq \pm 1$, possible configurations are shown in Figures 5d–5g.

In the case when $p_1 = p_2 = 1$, the limit position of critical trajectories is just a circle centers at $z = -1$ with radius 2 configuration and in the case when $p_1 = 1$, $p_2 = -1$ there is one critical trajectory which is an open interval from $z = -1$ to $z = 1$.

7. How parameters determine the type of domain configuration

Our goal in this section is to identify the ranges of the parameters p_1 and p_2 corresponding to topological types discussed in Section 6. For a fixed p_1 with $\Im p_1 \neq 0$, we will define four regions of the parameter p_2 . These regions and their boundary arcs will correspond to domain configurations with specific properties; see Figure 6.

It will be useful to introduce the following notation. For $a \in \mathbb{C}$ with $\Im a \neq 0$, by $L(a)$ and $H(a)$ we denote, respectively, an ellipse and hyperbola with foci at $z = 1$ and $z = -1$, which pass through the point $z = a$. If $\Im a \neq 0$, then the set $\mathbb{C} \setminus (L(a) \cup H(a))$ consists of four connected components, which will be denoted by $E_1^+(a)$, $E_1^-(a)$, $E_{-1}^+(a)$, and $E_{-1}^-(a)$. We assume here that $1 \in E_1^+(a)$, $-1 \in E_{-1}^+(a)$, $E_1^-(a) \cap \mathbb{R}_+ \neq \emptyset$, and $E_{-1}^-(a) \cap \mathbb{R}_- \neq \emptyset$. Furthermore, assuming that $\Im a \neq 0$, we define the following open arcs: $L^+(a) = (L(a) \cap \partial E_1^+(a)) \setminus \{a, \bar{a}\}$, $L^-(a) = (L(a) \cap \partial E_{-1}^+(a)) \setminus \{a, \bar{a}\}$, $H^+(a) = (H(a) \cap \partial E_1^+(a)) \setminus \{a, \bar{a}\}$, $H^-(a) = (H(a) \cap \partial E_{-1}^-(a)) \setminus \{a, \bar{a}\}$. Let $l_1(a)$ and $l_{-1}(a)$ be straight lines passing through the points 1 and \bar{a} and -1 and \bar{a} , respectively. Let $l_1^+(a)$ and $l_{-1}^+(a)$ be open rays issuing from the points $z = 1$ and $z = -1$, respectively, which pass through the point $z = \bar{a}$ and let $l_1^-(a)$ and $l_{-1}^-(a)$ be their complementary rays. The line $l_1(a)$ divides \mathbb{C} into two half-planes, we call them P_1 and P_2 and enumerate such that $P_1 \ni 2$. Similarly, the line $l_{-1}(a)$ divides \mathbb{C} into two half-planes P_3 and P_4 , where $P_3 \ni -2$.

Before we state the main result of this section, we recall the reader that the local structure of trajectories near a pole z_0 is completely determined by the leading coefficient of the Laurent expansion of $Q(z)$ at z_0 , see [21, Ch. 3]. In particular, for the quadratic differential $Q(z) dz^2$ defined by (6.1) we have

$$Q(z) = -\frac{1}{4} \frac{C_1}{(z-1)^2} + \text{higher degrees of } (z-1) \quad \text{as } z \rightarrow 1 \quad (7.1)$$

and

$$Q(z) = -\frac{1}{4} \frac{C_{-1}}{(z+1)^2} + \text{higher degrees of } (z+1) \quad \text{as } z \rightarrow -1.$$

Then, assuming that $p_1 \neq \pm 1, p_2 \neq \pm 1$, we find

$$C_1 = (p_1 - 1)(p_2 - 1) \neq 0 \quad \text{and} \quad C_{-1} = (p_1 + 1)(p_2 + 1) \neq 0. \quad (7.2)$$

A complete description of sets of pairs p_1, p_2 with $\Im p_1 > 0$ corresponding to all possible types of domain configurations discussed in Section 6 is given by the following theorem.

Theorem 4. *Let p_1 with $\Im p_1 > 0$ be fixed. Then the following holds.*

7.A. *The types of domain configurations \mathcal{D}_Q correspond to the following sets of the parameter p_2 .*

- (1) *If $p_2 = \bar{p}_1$, then the domain configuration \mathcal{D}_Q is of the type **6.1(c)**.*
- (2) *If $p_2 \in l_1^+(p_1) \setminus \{\bar{p}_1\}$, then \mathcal{D}_Q has the type **6.2** with circle domains $D_\infty \ni \infty$ and $D_1 \ni 1$. Furthermore, if $p_2 \in l_1^+(p_1) \cap E_1^+(p_1)$, then $p_1 \in \partial D_\infty$ and if $p_2 \in l_1^+(p_1) \cap E_{-1}^-(p_1)$, then $p_2 \in \partial D_\infty$.*

*If $p_2 \in l_{-1}^+(p_1) \setminus \{\bar{p}_1\}$, then \mathcal{D}_Q has the type **6.2** with circle domains $D_\infty \ni \infty$ and $D_{-1} \ni -1$. Furthermore, if $p_2 \in l_{-1}^+(p_1) \cap E_{-1}^+(p_1)$, then $p_1 \in \partial D_\infty$ and if $p_2 \in l_{-1}^+(p_1) \cap E_{-1}^-(p_1)$, then $p_2 \in \partial D_\infty$.*

- (3a) *If $p_2 \in L(a) \setminus \{p_1, \bar{p}_1\}$, then the domain configuration \mathcal{D}_Q has type **6.3(a)**. Furthermore, if $p_2 \in L^+(p_1)$, then there is a critical trajectory having one end point at p_2 , which in other direction approaches the pole $z = 1$. Similarly, if $p_2 \in L^-(p_1)$, then there is a critical trajectory having one end point at p_2 , which in other direction approaches the pole $z = -1$.*
- (3b1) *If $p_2 \in H(p_1) \setminus \{p_1, \bar{p}_1\}$, then \mathcal{D}_Q has type **6.3(b1)**. Furthermore, if $p_2 \in H^+(p_1)$, then there is a critical trajectory having both end points at p_1 . If $p_2 \in H^-(p_1)$, then there is a critical trajectory having both end points at p_2 .*
- (3b2) *In all remaining cases, i.e., if $p_2 \notin L(p_1) \cup H(p_1) \cup l_1^+(p_1) \cup l_{-1}^+(p_1) \cup \{-1, 1\}$, the domain configuration \mathcal{D}_Q belongs to type **6.3(b2)**. Furthermore, if $p_2 \in (E_1^+(p_1) \cup E_{-1}^+(p_1)) \setminus (l_1^+(p_1) \cup l_{-1}^+(p_1) \cup \{-1, 1\})$, then $p_1 \in \partial D_\infty$ and if $p_2 \in (E_1^-(p_1) \cup E_{-1}^-(p_1)) \setminus (l_1^+(p_1) \cup l_{-1}^+(p_1))$, then $p_2 \in \partial D_\infty$.*

In addition, if $p_2 \in E_1^+(p_1) \setminus (l_1^+(p_1) \cup \{1\})$, then the pole $z = 1$ attracts only one critical trajectory of the quadratic differential (6.1), which has its second end point at $z = p_2$ and if $p_2 \in E_{-1}^-(p_1) \setminus (l_1^+(p_1))$, then the pole $z = 1$ attracts only one critical trajectory of the quadratic differential (6.1), which has its second end point at $z = p_1$. If $p_2 \in E_{-1}^+(p_1) \setminus (l_{-1}^+(p_1) \cup \{-1\})$, then the pole $z = -1$ attracts only one critical trajectory of the quadratic differential (6.1), which has its second end point at $z = p_2$ and if $p_2 \in E_1^-(p_1) \setminus (l_{-1}^+(p_1))$, then the pole $z = -1$ attracts only one critical trajectory of the quadratic differential (6.1), which has its second end point at $z = p_1$.

7.B. *The local behavior of the trajectories near the poles $z = 1$ and $z = -1$ is controlled by the position of the zero p_2 with respect to the lines $l_1(p_1)$ and $l_{-1}(p_1)$. Precisely, we have the following possibilities.*

- (1) *If $p_2 \in l_1^-(p_1)$ or, respectively, $p_2 \in l_{-1}^-(p_1)$, then $Q(z) dz^2$ has radial structure of trajectories near the pole $z = 1$ or, respectively, near the pole $z = -1$.*
- (2) *If $p_2 \in P_1$ or, respectively, $p_2 \in P_2$, then the trajectories of $Q(z) dz^2$ approaching the pole $z = 1$ spiral counterclockwise or, respectively, clockwise.*
If $p_2 \in P_3$ or, respectively, $p_2 \in P_4$, then the trajectories of $Q(z) dz^2$ approaching the pole $z = -1$ spiral counterclockwise or, respectively, clockwise.

Proof. **7.A(1).** We have shown in Section 6 that a domain configuration \mathcal{D}_Q of the type **6.1(c)** occurs if and only if $p_2 = \bar{p}_1$. Thus, we have to consider cases **7.A(2)** and **7.A(3)**. We first prove statements about positions of zeros p_1 and p_2 for each of these cases. Then we will turn to statements about critical trajectories.

7.A(2). A domain configuration \mathcal{D}_Q contains exactly two circle domains centered at $z = \infty$ and $z = -1$ if and only if $C_{-1} > 0$ and C_1 is not a positive real number. This is equivalent to the following conditions:

$$\arg(p_1 + 1) = -\arg(p_2 + 1) \pmod{2\pi}, \tag{7.3}$$

$$\arg(p_1 - 1) \neq -\arg(p_2 - 1) \pmod{2\pi}. \tag{7.4}$$

Geometrically, equations (7.3) and (7.4) mean that the points p_1 and p_2 lie on the rays issuing from the pole $z = -1$, which are symmetric to each other with respect to the real axis. Furthermore, each ray contains one of these points and $p_1 \neq \bar{p}_2$.

Assuming (7.3), (7.4), we claim that $p_1 \in \partial D_\infty$ if and only if $|p_2 + 1| < |p_1 + 1|$. First we prove that the claim is true for all p_2 sufficiently close to $z = -1$ if p_1 is fixed. Arguing by contradiction, suppose that there is a sequence $s_k \rightarrow -1$ such that $\arg(s_k + 1) = -\arg(p_1 + 1)$ and $p_1 \in \partial D_{-1}^k$, $s_k \in \partial D_\infty^k$ for all $k = 1, 2, \dots$. Here $D_{-1}^k \ni -1$ and $D_\infty^k \ni \infty$ denote the corresponding circle domains of the quadratic differential

$$Q_k(z) dz^2 = -\frac{(z - p_1)(z - s_k)}{(z - 1)^2(z + 1)^2} dz^2. \tag{7.5}$$

Changing variables in (7.5) via $z = (s_k + 1)\zeta - 1$ and then dividing the resulting quadratic differential by $\delta_k = |s_k + 1|$, we obtain the following quadratic differential:

$$\widehat{Q}_k(\zeta) d\zeta^2 = \frac{\zeta - 1}{\zeta^2} \frac{|1 + p_1| - \delta_k^{-1}(s_k + 1)^2 \zeta}{(2 - (s_k + 1)\zeta)^2} d\zeta^2. \tag{7.6}$$

We note that the trajectories of $Q_k(z) dz^2$ correspond under the mapping $z = (s_k + 1)\zeta - 1$ to the trajectories of the quadratic differential $\widehat{Q}_k(\zeta) d\zeta^2$. Thus, $\widehat{Q}_k(\zeta) d\zeta^2$ has two circle domains $\widehat{D}_{k,\infty} \ni \infty$ and $\widehat{D}_{k,0} \ni 0$. The zeros of $\widehat{Q}_k(\zeta) d\zeta^2$ are at the points

$$\zeta'_k = 1 \in \partial \widehat{D}_{k,\infty}, \quad \zeta''_k = \delta_k |1 + p_1| (s_k + 1)^{-2} \in \partial \widehat{D}_{k,0}. \tag{7.7}$$

From (7.6), we find that

$$\widehat{Q}_k(\zeta) d\zeta^2 \rightarrow \widehat{Q}(\zeta) d\zeta^2 := \frac{|1+p_1|\zeta-1}{4\zeta^2} d\zeta^2, \tag{7.8}$$

where convergence is uniform on compact subsets of $\mathbb{C} \setminus \{0\}$. Since

$$\widehat{Q}(\zeta) = -(|1+p_1|/4)\zeta^{-2} + \dots \quad \text{as } \zeta \rightarrow 0$$

the quadratic differential $\widehat{Q}(\zeta) d\zeta^2$ has a circle domain \widehat{D} centered at $\zeta = 0$. Let $\widehat{\gamma}$ be a trajectory of $\widehat{Q}(\zeta) d\zeta^2$ lying in \widehat{D} and let $\widehat{\gamma}_k$ be an arbitrary trajectory of $\widehat{Q}_k(\zeta) d\zeta^2$ lying in the circle domain $\widehat{D}_{k,0}$. Since $\widehat{\gamma}_k$ is a \widehat{Q}_k -geodesic in its class and by (7.8) we have

$$|\widehat{\gamma}_k|_{\widehat{Q}_k} \leq |\widehat{\gamma}|_{\widehat{Q}} \rightarrow |\widehat{\gamma}|_{\widehat{Q}} = |1+p_1|^{1/2} \quad \text{as } k \rightarrow \infty. \tag{7.9}$$

On the other hand, conditions (7.7) imply that for every $R > 1$ there is k_0 such that for every $k \geq k_0$ there is an arc τ_k joining the circles $\{\zeta : |\zeta| = 1\}$ and $\{\zeta : |\zeta| = R\}$, which lies on regular trajectory of the quadratic differential $\widehat{Q}_k(\zeta) d\zeta^2$ lying in the circle domain $\widehat{D}_{k,0}$. Then, using (7.6), we conclude that there is a constant $C > 0$ independent on R and k such that

$$|\widehat{\gamma}_k|_{\widehat{Q}_k} \geq |\tau_k|_{\widehat{Q}_k} = \int_{\tau_k} |\widehat{Q}_k(\zeta)|^{1/2} |d\zeta| \geq C \int_1^R \frac{\sqrt{|\zeta|-1}}{|\zeta|} d|\zeta|$$

for all $k \geq k_0$. Since $\int_1^R x^{-1}\sqrt{x-1} dx \rightarrow \infty$ as $R \rightarrow \infty$, the latter equation contradicts equation (7.9). Thus, we have proved that if p_1 is fixed and p_2 is sufficiently close to $z = -1$ then $p_1 \in \partial D_\infty$ and $p_2 \in \partial D_{-1}$.

Now, we fix p_1 with $\Im p_1 \neq 0$ and consider the set A consisting of all points p'_2 on the ray $r = \{z : \arg(z+1) = -\arg(p_1+1)\}$ such that $p_1 \in \partial D_\infty(p_1, p_2)$ and $p_2 \in \partial D_{-1}(p_1, p_2)$ for all $p_2 \in r$ such that $|p_2+1| < |p'_2+1|$. Here $D_\infty(p_1, p_2)$ and $D_{-1}(p_1, p_2)$ are corresponding circle domains of the quadratic differential (6.1). Our argument above shows that $A \neq \emptyset$. Let $p_2^m \in r$ be such that

$$|p_2^m + 1| = \sup_{p_2 \in A} |p_2 + 1|.$$

Consider the quadratic differential $Q(z; p_1, p_2^m) dz^2$ of the form (6.1) with p_2 replaced by p_2^m . Let $D_\infty(p_1, p_2^m) \ni \infty$ and $D_{-1}(p_1, p_2^m) \ni -1$ be the corresponding circle domains of $Q(z; p_1, p_2^m) dz^2$. Since the quadratic differential (6.1) depends continuously on the parameters p_1 and p_2 , it is not difficult to show, using our definition of p_2^m , that both zeros of $Q(z; p_1, p_2^m) dz^2$ belong to the boundary of each of the domains $D_{-1}(p_1, p_2^m)$ and $D_\infty(p_1, p_2^m)$. But, as we have shown in part **6.2** of Section 6, in this case the domain configuration of $Q(z; p_1, p_2^m) dz^2$ must consist of three circle domains. Therefore, as we have shown in part **6.1** of Section 6, we must have $p_1^m = \bar{p}_1$.

Thus, we have shown that $p_2 \in \partial D_{-1}$ if p_1 and p_2 satisfy (7.3) and $|p_2+1| < |p_1+1|$. The Möbius map $w = \frac{3-z}{1+z}$ interchanges the poles $z = \infty$ and $z = -1$ of the quadratic differential (6.1) and does not change the type of its domain configuration. Therefore, our argument shows also that $p_1 \in \partial D_\infty$ if $|p_2+1| <$

$|p_1 + 1|$. This completes the proof of our claim that $p_1 \in \partial D_\infty$ if and only if $|p_2 + 1| < |p_1 + 1|$.

Similarly, if $Q(z) dz^2$ has exactly two circle domains $D_\infty \ni \infty$ and $D_1 \ni 1$, then $p_2 \in \partial D_1$ and $p_1 \in \partial D_\infty$ if and only if

$$\arg(p_1 - 1) = -\arg(p_2 - 1) \pmod{2\pi} \quad \text{and} \quad |p_2 - 1| < |p_1 - 1|.$$

7.A(3). In this part, we will discuss cases **6.3(a)**, **6.3(b1)**, and **6.3(b2)** discussed in Section 6. A domain configuration \mathcal{D}_Q contains exactly one circle domains centered at $z = \infty$ if and only if neither C_1 or C_{-1} is a positive real number. As we have found in Section 6, in this case there exist one or two strip domains G_1 and G_2 having their vertices at the poles $z = 1$ and $z = -1$. In what follows, we will use the notion of the *normalized height* h of a strip domain G , which is defined as

$$h = \frac{1}{2\pi} \Im \int_\gamma \sqrt{Q(z)} dz > 0,$$

where the integral is taken over any rectifiable arc $\gamma \subset G$ connecting the sides of G .

The sum of *normalized heights* in the Q -metric of the strip domains, which have a vertex at the pole $z = 1$ or at the pole $z = -1$ can be found using integration over circles $\{z : |z - 1| = r\}$ and $\{z : |z + 1| = r\}$ of radius r , $0 < r < 1$, as follows:

$$h_+ = \frac{1}{2\pi} \Im \int_{|z-1|=r} \sqrt{Q(z)} dz = \frac{1}{2} \Im \sqrt{C_1} = \frac{1}{2} \Im \sqrt{(p_1 - 1)(p_2 - 1)} \quad (7.10)$$

if $z = 1$ and

$$h_- = \frac{1}{2\pi} \Im \int_{|z+1|=r} \sqrt{Q(z)} dz = \frac{1}{2} \Im \sqrt{C_{-1}} = \frac{1}{2} \Im \sqrt{(p_1 + 1)(p_2 + 1)} \quad (7.11)$$

if $z = -1$. The branches of the radicals in (7.10) and (7.11) are chosen such that $h_+ \geq 0$, $h_- \geq 0$. Also, we assume here that if a strip domain has both vertices at the same pole then its height is counted twice.

Comparing h_+ and h_- , we find three possibilities:

- 1) If $h_+ = h_-$, then the domain configuration \mathcal{D}_Q has only one strip domain G_2 . This is the case discussed in parts **6.3(a)** and **6.3(b1)** in Section 6.
- 2) The case $h_+ > h_-$ corresponds to the configuration with two strip domains G_1 and G_2 discussed in part **6.3(b2)** in Section 6. In this case, the normalized heights h_1 and h_2 of the strip domains G_1 and G_2 can be calculated as follows:

$$h_1 = \frac{1}{2} (h_+ - h_-), \quad h_2 = h_-. \quad (7.12)$$

- 3) The case $h_+ < h_-$ corresponds to the configuration with two strip domains mentioned in part **6.3(b3)** in Section 6.

Next, we will identify pairs p_1, p_2 , which correspond to each of the cases **6.3(a)**, **6.3(b1)**, and **6.3(b2)**. The domain configuration \mathcal{D}_Q has exactly one strip

domain if and only if $h_+ = h_-$. Now, (7.10) and (7.11) imply that the latter equation is equivalent to the following equation:

$$\begin{aligned} & \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(\bar{p}_1 - 1)(\bar{p}_2 - 1)} \right)^2 \\ &= \left(\sqrt{(p_1 + 1)(p_2 + 1)} - \sqrt{(\bar{p}_1 + 1)(\bar{p}_2 + 1)} \right)^2. \end{aligned}$$

Simplifying this equation, we conclude that $h_+ = h_-$ if and only if p_1 and p_2 satisfy the following equation:

$$p_1 + \bar{p}_1 + p_2 + \bar{p}_2 + |p_1 - 1||p_2 - 1| - |p_1 + 1||p_2 + 1| = 0 \tag{7.13}$$

We claim that for a fixed p_1 with $\Im p_1 \neq 0$, the pair p_1, p_2 satisfies equation (7.13) if and only if $p_2 \in L(p_1)$ or $p_2 \in H(p_1)$. Indeed, $p_2 \in L(p_1)$ if and only if

$$|p_1 - 1| + |p_1 + 1| = |p_2 - 1| + |p_2 + 1|. \tag{7.14}$$

Similarly, $p_2 \in H(p_1)$ if and only if

$$|p_1 - 1| - |p_1 + 1| = |p_2 - 1| - |p_2 + 1|. \tag{7.15}$$

Multiplying equations (7.14) and (7.15), after simplification we again obtain equation (7.13). Therefore, $p_2 \in L(p_1)$ or $p_2 \in H(p_1)$ if and only if the pair p_1, p_2 satisfy equation (7.13). Thus, \mathcal{D}_Q has only one strip domain if and only if $p_2 \in L(p_1) \setminus \{p_1, \bar{p}_1\}$ or $p_2 \in H(p_2) \setminus \{p_1, \bar{p}_1\}$. This proves the first parts of statements **6.3(a)** and **6.3(b1)**.

Now, we will prove that $p_1 \in \partial D_\infty$ for all $p_2 \in E_{-1}^+(p_1)$. First, we claim that $p_1 \in \partial D_\infty$ for all p_2 sufficiently close to -1 . Arguing by contradiction, suppose that there is a sequence $s_k \rightarrow -1$ such that $s_k \in \partial D_\infty^k$ for all $k = 1, 2, \dots$. Here $D_\infty^k \ni \infty$ denotes the corresponding circle domain of the quadratic differential $Q_k(z) dz^2$ having the form (7.5). From (7.5) we find that

$$Q_k(z) dz^2 \rightarrow \widehat{Q}(z) dz^2 := -\frac{z - p_1}{(z + 1)(z - 1)^2} dz^2,$$

where convergence is uniform on compact subsets of $\mathbb{C} \setminus \{-1, 1\}$. Since the residue of $\widehat{Q}(z)$ at $z = \infty$ equals 1, the quadratic differential $\widehat{Q}(z) dz^2$ has a circle domain $\widehat{D}_\infty \ni \infty$ and if $\gamma \subset \widehat{D}_\infty$ is a closed trajectory of $\widehat{Q}(z) dz^2$, then $|\gamma|_{\widehat{Q}} = 2\pi$.

Let us show that the boundary of \widehat{D}_∞ consists of a single critical trajectory $\widehat{\gamma}_\infty$ of $\widehat{Q}(z) dz^2$, which has both its end points at $z = p_1$. Indeed, $\partial \widehat{D}_\infty$ consists of a finite number of critical trajectories of $\widehat{Q}(z) dz^2$, which have their end points at finite critical points. Therefore, if $-1 \in \partial \widehat{D}_\infty$, then $\partial \widehat{D}_\infty$ contains a critical trajectory, call it $\widehat{\gamma}_1$, which joins $z = -1$ and $z = p_1$. Some notations used in this part of the proof are shown in Figure 7a. This figure shows the limit configuration, which is, in fact, impossible as we explain below. In this case, $\partial \widehat{D}_\infty$ must contain a second critical trajectory, call it $\widehat{\gamma}_2$, which has both its end points at $z = p_1$. This implies that $z = 1$ is the only pole of $\widehat{Q}(z) dz^2$ lying in a simply connected domain, call it \widehat{D}_1 , which is bounded by critical trajectories. Hence, \widehat{D}_1 must be a circle domain of $\widehat{Q}(z) dz^2$. Furthermore, the domain configuration $\mathcal{D}_{\widehat{Q}}$ consists of

two circle domains $\widehat{D}_1, \widehat{D}_\infty$, which in this case must be the extremal domains of Jenkins module problem on the following maximum of the sum of reduced moduli:

$$m(B_\infty, \infty) + t^2 m(B_1, 1) \quad \text{with some fixed } t > 0,$$

where the maximum is taken over all pairs of simply connected non-overlapping domains $B_\infty \ni \infty$ and $B_1 \ni 1$. It is well known that such a pair of extremal domains is unique; see for example, [30]. Therefore, \widehat{D}_1 and \widehat{D}_∞ must be symmetric with respect to the real line (as is shown, for instance, in Figure 5d), which is not the case since $\widehat{Q}(z) dz^2$ has only one zero p_1 with $\Im p_1 > 0$.

Thus, $\partial \widehat{D}_\infty = \widehat{\gamma}_\infty \cup \{p_1\}$ and $z = -1$ lies in the domain complementary to the closure of \widehat{D}_∞ . Figure 7b illustrates notations used further on in this part of the proof.

Let $\tilde{\gamma}_{-1}$ denote the \widehat{Q} -geodesic in the class of all curves having their end points at $z = -1$, which separate the points $z = 1$ and $z = p_1$ from $z = \infty$. Since $-1 \notin \partial \widehat{D}_\infty$ it follows that

$$|\tilde{\gamma}_{-1}|_{\widehat{Q}} > |\widehat{\gamma}_\infty|_{\widehat{Q}} = 2\pi. \tag{7.16}$$

Let $\varepsilon > 0$ be such that

$$\varepsilon < \frac{1}{4} \left(|\tilde{\gamma}_{-1}|_{\widehat{Q}} - 2\pi \right). \tag{7.17}$$

Let $r > 0$ be sufficiently small such that

$$|[-1, -1 + re^{i\theta}]|_{\widehat{Q}} < \varepsilon/8 \quad \text{for all } 0 \leq \theta < 2\pi. \tag{7.18}$$

Now let $\tilde{\gamma}_r$ be the shortest in the \widehat{Q} -metric among all arcs having their end points on the circle $C_r(-1) = \{z : |z + 1| = r\}$ and separating the points $z = 1$ and $z = p_1$ from the point $z = \infty$ in the exterior of the circle $C_r(-1)$. It is not difficult to show that there is at least one such curve $\tilde{\gamma}_r$. It follows from (7.18) that

$$|\tilde{\gamma}_r|_{\widehat{Q}} > |\tilde{\gamma}_{-1}|_{\widehat{Q}} - \varepsilon/4. \tag{7.19}$$

Since $s_k \rightarrow -1$, $s_k \in \partial D_\infty^k$, and $p_1 \notin D_\infty^k$, it follows that for every sufficiently large k there is a regular trajectory $\gamma(k)$ of $Q_k(z) dz^2$ intersecting the circle $C_r(-1)$ and such that the arc $\gamma'(k) = \gamma(k) \setminus \{z : |z + 1| \leq r\}$ separates the points $z = 1$ and $z = p_1$ from $z = \infty$ in the exterior of $C_r(-1)$. Since $|\gamma(k)|_{Q_k} = 2\pi$ for all k and since every quadratic differential $Q_k(z) dz^2$ has second-order poles at $z = 1$ and $z = \infty$ it follows from (7.5) that there is $r_0 > 0$ small enough such that $\gamma'(k)$ lies on the compact set $K_0 = \{z : |z| \leq 1/r_0\} \setminus (\{z : |z - 1| < r_0\} \cup \{z : |z + 1| < r\})$ for all k sufficiently large. We note also that $Q_k(z) \rightarrow \widehat{Q}(z)$ uniformly on K_0 . This implies, in particular, that for all k the Euclidean lengths of $\gamma'(k)$ are bounded by the same constant and that

$$|\gamma'(k)|_{Q_k} \geq |\gamma'(k)|_{\widehat{Q}} - \varepsilon/4 \tag{7.20}$$

for all k sufficiently large.

Combining (7.16)–(7.20), we obtain the following relations:

$$\begin{aligned} 2\pi &= |\gamma(k)|_{Q_k} \geq |\gamma'(k)|_{Q_k} \geq |\gamma'(k)|_{\tilde{Q}} - \varepsilon/4 \geq |\tilde{\gamma}_r|_{\tilde{Q}} - \varepsilon/4 \\ &> |\tilde{\gamma}_{-1}|_{\tilde{Q}} - \varepsilon/2 > |\tilde{\gamma}_{-1}|_{\tilde{Q}} - \frac{1}{2}(|\tilde{\gamma}_{-1}|_{\tilde{Q}} - 2\pi) = \frac{1}{2}(|\tilde{\gamma}_r|_{\tilde{Q}} + 2\pi) > 2\pi, \end{aligned}$$

which, of course, is absurd. Thus, $p_2 \in \partial D_\infty$ for all p_2 sufficiently close to -1 .

Let $\Delta \neq \emptyset$ be the set of all $p_2 \in E_{-1}^+(p_1)$ such that $p_1 \in \partial D_\infty$. To prove that $\Delta = E_{-1}^+(p_1) \setminus \{-1\}$, it is sufficient to show that Δ is closed and open in $E_{-1}^+(p_1)$. Arguing by contradiction, we suppose that there is a sequence of poles $s_k := p_2^k \in E_{-1}^+(p_1)$, $k = 1, 2, \dots$, such that $s_k \rightarrow s_0 := p_2^0 \in E_{-1}^+(p_1)$ and $p_1 \in \partial D_\infty^k$ for all $k = 1, 2, \dots$ but $p_1 \notin \partial D_\infty^0$. In this part of the proof, the index $k = 0, 1, 2, \dots$, used in the notations D_∞^k , $\tilde{\gamma}_k$, etc., will denote domains, trajectories, and other objects corresponding to the quadratic differential $Q_k(z) dz^2$ defined by (7.5). Since ∂D_∞^0 contains a critical point and $p_1 \notin \partial D_\infty^0$, we must have $p_2^0 \in \partial D_\infty^0$.

Figure 7c illustrates some notations used in this part of the proof. In this case, the boundary ∂D_∞^0 consists of a single critical trajectory γ_∞^0 and its end points, each of which is at $z = p_2^0$. In addition, there is a critical trajectory of infinite Q^0 -length, called it $\hat{\gamma}$, which has one end point at p_2^0 and which approaches to the pole $z = -1$ or the pole $z = 1$ in the other direction. Let P_0 be a point on $\hat{\gamma}$ such that the Q^0 -length of the arc $\hat{\gamma}_0$ of $\hat{\gamma}$ joining p_2^0 and P_0 equals L , where $L > 0$ is sufficiently large. For $\delta > 0$ sufficiently small, let γ_1^\perp and γ_2^\perp denote disjoint open arcs on the orthogonal trajectory of $Q^0(z) dz^2$ passing through P_0 such that each of γ_1^\perp and γ_2^\perp has one end point at P_0 and each of them has Q^0 -length equal to δ . If δ is small enough, then there is an arc of a trajectory of $Q^0(z) dz^2$, call it $\tilde{\gamma}$, which connects the second end point of γ_1^\perp with the second end point of γ_2^\perp . Now, let $D(\delta)$ be the domain, the boundary of which consists of the arcs γ_∞^0 , $\hat{\gamma}_0$, γ_1^\perp , γ_2^\perp , and their end points. In the terminology explained in Section 3, the domain $D(\delta)$ is a Q^0 -rectangle of Q^0 -height δ .

If $\delta > 0$ is sufficiently small, then p_1 belong to the bounded component of $\mathbb{C} \setminus \overline{D(\delta)}$. Let $\tilde{\gamma}_1$ be the arc of a trajectory of $Q^0(z) dz^2$, which divide $D(\delta)$ into two Q^0 -rectangles, each of which has the Q^0 -height equal to $\delta/2$. Since $p_1 \in \partial D_k$ for all k and p_1 belongs to the bounded component of $\mathbb{C} \setminus \overline{D(\delta)}$, it follows that, for each $k = 1, 2, \dots$, there is a closed trajectory $\hat{\gamma}_k$ of $Q_k(z) dz^2$ lying in D_∞^k , which intersects $\tilde{\gamma}_1$ at some point $\tilde{z}_k \in D(\delta)$.

Since $Q_k(z) \rightarrow Q^0(z)$ it follows that, for all sufficiently large k , the trajectory $\hat{\gamma}_k$ has an arc $\tilde{\gamma}_k$ such that $\tilde{\gamma}_k \subset D(\delta)$ and $\tilde{\gamma}_k$ has one end point on each of the arcs γ_1^\perp and γ_2^\perp .

Now, since $Q_k(z) \rightarrow Q^0(z)$ uniformly on $\overline{D(\delta)}$ it follows that

$$|\hat{\gamma}_k|_{Q_k} \geq |\tilde{\gamma}_k|_{Q_k} \rightarrow |\tilde{\gamma}_1|_{Q^0} = |\gamma_\infty^0|_{Q^0} + 2|\hat{\gamma}_0|_{Q^0} = 2\pi + 2L,$$

contradicting to the fact that $|\hat{\gamma}_k|_{Q_k} = 2\pi$. The latter fact follows from the assumption that $\hat{\gamma}_k$ is a closed trajectory of $Q_k(z) dz^2$, which lies in a circle domain D_∞^k .

Thus, we have proved that Δ is closed in $E_{-1}^+(p_1)$. A similar argument can be used to show that Δ is open in $E_{-1}^+(p_1)$. The difference is that to construct a

domain $D(\delta)$, we now use an arc $\tilde{\gamma}_1$ of a critical trajectory $\hat{\gamma}_1$, which has one of its end points at the pole p_1 and not at the pole p_1^0 as we had in the previous case.

Therefore, we have proved that if $p_2 \in E_{-1}^+(p_1)$, then $p_1 \in \partial D_\infty$. The same argument can be used to prove that if $p_2 \in E_1^+(p_1)$, then $p_1 \in \partial D_\infty$.

Finally, if $p_2 \in E_1^-(p_1)$ or $p_2 \in E_{-1}^-(p_1)$, then we can switch the roles of the poles p_1 and p_2 in our previous proof and conclude that $p_2 \in \partial D_\infty$ in these cases. This proves the first part of statement **6.3(b2)**.

Now, possible positions of zeros p_1 and p_2 on boundaries of the corresponding circle and strip domains are determined for all cases. Next, we will discuss limiting behavior of critical trajectories. We will give a proof for the most general case when the domain configuration consists of a circle domain D_∞ and strip domains G_1 and G_2 . In all other cases proofs are similar.

Let Δ denote the set of pairs (p_1, p_2) , for which the limiting behavior of critical trajectories is shown in Fig. 4a or in more general case in Fig. 4b. That is when γ_1 joins $p_1 \in \partial D_\infty \cap \partial G_1$ and $z = 1$, γ_{-1} joins $p_2 \in \partial G_1 \cap \partial G_2$ and $z = -1$, and γ_0^+ and γ_0^- each joins p_2 and $z = 1$. First, we note that Δ is not empty since $(p_1, p_2) \in \Delta$ when $p_1 > 1$ and $-p_1 < p_2 < -1$. In this case the intervals $(p_2, -1)$ and $(1, p_1)$ represent critical trajectories γ_1 and γ_{-1} and critical trajectories γ_0^+ and γ_0^- connect a zero at p_2 with a pole at $z = 1$; see Fig. 4a.

We claim that Δ is open. To prove this claim, suppose that $(p_1^0, p_2^0) \in \Delta$ and that $(p_1^k, p_2^k) \rightarrow (p_1^0, p_2^0)$ as $k \rightarrow \infty$, $k = 1, 2, \dots$. Fix $\varepsilon > 0$ small enough and consider the arc $\gamma_1^0(\varepsilon) = \gamma_1^0 \setminus \{z : |z - 1| < \varepsilon\}$ of the critical trajectory γ_1^0 , which goes from p_1^0 to the pole $z = 1$. Since $(p_1^k, p_2^k) \rightarrow (p_1^0, p_2^0)$ it follows that for all k sufficiently big there is a critical trajectory γ_1^k having one point at p_1^k which has a subarc $\gamma_1^k(\varepsilon)$ which lies in the $\varepsilon/10$ -neighborhood of the arc $\gamma_1^0(\varepsilon)$. In particular, eventually, $\gamma_1^k(\varepsilon)$ enters the disk $\{z : |z - 1| < \varepsilon\}$. Therefore, it follows from the standard continuity argument and Lemma 4 that γ_1^k approaches the pole $z = 1$. The same argument works for all other critical trajectories of the quadratic differential (6.1) with $p_1 = p_1^k$, $p_2 = p_2^k$. Thus, we have proved that Δ is open.

Same argument can be applied to show that all other sets of points (p_1, p_2) responsible for different types of limiting behavior of critical trajectories mentioned in part **6.3(b2)** of Theorem 4 are also nonempty and open. The latter implies that each of these sets must coincide with some connected component of the set $\mathbb{C} \setminus (L(p_1) \cup H(p_1))$. This proves the desired statement in the case under consideration.

7.B. The local behavior of trajectories near second-order poles at $z = 1$ and $z = -1$ is controlled by Laurent coefficients C_1 and C_{-1} , respectively, which are given by formula (7.2). The radial structure near $z = 1$ or near $z = -1$ occurs if and only if $C_1 < 0$ or $C_{-1} < 0$, respectively. The latter inequalities are equivalent to the following relations:

$$\arg(p_1 - 1) = -\arg(p_2 - 1) + \pi \tag{7.21}$$

or

$$\arg(p_1 + 1) = -\arg(p_2 + 1) + \pi. \tag{7.22}$$

Now, statement (1) about radial behavior follows from (7.21) and (7.22).

Next, trajectories of $Q(z) dz^2$ approaching the pole $z = 1$ spiral clockwise if and only if $0 < \arg C_1 < \pi$. The latter is equivalent to the inequalities:

$$-\arg p_1 - 1 < \arg(p_2 - 1) < -\arg(p_1 - 1) + \pi,$$

which imply the desired statement for the case when trajectories of $Q(z) dz^2$ approaching $z = 1$ spiral clockwise. In the remaining cases the proof is similar.

The proof of Theorem 4 is now complete. \square

Remark 3. The case when $\Im p_1 = 0$ but $\Im p_2 \neq 0$ can be reduced to the case covered by Theorem 4 by changing numeration of zeros. In the remaining case when $\Im p_1 = 0$ and $\Im p_2 = 0$, the domain configurations are rather simple; they are symmetric with respect to the real axis as it is shown in Figures 1a, 1b, 2a, 3a, and some other figures.

8. Identifying simple critical geodesics and critical loops

Topological information obtained in Section 6 is sufficient to identify all critical geodesics and all critical geodesic loops of the quadratic differential (6.1) in all cases. In particular, we can identify all simple geodesics.

Cases **6.1(a)** and **6.1(b)**; see Figure 1a and Figure 1b. Let γ be a geodesic joining p_1 and p_2 . Since D_∞ , D_1 , and D_{-1} are simply connected and $p_1 \in \partial D_\infty \cap \partial D_1$ and $p_2 \in \partial D_\infty \cap \partial D_{-1}$ it follows from Lemma 4 that γ does not intersect D_∞ , D_1 , and D_{-1} . In this case, γ must be composed of a finite numbers of copies of γ_0 , a finite number of copies of γ_1 , and a finite number of copies of γ_{-1} . Therefore the only simple geodesic joining p_1 and p_2 in this case is the segment $\gamma_0 = [p_2, p_1]$.

In addition, by Lemma 5, γ_1 is the only simple non-degenerate geodesic from the point p_1 to itself and γ_{-1} is the only short geodesic from p_2 to p_2 .

Case **6.1(c)**; see Figure 1c. As in the previous case, any geodesic γ joining p_1 and p_2 must be composed of a finite number of copies of γ_0 , a finite number of copies of γ_1 , and a finite number of copies of γ_{-1} . Thus, in this case there exist exactly three simple geodesics joining p_1 and p_2 , which are γ_0 , γ_1 , and γ_{-1} . By Lemma 5, there are no geodesic loops in this case.

Case **6.2**; see Figures 2a, 2b. Suppose that \mathcal{D}_Q consists of circle domains D_∞ and D_{-1} and a strip domain G_1 . Let γ be a geodesic joining p_1 and p_2 . If γ contains a point $\zeta \in \gamma_{-1}$ or a point $\zeta \in \gamma_\infty$, then it follows from Lemma 4 that γ_{-1} or, respectively, γ_∞ is a subarc of γ . Thus, γ is not simple in these cases.

Suppose now that $\gamma \subset G_1 \cup \gamma_1^+ \cup \gamma_1^-$. Since G_1 is a strip domain the function $w = F(z)$ defined by

$$F(z) = \frac{1}{2\pi} \int_{p_1}^z \sqrt{Q(z)} dz, \tag{8.1}$$

with an appropriate choice of the radical, maps G_1 conformally and one-to-one onto the horizontal strip S_{h_1} , where $S_h = \{w : 0 < \Im w < h\}$, in such a way that the trajectory γ_∞ is mapped onto an interval $(x_1, x'_1) \subset \mathbb{R}$ with $x_1 = 0$ and

$x'_1 = 1$. Here h_1 is the normalized height of the strip domain G_1 defined by (7.12). Figure 8a and Figure 9a illustrate some notions relevant to Case **6.2**. To simplify notations in our figures, we will use the same notations for Q -geodesics (such as $\gamma_\infty, \gamma_{11}, \gamma'_{12}$, etc.) in the z -plane and for their images under the mapping $w = F(z)$ in the w -plane.

The indefinite integral $\Phi(z) = \frac{1}{2\pi} \int \sqrt{Q(z)} dz$ can be expressed explicitly in terms of elementary functions as follows:

$$\begin{aligned} \Phi(z) = & \frac{1}{4\pi i} \left(\sqrt{(p_1 - 1)(p_2 - 1)} \log(z - 1) - \sqrt{(p_1 + 1)(p_2 + 1)} \log(z + 1) \right. \\ & + 4 \log(\sqrt{z - p_1} + \sqrt{z - p_2}) \\ & + 2\sqrt{(p_1 + 1)(p_2 + 1)} \log(\sqrt{(p_1 + 1)(z - p_2)} - \sqrt{(p_2 + 1)(z - p_1)}) \\ & \left. - 2\sqrt{(p_1 - 1)(p_2 - 1)} \log(\sqrt{(p_1 - 1)(z - p_2)} - \sqrt{(p_2 - 1)(z - p_1)}) \right). \end{aligned} \tag{8.2}$$

Equation (8.2) can be verified by straightforward differentiation. Alternatively, it can be verified with *Mathematica* or *Maple*. With (8.2) at hands, the function $F(z)$ can be written as

$$F(z) = \Phi(z) - \Phi(p_1), \tag{8.3}$$

where

$$\Phi(p_1) = \frac{1}{4\pi i} \left(2 + \sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \log(p_1 - p_2). \tag{8.4}$$

Calculating $\Phi(p_2)$, after some algebra, we find that:

$$F(p_2) = \frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right). \tag{8.5}$$

Of course, all branches of the radicals and logarithms in (8.2)–(8.5) have to be appropriately chosen.

To explain more precisely our choice of branches of multi-valued functions in (8.2)–(8.5), we note that the points p_1, p_2 and points of the arcs γ_1^+ and γ_1^- each represents two distinct boundary points of G_1 and therefore every such point has two images under the mapping $F(z)$. These images will be denoted by $x_1(\zeta)$ and $x'_1(\zeta)$ if $\zeta \in \gamma_1^+ \cup \{p_1\}$ and by $x_2(\zeta) + ih_1$ and $x'_2(\zeta) + ih_1$ if $\zeta \in \gamma_1^- \cup \{p_2\}$. We assume here that $x_1(\zeta) < x'_1(\zeta)$ for all $\zeta \in \gamma_1^+ \cup \{p_1\}$ and $x_2(\zeta) < x'_2(\zeta)$ for all $\zeta \in \gamma_1^- \cup \{p_2\}$. In accordance with our notation above, $x_1(p_1) = x_1 = 0$ and $x'_1(p_1) = x'_1 = 1$. We also will abbreviate $x_2(p_2)$ and $x'_2(p_2)$ as x_2 and x'_2 , respectively.

For every $\zeta \in \gamma_1^+$, the segments $[x_1(\zeta), x_1]$ and $[x'_1, x'_1(\zeta)]$ are the images of the same arc on γ_1^+ . Therefore they have equal lengthes. Similarly, the segments $[x_2(\zeta) + ih_1, x_2 + ih_1]$ and $[x'_2 + ih_1, x'_2(\zeta) + ih_1]$ have equal lengthes. Thus, for every $\zeta \in \gamma_1^+$ and every $\zeta \in \gamma_1^-$, we have, respectively:

$$x_1 - x_1(\zeta) = x'_1(\zeta) - x'_1 \quad \text{and} \quad x_2 - x_2(\zeta) = x'_2(\zeta) - x'_2. \tag{8.6}$$

We know that the preimage under the mapping $F(z)$ of every straight line segment is a geodesic. This immediately implies that in the case under consideration there exist four simple critical geodesics, which are the following preimages:

$$\begin{aligned} \gamma_{12} &= F^{-1}((x_1, x_2 + ih_1)), & \gamma'_{12} &= F^{-1}((x_1, x'_2 + ih_1)), \\ \gamma_{21} &= F^{-1}((x'_1, x_2 + ih_1)), & \gamma'_{21} &= F^{-1}((x'_1, x'_2 + ih_1)). \end{aligned} \tag{8.7}$$

The geodesic loops γ_∞ and γ_{-1} are the following preimages:

$$\gamma_\infty = F^{-1}((x_1, x_1)), \quad \gamma_{-1} = F^{-1}((x_2 + ih_1, x'_2 + ih_1)). \tag{8.8}$$

We claim that there is no other simple geodesic joining the points p_1 and p_2 . Figure 9a illustrates some notation used in the proof of this claim. Suppose that τ is a geodesic ray issuing from p_1 into the region G_1 . Let $\tau_k, k = 1, \dots, N$, be connected components of the intersection $\tau \cap G_1$ enumerated in their natural order on τ . In particular, τ_1 starts at p_1 . We may have finite or infinite number of such components. Thus, N is a finite number or $N = \infty$. Let $l_k = F(\tau_k)$. Since all τ_k lie on the same geodesic it follows that l_k are parallel line intervals in S joining the real axis and the horizontal line L_{h_1} , where $L_h = \{w : \Im w = h\}$. Let v'_k and v''_k be the initial point and terminal point of l_k , respectively. Then $v'_k = e'_k$ and $v''_k = e''_k + ih_1$ with real e'_k and e''_k if k is odd and $v'_k = e'_k + ih_1, v''_k = e''_k$ with real e'_k and e''_k if k is even.

The interval l_1 may start at x_1 or at x'_1 . To be definite, suppose that $e'_1 = x_1$. For the position of e''_1 we have the following possibilities:

- (a) $e''_1 = x_2$ or $e''_1 = x'_2$. In this case, $\tau_1 = \gamma_{12}$ or $\tau_1 = \gamma'_{12}$. Thus we obtain two out of four geodesics in (8.7).
- (b) $x_1 < e''_1 < x'_1$. In this case, τ_1 has its end point on γ_{-1} . By Lemma 4, the continuation of τ_1 as a geodesic will stay in D_{-1} and will approach to the pole $z = -1$. Thus, τ is not a geodesic from p_1 to p_2 or a geodesic loop from p_1 to itself in this case.
- (c) $e''_1 > x'_2$. Let $d = e''_1 - x'_2$. It follows from (8.6) that $e'_2 = x_2 - d$. Then $e''_2 = x_1 - d$. In general, $e'_{2k-1} = x'_1 + (k-1)d, e''_{2k-1} = x'_2 + kd$ for $k = 1, 2, \dots$, and $e'_{2k} = x_2 - kd, e''_{2k} = x_1 - kd$ for $k = 1, 2, \dots$. Thus, τ cannot terminate at p_1 or p_2 . Instead, τ approaches to the pole at $z = 1$ as a logarithmic spiral.
- (d) $e''_1 < x_2$. Let $d_0 = x_2 - e''_1$. Then $e'_2 = x'_2 + d_0$ by (8.6). For the position of e''_2 we have three possibilities.
 - (α) $x_1 < e''_2 < x'_1$. In this case by Lemma 4, the continuation of τ_2 as a geodesic ray will stay in D_∞ and will approach to the pole $z = \infty$. Thus, τ is not a geodesic from p_1 to p_2 or a geodesic loop in this case.
 - (β) $e''_2 = x'_1$. In this case, τ is a critical geodesic loop $\gamma_{11} = F^{-1}((x_1, v''_1] \cup [v'_2, x'_1))$ from p_1 to itself. We emphasize here, that since the segments l_1 and l_2 are parallel a critical geodesic loop from p_1 to itself occurs if and only if $|\gamma_\infty|_Q = x'_1 - x_1 > x'_2 - x_2 = |\gamma_{-1}|_Q$. If $|\gamma_\infty|_Q < |\gamma_{-1}|_Q$, then there is a critical geodesic loop γ_{22} with end points at p_2 .

(γ) $e''_2 > x'_1$. Let $d = e''_2 - x'_1$. Then, as in the case c), we obtain that $e'_{2k+1} = x_1 - kd$, $e''_{2k+1} = x_2 - d_0 - kd$ for $k = 1, 2, \dots$, and $e'_{2k} = x'_2 + d_0 + kd$, $e''_{2k} = x'_1 + kd$ for $k = 1, 2, \dots$. Therefore, τ does not terminate at p_1 or p_2 . Instead, τ approaches to the pole at $z = 1$ as a logarithmic spiral.

If l_1 has its initial point at x'_1 , the same argument shows that there are exactly two geodesics joining p_1 and p_2 , which are the geodesics γ_{21} and γ'_{21} defined by (8.7).

Combining our findings for Case **6.2**, we conclude that in this case there exist exactly four distinct geodesics joining p_1 and p_2 , which are given by (8.7). The geodesic loops γ_∞ and γ_{-1} are given by (8.8). In addition, if $|\gamma_\infty|_Q \neq |\gamma_{-1}|_Q$, then there is exactly one geodesic loop containing the pole $z = 1$ in its interior domain, which has its end points at a zero of $Q(z) dz^2$. This loop has the pole $z = 1$ in its interior domain, which does not contain other critical points of $Q(z) dz^2$, and has both its end points at p_1 or at p_2 , if $|\gamma_\infty|_Q > |\gamma_{-1}|_Q$ or $|\gamma_\infty|_Q < |\gamma_{-1}|_Q$, respectively.

Finally, if $|\gamma_\infty|_Q = |\gamma_{-1}|_Q$, then the geodesics γ_{12} and γ'_{21} together with points $z = p_1$ and p_2 form a boundary of a simply connected bounded domain, which contains the pole $z = 1$ and does not contain other critical points of $Q(z) dz^2$. There are no geodesic loops containing $z = 1$ in its interior domain in this case.

The argument based on the construction of parallel segments divergent to ∞ , which was used above to prove non-existence of some geodesics, will be used for the same purpose in several other cases considered below. Since the detailed construction is rather lengthy, the detailed exposition will be given for one more case when we have two strip domains. In other cases, we will just refer to this argument (which actually is rather standard, see [33, Ch. IV]) and call it the “proof by construction of divergent geodesic segments”.

Case **6.3(a)**; see Figure 8b. In this case, the domain configuration \mathcal{D}_Q consists of a circle domain D_∞ and a strip domain G_2 having its vertices at the poles $z = 1$ and $z = -1$. The function $F(z)$ defined by (8.1) maps G_2 conformally and one-to-one onto the strip S_{h_1} such that the trajectory γ_∞^+ is mapped onto the interval $(x_1, x_2) \subset \mathbb{R}$ with $x_1 = 0$ and some x_2 , $0 < x_2 < 1$. The points $z = p_1$ and $z = p_2$ each has two images under the mapping $F(z)$. Let $x_1 = 0$ and $x'_1 + ih_1$ with some real x'_1 be the images of p_1 and let x_2 and $x'_2 + ih_1$ with $x'_2 = x'_1 + (1 - x_2)$ be the images of p_2 . Arguing as in Case **6.2**, one can easily find four distinct simple geodesics joining the points p_1 and p_2 . These geodesics are:

$$\begin{aligned} \gamma_{12} &= F^{-1}((x_1, x_2)) = \gamma_\infty^+, & \gamma'_{12} &= F^{-1}((x'_1 + ih_1, x'_2 + ih_1)) = \gamma_\infty^-, \\ \gamma_{21} &= F^{-1}((x_1, x'_2 + ih_1)), & \gamma'_{21} &= F^{-1}((x_2, x'_1 + ih_1)). \end{aligned}$$

In addition, there are two critical geodesic loops:

$$\gamma_{11} = F^{-1}((x_1, x'_1 + ih_1)) \quad \text{and} \quad \gamma_{22} = F^{-1}((x_2, x'_2 + ih_1)).$$

It follows from Lemma 5 that there are no other such loops.

Using the proof by construction of divergent geodesic segments as in Case **6.2**, we can show that there are no other simple geodesics joining p_1 and p_2 .

Case **6.3(b1)**; see Figure 8c. We still have a circle domain D_∞ and a strip domain G_2 . In this case, the function $F(z)$ defined by (8.1) as in Case **6.2** maps G_2 conformally and one-to-one onto S_{h_1} such that γ_∞ is mapped onto the interval $(x_1, x'_1) \subset \mathbb{R}$, where $x_1 = 0$ and $x'_1 = 1$. The difference is that now the point p_2 represents three boundary points of G_2 . Two of them belong to the side l_2 and the third point belongs to the side l_1 . Accordingly, there are three images of p_2 under the mapping $F(z)$, which we will denote by $x_2 + ih_1$, x'_2 , and x''_2 . Here x_2 may be any real number while x'_2 and x''_2 satisfy the following conditions:

$$x'_2 > x'_1, \quad x''_2 < x_1, \quad \text{and} \quad x'_2 - x'_1 = x_1 - x''_2.$$

In this case, there are three short geodesics, which are the following preimages:

$$\gamma_0 = F^{-1}((x''_1, x_1)) = F^{-1}((x'_1, x'_2))$$

and

$$\gamma_{12} = F^{-1}((x_1, x_2 + ih_1)), \quad \gamma'_{12} = F^{-1}((x'_1, x_2 + ih_1)).$$

In addition, there are three geodesic loops:

$$\gamma_\infty = F^{-1}((x_1, x'_1)), \quad \gamma'_{22} = F^{-1}((x_2 + ih_1, x'_2)), \quad \gamma''_{22} = F^{-1}((x_2 + ih_1, x''_2)).$$

Using the proof by construction of divergent segments as above, it is not difficult to show that there are no other simple geodesics joining the points p_1 and p_2 .

Case **6.3(b2)**. This is the most general case with many subcases illustrated in Figures 10a–10i. In this case we have a circle domain D_∞ and two strip domains G_1 and G_2 . We assume that \mathcal{D}_Q has topological type shown in Figure 4b. In other cases the proof follows same lines. The function $F(z)$ defined by (8.1) maps G_1 conformally and one-to-one onto the strip S_{h_1} such that γ_∞ is mapped onto the interval $(x_1, x'_1) \subset \mathbb{R}$, where $x_1 = 0$ and $x'_1 = 1$. The point p_2 represents one boundary point of G_1 and two boundary points of G_2 . Let $x_2 + ih_1$ be the image of p_2 considered as a boundary point of G_1 . Then the trajectory γ_0^+ considered as boundary arc of G_1 is mapped onto the ray $r_1 = \{w = t + ih_1 : t < x_2\}$, while the trajectory γ_0^- is mapped onto the ray $r_2 = \{w = t + ih_1 : t > x_2\}$. The function $F(z)$ can be continued analytically through the trajectory γ_0^+ . The continued function (for which we keep our previous notation $F(z)$) maps G_2 conformally and one-to-one onto the strip $S(h_1, h) = \{w : h_1 < \Im w < h\}$ with $h = h_1 + h_2$, where h_1 and h_2 are defined by (7.12). Two boundary points of G_2 situated at p_2 are mapped onto the points $x_2 + ih_1$ and $x'_2 + ih$ with some $x'_2 \in \mathbb{R}$. Thus, the domain $\tilde{D} = G_1 \cup G_2 \cup \gamma_0^+$ is mapped by $F(z)$ conformally and one-to-one onto the slit strip $\hat{S}(h_1, h) = \{w : 0 < \Im w < h\} \setminus \{w = t + ih_1 : t \geq x_2\}$.

We note that every boundary point $\zeta \in \gamma_1 \cup \gamma_{-1} \cup \gamma_0^-$ under the mapping $F(z)$ has two images $w_1(\zeta)$ and $w_2(\zeta)$, which satisfy the following conditions similar to

conditions (8.6):

$$x_1 - w_1(\zeta) = w_2(\zeta) - x'_1 > 0 \quad \text{if } \zeta \in \gamma_1, \tag{8.9}$$

$$w_1(\zeta) = u_1(\zeta) + ih, \quad w_2(\zeta) = u_2(\zeta) + ih_1, \tag{8.10}$$

where $x'_2 - u_1(\zeta) = u_2(\zeta) - x_2 > 0$ if $\zeta \in \gamma_0^-$, and

$$w_1(\zeta) = u_1(\zeta) + ih, \quad w_2(\zeta) = u_2(\zeta) + ih_1,$$

where $u_1(\zeta) - x'_2 = u_2(\zeta) - x_2 > 0$ if $\zeta \in \gamma_{-1}$.

Consider four straight lines P_k , $k = 1, 2, 3, 4$, where P_2 passes through x'_1 and $x_2 + ih_1$, P_3 passes through x_1 and $x_2 + ih_1$, P_1 passes through x_1 and is parallel to P_2 , and P_4 passes through x'_1 and is parallel to P_3 . Let $u_k + ih$ denote the point of intersection of P_k and the horizontal line $L(h)$, where $L(m)$ stands for the line $\{w : \Im w = m\}$. Then the points $u_k + ih$, $k = 1, 2, 3, 4$, are ordered in the positive direction on $L(h)$; see Figure 10a.

Next, we consider five possible positions for x'_2 , which correspond to “non-degenerate” cases and four positions corresponding to “degenerate” cases. Figures 10a–10i illustrate our constructions of critical geodesics and critical geodesic loops in all these cases. First, we will work with non-degenerate cases, which are cases (a), (c), (e), (g), and (i) and after that we will briefly mention degenerate cases (b), (d), (f), and (h).

- (a) $x'_2 < u_1$. Then the slit strip S_1 contains four intervals: $(x_1, x_2 + ih_1)$, $(x'_1, x_2 + ih_1)$, $(x_1, x'_2 + ih)$, and $(x'_1, x'_2 + ih)$. Therefore the preimages of these intervals under the mapping $F(z)$ provide four distinct geodesics joining the points p_1 and p_2 :

$$\begin{aligned} \gamma_{12} &= F^{-1}((x_1, x_2 + ih_1)), & \gamma'_{12} &= F^{-1}((x'_1, x_2 + ih_1)), \\ \gamma_{21} &= F^{-1}((x_1, x'_2 + ih)), & \gamma'_{21} &= F^{-1}((x'_1, x'_2 + ih)). \end{aligned} \tag{8.11}$$

In addition, there are two critical geodesic loops:

$$\gamma_\infty = F^{-1}((x_1, x'_1)) \quad \text{and} \quad \gamma_{22} = F^{-1}((x_2 + ih_1, x'_2 + ih)). \tag{8.12}$$

The curve $\gamma_{22} \cup \{p_2\}$ bounds a simply connected domain, call it D_{-1} , which contains the trajectory γ_2 and the pole $z = -1$.

One more critical geodesic loop can be found as follows. Let P_5 be the line through $x'_2 + ih$ that is parallel to P_1 and let u'_5 be the point of intersection of P_5 with the real axis. It follows from elementary geometry that there exists a point u_5 , $u'_5 < u_5 < x_1$ such that the line segments $[x'_2 + ih, u_5]$ and $[u_6, x_2 + ih_1]$ with $u_6 = x'_1 + x_1 - u_5$ are parallel to each other. Therefore, it follows from equation (8.9) that the preimage $\gamma'_{22} = F^{-1}((x'_2 + ih, u_5) \cup [u_6, x_2 + ih_1])$ is a geodesic loop from p_2 to p_2 containing the pole $z = 1$ in its interior domain.

We claim that there no other simple critical geodesics in this case. The proof is by the method of construction of divergent geodesic segments. An example of such construction for the case under consideration is shown in Figure 9b.

Suppose that τ is a geodesic ray issuing from p_1 into the region \tilde{G} . Let $\tau_k, k = 1, \dots, N$, where N is a finite integer or $N = \infty$, be connected component of $\tau \cap \tilde{G}$ enumerated in the natural order on τ . Let $l_k = F(\tau_k)$ and let e'_k and e''_k be the initial and terminal points of l_k , respectively.

The interval l_1 may start at x_1 or at x'_1 . To be definite, assume that $e'_1 = x_1$. Then for e''_1 we have the following cases:

- (α) $e''_1 = x'_2 - d_1 + ih$ with some $d_1 > 0$,
- (β) $e''_1 = x'_2 + d_1 + ih$ with some $d_1 > 0$,
- (γ) $e''_1 = x_2 + d_1 + ih_1$ with some $d_1 > 0$.

We give a proof for the case α). In two other case the proof is similar. By (8.10), $e'_2 = x_2 + d_1 + ih_1$ and $e''_2 > x'_1$. Let $d = e''_2 - x'_1$. Continuing, we find the following expressions for the end points of the segments l_k :

$$\begin{aligned} e'_{2k-1} &= x_1 + (k-1)d, & e''_{2k-1} &= x'_2 + d_1 + (k-1)d + ih, \\ e'_{2k} &= x_2 + d_1 + (k-1)d + ih_1, & e''_{2k} &= x'_1 + kd. \end{aligned}$$

Thus, in this case τ cannot terminate at p_2 . Instead, it approaches to the pole $z = 1$ as a logarithmic spiral.

- (c) $u_1 < x'_2 < u_2$. In this case we still have geodesics (8.11) and loops (8.12). The only difference is that we cannot construct the loop γ'_{22} as in part (a). Instead, we can construct a loop γ'_{11} from p_1 to p_1 . Indeed, using elementary geometry, we easily find that there is a point $u_7 + ih$ with $u_7 < x'_2$ such that the segments $[x_1, u_7 + ih]$ and $[u_8 + ih_1, x'_1]$ with $u_8 = x_2 + x'_2 - u_7$ are parallel. Therefore using (8.10), we conclude that $\gamma'_{11} = F^{-1}((x_1, u_7 + ih) \cup [u_8 + ih_1, x_1])$ is a critical geodesic loop.
- (e) $u_2 < x'_2 < u_3$. We still have geodesics $\gamma_{12}, \gamma'_{12}$, and γ_{21} given by (8.11) and the loops $\gamma_\infty, \gamma_{22}$, and γ'_{11} as in the case c). But the geodesic γ'_{21} in (8.11) should be replaced with a geodesic constructed as follows. From elementary geometry we find that there is $u_9 > x_2$ such that the segments $[x'_1, u_9 + ih_1]$ and $[u_{10} + ih, x_2 + ih_1]$ with $u_{10} = x'_2 - u_9 + x_2$ are parallel. Using (8.10), we conclude that the arc $\gamma'_{21} = F^{-1}((x'_1, u_9 + ih_1) \cup [u_{10} + ih, x_2 + ih_1])$ is a geodesic from p_1 to p_2 .
- (g) $u_3 < x'_2 < u_4$. The geodesics $\gamma_{12}, \gamma'_{12}$, and γ'_{21} and all three critical geodesic loops can be constructed as in part (e). The geodesic γ_{21} in this case can be constructed as follows. Using elementary geometry one can find that there is $u_{11} > x_2$ such that the segments $[x_1, u_{11} + ih_1]$ and $[u_{12} + ih, x_2 + ih_1]$ with $u_{12} = x'_2 + x_2 - u_{11}$ are parallel. Using (8.10) we conclude that the arc $\gamma_{21} = F^{-1}((x_1, u_{11} + ih_1) \cup [u_{12} + ih, x_2 + ih_1])$ is a geodesic from p_1 to p_2 .
- (i) $x'_2 > u_4$. The geodesics from p_1 to p_2 can be constructed as in case (g). Of course, we still have loops (8.12). The third geodesic critical loop can be obtained as follows. For $u_{13} < x_1 = 0$, let l^1 be the line segment joining the real axis and the line $L(h)$, which has its initial point at $z = u_{13}$ and passes through $z = x_2 + i$. Let $z = u_{14} + ih$ be the terminal point of l^1 on $L(h)$. We consider only those values of u_{13} , for which $u_{14} < x'_2$. Let $d = x'_2 - u_{14}$ and let l^2 be a line segment joining the real axis and $L(h_1)$, which is parallel to

l^1 and has its initial point at $u_{15} = x'_1 + d$. Let $z = u_{16} + ih_1$ be the terminal point of l^2 on $L(h_1)$. It follows from elementary geometry that we can find a unique value of u_{13} such that for this value $u_{16} - x_2 = x'_2 - u_{14}$.

It follows from our construction and from the identification properties (8.9) and (8.10) that the preimage

$$\gamma'_{22} = F^{-1}([u_{13}, x_2 + ih_1] \cup (x_2 + ih_1, u_{14} + ih) \cup [u_{15}, u_{16} + ih_1])$$

is a geodesic loop from the point p_2 to itself. In addition, this loop contains the pole $z = 1$ in its interior, which does not contain other critical points.

Now we consider four “degenerate” cases.

- (b) If $x'_2 = u_1$, then we still have critical geodesics (8.11) and critical geodesic loops (8.12). But there is no critical geodesic loop separating the pole $z = 1$ from other critical points. Instead, the boundary of a simply connected domain having $z = 1$ inside and bounded by critical geodesics will consist of geodesics γ'_{12} and γ_{22} .
- (d) If $x'_2 = u_2$, then we have all critical geodesic loops and geodesics γ_{12} , γ'_{12} , and γ_{21} as in the case $u_1 < x'_2 < u_2$ but instead of geodesic γ'_{21} we have a non-simple geodesic, which is the union $\gamma'_{12} \cup \gamma_{22}$.
- (f) If $x'_2 = u_3$, then we have all critical geodesic loops and geodesics γ_{12} , γ'_{12} , and γ'_{21} as in the case $u_2 < x'_2 < u_3$ but instead of geodesic γ_{21} we have a non-simple geodesic, which is the union $\gamma_{12} \cup \gamma_{22}$.
- (h) If $x'_2 = u_4$, then we have all geodesics and loops γ_∞ , γ_{22} constructed as in the case $u_3 < x'_2 < u_4$ but instead of the loop γ'_{11} we will have non-simple critical geodesic separating the pole $z = 1$ from all other critical points. This non-simple critical geodesic is the union $\gamma_{12} \cup \gamma'_{21}$.

Using the proof by construction of divergent geodesic segments one can show that in all cases considered above there are no any other critical geodesics or critical geodesic loops.

Quadratic differentials defined by formula (6.1) depend on four real parameters which are real parts and imaginary parts of zeroes p_1 and p_2 . As the reader may noticed in the generic case configurations shown in Figures 10 also depend on four real parameters which are x_2 , x'_2 , h_1 , and h . This is not a coincidence; in fact, the set of pairs (p_1, p_2) is in a one-to-one correspondence with the set of these diagrams. To explain how this one-to-one correspondence works, we will show three basic steps. To be definite, we assume that the domain configuration consists of a circle domain D_∞ and strip domains G_1 and G_2 . Thus, we will consider diagrams shown in Figures 10.

- As we described above, for any given p_1 and p_2 , the function $F(z)$ defined by (8.1) maps G_1 and G_2 onto horizontal strips shown in Figures 10. Furthermore, for fixed p_1 and p_2 , the values of the parameters x_2 , x'_2 , h_1 , and h are uniquely defined via function $F(z)$.

- To prove that different pairs (p_1, p_2) define different diagrams, we argue by contradiction. Suppose that mappings $F_1(z)$ and $F_2(z)$ constructed by formula (8.1) for distinct pairs (p_1^1, p_2^1) and (p_1^2, p_2^2) produce identical diagrams of the form shown in Figures 10. Then the composition $\varphi = F_1^{-1} \circ F_2$ is well defined and defines a one-to-one meromorphic mapping from $\overline{\mathbb{C}}$ onto itself. Since $\varphi(1) = 1$, $\varphi(-1) = -1$, and $\varphi(\infty) = \infty$ we conclude that φ is the identity mapping. Thus, $\varphi(z) \equiv z$ and therefore $p_1^1 = p_1^2$ and $p_2^1 = p_2^2$.
- Now, we want to show that every diagram of the form shown in Figures 10a–10i corresponds via a mapping defined by formula (8.1) to a quadratic differential of the form (6.1) with some p_1 and p_2 .

To show this, we will construct a compact Riemann surface \mathcal{R} using identification of appropriate edges of the diagram. For more general quadratic differentials, similar construction was used in [31].

To be definite, we will give detailed construction for the diagram shown in Figure 10a. In all other cases constructions of an appropriate Riemann surface follow same lines. Consider a domain Ω defined by

$$\Omega = \{w : x_1 < \Re w < x'_1, \Im w \leq 0\} \cup \{w : 0 < \Im w < h\} \setminus \{w = t + ih_1 : t \geq x_2\}.$$

Thus, Ω is a slit horizontal strip shown in Figure 10a with a vertical half-strip $\{w : x_1 < \Re w < x'_1, \Im w \leq 0\}$ attached to this horizontal strip along the interval (x_1, x'_1) ; see Figure 11. To construct a Riemann surface \mathcal{R} mentioned above, we identify boundary points of Ω as follows:

$$\begin{aligned} iy &\simeq 1 + iy && \text{for } y \leq 0, \\ -x &\simeq 1 + x && \text{for } x \geq 0, \\ x + x_2 + i(h_1 - 0) &\simeq -x + x'_2 + ih && \text{for } x \geq 0, \\ x + x_2 + i(h_1 + 0) &\simeq x + x'_2 + ih && \text{for } x \geq 0. \end{aligned} \tag{8.13}$$

After identifying points by rules (8.13), we obtain a surface, which is homeomorphic to a complex sphere $\overline{\mathbb{C}}$ punctured at three points. These punctures correspond boundary points of Ω situated at ∞ . One puncture corresponds to the point of $\partial\Omega$, we call it b_1 , which is accessible along the path $\{z = \frac{1}{2} + it\}$ as $t \rightarrow -\infty$. Second puncture corresponds to a point b_2 in $\partial\Omega$, which is accessible along the path $\{z = t + i\frac{h_1+h}{2}\}$ as $t \rightarrow \infty$. The third puncture corresponds to two boundary points of Ω ; one of them, we call it b_3^1 , is accessible along the path $\{z = t + ih_1\}$ as $t \rightarrow -\infty$ and the other one, we call it b_3^2 , is accessible along the path $\{z = t + \frac{h_1}{2}\}$ as $t \rightarrow \infty$. Adding these three punctures, we obtain a compact surface \mathcal{R} which is homeomorphic to a sphere $\overline{\mathbb{C}}$.

Next, we introduce a complex structure on \mathcal{R} as follows. Every point of \mathcal{R} corresponding to a point of Ω inherits its complex structure from Ω as a subset of \mathbb{C} . A point of \mathcal{R} corresponding to iy inherits its complex structure from two half-disks $\{z : |z - iy| < \varepsilon, -\pi/2 \leq \arg(z - iy) \leq \pi/2\}$ and $\{z : |z - (1 + iy)| < \varepsilon, \pi/2 \leq \arg(z - iy) \leq 3\pi/2\}$. Similarly, every

point of \mathcal{R} corresponding to a finite boundary point of Ω , except those which corresponds to the points x_1 , and $x_2 + ih_1$, inherits its complex structure from the corresponding boundary half-disks.

Now we assign complex charts for five remaining special points. For a point $x_1 \simeq x'_1$ a complex chart can be assigned as follows:

$$\zeta = \begin{cases} (w - 1)^{\frac{2}{3}} & \text{if } |w - 1| < \varepsilon, 0 \leq \arg w \leq \frac{3\pi}{2}, \\ (-w)^{\frac{2}{3}} & \text{if } |w| < \varepsilon, -\frac{\pi}{2} \leq \arg w \leq \pi, \end{cases} \quad (8.14)$$

where the branches of the radicals are taken such that $\zeta(w) > 0$ when w is real such that $w > 1$ or $w < 0$.

Similarly, to assign a complex chart to a point $x_2 + ih_1 \simeq x'_2 + ih$, we use the following mapping:

$$\zeta = \begin{cases} (w - (x_2 + ih_1))^{\frac{2}{3}} & \text{if } |w - (x_2 + ih_1)| < \varepsilon, \\ & 0 \leq \arg(w - (x_2 + ih_1)) \leq 2\pi, \\ (w - (x'_2 + ih))^{\frac{2}{3}} & \text{if } |w - (x'_2 + ih)| < \varepsilon, \\ & \pi \leq \arg(w - (x'_2 + ih)) \leq 2\pi, \end{cases} \quad (8.15)$$

with appropriate branches of the radicals.

To a point of \mathcal{R} corresponding to an infinite boundary point b_1 , a complex chart can be assigned via the function

$$\zeta = \exp(-2\pi iw) \quad \text{for } w \text{ such that } 0 \leq \Re w \leq 1, \Im w < 0, \quad (8.16)$$

which maps the half-strip $\{w : 0 \leq \Re w \leq 1, \Im w < 0\}$ onto the unit disc punctured at $\zeta = 0$. This mapping respects the first identification rule in (8.13) and the origin $\zeta = 0$ represents the point b_1 .

To assign a complex chart to a puncture corresponding to a pair of boundary points b_3^1 and b_3^2 , we will work with horizontal half-strips H_3^1 and H_3^2 defined as follows. The boundary of H_3^1 consists of two horizontal rays $\{w : w = t : t \geq u_6\}$ and $\{w = t + ih_1 : t \geq x_2\}$ and a line segment $[u_6, x_2 + ih_1]$; the boundary of H_3^2 consists of two horizontal rays $\{w : w = t : t \leq u_5\}$ and $\{w = t + ih : t \leq x'_2\}$ and a line segment $[u_5, x'_2 + ih]$. To construct a required chart, we rotate the half-strip H_3^1 by angle π with respect to the point $w = 1/2$ and then we glue the result to the half-strip H_3^2 along the interval $(-\infty, u_5)$. As a result, we obtain a wider half-strip \tilde{H}_3 the boundary of which consists of horizontal rays $\{w = t + ih : t < x'_2\}$ and $\{w = t - ih_1 : t < 1 - x_2\}$ and a line segment $[1 - x_2 - ih_1, x'_2 + ih]$. After that we map an obtained wider half-strip \tilde{H}_3 conformally onto the unit disk in such a way that horizontal rays are mapped onto appropriate logarithmic spirals. The conformal mapping just described can be expressed explicitly in the following form:

$$\zeta = \begin{cases} \exp(2\pi i C_3(1 - u_5 - w)) & \text{if } w \in H_3^1, \\ \exp(2\pi i C_3 w) & \text{if } w \in H_3^2, \end{cases} \quad (8.17)$$

where

$$C_3 = \frac{(x_2 + x'_2 - 1) - i(h + h_1)}{|(x_2 + x'_2 - 1) - i(h + h_1)|^2}.$$

In a similar way we can assign a complex chart to the puncture corresponding to the boundary point b_2 . In this case, we use the following mapping from the horizontal half-strip H_2 , the boundary of which consists of the rays $\{w = t + ih_1 : t \geq x_2\}$ and $\{w = t + ih : t \geq x'_2\}$ and a line segment $[x_2 + ih_1, x'_2 + ih]$, onto the unit disk:

$$\zeta = \exp(-2\pi i C_2(w - (x_2 + ih_1))) \quad \text{for } w \in H_2, \tag{8.18}$$

where

$$C_2 = \frac{(x'_2 - x_2) - i(h - h_1)}{|(x'_2 - x_2) - i(h - h_1)|^2}.$$

Now, our compact surface \mathcal{R} with conformal structure introduced above is conformally equivalent to the Riemann sphere $\overline{\mathbb{C}}$. Let $\Phi(w)$ be a conformal mapping from \mathcal{R} onto $\overline{\mathbb{C}}$ uniquely determined by conditions

$$\Phi(b_1) = \infty, \quad \Phi(b_2) = 1, \quad \Phi(b_3^1) = \Phi(b_3^2) = -1.$$

Next, we consider a quadratic differential $\mathcal{Q}(w) dw^2$ on \mathcal{R} defined by

$$\mathcal{Q}(w) dw^2 = 1 \cdot dw^2 \tag{8.19}$$

if w is finite and $w \neq x_1$ and $w \neq x_2 + ih_1$. This quadratic differential can be extended to the points $w = x_1$ and $w = x_2 + ih_1$ as a quadratic differential having simple zeroes at these points in terms of the local parameters defined by formulas (8.14) and (8.15), respectively.

Similarly, using local parameters defined by formulas (8.16), (8.17), and (8.18), we can extend quadratic differential (8.19) to the points of \mathcal{R} corresponding to the infinite boundary points of Ω situated at b_1 , b_2 , and $b_3^1 \simeq b_3^2$, respectively.

We note that the horizontal strips $\{w : 0 < \Im w < h_1\}$ and $\{w : h_1 < \Im w < h\}$ are strip domains of the quadratic differential (8.19), while the half-strip $\{w : 0 \leq \Re w \leq 1, \Im w < 0\}$, which boundary points are identified by the first rule in (8.13), defines a circle domain of this quadratic differential.

Now, when the quadratic differential (8.19) have been extended to a quadratic differential defined on the whole Riemann surface \mathcal{R} , we may use conformal mapping $z = \Phi(w)$ to transplant this quadratic differential to get a quadratic differential $\widehat{Q}(z) dz^2$ defined on $\overline{\mathbb{C}}$. Since critical points of a quadratic differential are invariant under conformal mapping, it follows that $\widehat{Q}(z) dz^2$ has second-order poles at the points $z = \infty$, $z = 1$ and $z = -1$ and it has simple zeroes at the images $\Phi(x_1)$ and $\Phi(x_2 + ih_1)$ of the points $w = x_1$ and $w = x_2 + ih_1$.

Furthermore, the pole $z = \infty$ belongs to a circle domain of $\widehat{Q}(z) dz^2$ and every trajectory in this circle domain has length 1. Using the above information, we conclude that $\widehat{Q}(z) dz^2 = \frac{1}{4\pi^2} Q(z) dz^2$, where $Q(z) dz^2$ is given by formula (6.1) with $p_1 = \Phi(x_1)$ and $p_2 = \Phi(x_2 + ih_1)$.

Combining our observations made in this section, we conclude the following:

Every quadratic differential of the form (6.1) having two strip domains generates a diagram of the type shown in Figures 10a–10i and every diagram of this type corresponds to one and only one quadratic differential with two strip domains in its domain configuration of the form (6.1).

9. How parameters count critical geodesics and critical loops

In Section 8, we described Q -geodesics corresponding to the quadratic differential (6.1) in terms of Euclidean geodesics in the w -plane. In this section, we explain how this information can be used to find the number of short geodesics and geodesic loops for each pair of zeros p_1 and p_2 .

To be definite, we will work with the case **6.3(b2)** of Theorem 4 assuming that

$$\Im p_1 > 0, \quad \text{and } p_2 \in E_{-1}^+(p_1). \tag{9.1}$$

In all other cases, the number of short geodesics and geodesic loops can be found similarly.

Under conditions (9.1), the domain configuration of the quadratic differential (6.1) consists of domains D_∞ , G_1 , and G_2 as it is shown in Figure 4a and Figure 4b and possible configurations of images of G_1 and G_2 under the mapping (8.1) are shown in Figures 10a–10i.

Let $\varepsilon > 0$ be sufficiently small and let dz_ε^+ denote a tangent vector to the trajectory of the quadratic differential (6.1) at $z = 1 + \varepsilon$, which can be found from the equation $Q(z) dz^2 > 0$. Using (7.1) and (7.2), we find that

$$\arg(dz_\varepsilon^+) = \frac{\pi}{2} - \frac{1}{2} \arg C_1 + o(1) = \frac{\pi}{2} - \frac{1}{2} \arg((p_1 - 1)(p_2 - 1)) + o(1), \tag{9.2}$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We assume here that $-\frac{\pi}{2} \leq \arg(dz_\varepsilon^+) \leq \frac{\pi}{2}$.

If $1 + \varepsilon \in \gamma_1$ then the tangent vector dz_ε^+ corresponds to the direction on γ_1 from $z = 1$ to $z = p_1$. Let $\alpha_\varepsilon^+ = \alpha^+ + o(1)$, where α^+ is a constant such that $0 \leq \alpha^+ \leq \pi$, denote the angle formed at the point $1 + \varepsilon \in \gamma_1$ by dz_ε^+ and the vector $\vec{v} = -i$, which is tangent to the circle $\{z : |z - 1| = \varepsilon\}$ at $z = 1 + \varepsilon$. It follows from (9.2) that

$$\alpha^+ = \pi - \frac{1}{2} \arg C_1 = \pi - \frac{1}{2} \arg((p_1 - 1)(p_2 - 1)). \tag{9.3}$$

Similarly, if dz_ε^- denote the tangent vector to the trajectory of the quadratic differential (6.1) at $z = -1 + \varepsilon$, then

$$\arg(dz_\varepsilon^-) = \frac{\pi}{2} - \frac{1}{2} \arg C_{-1} + o(1) = \frac{\pi}{2} - \frac{1}{2} \arg((p_1 + 1)(p_2 + 1)) + o(1). \tag{9.4}$$

Suppose that $1 + \varepsilon \in \gamma_{-1}$ and that d_ε^- shows direction on γ_{-1} from $z = -1$ to $z = p_2$. As before we can find constant α^- , $0 \leq \alpha^- \leq \pi$, such that the angle

formed at $z = -1 + \varepsilon \in \gamma_{-1}$ by the vectors dz_ε^+ and $\vec{v} = -i$ is equal to $\alpha^- + o(1)$, where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$\alpha^- = \pi - \frac{1}{2} \arg C_{-1} = \pi - \frac{1}{2} \arg((p_1 + 1)(p_2 + 1)). \tag{9.5}$$

To relate angles α^+ and α^- to geometric characteristics of diagrams in Figures 10a–10i, we recall that geodesics are conformally invariant and that for small $\varepsilon > 0$ a geodesic loop γ_ε^+ which passes through the point $z = 1 + \varepsilon$ and surrounds the pole $z = 1$ is an infinitesimal circle. Therefore the angle formed by the vector dz_ε^+ and the tangent vector to γ_ε^+ at $z = 1 + \varepsilon$ equals $\alpha^+ + o(1)$.

Similarly, the angle formed by the vector dz_ε^- and the tangent vector to the corresponding geodesic loop $\gamma_\varepsilon^- \ni -1 + \varepsilon$ surrounding the pole at $z = -1$ is equal to $\alpha^- + o(1)$.

Since geodesics are conformally invariant and since conformal mappings preserve angles, we conclude that trajectories of the quadratic differential $Q(w) dw^2$ defined in Section 8 (see formula (8.19)) form angles of opening α^+ or α^- with the images of the corresponding geodesic loops γ_ε^+ or γ_ε^- , respectively. Since the metric defined by the quadratic differential (8.19) is Euclidean, it follows that the corresponding images of geodesic loops are line segments joining pairs of points identified by relations (8.13).

Using this observation and identification rule $-x + x'_2 + ih \simeq x + x_2 + ih_1$, we conclude that the segment $[x_2 + ih_1, x'_2 + ih]$ forms an angle $\pi - \alpha^-$ with the positive real axis; i.e.,

$$\pi - \alpha^- = \arg((x'_2 - x_2) + i(h - h_1)). \tag{9.6}$$

To find an equation for the angle α^+ , we will use the half-strip \tilde{H}_3 constructed at the end of Section 8, which is related to a conformal mapping defined by formula (8.17). In this case, $\pi - \alpha^+$ is equal to the angle formed by the segment $[1 - x_2 - ih_1, x'_2 + ih]$ with the positive real axis; i.e.,

$$\pi - \alpha^+ = \arg((x_2 + x'_2 - 1) + i(h + h_1)). \tag{9.7}$$

Equating the right-hand sides of equations (9.3) and (9.4) to the right-hand sides of equations (9.7) and (9.6), respectively, we obtain two equations, which relate parameters x_2, x'_2, h_1 , and h . Combining this with equations (7.10)–(7.12), we obtain the following system of four equations:

$$\begin{aligned} \arg((x_2 + x'_2 - 1) + i(h + h_1)) &= \frac{1}{2} \arg((p_1 - 1)(p_2 - 1)) \\ \arg((x'_2 - x_2) + i(h - h_1)) &= \frac{1}{2} \arg((p_1 + 1)(p_2 + 1)) \\ h_1 &= \frac{1}{4} \Im \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \\ h &= \frac{1}{4} \Im \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right). \end{aligned}$$

This system of equations can be solved to obtain the following:

$$\begin{aligned} x_2 + ih_1 &= \frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right), \\ x'_2 + ih &= \frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right). \end{aligned} \quad (9.8)$$

Now, when the points $x_2 + ih_1$ and $x'_2 + ih$ are determined, we can give explicit conditions on the zeros p_1 and p_2 which correspond to all subcases (a)–(i) of the case **6.3(b2)** discussed in Section 8.

Theorem 5. *Suppose that zeros p_1 and p_2 satisfy conditions (9.1). Then the number of short geodesics and geodesic loops and their topology are determined by the following inequalities, which corresponds to the subcases (a)–(i) of Case **6.3(b2)** described in Section 8 and shown in Figures 10a–10i:*

Case (a) with four short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) \\ &< \arg \left(\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) < \pi. \end{aligned}$$

Case (b) with four short geodesics and two critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) \\ &= \arg \left(\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) < \pi. \end{aligned}$$

Case (c) with four short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg \left(\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) \\ &< \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) < \pi, \\ 0 &< \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) \\ &< \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) < \pi. \end{aligned}$$

Case (d) with three short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} - \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) \\ &= \arg \left(-\frac{1}{2} + \frac{1}{4} \left(\sqrt{(p_1 - 1)(p_2 - 1)} + \sqrt{(p_1 + 1)(p_2 + 1)} \right) \right) < \pi. \end{aligned}$$

Case (e) with four short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg\left(-\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &< \arg\left(-\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi, \\ 0 &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi. \end{aligned}$$

Case (f) with three short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &= \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi. \end{aligned}$$

Case (g) with four short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi, \\ 0 &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &< \arg\left(-\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi. \end{aligned}$$

Case (h) with four short geodesics and two critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &= \arg\left(-\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi. \end{aligned}$$

Case (i) with four short geodesics and three critical geodesic loops occurs if the following conditions are satisfied:

$$\begin{aligned} 0 &< \arg\left(-\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} + \sqrt{(p_1+1)(p_2+1)}\right)\right) \\ &< \arg\left(\frac{1}{2} + \frac{1}{4}\left(\sqrt{(p_1-1)(p_2-1)} - \sqrt{(p_1+1)(p_2+1)}\right)\right) < \pi. \end{aligned}$$

10. Some related questions

Our results presented in Sections 6–9 provide complete information concerning critical trajectories and Q -geodesic of the quadratic differential (6.1). This allows us to answer many related questions. As an example, we will discuss three questions originated in the study of limiting distributions of zeros of Jacobi polynomials.

Below, we suppose that $p_1, p_2 \in \mathbb{C}$ are fixed. Then we consider the family of quadratic differentials $Q_s(z) dz^2$ depending on the real parameter s , $0 \leq s < 2\pi$, such that

$$Q_s(z) dz^2 := e^{-is} Q(z) dz^2 = -e^{-is} \frac{(z - p_1)(z - p_2)}{(z - 1)^2(z + 1)^2} dz^2. \tag{10.1}$$

- 1) For how many values of s , $0 \leq s < 2\pi$, the quadratic differential $Q_s(z) dz^2$ has a trajectory loop with end points at p_1 and for how many values of s $Q_s(z) dz^2$ has a trajectory loop with end points at p_2 ?
- 2) For how many values of s , $0 \leq s < 2\pi$, the corresponding quadratic differential $Q_s(z) dz^2$ has a short critical trajectory?
- 3) How we can find the values of s , $0 \leq s < 2\pi$, mentioned in questions stated above?

To answer these questions we need two simple facts:

- (a) First, we note that γ is a short trajectory loop or, respectively, a short critical trajectory for the quadratic differential (10.1) with some s if and only if γ is a short geodesic loop or, respectively, a short geodesic joining points p_1 and p_2 for the quadratic differential (6.1). Thus, the numbers of values s in question 1) and question 2), respectively, are bounded by the number of short geodesic loops and the number of short geodesics, respectively. In the most general case with one circle domain and two strip domains, these short geodesic loops and short geodesics were described in Theorem 5 and their images under the canonical mapping were shown in Figures 10a–10i. Of course, one value of s can correspond to more than one short geodesic loop and more than one short geodesic.
- (b) To find the values of s in question 3), we use the following observation. If l is a straight line segment in the image domain Ω forming an angle α , $0 \leq \alpha < \pi$, with the direction of the positive real axis, then l is an image under the canonical mapping (8.1) of an arc of a trajectory of the quadratic differential (10.1) with

$$s = 2\alpha. \tag{10.2}$$

We will use (10.2) to find values of s which turn short geodesic loops and short geodesics into short trajectory loops and short trajectories, respectively. It is convenient to introduce notations $\alpha_\infty, \alpha_{12}, \alpha'_{12}, \alpha_{22}, \alpha'_{22}, \alpha''_{22}$, and so on, to denote the angles formed by corresponding geodesics $\gamma_\infty, \gamma_{12}, \gamma'_{12}, \gamma_{22}, \gamma'_{22}, \gamma''_{22}$, and so on (considered in the w -plane) with the positive direction of the real axis. Furthermore, we will use notations $\mathcal{A}(6.1), \mathcal{A}(6.1(a)), \mathcal{A}(6.2), \mathcal{A}(6.3(a)), \mathcal{A}(6.3(b1)), \mathcal{A}(6.3(b2)(a))$, and so on, to denote the sets of all angles introduced above in the

cases under consideration; i.e., in the cases **6.1**, **6.2**, **6.3(a)**, **6.3(b₁)**, **6.3(b₂)(a)**, and so on.

Now, we are ready to answer questions stated above. We proceed with two steps. First, we identify the type of domain configuration \mathcal{D}_Q . This will provide us with the first portion of necessary information. We recall that in general there are at most three geodesic loops centered at $z = \infty$, $z = 1$, and $z = -1$. Thus, the maximal number of values s in question 1) is at most three. Then we identify which of the schemes corresponds to the parameters p_1, p_2 (in the most general case these schemes are shown in Figures 10a–10i). This will provide us with the remaining portion of necessary information.

- Suppose that \mathcal{D}_Q has type **6.1**. Then we already have three circle domains and therefore $s = 0$ is the only value for which $Q_s z) dz^2$ may have short trajectory loops. In case **6.1(a)**, we have short trajectory loops centered at $z = 1$ and $z = -1$ and no other such loops. In case **6.1(b)** with $1 < p_2 < p_1$ (respectively with $p_1 < p_2 < -1$), we have short trajectory loops centered at $z = \infty$ and $z = 1$ (respectively, at $z = \infty$ and $z = -1$). In case **6.1(c)**, there are no short geodesic loops.

As concerns short critical trajectories for domain configuration of type **6.1**, again $s = 0$ is the only value for which there are such trajectories. This follows from the fact discussed in Section 8 that in case **6.1** there are no other simple geodesics joining p_1 and p_2 . In cases **6.1(a)** and **6.1(b)**, there is a single short critical trajectory which is the interval $\gamma_0 = (p_2, p_1)$. In case **6.1(c)**, there are three short critical trajectories which are arcs γ_0, γ_1 , and γ_{-1} shown in Figure 1c.

- Next, we consider the case when \mathcal{D}_Q has type **6.2**. For $s = 0$, we have two short trajectory loops. As before, we assume that these loops surround points $z = -1$ and $z = \infty$. In other cases discussion is similar, we just have to switch roles of the poles of the quadratic differential (10.1).

In this case, $\mathcal{A}(6.2) = \{0, \alpha_{11}, \alpha_{12}, \alpha'_{12}, \alpha_{21}, \alpha'_{21}\}$. One more value of s , for which we may have a short trajectory loop (centered at $z = 1$) may occur for $s = 2\alpha_{11} = -\arg((1 - p_1)(1 - p_2))$. If $|\gamma_\infty|_Q > |\gamma_{-1}|_Q$ then we will have a short geodesic loop from p_1 to p_1 . This loop corresponds to a geodesic γ_{11} in Figure 8a. If $|\gamma_\infty|_Q < |\gamma_{-1}|_Q$, then we will have a similar short geodesic loop from p_2 to p_2 . In the case $|\gamma_\infty|_Q = |\gamma_{-1}|_Q$, we have $\alpha_{11} = \alpha_{12} = \alpha'_{21}$. In this case, we do not have the third short geodesic loop. Instead, we have two short critical trajectories joining p_1 and p_2 .

By (10.2), the value of s , which corresponds to the third loop (if it exists) is equal to $2\alpha_{11}$. As concerns values of s corresponding to short critical trajectories, in case **6.2** with $|\gamma_\infty|_Q \neq |\gamma_{-1}|_Q$ we have four such values. These values are $2\alpha_{12}, 2\alpha'_{12}, 2\alpha_{21}$, and $2\alpha'_{21}$ (see Fig. 8a).

If $|\gamma_\infty|_Q = |\gamma_{-1}|_Q$, then there are three values of s , which produce short geodesics from p_1 to p_2 . Two of these values, $s = 2\alpha'_{12}$ and $s = 2\alpha_{21}$, generate one short critical trajectory each. The third value $s = 2\alpha_{12}$ generates two short critical trajectories.

- Turning to the most general case **6.3**, we will give detailed account for subcases **6.3(b1)** and **6.3(b2)(i)**, in all other subcases consideration is similar.

First, we consider the subcase **6.3(b1)** when the domain configuration \mathcal{D}_Q consists of one circle domain and one strip domain; see Figures 3a–3e. In this case, $\mathcal{A}(6.3(b1)) = \{0, \alpha'_{22}, \alpha''_{22}, \alpha_{12}, \alpha'_{12}\}$. The value $s = 0$ generates one short trajectory loop and one short trajectory. The values $s = 2\alpha'_{22}$ and $s = 2\alpha''_{22}$ generate one short trajectory loop each and the values $s = 2\alpha_{12}$ and $s = 2\alpha'_{12}$ generate one short trajectory each.

Let us consider case **6.3(b2)(i)** shown in Figure 10i. We have $\mathcal{A}(6.3(b2)(i)) = \{0, \alpha_{22}, \alpha'_{22}, \alpha_{12}, \alpha'_{12}, \alpha_{21}, \alpha'_{21}\}$ where all angles are distinct. The values $s = 0$, $s = 2\alpha_{22}$, and $s = 2\alpha'_{22}$ generate short trajectory loops γ_∞ , γ_{22} , and γ''_{22} , respectively. Remaining values $s = 2\alpha_{12}$, $s = 2\alpha'_{12}$, $s = 2\alpha_{21}$, $s = 2\alpha'_{21}$ generate short trajectories γ_{12} , γ'_{12} , γ_{21} , and γ'_{21} , respectively.

Finally, we note that position of points x_1 , x'_1 , $x_2 + ih_1$, and $x'_2 + ih$ are given explicitly; see formulas (9.8). Using these formulas one can find explicit expressions for all angles α_{12} , α'_{12} , α_{21} , α'_{21} , and so on, in all possible cases.

11. Figures Zoo

This section contains all our figures. For convenience, we divide the set of all figures in eleven groups.

I. Configurations with three circle domains.

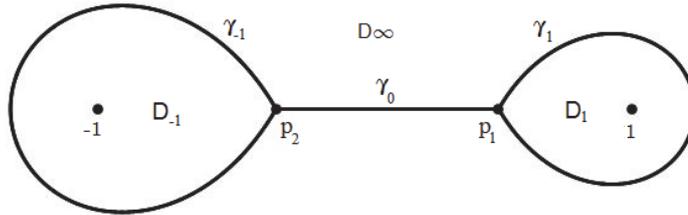


FIG. 1a. Three circle domains. Case **6.1(a)**.

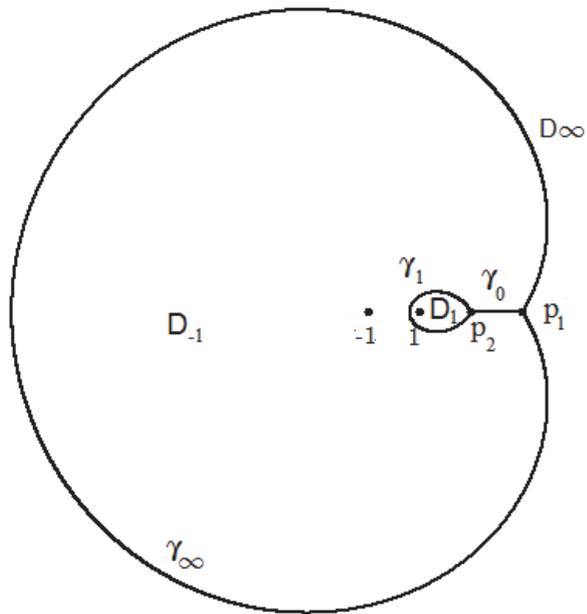


FIG. 1b. Three circle domains. Case **6.1(b)**.

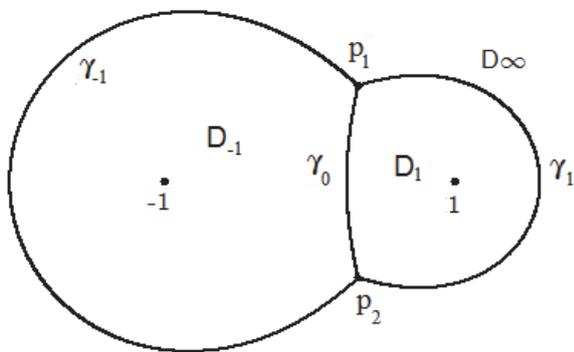


FIG. 1c. Three circle domains. Case **6.1(c)**.

II. Configurations with two circle domains.

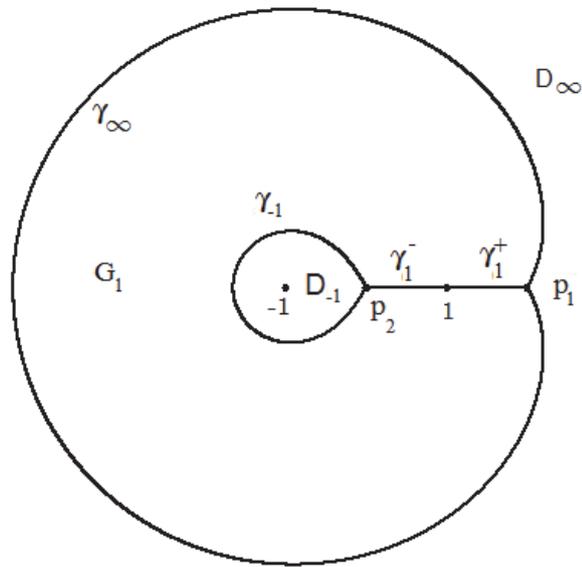


FIG. 2a. Two circle domains. Case 6.2 with symmetric domains.

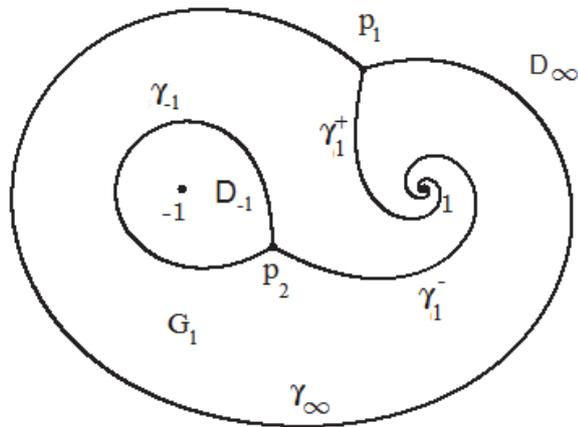


FIG. 2b. Two circle domains. Case 6.2 with non-symmetric domains.

III. Configurations with one circle domain and one strip domain.

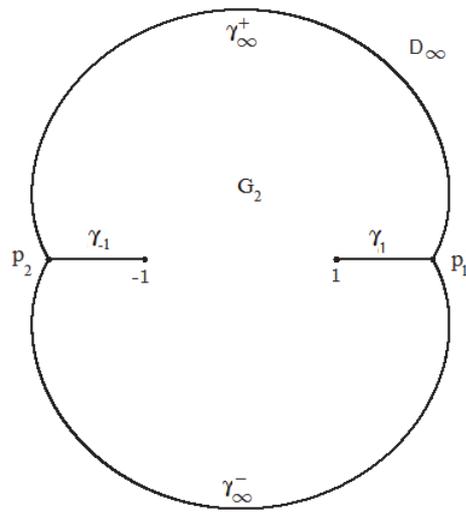


FIG. 3a. One circle domain. Case 6.3(a) with axial symmetry.

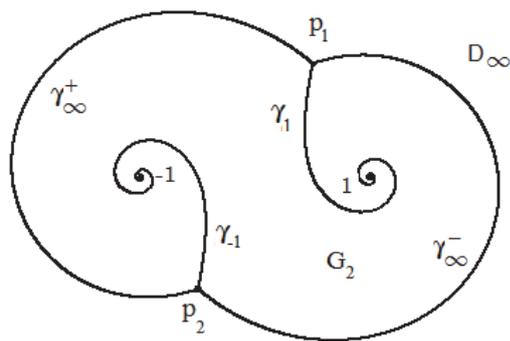


FIG. 3b. One circle domain. Case 6.3(a) with central symmetry.

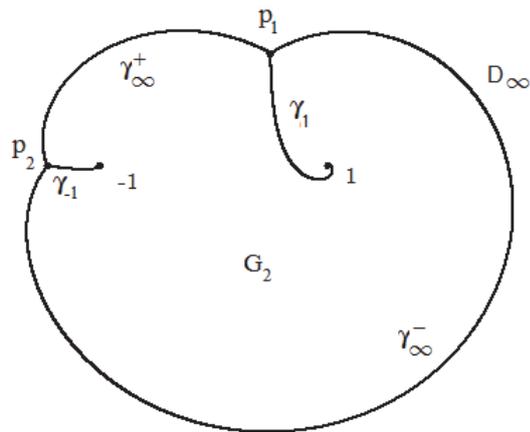


FIG. 3c. One circle domain. Case **6.3(a)** with non-symmetric domains.

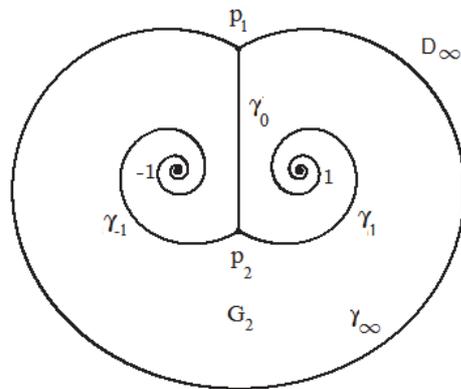


FIG. 3d. One circle domain. Case **6.3(b1)** with symmetric domains.

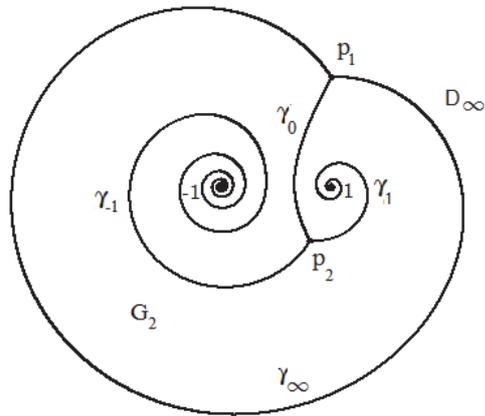


FIG. 3e. One circle domain. Case **6.3(b1)** with non-symmetric domains.

IV. Configurations with one circle domain and two strip domains.

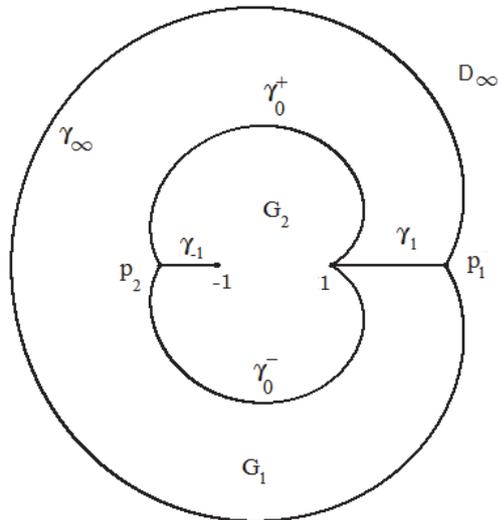


FIG. 4a. One circle domain. Case **6.3(b2)** with symmetric domains.

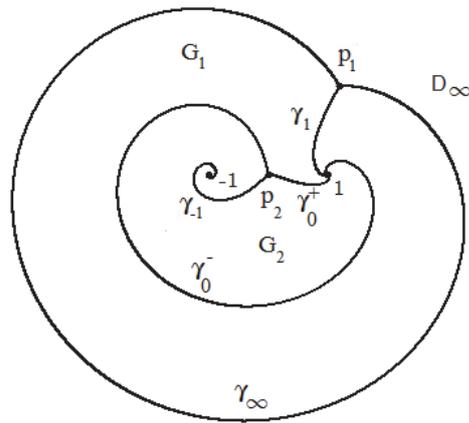


FIG. 4b. One circle domain. Case **6.3(b2)** with non-symmetric domains.

V. Degenerate configurations.

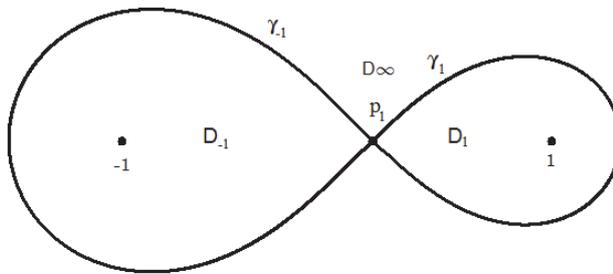


FIG. 5a. Degenerate case with $-1 < p_1 = p_2 < 1$.

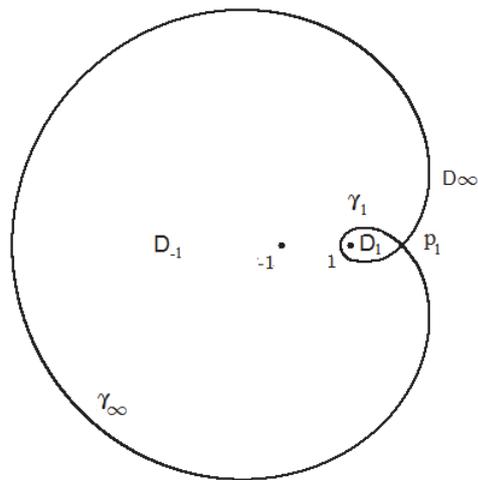


FIG. 5b. Degenerate case with $p_1 = p_2 > 1$.

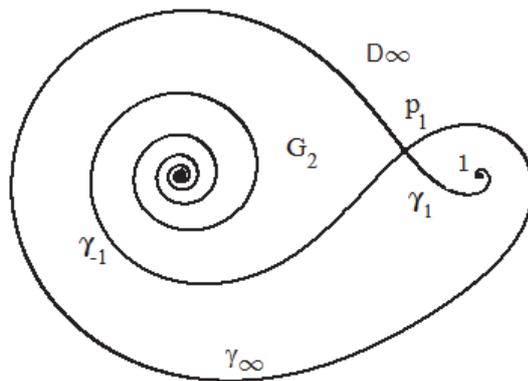


FIG. 5c. Degenerate case with $p_1 = p_2, \Im p_1 > 0$.

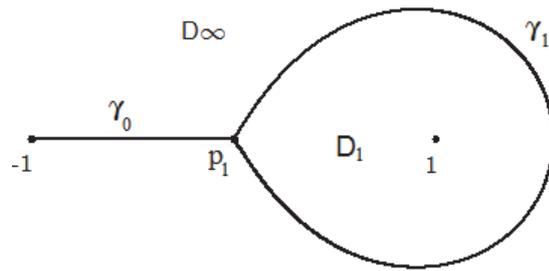


FIG. 5d. Degenerate case with $p_2 = -1$, $-1 < p_1 < 1$.

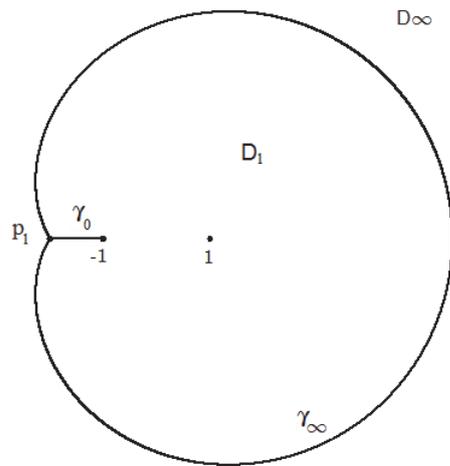


FIG. 5e. Degenerate case with $p_2 = -1$, $p_1 < -1$.

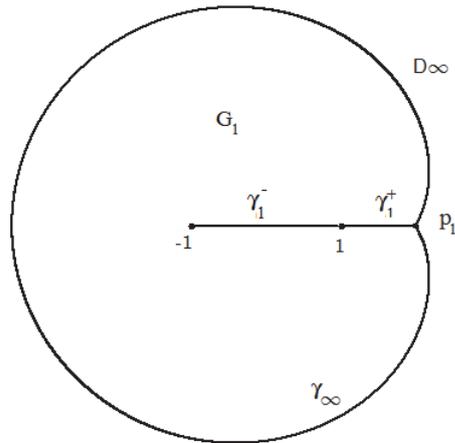


FIG. 5f. Degenerate case with $p_2 = -1$, $p_1 > 1$.

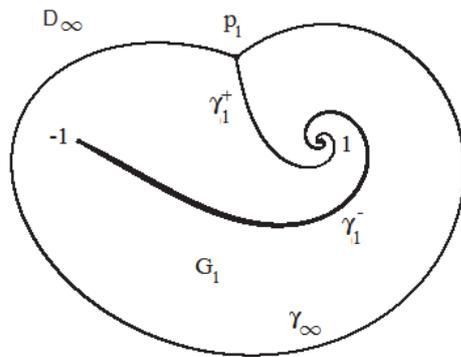


FIG. 5g. Degenerate case with $p_2 = -1$, $\Im p_1 > 0$.

VI. Type regions.

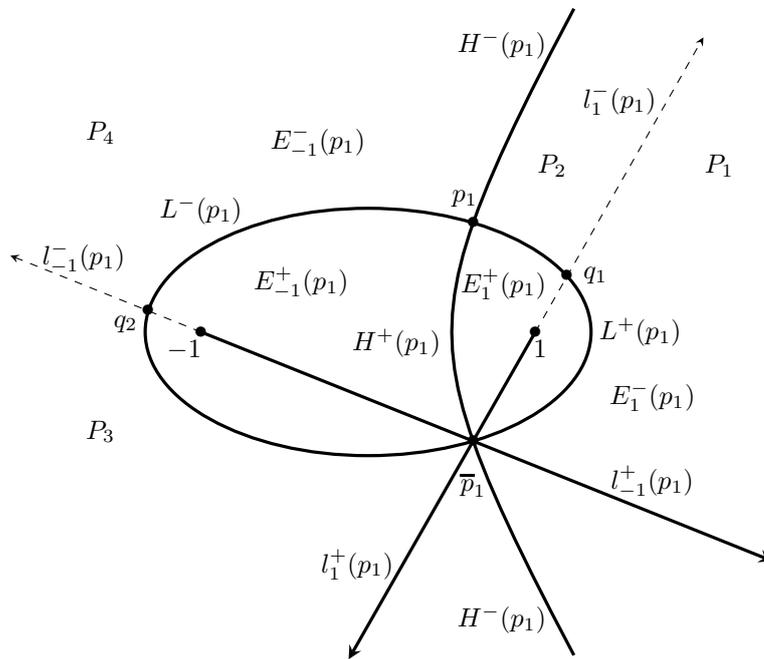


FIG. 6. Type regions.

VII. Figures for the proof of Theorem 4.

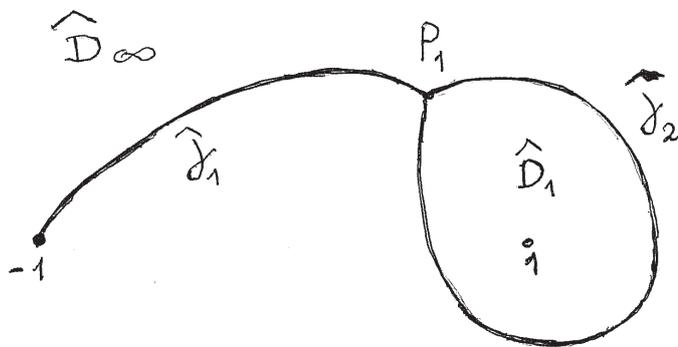


FIG. 7a. Proof of Theorem 4: Impossible limit configuration.

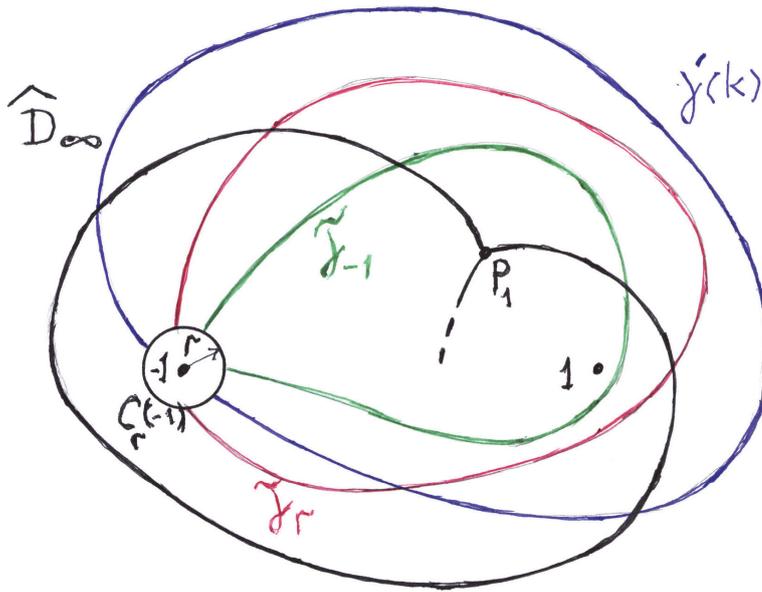


FIG. 7b. Proof of Theorem 4: Limit configuration.

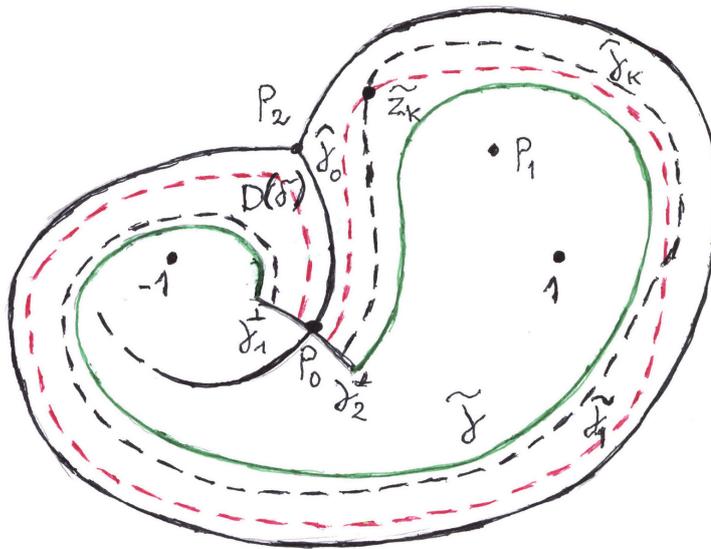


FIG. 7c. Proof of Theorem 4: Q^0 -rectangle $D(\delta)$ with trajectories.

VIII. Geodesics and loops in simple cases.

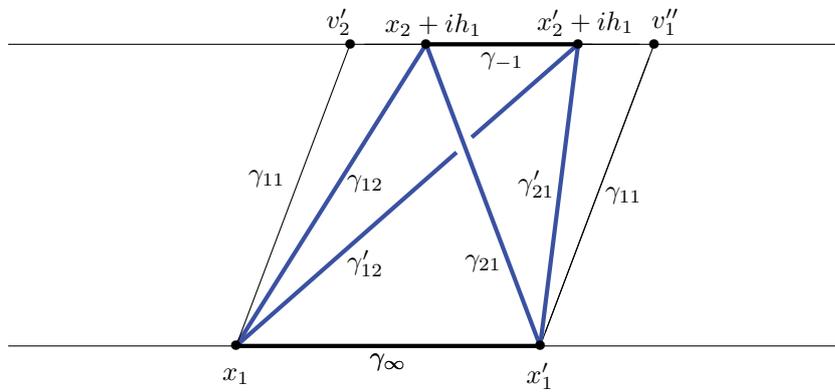


FIG. 8a. Geodesics and loops. Case **6.2**.

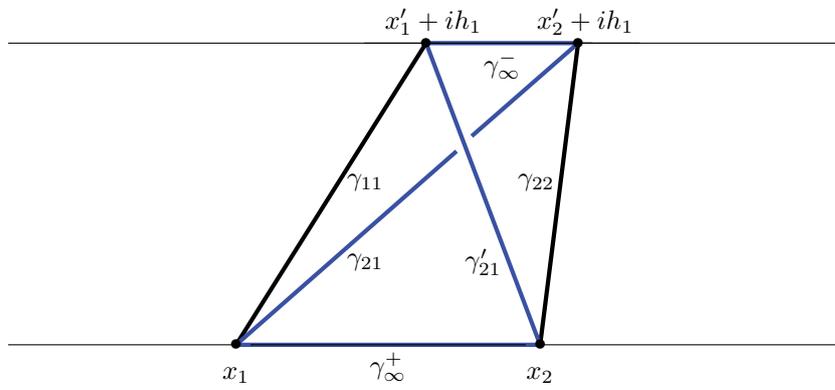


FIG. 8b. Geodesics and loops. Case **6.3(a)**.

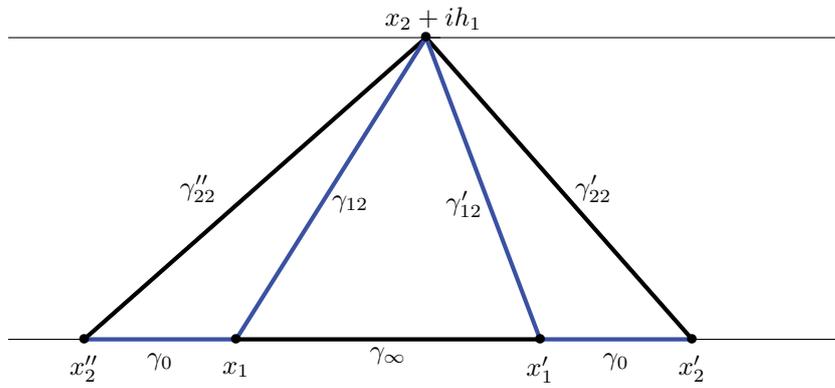


FIG. 8c. Geodesics and loops. Case **6.3(b1)**.

IX. Divergent segments.

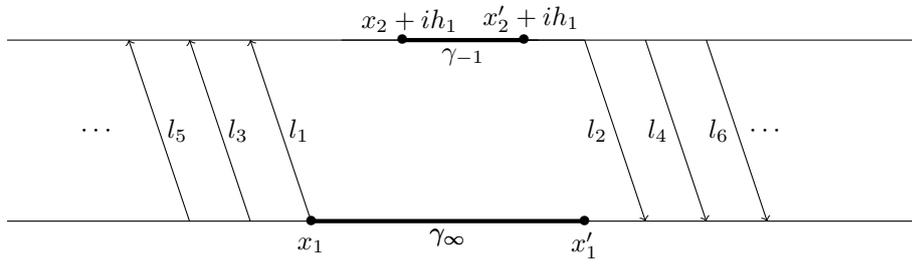


FIG. 9a. Divergent segments. Case 6.2.

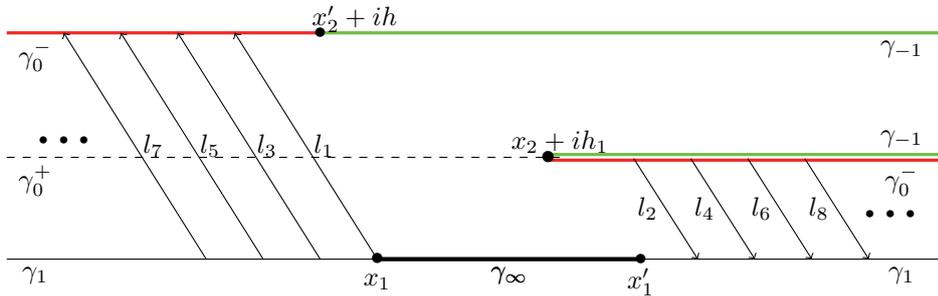


FIG. 9b. Divergent segments. Case 6.3(b2).

X. Geodesics and loops in the most general case.

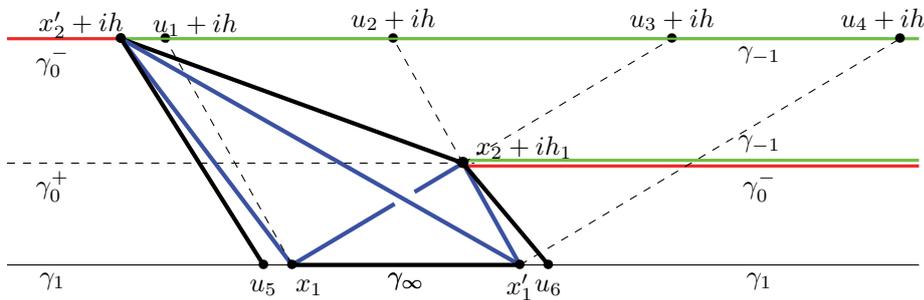


FIG. 10a. Critical geodesics and loops. Case 6.3(b2)(a).

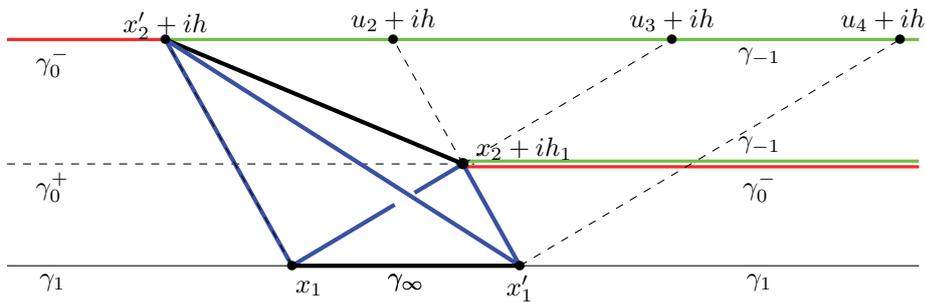


FIG. 10b. Critical geodesics and loops. Case **6.3(b2)(b)**.

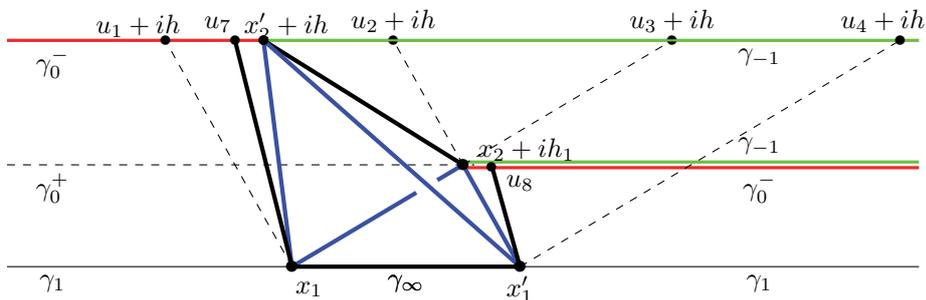


FIG. 10c. Critical geodesics and loops. Case **6.3(b2)(c)**.

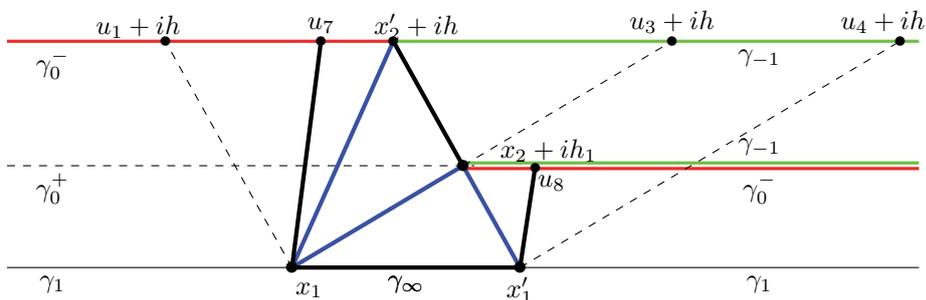


FIG. 10d. Critical geodesics and loops. Case **6.3(b2)(d)**.

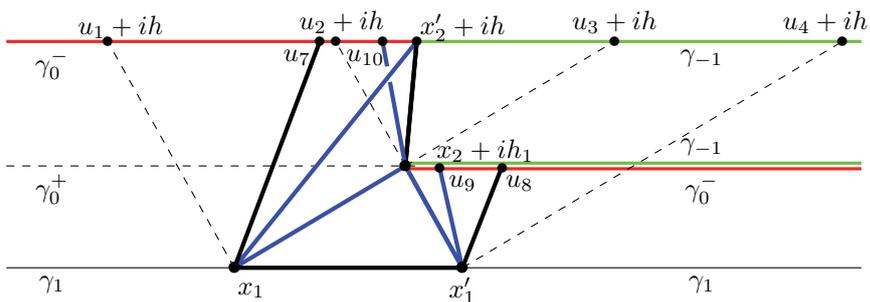


FIG. 10e. Critical geodesics and loops. Case **6.3(b2)(e)**.

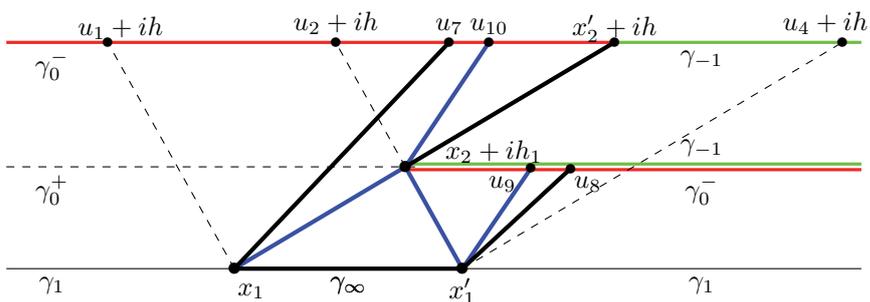


FIG. 10f. Critical geodesics and loops. Case **6.3(b2)(f)**.

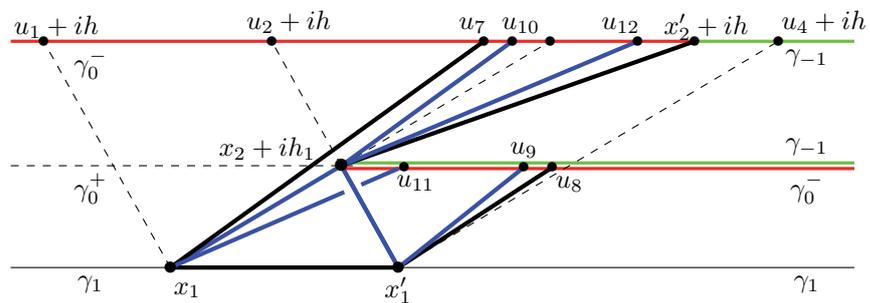


FIG. 10g. Critical geodesics and loops. Case **6.3(b2)(g)**.

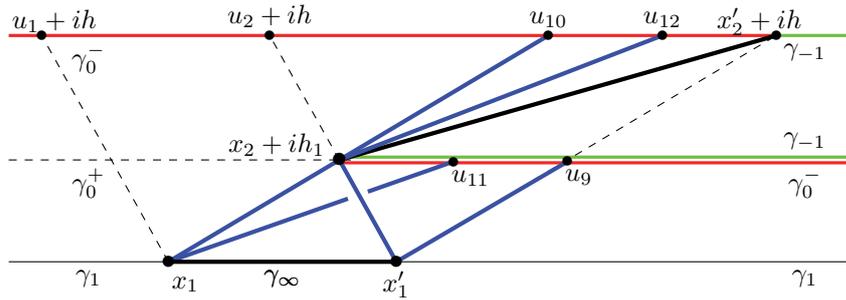


FIG. 10h. Critical geodesics and loops. Case **6.3(b2)(h)**.

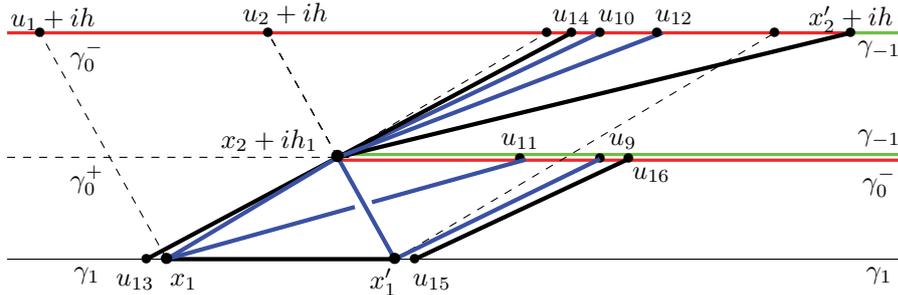


FIG. 10i. Critical geodesics and loops. Case **6.3(b2)(i)**.

XI. Identification rules.

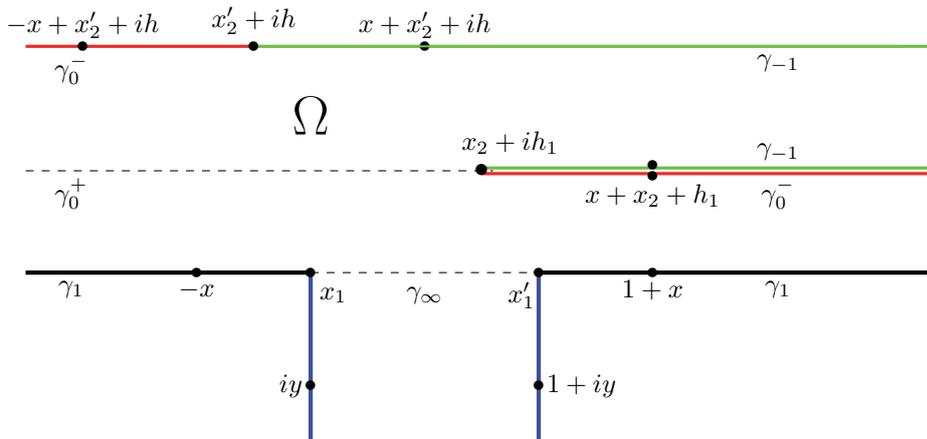


FIG. 11. Domain Ω and identification rules.

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