ASYMPTOTICS OF RODRIGUES’ DESCENDANTS OF A POLYNOMIAL

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Abstract. Motivated by the classical Rodrigues’ formula, we study below the root asymptotics of the polynomial sequence

\[ R_{\lfloor \alpha n \rfloor, n, P}(z) = \frac{d^n}{dz^n} P_n(z), \quad n = 0, 1, \ldots \]

where \( P(z) \) is a given polynomial, \( \alpha < \deg P \) is a fixed positive number, and \( \lfloor \alpha n \rfloor \) stands for the integer part of \( \alpha n \).

Our answer is expressed in terms of an explicit harmonic function determined by a certain plane rational curve. This curve is obtained from the sequence of symbols of the linear homogeneous differential equations satisfied by \( \{R_{\lfloor \alpha n \rfloor, n, P}(z)\} \) using a natural limiting procedure when \( n \to \infty \). In particular, this curve provides the algebraic equation satisfied by the Cauchy transform of the asymptotic root-counting measure for the latter sequence.

1. Introduction

In 1816 (Benjamin) Olinde Rodrigues\(^1\) discovered his famous formula

\[ P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left( (z^2 - 1)^n \right) \]  

(1.1)

for the Legendre polynomials which undoubtedly became a standard tool in the toolbox of the theory of classical orthogonal polynomials and special functions, see e.g. [AbSt].

Later this formula was also rediscovered by Sir J. Ivory and C. G. Jacobi, see [As]. Among other properties, the \( n \)-th Legendre polynomial \( P_n(z) \) satisfies the linear differential equation

\[ (1 - z^2)y'' - 2zy' + n(n+1)y = 0. \]  

(1.2)

Imitating Rodrigues’ approach, one can, for any given polynomial \( P \) of degree \( d \geq 1 \), consider a double-indexed family of polynomials given by the Rodrigues-like expression

\[ R_{m, n, P}(z) := \frac{d^m}{dz^m} (P^n(z)), \quad n = 0, 1, \ldots \text{ and } m = 0, 1, \ldots, nd, \]

\(^1\)Born in a Jewish family of sephardic origin in Bordeaux on October 6, 1795, O. Rodrigues, thanks to Napoleon’s measures ensuring equality of rights for different religious minorities, was able to attend Lyceé Imperial which he joined in 1808 at the age of 14. Besides his mathematical interests, he had another passion: banking and its usage for social purposes. He has been a close friend and supporter of Saint-Simon and a very peculiar philanthropic figure with strong socialist undertones, see more details in [Al] and [AlOr].

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see an illustration in Fig. 3. If $P = (z^2 - 1)$ and $m = n$, we get the above classical case of the Legendre polynomials up to a scalar factor. Analogously, for a meromorphic function $f$ given in some open domain $\Omega \subseteq \mathbb{C}$, one can define its $(m,n)$-th Rodrigues’ descendant in $\Omega$ as

$$R_{m,n,f}(z) := \frac{d^m f^n(z)}{dz^m}.$$  

In the present paper, we study the asymptotic root behavior for the sequences of Rodrigues’ descendants when $f$ is a polynomial postponing the case when $f$ is a rational function for a future publication, see [BHS].

**Main results.** In what follows, we will always assume that a polynomial $P(z)$ under consideration satisfies the condition $d := \deg P \geq 2$. The remaining case $d = 1$ is trivial.

For any polynomial $P$ and its Rodrigues’ descendant $R_{m,n,P}(z)$, denote by $\mu_{m,n,P}$ the root-counting measure of $R_{m,n,P}(z)$ and by

$$C_{m,n,P}(z) := \frac{R'_{m,n,P}(z)}{(dn - m) \cdot R_{m,n,P}(z)}$$

its Cauchy transform. Notice that $dn - m = \deg R_{m,n,P}$. (For the used basic definitions from potential theory consult § 2.2 and [Ra].)

**Theorem 1.** For any polynomial $P$ and a given positive number $\alpha < \deg P$, there exists a weak limit

$$\mu_{\alpha,P} = \lim_{n \to \infty} \mu_{[\alpha n],n,P}.$$  

Moreover, its Cauchy transform $C_{\alpha,P}$ defined as the pointwise limit

$$C := C_{\alpha,P}(z) := \lim_{n \to \infty} C_{[\alpha n],n,P}(z)$$

exists almost everywhere (a.e.) in $\mathbb{C}$ and satisfies the algebraic equation:

$$\sum_{k=0}^{d} \frac{\alpha^{k-1}(\alpha - k)(d - \alpha)^{d-k}}{k!} P^{(k)} C^{d-k} = 0. \quad (1.3)$$
Corollary 1. The scaled Cauchy transform $W := W_{\alpha,P} := \frac{d-\alpha}{\alpha} C_{\alpha,P}$ satisfies a simpler algebraic equation:

$$\sum_{k=0}^{d} \frac{\alpha - k}{k!} P^{(k)} W^{d-k} = 0. \quad (1.4)$$

Corollary 2. Equations (1.3) and (1.4) define rational affine curves which are irreducible if and only if all roots of $P$ are simple. If $P$ has roots of multiplicity at least 2, then (1.4) admits a finite number of “trivial” factors of the form $W = (b - z)^{-1}$, where $b$ is such a root. The remaining factor of (1.4) is irreducible and can be written as

$$\alpha W = \frac{P'(z + W^{-1})}{P(z + W^{-1})} = \frac{d \log P(z + W^{-1})}{dz}. \quad (1.5)$$

Remark 1. For the original Cauchy transform, (1.5) implies the relation

$$(d - \alpha) C = \frac{d \log P(z + \frac{\alpha}{d-\alpha} C^{-1})}{dz}. \quad (1.6)$$

Remark 2. Observe that, for any $\alpha < d$, the support $S_{\alpha}$ of $\mu_{\alpha}$ is contained in the convex hull $\text{Conv}_P$ of the zero locus $Z_P$ of polynomial $P$.

Theorem 1 together with the description of the the logarithmic transform as a piecewise harmonic function in Theorem 14 gives the following consequence, through the analysis of supports of the Laplacian of a maximum of harmonic functions in Theorem 4 and using results from [BB, BBB]. Konstig fras och nsta korrolarium!

Corollary 3. For any polynomial $P$ and $0 < \alpha < d$, the support $S_{\alpha,P}$ of $\mu_{\alpha,P}$ consists of finitely many compact semi-analytic curves and points, see Fig. 2. That the measure has point support is precisely the case when $\alpha < 1$. 

Figure 2. The zeros of $R_{m,n,P}(z)$ (shown by small dots), where $P = (z - 1)(z - 6)(z - 3i)$, $n = 60$. Here, $m = 3$ (top left), $m = 18$ (top right), $m = 60$ (bottom left), and $m = 120$ (bottom right). The larger dots are the zeros of $P$, and the triangle is the center of mass of the zero locus of $P$. 
Figure 3. The zeros of $(P^n)^{(m)}$ for $m \in \{0, 1, \ldots, (3 + j)n\}$, (shown by small dots), where $P = z^j(z - 2 - 8i)(z - 8 - 7i)(z - 12)$, $n = 20$, and $j = 1, 2, 3, 4$. In each case, the larger dots are the zeros of $P$ and $P'$, and the triangle is the center of mass of the zero locus $Z(P)$.

**Examples.**

(i) For $P = z^2 + az + b$, equation (1.3) reduces to

$$(2 - \alpha)^2(z^2 + z + b)C^2 + (\alpha - 1)(2 - \alpha)(2z + a)C + \alpha(\alpha - 2) = 0.$$  \hspace{1cm} (1.7)

If $\alpha = 1$, and $P$ is a monic polynomial, then

(ii) For $d = 2$, we get

$$PC^2 - 1 = 0.$$  

(iii) For $d = 3$, we get

$$PC^3 - \frac{P''}{2!} 2^2 C - \frac{2}{2^3} = 0.$$  

(iv) For $d = 4$, we get

$$PC^4 - \frac{P''}{2!} 3^2 C^2 - \frac{2P'''}{3!} 3^3 C - \frac{3P^{(iv)}}{4!} 3^4 = 0.$$  

(v) For $d = 5$, we get

$$PC^5 - \frac{P''}{2!} 4^2 C^3 - \frac{2P'''}{3!} 4^3 C^2 - \frac{3P^{(iv)}}{4!} 4^4 C - \frac{4P^{(v)}}{5!} 4^5 = 0.$$
1.1. If Olinde Rodrigues were a coauthor... A very basic example of the considered situation is related to Legendre polynomials and the original Rodrigues’ formula in which case we will exemplify our results explicitly. Then $P(z) = z^2 - 1$ and for the sequence of Legendre polynomials (1.1), the density of their asymptotic root distribution equals

$$\rho(x) = \frac{2}{\pi} \sqrt{1-x^2} dx, \; x \in [-1, 1].$$

More generally, for a given $0 < \alpha < 2$, considering the sequence $R_n^{(\alpha)}(x) := \frac{d[(\alpha n)!]}{d[(\alpha n)!]}(x^2 - 1)^n$, the smooth rational plane algebraic curve $D$ that plays an important role in our calculations is defined by the equation

$$P'(u)(u-z) = \alpha P(u) \iff (2-\alpha)u^2 - 2uz + \alpha = 0 \iff u = \frac{1}{2-\alpha}(z \pm \sqrt{z^2 + \alpha^2 - 2\alpha}).$$

It has branch points corresponding to $z = \pm \sqrt{\alpha(2-\alpha)}$. Also the pluriharmonic function $H$ is

$$H(u, z) = \frac{1}{2 - \alpha}(\log|u^2-1| - \alpha \log|u-z|) = \frac{1}{2 - \alpha}(\log\frac{2}{\alpha} + \log|u| + (1-\alpha)\log|u-z|).$$

We get that for $2 > \alpha \geq 1$ the asymptotic root distribution will be supported on the interval $I_\alpha = [-\sqrt{\alpha(2-\alpha)}, \sqrt{\alpha(2-\alpha)}]$, with the density given by

$$\rho^{(\alpha)}(x) = \frac{2}{\pi} \sqrt{2\alpha - \alpha^2 - x^2}, \; x \in I_\alpha,$$

while for $1 > \alpha > 0$ the asymptotic root distribution will be supported on the two disjoint intervals $I^\pm_\alpha = [\pm 1, \pm \sqrt{\alpha(2-\alpha)}]$ with the density given by

Hence we may intuitively describe what happens when $\alpha$ increases as follows. The two point measures move out of $\pm 1$, until the measure has support on the whole interval at $\alpha = 1$, to then contract to the midpoint of the interval (and vanish at $\alpha = 2$). See figure ??.

THE FOLLOWING IS NOT TRUE: we get that the asymptotic root distribution will be supported on the interval $I_\alpha = [-\sqrt{\alpha(2-\alpha)}, \sqrt{\alpha(2-\alpha)}]$ with the density given by

$$\rho^{(\alpha)}(x) = \frac{2}{\pi} \sqrt{2\alpha - \alpha^2 - x^2}, \; x \in I_\alpha.$$

In particular, for all $0 < \alpha < 2$, $I_\alpha$ is contained in $[-1, 1]$ and is strictly smaller than $[-1, 1]$ for all $\alpha \neq 1$.

1.2. Techniques. Most of the above mentioned results are obtained as an application of a general framework which we develop in §3 and which can be quite useful in various asymptotic questions involving polynomial sequences originating from a family of linear ordinary differential equations.

Namely, given $\mathbb{C}^2$ with coordinates $(z, w)$, we define and study a special class of plane curves which we call affine Boutroux curves; such a curve is characterized by the fact that the standard 1-form $w dz$ has only imaginary periods on the normalization of its compactification in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Special type of Boutroux curves was earlier introduced in [BM] where also the term “Boutroux curves” was coined; it was further elaborated in [Be] and later used by a number of authors. For every
affine Boutroux curve, we define a special harmonic function on this curve as well as its push-forward to $\mathbb{CP}^1$. This push-forward is a piecewise harmonic function whose Laplacian (considered as a 2-current) is a signed measure on $\mathbb{CP}^1$ supported on a finite union of semi-analytic curves and isolated points. The most essential property of this measure is that its Cauchy transform satisfies a.e. in $\mathbb{CP}^1$ the same algebraic equation which defines the initial affine Boutroux curve!

The structure of the paper is as follows. After recalling some basic notions in § 2, we introduce in § 3 affine Boutroux curves and related harmonic functions and measures. In § 4, we settle Proposition 8 and Corollaries 4, 5, and 6. In § 5, we prove that algebraic curves given by (1.3) and (1.5) are affine Boutroux curves. In § 6, we settle Theorem 1 by using a special version of the saddle-point method. Finally, in § 7, we suggest a number of open problems related to this topic.

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2. SOME BASICS OF POTENTIAL THEORY IN THE COMPLEX PLANE

For the convenience of our readers, let us briefly recall some basic notions and facts used throughout the text. Let $\mu$ be a finite compactly supported complex measure in the complex plane $\mathbb{C}$. Define the logarithmic potential of $\mu$ as

$$L_\mu(z) := \int_{\mathbb{C}} \ln |z - \xi| \, d\mu(\xi)$$

and the Cauchy transform of $\mu$ as

$$C_\mu(z) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi}.$$

Standard facts about the logarithmic potential and the Cauchy transform include:

- $C_\mu$ and $L_\mu$ are locally integrable; in particular they define distributions on $\mathbb{C}$ and therefore can be acted upon by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.
- $C_\mu$ is analytic in the complement in $\mathbb{C} \cup \{\infty\}$ to the support of $\mu$. For example, if $\mu$ is supported on the unit circle (which is the most classical case), then $C_\mu$ is analytic both inside the open unit disc and outside the closed unit disc.
- the relations between $\mu$, $C_\mu$ and $L_\mu$ are as follows:

$$C_\mu = 2 \frac{\partial L_\mu}{\partial z} \quad \text{and} \quad \mu = \frac{1}{\pi} \frac{\partial C_\mu}{\partial \bar{z}} = \frac{2}{\pi} \frac{\partial^2 L_\mu}{\partial z \partial \bar{z}} = \frac{1}{2\pi} \left( \frac{\partial^2 L_\mu}{\partial x^2} + \frac{\partial^2 L_\mu}{\partial y^2} \right).$$

They should be understood as equalities of distributions.
- the Laurent series of $C_\mu$ around $\infty$ is given by

$$C_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \ldots,$$

where

$$m_k(\mu) = \int_{\mathbb{C}} z^k \, d\mu(z), \, k = 0, 1, \ldots.$$
are the harmonic moments of measure $\mu$.

Given a polynomial $p$, we associate to $p$ its standard \textit{root-counting measure}

$$\mu_p = \frac{1}{\deg p} \sum_i m_i \delta(z - z_i),$$

where the sum is taken over all distinct roots $z_i$ of $p$ and $m_i$ is the multiplicity of $z_i$.

One can easily check that the Cauchy transform of $\mu_p$ is given by

$$\mathcal{C}_{\mu_p} = \frac{1}{\deg p} \cdot \frac{p'}{p}.$$

For more relevant information on the Cauchy transform we will probably recommend a short and well-written treatise [Ga].

The above notions of a complex measure $\mu$ compactly supported in $\mathbb{C}$, its logarithmic potential $L_\mu$, and its Cauchy transform $\mathcal{C}_\mu$ have natural extensions to similar objects $\bar{\mu}$, $\bar{L}_\mu$, $\bar{\mathcal{C}}_\mu$ defined on $\mathbb{C}P^1 \supset \mathbb{C}$ and such that the main relations between these objects are preserved. They are constructed as follows.

(i) For a finite complex measure $\mu$ compactly supported in $\mathbb{C}$, we introduce the signed measure $\bar{\mu}$ of total mass 0 defined on $\mathbb{C}P^1$ by adding to $\mu$ the point measure $-m \cdot \delta(\infty)$ placed at $\infty$, where $m = \int_C d\mu$. (It is natural to think of $\bar{\mu}$ as an exact 2-current on $\mathbb{C}P^1$.)

(ii) The logarithmic potential $L_\mu$ is defined as a function on $\mathbb{C} \subset \mathbb{C}P^1$ with a logarithmic singularity at $\infty$. In terms of a local coordinate $w = 1/z$ at $\infty$ the logarithmic potential is $L^{1}_{\text{loc}}$ near $\infty$, and this implies that we may talk about its derivatives. In the following when we think of $L_\mu$ as an object on $\mathbb{C}P^1$ we denote it by $\bar{L}_\mu$.

Recall that on any complex manifold the exterior differential $d$ (acting on currents) is standardly decomposed as $d = d' + d''$, where $d'$ is the holomorphic part and $d''$ is the anti-holomorphic part. For a function $f$ on a Riemann surface with a local coordinate $z$, we get

$$d'f = \frac{\partial f}{\partial z} dz \quad \text{and} \quad d''f = \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$ 

The above quantities $\bar{\mu}$ and $\bar{L}_\mu$ satisfy the relation

$$\bar{\mu} dx \wedge dy = \frac{i}{\pi} d'd'' \bar{L}_\mu.$$ 

More explicitly, we have that

$$\bar{\mu} dx \wedge dy = \frac{1}{2\pi} \left( \frac{\partial^2 \bar{L}_\mu}{\partial x^2} + \frac{\partial^2 \bar{L}_\mu}{\partial y^2} \right) dx \wedge dy = \frac{2}{\pi} \frac{\partial^2 \bar{L}_\mu}{\partial z \partial \bar{z}} dz \wedge d\bar{z},$$

where $\frac{\partial^2 \bar{L}_\mu}{\partial z \partial \bar{z}}$ is understood as a distribution on $\mathbb{C}P^1$.

(iii) Finally, the Cauchy transform $\bar{\mathcal{C}}_\mu$ should be understood as an 1-current given by the relation

$$\bar{\mathcal{C}}_\mu = 2d' \bar{L}_\mu = 2 \frac{\partial \bar{L}_\mu}{\partial z} dz.$$ 

Then

$$\bar{\mu} dx \wedge dy = \frac{i}{\pi} d'd'' \bar{L}_\mu = -\frac{i}{2\pi} d'' \bar{\mathcal{C}}_\mu = \frac{i}{2\pi} \frac{\partial \bar{\mathcal{C}}_\mu}{\partial z} dz \wedge d\bar{z}.$$
3. Differentials with imaginary periods, affine Boutroux curves, subharmonic functions, and signed measures on $\mathbb{C}P^1$

To settle Theorem 1, we introduce a special class of plane algebraic curves and show how they give rise to measures on open subsets of $\mathbb{C}P^1$. This is a special case of a more general construction which can be carried out for Riemann surfaces endowed with a meromorphic 1-differential and a meromorphic function.

3.1. Differentials with imaginary periods. Usually multi-valued harmonic and subharmonic functions on Riemann surfaces are constructed by integrating differential 1-forms. For some special types of differentials, one can get uni-valued functions instead of multi-valued. This is the situation which we want to capture. By a period of a meromorphic 1-form on a Riemann surface $Y$ we mean the integral of this form along a piecewise smooth closed curve in $Y$.

**Definition 1.** A meromorphic 1-form $\omega$ defined on a compact orientable Riemann surface $Y$ is said to have purely imaginary periods if all its periods are purely imaginary complex numbers.

**Remark 3.** Observe that the periods of $\omega$ can be roughly subdivided into two different types: a) periods related to the poles of $\omega$, i.e. integrals of $\omega$ over small loops surrounding the poles and b) periods related to the 1-dimensional homology classes of $Y$, i.e. integrals of $\omega$ over global cycles in $H_1(Y, \mathbb{R})$. (Observe however that these two types of periods are, in general, dependent.)

Note that the first type of periods are purely imaginary if and only if all residues of $\omega$ are real and that the second type of periods do not appear if $Y$ has genus 0.

**Remark 4.** Definition 1 clearly makes sense even if $Y$ is not necessarily a compact Riemann surface. For our purposes, it suffices that $Y$ is an open subset of a compact Riemann surface $\tilde{Y}$ such that $\tilde{Y} \setminus Y$ consists of a finite number of points. This will always be the case for e.g. smooth quasi-affine plane algebraic curves. Note that a meromorphic 1-differential $\omega$ on $\tilde{Y}$ has purely imaginary periods if and only if the restriction of $\omega$ to $Y$ has purely imaginary periods.

For a meromorphic 1-form $\omega$ with purely imaginary periods given on a compact Riemann surface $Y$, denote by $Pol^-_{\omega} \subset Y$ (resp. $Pol^+_{\omega} \subset Y$) the set of all poles of $\omega$ with negative (resp. positive) residues.

Meromorphic 1-forms (i.e. abelian differentials) with purely real periods have been earlier considered in e.g. [GrKr] where they were used to study the moduli spaces of Riemann surfaces with marked points. (Here we consider purely imaginary periods, but the translation is trivial.) One of the main results of [GrKr] is as follows, see Proposition 3.4 in loc. cit.

**Proposition A.** For any $(Y, p_1, \ldots, p_n) \in M_{g,n}$, any set of positive integers $h_1, \ldots, h_n$, any choice of $h_i$-jets of local coordinates $z_i$ in the neighborhood of marked points $p_i$, and any singular parts (i.e., for $i = 1 \ldots n$, the choice of Taylor coefficients $c_i^1, \ldots, c_i^{h_i}$, with all imaginary residues $c_i^1 \in i\mathbb{R}$ and the sum of the residues $\sum c_i^1$ vanishing), there exists a unique differential $\Psi$ on $Y$ with purely real periods and prescribed singular parts. In other words, in a neighborhood $U_i$ of each $p_i$ the
differential $\Psi$ satisfies the condition
$$\Psi|_{U_i} = \sum_{j=1}^{h_i} c_i \frac{dz}{z^j} + O(1).$$

Proposition A implies that on an arbitrary compact Riemann surface $Y$ there exists a large class of meromorphic 1-forms with purely imaginary periods. Further we can associate to each meromorphic differential with imaginary periods on $Y$ a real-valued function $Y \setminus Pol_\omega \to \mathbb{R}$. Namely, fix a point $p_0 \in Y \setminus Pol_\omega$ and consider
$$\Psi(p) := \int_{p_0}^p \omega.$$

$\Psi(p)$ is a well-defined uni-valued function on the universal covering of $Y \setminus Pol_\omega$. The next statement is trivial.

**Lemma 2.** In the above notation, $\omega$ has purely imaginary periods if and only if the multi-valued primitive function $\Psi(p)$ has a uni-valued real part $\text{Re} \, \Psi(p)$. In other words,
$$H_\omega(p) := \text{Re} \, \Psi(p)$$
is uni-valued already on $Y \setminus Pol_\omega$.

Note that $H_\omega(p)$ is continuous and harmonic in a neighborhood of any point in $Y \setminus Pol_\omega$. The local behavior of $H_\omega$ around a pole $p$ is determined by the sign of the residue of $\omega$ at this point. Namely, let $\omega$ be a meromorphic 1-form with purely imaginary periods and only simple poles. For $p \in Pol_\omega$, let $z$ be a local coordinate at $p$, and denote by $c$ the residue of $\omega$ at $p$.

**Lemma 3.** In the above notation, $H_\omega$ is a subharmonic $L^1_{loc}$-function on $Y \setminus Pol_\omega$. Locally, for the restriction of $H_\omega$ to a suitable neighborhood of $p$, the following holds:

1. $H_\omega(z) = c \log |z| + \tilde{H}_{\omega}(z)$, where $\tilde{H}_{\omega}(z)$ is harmonic in a neighborhood of $p$. Consequently, $\frac{\partial^p H_\omega}{\partial z} = \frac{c}{z^2} \delta_p$, where $\delta_p$ is the Dirac measure at $p$, and the derivatives are taken in the sense of distributions.
2. If $p \in Pol_\omega^+$, then there is a neighborhood of $p$ in which $H_\omega$ is a well-defined subharmonic function. As such, $H_\omega(p) = -\infty$.
3. If $p \in Pol_\omega^-$, then $\lim_{z \to p} H_\omega(z) = \infty$.
4. $\frac{\partial H_\omega(z)}{\partial z} = \frac{1}{2} \omega$.

**Proof.** (1) is a consequence of the fact that $\omega$ has a simple pole at $p$ and hence locally can be written as $\omega = \frac{c}{z} dz + \tilde{\omega}$, where $\omega$ is holomorphic at $p$. Then (2) and (3) follow, while (4) follows from the standard relation
$$\frac{\partial \Psi}{\partial z} = \omega = 2 \frac{\partial (\text{Re} \, \Psi)}{\partial z}.$$
3.2. Tropical trace. Given a branched covering $\nu : Y \to Y'$ of Riemann surfaces and a function $f : Y \to \mathbb{R}$, we will define the induced function on $Y'$ by taking the maximum of the values of $f$ on each fibre (instead of the summation or the integration over the fiber as it is done in the case of the usual trace). It seems that this construction which frequently occurs in the study of the root asymptotics for sequences of polynomials has not been given any special name yet.

**Definition 2.** Given a real-valued function $f : Y \to \mathbb{R}$, we define its *tropical trace* $\nu_* f$ as

$$\nu_* f(z) = \max_{q \in \nu^{-1}(z)} f(q).$$

The same definition extends to real-valued functions $f$ defined on $Y \setminus S$, where $S$ is a discrete set such that for any $s \in S$, $\lim_{z \to s} f(z)$ exists as a real number or $\pm \infty$. (In other words, we allow $f$ to attain values $\pm \infty$.)

**Example 1.** Let $H_i$, $i \in \{1, 2, ..., n\} := [n]$ be an $n$-tuple of real harmonic functions on $Y'$. They define a harmonic function $H$ on $Y := Y' \times [n]$ by

$$H(z, i) = H_i(z).$$

For the canonical projection $\nu : Y \to Y'$ given by $\nu(z, i) = z$, we get

$$\nu_* H(z) = \max_{i \in [n]} H_i(z).$$

Furthermore, the Laplacian of $\nu_* H$ is supported on the union of the real semi-analytic curves $H_i(z) = H_j(z)$, $i \neq j$. In fact, this Laplacian considered as a measure is explicitly given by the Plemelj-Sokhotski formula, see e.g. [BB].

Definition 2 works for an arbitrary finite map between complex manifolds. The elementary fact that the maximum of a finite number of subharmonic functions is subharmonic, implies certain consequences on the support of the Laplacian of $\nu_* f$ which makes Definition 2 useful. In particular, the trace of a subharmonic function is subharmonic. We describe this in more detail in Theorem 4 below.

Let $Y$ be a (not necessarily compact) Riemann surface and let $\nu : Y \to Y'$ be a branched covering. Let $f : Y \to \mathbb{R}$ be a real-valued function harmonic except for a finite number of points $P$ where it has logarithmic singularities. In other words, $f(z) = c \log |z| + \tilde{f}(z)$, where $z$ is a local coordinate and $\tilde{f}(z)$ is harmonic in a neighborhood of $p$. Let $P_-$ (resp. $P_+$) be the set of those $p \in P$ at which the residue $c$ is negative (resp. $c > 0$). Then $f$ is subharmonic in $Y \setminus P_-$. Note that $P$ supports all the point masses of the Laplacian of $f$ considered as a measure.

**Theorem 4.** Under the above conditions, the tropical trace $\nu_* f$ is harmonic in the open set $Y' \setminus \nu(P)$, subharmonic in $U = Y' \setminus \nu(P^-)$, and it has at most logarithmic singularities. The Laplacian of $\nu_* f$ in $U$ is supported on a finite union of real semi-analytic curves and points; the latter set is contained in the set of the images of all poles of $f$ under $\nu$.

*Proof.* Note first that the maximum $h(z) = \max f_i(z)$ of a finite number of harmonic functions $f_i$, $i = 1, ..., r$, defined on an open set $V \subset Y'$ is subharmonic; its Laplacian is supported on (some parts of) the level curves $f_i = f_j$, $i \neq j$. Furthermore these level curves are real semi-analytic. Assume that all $f_i$s are subharmonic and harmonic except for the certain points $p_i$ where they might have logarithmic singularities. In other words, $f_i(z) = c_i \log |z| + \tilde{f}_i(z)$, where $c_i \geq 0$ and $\tilde{f}_i$ is harmonic.
In such a situation $h(z)$ is subharmonic. Only in the case when $f_i(p) = -\infty$ for all $i = 1, 2, \ldots, r$, one gets $h(p) = -\infty$. Then, in addition to a possible measure supported on a real semi-analytic curve, the Laplacian of $h$ will have a point mass $(\min_i c_i)\delta_p$ at $p$. If some $c_i < 0$, then $h$ will not be subharmonic at $p$, but still it has a logarithmic singularity there and its Laplacian contains a negative point mass at $p$.

Let $BR \subset Y$ denote the set of all branch points of $\nu$ and set $D := \nu(BR) \subset Y$. If $V \subset Y' \setminus D$ is a simply connected open set, there exists a disjoint union $\nu^{-1}(V) = \bigcup V'_i$, such that the restriction $\nu : V'_i \to V$ is a locally biholomorphic map which we denote by $\nu_i$. Then $\nu_i f(z) := \max_i f(\nu_i^{-1}(z))$, $z \in V$ is a subharmonic function, and by the previous example, its Laplacian is supported on a union of some real semi-analytic curves and (possibly) at some points in $\nu(P^+)$. A similar argument works in the case $p \in D$, in suitable local coordinates on $Y$ and $Y'$ resp., $\nu$ can be written as $w \mapsto z = \nu(w) = w^r$. The rest of Lemma follows since it is always possible to cover $Y' \setminus D$ by a finite number of open simply-connected sets such that the preceding argument holds. \hfill \Box

### 3.3. Affine Boutroux curves

Consider an irreducible affine algebraic curve $\Gamma \subset \mathbb{C} \times \mathbb{C}$, where the product $\mathbb{C} \times \mathbb{C}$ is equipped with a coordinate system $(w, z)$.

Denote by $\hat{\Gamma} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ the closure of $\Gamma$ in the product $\mathbb{C}P^1 \times \mathbb{C}P^1$ and denote by $\tilde{\Gamma}$ the normalisation of $\hat{\Gamma}$. We will use notation $\mathbb{C}_w, \mathbb{C}_z, \mathbb{C}_w^\perp$, and $\mathbb{C}_z^\perp$ to indicate with which coordinate the corresponding coordinate (projective) line is associated. We will denote by $\pi_w : \mathbb{C}_w \times \mathbb{C}_z \to \mathbb{C}_w$ or, similarly, by $\pi_w : \mathbb{C}_w^\perp \times \mathbb{C}_z^\perp \to \mathbb{C}_w^\perp$ the standard projection onto the first coordinate and by $\pi_z$ the projection onto the second coordinate.

Denote by $n : \tilde{\Gamma} \to \hat{\Gamma}$ the standard normalisation map. Recall that the smooth compact Riemann surface $\hat{\Gamma}$ is birationally equivalent to $\Gamma$. Now consider on $\mathbb{C}_w \times \mathbb{C}_z$ (resp. on $\mathbb{C}_w^\perp \times \mathbb{C}_z^\perp$) the standard meromorphic 1-form

$$\Omega := wdz.$$ One can easily show that the zero divisor of $\Omega$ on the latter space is a copy of $\mathbb{C}P^1$ denoted by $H^0_w$ which is given by $w = 0$; (the closure of) its pole divisor is the union of two intersecting copies $H^\infty_w \cup H^\infty_z$ of $\mathbb{C}P^1$ given by $w = \infty$ and $z = \infty$.

Given a curve $\Gamma \subset \mathbb{C}_w \times \mathbb{C}_z$ as above, consider the meromorphic 1-form

$$\Omega_\Gamma := \Omega|_\Gamma$$ resp. $\Omega_{\hat{\Gamma}} := \Omega|_{\hat{\Gamma}}$

obtained by the restriction of $\Omega$ to $\Gamma$ (resp. to $\hat{\Gamma}$). Denote by $\tilde{\Omega}$ the pullback of $\Omega_{\hat{\Gamma}}$ to $\Gamma$ under the normalisation map $n$. This form will be the key ingredient below.

The zero divisor of $\Omega_{\hat{\Gamma}}$ consists of the intersection points $H^0_w \cap \tilde{\Gamma}$ and all singularities of $\hat{\Gamma} \subset \mathbb{C}P^1_w \times \mathbb{C}P^1_z$.

The pole divisor of $\Omega_{\hat{\Gamma}}$ consists of non-singular points of the intersection

$$(H^\infty_w \cup H^\infty_z) \cap \tilde{\Gamma}.$$
Given an irreducible affine curve $\Gamma \subset \mathbb{C}_w \times \mathbb{C}_z$ as above and the corresponding meromorphic 1-form $\tilde{\Omega}$ on $\tilde{\Gamma}$, consider the multi-valued primitive function

$$\Psi(p) = \int_{p_0}^{p} \tilde{\Omega}.$$ 

$\Psi(p)$ is a well-defined uni-valued function on the universal covering of $\tilde{\Gamma} \setminus Pol$, where $Pol \subset \tilde{\Gamma}$ is the set of all poles of $\tilde{\Omega}$ and $p_0 \in \tilde{\Gamma} \setminus Pol$ is some fixed base point. The next statement is trivial, cf. 3.1.

**Lemma 5.** In the above notation, $\tilde{\Omega}$ has purely imaginary periods if and only if the multi-valued primitive function $\Psi(p)$ has a uni-valued real part $\text{Re} \Psi(p)$. In other words, $\text{Re} \Psi(p)$ is uni-valued already on $\tilde{\Gamma} \setminus Pol$.

**Definition 3.** A plane affine irreducible curve $\Gamma \subset \mathbb{C}_w \times \mathbb{C}_z$ is called an affine Boutroux curve (aBc, for short) if its associated meromorphic 1-form $\tilde{\Omega}$ has purely imaginary periods on $\tilde{\Gamma}$.

We can reformulate the latter definition as follows. Let $\Gamma_{sm} \subseteq \Gamma$ be the smooth part of $\Gamma$. Then $\Gamma$ is an aBc if and only if the restriction of $wdz$ to $\Gamma_{sm}$ has purely imaginary periods on $\Gamma_{sm}$. In fact, this is equivalent to the requirement that $wdz$ has purely imaginary periods on any smooth $\Gamma_1 \subseteq \tilde{\Gamma}$ such that $\tilde{\Gamma} \setminus \Gamma_1$ is finite.

### 3.4. How to construct affine Boutroux curves.

The following result describes a mechanism how to obtain affine Boutroux curves. In Section 5 we will prove that the symbol curve (1.3) is an aBc curve and show that this curve is an instance of our present construction.

We start with a real-valued harmonic function $H(w, z)$ on $\mathbb{C}^2 \setminus S$, where $S$ is a finite set, such that the holomorphic differential $d'H = Pdz + Qdw$ has non-trivial rational functions $P(w, z)$ and $Q(w, z)$ as coefficients. Write $P = P_1/R_1$ and $Q = Q_1/R_2$ with relatively prime polynomial numerators and denominators. Let $R$ be the least common multiple of $R_1$ and $R_2$ and assume that

\[ (*) \quad Q_1 \notin \mathbb{C}[z] \text{ is irreducible and relatively prime to } R. \]

Denote by $U$ the complement to the zero locus of $R$. Let $\mathcal{D} \subset U \subset \mathbb{C}^2$ be a curve given by $\{Q = 0\}$, and let $\pi_z : \mathcal{D} \rightarrow \mathbb{C}$ be the canonical projection sending $(w, z) \rightarrow z$. The affine ring of functions on $U$ coincides with the ring $\mathbb{C}[w, z]_R$ of rational functions whose denominators are powers of $R$. The affine ring of functions on $\mathcal{D}$ is the domain $\mathbb{C}[w, z]_R/(Q) = (\mathbb{C}[w, z]/(Q_1))_R$.

By $(*)$, $Q_1$ is a non-constant function no factor of which is a polynomial in the single variable $z$ and is also relatively prime with respect to $R$. Therefore the maps of commutative rings

$$\mathbb{C}[z] \subset \frac{\mathbb{C}[P, z]_R}{(Q)} \subset \frac{\mathbb{C}[w, z]_R}{(Q)}$$

are inclusions and induce a factorization $\pi_z = \pi_1 \circ \pi_2$

$$\mathcal{D} \rightarrow \mathcal{C} := \text{spec } \frac{\mathbb{C}[P, z]_R}{(Q)} \rightarrow \mathbb{C} = \text{spec } \mathbb{C}[z]$$

by maps with finite fibers. KONSTIG FRAS? There is a map $\nu : \mathcal{C} \rightarrow \mathbb{C}^2$ induced by the map $\mathbb{C}[u, z] \rightarrow \mathbb{C}[P, z]$ sending $u \rightarrow P$. 

Proposition 6. In the above notation, the curve $C$ is an aBc.

Proof. Since $Q_1$ is an irreducible polynomial, $C$ is an irreducible affine variety. Except for a finite number of points $S$, both $z$ and $u$ are local coordinates and $\nu$ is a covering map. By the remark following the definition of affine Bouttroux curves, it suffices to check that the periods of $wdz$ on the complement of $S$ are purely imaginary. Let $H$ be the restriction of $H(w,z)$ to $C$, and note that $dH = Pdz = \frac{1}{2} udz$. Proposition 6 then follows from the fact that the real part of a primitive function of $wdz$ (which exists on the universal cover) is given by the uni-valued function $H$ defined on $C$. \hfill \Box

Theorem 7 below will explain our interest in affine Bouttroux curves.

3.5. Branched pushforward and piecewise-analytic forms.

Definition 4. Given a branched covering $\nu : Y \to Y'$, where $Y$ and $Y'$ are compact Riemann surfaces, by a uni-valued branch of $\nu$ we mean an open subset $U \subset Y$ such that $\nu$ maps $U$ diffeomorphically onto its image $\nu(U)$ and additionally, $\nu(U) = Y' \setminus C$, where $C$ is a finite union of smooth compact curves and points in $Y'$.

An easy way to get several uni-valued branches simultaneously is to fix a cut $\mathcal{C}$ such that

(i) $\mathcal{C}$ contains all the branch points of $\nu$;
(ii) $Y' \setminus \mathcal{C}$ consists of contractible connected components.

Then the surface $Y \setminus \nu^{-1}(\mathcal{C})$ splits into $d$ disjoint uni-valued sheets which are uniquely defined in case when $Y' \setminus \mathcal{C}$ is connected.

Definition 5. Given a meromorphic 1-form $\omega$ on a compact Riemann surface $Y$ and a branched covering $\nu : Y \to Y'$ of some degree $d$, where $Y'$ is a compact Riemann surface, we define the branched push-forward $\nu_* \omega$ as a $d$-valued 1-form on $Y'$ obtained by assigning to a tangent vector $v$ at any point $p \in Y' \setminus D_{\nu}$ one of $d$ possible values $\omega(\nu_j^{-1}(v))$, $j = 1, \ldots, d$. Here $\nu^{-1}(v)$ is one of $d$ possible pull-backs of $v$ to the tangent bundle of $Y$ and $D_{\nu}$ is the set of all discriminantal points of $\nu$. (Observe that $\nu$ is a local diffeomorphism near any point of $Y$ which is not a critical point.)

Using fancier language, the above definition means that we are considering a (set-theoretic) section $\theta : Y' \to Y$ of the covering $\nu : Y \to Y'$, which chooses at each point in $Y'$ (with some finite number of exceptions) one of the $d$ possible points in the fiber. This operation induces a branched push-forward $\nu_* \omega$ as a (set-theoretical) section of the bundle of meromorphic 1-forms on $Y'$. We use set-theoretical sections, since we are not requiring any differentiability of $\theta$.

We want $\theta$ to satisfy the condition similar to the one which results from providing a cut of $Y'$ in order to obtain a $d$-fold covering by disjoint sheets, as described above. In other words, we want to remove from $Y'$ a cut which is a set of Lebesgue measure 0, and decompose the remaining surface into open domains on each of which $\theta$ is biholomorphic. More precisely, assume that $Y' \setminus D_{\nu} = \cup_{i=1}^n U_i \cup E$, where $U_i$s are disjoint open sets and $E$ has Lebesgue measure 0 such that $\theta$ is biholomorphic on each $U_i$ and is a section of $\nu$. In this case we will say that the associated $\nu_* \omega$ is piecewise-analytic.

The Cauchy transform of the asymptotic root-counting measure which we will construct later will be piecewise-analytic in the above sense. Recall that we think...
of the Cauchy transform as a 1-current, i.e. as a generalized 1-form. The piecewise analytic character of our construction stems from the fact that the Cauchy transform will be associated to a section of a finite cover.

3.6. Affine Boutroux curves, induced signed measures on $\mathbb{C}P^1$, and their Cauchy transforms. In this subsection we will show that under some additional conditions, given an aBc $\Gamma$, we can construct a signed measure on $\mathbb{C}P^1$ whose Cauchy transform satisfies the algebraic equation defining $\Gamma$.

Indeed, given a plane curve $\Gamma \subset \mathbb{C}_w \times \mathbb{C}_\tau$ as above, we have a natural meromorphic function $\tau: \hat{\Gamma} \to \mathbb{C}P^1_{\bar{w}}$ induced by the composition of the normalisation map $n: \hat{\Gamma} \to \hat{w}$ with the projection of $\pi_2: \hat{\Gamma} \subset \mathbb{C}P^1_w \times \mathbb{C}P^1_{\bar{w}}$ along the $\bar{w}$-axis.

Theorem 7. In the above notation, given an aBc $\Gamma \subset \mathbb{C}_w \times \mathbb{C}_\tau$, such that the restriction of $wdz$ to $\Gamma$ has at most simple poles, there exists a signed measure $\mu_{\Gamma}$ of total mass 0 supported in $\mathbb{C}P^1_{\bar{w}}$ with the following properties:

(i) its support $S_{\bar{w}} := \text{supp}(\mu_{\Gamma}) \subset \mathbb{C}P^1_{\bar{w}}$ consists of finitely many compact real semi-analytic curves and isolated points;

(ii) its Cauchy transform $\hat{\mu}_{\Gamma}$ (considered as a 1-current) coincides with a uni-valued piecewise analytic branch of $\tau_\Omega$ in $\mathbb{C}P^1_{\bar{w}} \setminus S_{\bar{w}}$. In other words, if we represent $\hat{\mu}_{\Gamma} = C(z)dz$ in the affine chart $\mathbb{C}_z \subset \mathbb{C}P^1_z$, where $C(z)$ is a distribution, then $C(z)$ satisfies in $\mathbb{C}_z \setminus S_{\bar{w}}$ the algebraic equation defining $\Gamma$;

(iii) the support $S_{\bar{w}}$ of the negative part of $\mu_{\Gamma}$ coincides with $\tau(Pol^-) \subset \mathbb{C}P^1_{\bar{w}}$.

Remark 5. Observe that the converse to the statement of Theorem 7 is, in general, false, i.e. there exist curves for which conditions (i), (ii) and (iii) still hold but which are not affine Boutroux curves, see e.g. § 4 of [BoSh]. Thus being an aBc only provides a sufficient condition for (i) – (iii). Observe additionally that if we remove condition (iii), then there exists situations in which $\mu_{\Gamma}$ is not unique, see e.g. Theorem 4 of [STT]. We also want to point out a close connection of Theorem 7 with some results of [BaSh] where condition (iii) was mentioned as the existence of clean poles.

Proof. Choose an arbitrary point $p_0 \in \hat{\Gamma} \setminus Pol$ and, as in Lemma 3, consider the function

$$\text{Re } \Psi(p) = \text{Re } \left[ \int_{p_0}^p \bar{\Omega} \right],$$

where $\Omega = wdz$. Since $\Gamma$ is an aBc, then $\bar{\Omega}$ has purely imaginary periods and $\text{Re } \Psi(p)$ is a uni-valued harmonic function on $\hat{\Gamma} \setminus Pol$. (One can consider $\text{Re } \Psi(p)$ as defined on the whole $\hat{\Gamma}$ if one allows it to attain values $\pm \infty$.) Let $BR$ be the branch locus of the projection $\tau: \hat{\Gamma} \to \mathbb{C}P^1_{\bar{w}}$ and $\tau(Pol)$ be the image of the set $Pol$ of all poles of $\Omega_{\Gamma}$. Recall that the branching locus $BR \subset \mathbb{C}P^1_{\bar{w}}$ is defined as the set of all critical values, i.e. all branch points of the meromorphic function $\tau: \hat{\Gamma} \to \mathbb{C}P^1_{\bar{w}}$. Obviously, $BR$ is a finite set.

Now, for any $z \in \mathbb{C}P^1_{\bar{w}} \setminus (BR \cup \tau(Pol))$, define the function $\Theta$ on $\mathbb{C}P^1_{\bar{w}}$ given by

$$\Theta(z) := \max_{q_i \in \tau^{-1}(z)} \{ \text{Re } \Psi(q_i) \}.$$
In other words, $\Theta(z)$ is the tropical trace of the projection $\tau$ of $\Re \Psi$ to $\mathbb{C}P^1$.

Observe that if $z$ lies in $\mathbb{C}P^1 \setminus (\mathcal{B} \cup \tau(Pol))$, then it is a local parameter on every branch of $\tilde{\Gamma}$ near each point from the fiber $\tau^{-1}(z)$, which implies that each function $H_i(z) := \Re 2\Psi(p_i)$ is a well-defined harmonic function near $z$. Moreover, outside of its poles, $\Theta(z)$ is a continuous subharmonic function.

The above definition of $\Theta(z)$ also makes sense if $z$ is a branch point or the image of a pole; in the latter case $\Theta(z)$ might attain infinite values. Namely, if $p \in \tilde{\Gamma}$ is a pole with residue $\kappa_i$ and $z_p = \tau(p)$, then locally near $z_p$ the corresponding $H_i(z)$ has the asymptotics $\kappa_i \log |z - z_p|$. Hence, if $\kappa_i$ is positive, then $\lim_{z \to z_p} H_i(z) = -\infty$ in a sufficiently small neighbourhood of $z_p$. Analogously, if $\kappa_i$ is negative, then $\lim_{z \to z_p} H_i(z) = +\infty$. Finally, $\Theta(z) = -\infty$ if and only if every point in $\tau^{-1}(z)$ is a pole of $\tilde{\Omega}$ with a positive residue.

Now define the 2-current $\tilde{\mu}_\Gamma$ on $\mathbb{C}P^1$ given by

$$\tilde{\mu}_\Gamma := \frac{1}{2\pi} \left( \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} \right) dxdy + \frac{i}{\pi} \frac{\partial^2 \Theta}{\partial z \partial \bar{z}} dzd\bar{z},$$

(3.1)

where $(x, y)$ are the real and the imaginary parts of the local coordinate $z$. We will call the function $\Theta$ the logarithmic potential of the 2-current $\tilde{\mu}_\Gamma$.

The measure $\tilde{\mu}_\Gamma$ given by (3.1) satisfies conditions (i)-(iii) of Theorem 7. Conditions (i) and (iii) follow from Lemma 4, which says that $\tilde{\mu}_\Gamma$ is a signed measure on $\mathbb{C}P^1$ supported on finitely many semi-analytic curves belonging to the level sets $\{\Re \Psi(p_i) = \Re \Psi(p_j)\}, i \neq j$ and finitely many isolated points including $\tau(Pol^-)$ and possibly some part of $\tau(Pol^+)$. For (ii), assume that $V$ is a connected and simply connected subset of the complement $U$ to $\text{supp} \tilde{\mu}_\Gamma \cup \mathcal{B} \cup \tau(Pol)$. Then $\Theta(z) = \Psi(p_V(z))$ for a certain choice of a branch $p_V : V \rightarrow \Gamma \subset \mathbb{C}^2$. Set $p_V(z) = (z, w(z))$ where $w$ is a branch of the algebraic function defined by $\Gamma$. Clearly $z$ is a local coordinate both in $V$ and in $p_V(V)$ and therefore

$$d'(\Theta(z)) = \frac{\partial \Theta(z)}{\partial z} dz = w(z)dz.$$

Observe that since $\tilde{\mu}_\Gamma$ has a potential, it must necessarily be exact, i.e.

$$\int_{\mathbb{C}P^1} \tilde{\mu}_\Gamma = 0.$$

By construction, the negative part of $\tilde{\mu}_\Gamma$ is supported on $\tau(Pol^-)$. \qed

4. Differential equations satisfied by Rodrigues’ descendants

In this and the following sections we will prove our main results formulated in the Introduction. First, we need to obtain linear differential equations satisfied by the Rodrigues’ descendants.

For future applications we will deduce linear differential equations satisfied by the Rodrigues’ descendants of the meromorphic functions of the form

$$f(z) := P(z)e^{T(z)}/Q(z),$$

where $P(z) \neq 0, Q(z) \neq 0$ and $T(z)$ are polynomials with $\gcd(P, Q) = 1$. In case $T \equiv 0$ and $Q \equiv 1$, we obtain the polynomial situation considered in the present paper, see Corollary 5.
Proposition 8. In the above notation and for $d := \deg P + \deg Q + \deg T$, the Rodrigues’ descendant $R_{m,n,P+T/Q}(z)$ satisfies the linear homogeneous differential equation

$$
\sum_{i=0}^{d} \sum_{j=0}^{i} \sum_{k=0}^{j} \frac{(m+d-i+n(2j-i))\delta_{k,0} - nkT^{(k)}}{(m+d-i)!(i-j)!(j-k)!k!} P^{(i-j)} Q^{(j-k)} y^{(d-i)} = 0 \quad (4.1)
$$
of order $d$.

As special cases of the latter statement we obtain the following three corollaries.

Corollary 4. The Rodrigues’ descendant $R_{m,n,P+T/Q}(z)$ satisfies the linear homogeneous differential equation

$$
\sum_{j=0}^{d} \sum_{k=0}^{j} m + d + (n-1)j - 2nk P^{(k)} Q^{(j-k)} y^{(d-j)} = 0 \quad (4.2)
$$
of order $d = \deg P + \deg Q$.

Corollary 5. The Rodrigues’ descendant $R_{m,n,P}(z)$ satisfies the linear homogeneous differential equation

$$
\sum_{k=0}^{d} (d+n-1)! [(d-1) - (k-1)(n+1)] P^{(k)} y^{(d-k)} = 0 \quad (4.3)
$$
of order $d = \deg P$.

The original Rodrigues’ formula inspires the following consequence of Corollary 5.

Corollary 6. The Rodrigues’ descendant $R_{n,n,P}(z) := \frac{d^n}{dz^n}(P^n(z))$ satisfies the linear differential equation

$$
\sum_{k=0}^{d} (d+n-1)! [(d-1) - (k-1)(n+1)] P^{(k)} y^{(d-k)} = 0 \quad (4.4)
$$
of order $d = \deg P$.

Remark 6. A differential equation satisfied by $R_{n,n,P}(z)$ similar to $(4.4)$ was first obtained in [Ho].

Proof of Proposition 8. Consider the first-order differential equation

$$
PQw' + n(PQ' - P'Q - PQT')w = 0, \quad (4.5)
$$

Clearly, if $f = Pe^T/Q$, then $w = f^n$ satisfies $(4.5)$. By differentiating both sides of $(4.5)$ $\ell \geq d-1$ times (or $\ell > d-1$ times if $d = 0$) and using Leibniz’s rule for the derivative of a product, we get

$$
\sum_{i=0}^{\ell} \frac{\ell!}{i!(\ell-i)!} U^{(i)} w^{(\ell+1-i)} + n \sum_{i=0}^{\ell} \frac{\ell!}{i!(\ell-i)!} V^{(i)} w^{(\ell-i)} - n \sum_{i=0}^{\ell} \frac{\ell!}{i!(\ell-i)!} W^{(i)} w^{(\ell-i)} = 0, \quad (4.6)
$$

where $U := PQ$, $V := PQ' - P'Q$ and $W := PQT'$. In the first sum, remove the first term and replace $i$ by $r+1$ in the remaining sum. In the second and third
sums, replace \( i \) by \( r \) and remove the last terms. By combining the three resulting sums and simplifying, equation (4.6) becomes

\[
U w^{(\ell+1)} + n V^{(\ell)} w - n W^{(\ell)} w + \sum_{r=0}^{\ell-1} \left( \frac{\ell!}{(\ell-r)! r!} U^{(r+1)} + n V^{(r)} - n W^{(r)} \right) w^{(\ell-r)} = 0.
\]

(4.7)

By changing the upper limit of summation in (4.7) from \( \ell - 1 \) to \( \ell \), the terms \( n V^{(\ell)} w \) and \( -n W^{(\ell)} w \) are encompassed by the sum. Since \( U, V \) and \( W \) are polynomials of degrees at most \( d, d - 1 \) and \( d - 1 \), respectively (where we let \( \deg 0 := 0 \)), and \( \ell \geq d - 1 \), we can change the upper limit of summation further to \( d - 1 \), since higher terms vanish. That is, we obtain the equation

\[
U w^{(\ell+1)} + \sum_{r=0}^{d-1} \frac{\ell!}{(\ell-r)! r!} U^{(r+1)} + n V^{(r)} - n W^{(r)} \right) w^{(\ell-r)} = 0,
\]

or equivalently, if we replace \( r \) by \( i - 1 \), change the lower index of summation to \( i = 0 \), and define \( 0 \cdot V^{(-1)} = 0 \cdot W^{(-1)} = 0 \) as to not introduce any new terms,

\[
\sum_{i=0}^{d} \frac{\ell!}{(\ell-i+1)! i!} \left( (\ell-i+1)U^{(i)} + n V^{(i-1)} - n W^{(i-1)} \right) w^{(\ell-i+1)} = 0. \tag{4.8}
\]

Since \( U \) and \( V \) each contain two factors, while \( W \) contains three factors, we expand their derivatives using Leibniz’s rule as follows:

\[
U^{(i)} = (P \cdot Q \cdot 1)^{(i)} = PQ^{(i)} + \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{j!}{(i-j)!(j-k)!k!} P^{(i-j)} Q^{(j-k)} \delta_{k,0}, \tag{4.9}
\]

\[
V^{(i-1)} = (P \cdot Q' \cdot 1)^{(i-1)} - (P' \cdot Q \cdot 1)^{(i-1)}
\]

\[
= \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{(i-1)!}{(i-j-1)!(j-k)!k!} \left( P^{(i-j-1)} Q^{(j-k+1)} - P^{(i-j)} Q^{(j-k)} \right) \delta_{k,0}, \tag{4.10}
\]

\[
W^{(i-1)} = (P \cdot Q \cdot T')^{(i-1)} = \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{(i-1)!}{(i-j-1)!(j-k)!k!} P^{(i-j-1)} Q^{(j-k)} T^{(k+1)}. \tag{4.11}
\]

By inserting the expressions in (4.9)-(4.11) into (4.8) and simplifying, we see that

\[
\sum_{i=0}^{d} \frac{\ell!}{(\ell-i)!} PQ^{(i)} w^{(\ell-i+1)} + \sum_{i=0}^{d} \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{\ell!}{(\ell-i+1)!(i-j)!(j-k)!k!} \left[ (\ell-i+1)P^{(i-j)} Q^{(j-k)} \delta_{k,0}
\]

\[
+ n(i-j) \left( P^{(i-j-1)} Q^{(j-k+1)} - P^{(i-j)} Q^{(j-k)} \right) \delta_{k,0} - P^{(i-j-1)} Q^{(j-k)} T^{(k+1)} \right) \right] w^{(\ell-i+1)} = 0. \tag{4.12}
\]

By changing the upper index of summation from \( i - 1 \) to \( i \) in (4.12), and using the convention that \( 0 \cdot P^{(-1)} = 0 \) as previously, the first sum is encompassed by the triple sum. Next, reverse the order of summation in the outer sum, and let
\[ m := \ell - d + 1, \] which gives that
\[
\sum_{i=0}^{d} \sum_{j=0}^{d-i} \sum_{k=0}^{j} \frac{(m+d-1)!}{(m+i)!(d-i-j)!(j-k)!k!} \left[ (m+i)P^{(d-i-j)}Q^{(j-k)}T_{k,0} + n(d-i-j) \times \left( \left( P^{(d-i-j-1)}Q^{(j-k+1)} - P^{(d-i-j)}Q^{(j-k)} \right) T_{k,0} - P^{(d-i-j-1)}Q^{(j-k)}T^{(k+1)} \right) \right] W^{(m+i)} = 0,
\]
for all \( \ell = m + d - 1 \geq d - 1 \Leftrightarrow m \geq 0 \). Now let (*) denote the equation obtained by replacing \( W^{(m+i)} \) by \( y^{(i)} \) in (4.13). Clearly, \( y = W^{(m)} = (f^n)^{(m)} = [(P/Q)e^T]^n \) satisfies (*). Thus, by reversing the order of summation in (*) and simplifying, the proposition follows. \( \square \)

Corollaries 4, 5, and 6 are immediate consequences of Proposition 8.

5. The Symbol Curves (1.3) and (1.4) Are Affine Boutroux Curves

To prove Theorem 1 using the above Theorem 7, we need to study in detail the algebraic curves given by (1.3) and (1.4). In what follows we will denote
(i) by \( \Gamma_{\alpha,P} \subset \mathbb{C}_C \times \mathbb{C}_z \) the affine algebraic curve given by (1.3).

Our goal is to prove that any irreducible \( \Gamma_{\alpha,P} \) is a affine Boutroux curve. In fact, we will prove this property for the curve \( \Lambda_{\alpha,P} \) defined by
(ii) \( \Lambda_{\alpha,P} \subset \mathbb{C}_W \times \mathbb{C}_z \) is the affine algebraic curve given by (1.5).

Since \( \Gamma_{\alpha,P} \) is obtained from \( \Lambda_{\alpha,P} \) by a real scaling of the first coordinate, the claim that \( \Gamma_{\alpha,P} \) is an aBc follows from that of \( \Lambda_{\alpha,P} \).

Remark 7. Observe that (1.3) defines the closure \( \hat{\Gamma}_{\alpha,P} \subset \mathbb{C}P^1_1 \times \mathbb{C}P^1_2 \) of the bidegree \((d,d)\). By the adjunction formula, a smooth curve in \( \mathbb{C}P^1_1 \times \mathbb{C}P^1_2 \) of the bidegree \((d,d)\) has genus \((d-1)^2\). However, the curve given by (1.3) is rational and therefore highly singular.

The next technical theorem describes the algebraic geometric properties of the symbol curve and its canonical differential \( Wdz \) which are central for the application of the tropical trace to our problem. In Theorem 9, \( \Lambda \) stands for \( \Lambda_{\alpha,P} \), \( \hat{\Lambda} \) for \( \hat{\Lambda}_{\alpha,P} \) which is the closure of \( \Lambda_{\alpha,P} \subset \mathbb{C}P^1_1 \times \mathbb{C}P^1_2 \), and \( \tilde{\Lambda} \) for \( \tilde{\Lambda}_{\alpha,P} \) which is the normalisation of \( \hat{\Lambda}_{\alpha,P} \).

Theorem 9. Let \( P \) be a polynomial of degree \( d \geq 2 \) and let \( 0 < \alpha < d \) be a positive number. Assume that \( P \) and \( P' \) have only simple zeros. Then the algebraic curve \( \Lambda \subset \mathbb{C}_W \times \mathbb{C}_z \) given by (1.5) is an aBc.

More exactly, the following properties hold:
(i) \( \Lambda \) is an irreducible rational curve.
(ii) The inverse image \( \pi^{-1}_z(\infty) \subset \hat{\Lambda} \subset \mathbb{C}P^1_1 \times \mathbb{C}P^1_2 \) consists only of \((0,\infty)\), that is \( \infty \in \mathbb{C}P^1_2 \) is a complete ramification point of the function \( \tau : \hat{\Lambda} \to \mathbb{C}P^1_2 \).
(iii) The equation defining the slopes \( v \) of different branches of \( \hat{\Lambda} \) at \( \infty \in \mathbb{C}P^1_2 \) is given by
\[
(v + 1)^{d-1}(\alpha(v + 1) - d) = 0
\]
and, therefore, gives the essential slope \( v = \frac{d}{\alpha} - 1 \) and another slope \( v = -1 \) of multiplicity \( d - 1 \).
(iv) The only singularity of $\hat{\Lambda} \subset \mathbb{C}P^1_W \times \mathbb{C}P^1_z$ is $(0, \infty)$. As a consequence, the normalisation map $n : \hat{\Lambda} \to \tilde{\Lambda}$ is $1 - 1$ at all points except for $(0, \infty) \in \hat{\Gamma}$ whose preimage consists of $d$ points of $\hat{\Lambda}$.

(v) All $d$ local branches of $\hat{\Lambda}$ at the point $(\infty, 0)$ are smooth.

Finally, the set of all poles of $\tilde{\Omega}$ is presented in (vi)-(vii) below.

(vi) $\tilde{\Omega}$ has a simple pole at each of the points $p_i \in \tilde{\Lambda}$, $i = 1, \ldots, d$ whose image $n(p_i) = (\infty, z_i) \in \hat{\Lambda} \subset \mathbb{C}P^1_W \times \mathbb{C}P^1_z$, where $z_i$ runs over the set of zeros of $P$. At each such point $p_i$ the residue of $\tilde{\Omega}$ equals $1 - \alpha$.

(vii) $\tilde{\Omega}$ has a pole with a real residue at each of $d$ preimages of the singular point $(0, \infty) \in \hat{\Lambda}$ under the normalization map $n : \tilde{\Lambda} \to \hat{\Gamma}$. This residue equals $1$ for each of the $d - 1$ preimages coming from the branches with the slope $-1$ at $(0, \infty)$ and the remaining residue for the preimage coming from the essential branch equals $1 - \frac{d}{\alpha} < 0$.

Proof. To prove (i), observe that the global rational change of variables $W = W$, $\tilde{z} = z + W^{-1}$ transforms (1.5) into

$$\alpha W = \frac{P'(\tilde{z})}{P(\tilde{z})}. \quad (5.1)$$

Since $\alpha \neq 0$, this equation allows us to consider $W$ as the graph of a rational function in the variable $\tilde{z}$. The latter fact implies that $\Lambda$ is a rational curve, and in addition, since it is a graph, $\Lambda$ is irreducible.

To prove (ii), we argue as follows. Assuming that $P(z) = (z - z_1) \ldots (z - z_d)$ has all simple zeros, we obtain

$$\alpha W = \frac{P'(z + W^{-1})}{P(z + W^{-1})} = \sum_{i=1}^{d} \frac{1}{z + W^{-1} - z_i}. \quad (5.2)$$

Substituting $z = \frac{1}{y}$ in (5.2) and clearing the denominators, we get

$$\alpha \tilde{P} = y \sum_{j=1}^{d} \tilde{P}_j, \quad (5.3)$$

where

$$\tilde{P} = \prod_{i=1}^{d} (y + W - z_i y W), \quad \tilde{P}_j = \frac{\tilde{P}}{y + W - z_j y W}.$$

To obtain the fiber over $z = \infty \in \mathbb{C}P^1_z$, i.e. over $y = 0$, one should substitute $y = 0$ in (5.3). One can easily check that the result of this substitution is $\alpha W^d = 0$, implying that the only point in the fiber $\tau^{-1}(\infty)$ is $W = 0$. The argument works, even if $P$ does not have simple zeros.

To settle (iii), we need to calculate the slopes of the branches of $\Lambda_{\alpha,P}$ at $(0, \infty)$, for which one should substitute $W = v(y)y$ in (5.3). The slope coincides with $v := v(0)$. After the substitution of $W = v(y)y$ in (5.3), one cancels out the factor $y^{d-1}$ in both sides and then sets $y = 0$. The result is as follows

$$\alpha \prod_{i=1}^{d} (1 + v) = d \prod_{i=1}^{d-1} (1 + v).$$
or, equivalently,

\[ \alpha(1 + v)^d = d(1 + v)^{d-1} \iff (1 + v)^{d-1}(\alpha(v + 1) - d) = 0, \]

which is the required statement.

To prove (iv), we need to show that there are no singularities of \( \hat{\Lambda} \) above the affine part of \( \mathbb{C}P^1 \), i.e. for all \( z \neq \infty \) and \( W \in \mathbb{C}P^1 \). Note first that \( W = 0 \) is impossible for finite \( z \). \( W = 0 \) is equivalent to \( W^{-1} = \infty \). Rewrite (5.2) as

\[ G(W^{-1}, z) = \alpha P(z + W^{-1}) - W^{-1}P'(z + W^{-1}) = 0. \]  

(5.4)

A calculation shows that the coefficient at the highest power of \( W^{-1} \) is \( \alpha - d \), which is non-zero by the assumptions of the theorem. Hence, \( z \) finite and \( W = 0 \) is impossible. In other words, the curve \( \hat{\Lambda} \) intersects the coordinate space \( \mathbb{C}P^1 \) only at \( z = \infty \), and the part of \( \hat{\Lambda} \) in the finite plane \( \mathbb{C}_W \times \mathbb{C}_z \) is contained in the open set \( U \) given by \( W \neq 0 \).

Secondly, observe that the change of coordinates in the finite affine plane \( (W, z) \mapsto (W, \tilde{z}) \) where \( \tilde{z} = z + W^{-1} \) is a diffeomorphism between \( U : W \neq 0 \) and \( V : W \neq 0 \) (in the respective coordinate plane). The curve given by (5.1) in the coordinates \((w, \tilde{z})\) is clearly smooth when \( \tilde{z} \) is not a root of \( P \). Additionally, \( W = \infty \) at any root of \( P \) implying that our curve is smooth in the whole of \( V \). Any diffeomorphism preserves the smoothness property, and hence \( \hat{\Lambda} \) is smooth in \( U \). By the first observation, it is smooth at all points in the finite \((W, z)\)-plane.

It remains to check the points where \( W = \infty \), which occurs exactly at the roots of \( P \). This can be verified by setting \( W^{-1} = 0 \) in (5.2). Assume that \( p \in \mathbb{C}P^1 \times \mathbb{C}P^1 \) given by \( W^{-1} = 0 \) and \( z = z_i \) is a singular point of our curve where \( P(z_i) = 0 \). Then at \( p \), the partial derivatives of the left-hand side \( G(W^{-1}, z) \) of (5.4) with respect to the variables \( W^{-1} \) and \( z \) must vanish. A short calculation leads to

\[ \frac{\partial G(W^{-1}, z)}{\partial W^{-1}} - \frac{\partial G(W^{-1}, z)}{\partial z} = -P'(W^{-1}, z). \]

Since \( W^{-1} = 0 \) and \( P(z_i) = 0 \) at \( p \) and we have assumed that \( P(z) \) has only simple roots, we get that \( P'(z_i) \neq 0 \) which implies that the latter difference between the partial derivatives can not vanish at \( p \), a contradiction.

To prove (v), we first consider the essential branch at \( \infty \) whose slope is given by \( v = \frac{d}{\alpha} - 1 \). By our assumption, \( 0 < \alpha < d \), which, in particular, implies that this slope differs from \( -1 \) which is the slope for all other branches. By the implicit function theorem, the essential branch is smooth at \((0, \infty)\).

Let us now consider the remaining cases for which

\[ W = yv(y) = y(-1 + v_1y + v_2y^2 + \ldots) = -y + y^2u(y). \]  

(5.5)

We will first show that if \( P' \) has simple roots, then there are \( d-1 \) distinct solutions for \( v_1 \). Rewriting the equation in terms of \( v(y) \) corresponds to the blow-up of the curve (at the origin), and then rewriting it in terms of \( u = u(y) \) corresponds to still another blow-up.

Note that

\[ y^{-1} + W^{-1} = \frac{u}{yu - 1}, \]
and so substituting (5.5) in the equation (5.2) results in
\[
\alpha y = \sum_{i=1}^{d} \frac{1}{u - z_i(yu - 1)}.
\] (5.6)

If we insert \( y = 0 \), this equation becomes
\[
0 = \sum_{i=1}^{d} \frac{1}{u + z_i} = \frac{P'(-u)}{P(-u)}. \quad (5.7)
\]

Now
\[
\frac{P'(u)}{P(u)} = 0 \iff P'(u) = 0.
\]

The last equation has degree \( d - 1 \) in \( u \) and its solutions are exactly the zeros of \( P'(u) \). Thus there are \( d - 1 \) solutions \( u(0) = v_1 \) to (5.7), and these are distinct by the assumption that \( P' \) has only simple roots. We can say even more. The equation (5.6) defines a curve \( \alpha y = F'(u, y) \) in \( \mathbb{C}^2_{u, y} \) with coordinates \( u \) and \( y \). This curve will be smooth and transversal to \( y = 0 \) at a point \( (v, 0) \) if
\[
\left. \frac{\partial F(u, y)}{\partial u} \right|_{(v, 0)} = \sum_{i=1}^{d} \frac{1}{(v + z_i)^2} = \left( \frac{P'(-v)}{P(v)} \right)' \neq 0.
\]

On the other hand,
\[
\left( \frac{P'(-v)}{P(v)} \right)' = -\frac{P''(-v)P(-v) - (P'(-v))^2}{P^2(-v)} = -\frac{P''(-v)P(-v)}{P^2(-v)} \neq 0,
\]
if we assume that \( v = v_1 \) is one of the distinct roots of \( P'(-v) \).

This argument proves that in a neighbourhood of \( y = 0 \) there are \( d - 1 \) branches of the affine curve (5.6) in \( \mathbb{C}^2_{u, y} \) with coordinates \( (u, y) \), intersecting \( y = 0 \) in the \( d - 1 \) different smooth points \( (v_i, 0) \), where each \( v_i \) is a root of \( P'(-v) \). If we consider these branches to \( \mathbb{C}^2_{W, y} \) with coordinates \( (W, y) \) using the coordinate change \( \Theta : (u, y) \mapsto (W, y) = (-y + y^2u, y) \), then an easy calculation shows that they will be mapped onto all \( d - 1 \) distinct branches all of which having the slope \(-1\). This proves that these branches are smooth at \( y = 0 \). Note that \( \Theta \) is the composition of two blow-ups: \( (u, y) \mapsto (-1 + yu, y) = (\tilde{u}, y) \) (blowing up the point \((-1, 0)) and \((\tilde{u}, y) \mapsto (iy, y) \) (blowing up the origin \((0, 0)) \). We have deduced the desired results from the strict transform given by (5.6).

Thus besides the smooth essential branch, we get \( (d - 1) \) additional smooth branches at the complete ramification point with the same slope \(-1\) and distinct coefficients at \( y^2 \).

To prove (vi), observe that for \( z \neq \infty \), the poles of \( \bar{\Omega} \) (which is the pullback to \( \hat{\Lambda} \) of the form \( Wdz \) restricted to \( \hat{\Lambda} \) under the normalisation map \( n \)) occur at (the pullbacks of) the non-singular points of \( \hat{\Lambda} \cap H^\infty_W \), where \( H^\infty_W \subset \mathbb{C}P^1 \times \mathbb{C}P^1 \) is given by \( W = \infty \). Since \( W = \infty \) corresponds to \( W^{-1} = 0 \) and \( \alpha \neq 0 \), then for \( z \neq \infty \), we immediately observe from (5.2) that the poles of \( Wdz \) restricted to \( \hat{\Lambda} \) occur at the points of the form \((\infty, z_i) \), where \( z_i \) is any root of \( P \).

Using (5.2), let us calculate the residues of \( W(z)dz \) restricted to \( \hat{\Lambda} \) at each \((\infty, z_i) \). Dividing equation (5.2) by \( W(z) \) and introducing the local coordinate \( \xi_i = z - z_i \),
we get
\[ \alpha = \frac{W^{-1}(\xi_i)}{W^{-1}(\xi_i) + \xi_i} + \sum_{j \neq i} \frac{W^{-1}(\xi_i)}{W^{-1}(\xi_i) + \xi_i - (z_j - z_i)}. \]

Assuming that \( W^{-1}(\xi_i) = \kappa_i \xi_i + \ldots \) and taking the limit of the right-hand side of the latter equation when \( \xi_i \to 0 \), we get
\[ \alpha = \frac{\kappa_i \xi_i}{\kappa_i \xi_i + \xi_i} = \frac{\kappa_i}{\kappa_i + 1} \]
which immediately implies \( \kappa_i = \frac{\alpha}{1-\alpha} \). Thus
\[ W(\xi_i) = \frac{1-\alpha}{\alpha \xi_i} + \ldots \Rightarrow \text{Res}_{(\infty, z_i)} Wdz = \frac{1-\alpha}{\alpha}. \]

To settle (vii) and to study the behavior of \( \tilde{\Omega} = Wdz \) at the singular point \((0, \infty)\), we need to change the variable \( z = \frac{1}{y} \). Then \( \tilde{\Omega} = -W \frac{dy}{y} \). Observe that under the assumptions of (v), each local branch of \( \hat{\Lambda} \) at \((0, \infty)\) is smooth which implies that the normalisation map is a local diffeomorphism of the corresponding small neighborhood of \( \hat{\Lambda} \) with this branch. Then we have the following expansion of \( W(y) \) for each local smooth branch with the slope \(-1\) and the residue of \( Wdz \) restricted to this branch:
\[ W(y) = -y + \ldots \Rightarrow -W(y) \frac{dy}{y^2} = \left(1 + \ldots\right) \frac{dy}{y} \Rightarrow \text{Res}_{(\infty, 0)} \left(-W(y) \frac{dy}{y^2}\right) = 1. \]

Analogously, for the essential branch whose slope equals \( \frac{d}{\alpha} - 1 \), we get
\[ W(y) = \left(\frac{d}{\alpha} - 1\right)y + \ldots \Rightarrow \text{Res}_{(\infty, 0)} \left(-W(y) \frac{dy}{y^2}\right) = 1 - \frac{d}{\alpha}. \]

\[ \square \]

**Remark 8.** Under the assumptions of (vi), (vii) above, the total number of poles of \( \tilde{\Omega} = Wdz \) equals \( 2d - 2 \deg P \), all having real residues. Observe that in case of simple zeros of \( P \), the point \((0, \infty)\) reduces the genus by \((d-1)^2\) which means that this point is a rather complicated singularity. Under the above assumptions, the cardinality of \( CR \) equals \( 2d - 2 \) which implies the standard identity that the number of poles minus the number of zeros of \( \tilde{\Omega} \) equals \( 2 \). (Poles and zeros are counted with multiplicities.)

**Remark 9.** The sum of all residues of any meromorphic form on any compact Riemann surface must vanish. Our count gives the following sum
\[ \Sigma = \frac{1-\alpha}{\alpha} \cdot d + d - \frac{d}{\alpha} = 0. \]

**Remark 10.** Observe that the equation (1.3) defining the curve \( \Gamma_{\alpha, P} \) belongs to the class of the so-called balanced algebraic functions defined in [BoSh], § 3. For balanced algebraic functions it has been conjectured in § 3 of [BoSh] that there always exists a probability measure with required properties. However not all balanced algebraic functions correspond to affine Boutroux curves.
6. Proof of Theorem 1

We will use the classical saddle point method as presented in [Os], see also [Bi], § 7.3.11 and [Br]. Let \( P \) be a monic polynomial, as in Theorem 1, and let \( \mu_n \) be the root-counting measure of the polynomial

\[
q_n(z) := \mathcal{R}_{[\alpha n] - 1,n,R}(z) = (P^n)_{([\alpha n] - 1)}(z),
\]

where \( \alpha \in (0, d) \). (Note that our indexing here is slightly different from that in the Introduction, but this will not effect the final result.)

The sketch of the proof of Theorem 1 is as follows. For any \( z_0 \in \mathbb{C} \), Cauchy's formula for high order derivatives gives

\[
q_n(z_0) = \frac{([\alpha n] - 1)!}{2\pi i} \int_{c} \frac{P^n(z)dz}{(z - z_0)^{\alpha n}},
\]

where \( c \) is any simple closed curve in the \( z \)-plane encircling \( z_0 \) once counterclockwise. (Notice that \( P \) has no poles.)

The saddle point method allows us to analyze the asymptotics of (6.1) when \( n \to \infty \). The degree of the polynomial \( q_n(z) \) is given by \( d_n := dn - [\alpha n] + 1 \). We are able to find the limit of the sequence \( \{L_{\mu_n}(z)\} \) of the logarithmic potentials, where \( L_{\mu_n}(z) = \log |q_n(z)|/a_n^{1/dn} \) and \( a_n \) is the leading coefficient of \( q_n(z) \). It turns out that the saddle points of the integrand of (6.1) are given by the algebraic equation (1.3). This fact enables us to identify the limit \( \Psi(z) := \lim_{n \to \infty} L_{\mu_n}(z) \) with the tropical trace of a natural harmonic function on the associated plane curve. Finally, applying the Laplace operator, we obtain that \( \mu := \lim_{n \to \infty} \mu_n \) exists, and obtain the equation for its Cauchy transform. Now let us provide the relevant details.

6.1. Root asymptotics via the saddle point method. Given \( \alpha > 0 \), set

\[
n = \frac{[\alpha n]}{\alpha} + s_n,
\]

where \( 0 \leq s_n < 1/\alpha \) and set \( m := [\alpha n] \). Consider the integral \( I_P = \int_{\gamma} \frac{P^n(z)dz}{(z - z_0)^{\alpha n}} \) over a (sufficiently short) curve segment \( \gamma \). Take a neighborhood \( U \) of \( \gamma \) and a suitable choice of a branch of the logarithm, so that \( (P^{1/\alpha})^m P^s_n = P^n \) in \( U \), where we have used the logarithm to define powers. Then,

\[
I_P(m, s_n, \gamma) := \int_{\gamma} \left( \frac{P^{1/\alpha}(z)}{(z - z_0)} \right)^m P^s_n(z)dz = \int_{\gamma} e^{k(z)m} P^s_n(z)dz,
\]

where

\[
k(z) = \frac{1}{\alpha} \log P(z) - \log(z - z_0). \tag{6.4}
\]

Clearly \( k(z) \) is holomorphic if \( \gamma \) avoids the zeros of \( P \) and \( z_0 \). A saddle point is a zero of \( k'(z) \).

The version of the saddle point method which we use here is as follows. Assume that

(i) \( k(z) \) is any function holomorphic in a neighbourhood \( U \) of a simple curve \( \gamma \);
(ii) \( u_0 \in \gamma \) is a saddle point of \( k(z) \) lying in the interior of \( \gamma \);
(iii) \( \forall z \in \gamma, \) such that \( z \neq u_0, \) \( \Re k(z) < \Re k(u_0) \).

Let \( \ell \geq 2 \) be the order of the saddle point \( u_0 \), i.e.

\[
P(z) = P(u_0) - P_0(z - u_0)^{\ell}(1 - \phi(z)), \tag{6.5}
\]
Lemma 10. For \( m \in \mathbb{N} \) and \( 0 \leq s \leq A < \infty \), consider
\[
I(m, s, \gamma) := \int_{\gamma} e^{k(z)m} P^s(z) dz.
\]
Then,
\[
I(m, s, \gamma) = e^{k(u_0)m} \left( \Gamma(\ell^{-1}) \frac{\alpha_0 (\epsilon_1 - \epsilon_2)}{m^\ell} + O \left( \frac{K(P)}{m^\ell} \right) \right),
\]
where \( \epsilon_1 \) and \( \epsilon_2 \) are two distinct \( \ell \)-th roots of unity depending only on \( \gamma \), and \( K(P) \) is an upper bound for \( |P^s(z)| \) in \( U \). The implicit constant in the ordo-term \( O(\ldots) \) is independent of \( P, s \) and \( m \). The constant \( \alpha_0 \) is given by
\[
\alpha_0 = \frac{1}{\ell} \cdot \frac{1}{P_0} \cdot \frac{1}{P(u_0)}.
\]

Proof. Apply Theorem 1.2 and Corollary 1.4 of [Os] for \( S = 1 \). □

Under the hypothesis of Lemma 10, we have the following immediate result.

Corollary 7.
\[
\lim_{m \to \infty} |I(m, s, \gamma)|^{1/m} = e^{Re \, k(u_0)},
\]
uniformly in \( 0 \leq s \leq A \).

Proof. The uniformity in \( s \) follows from the fact that, by definition, \( K(P) \) is uniformly bounded in \( s \). The rest is a standard limit argument. □

Hence, for any sequence \( 0 \leq s_m \leq 1/\alpha \),
\[
|I_{R}(m, s_m, \gamma)|^{1/m} \to e^{Re \, k(u_0)},
\]
given that the Cauchy integral (6.3) and its saddlepoint \( u_0 \) satisfy the assumptions i-iii) above.

6.2. Deformation of the contour. The saddle points of \( k(u) \) in (6.4) are the points \( u \) such that
\[
k'(u) = \frac{P'(u)}{\alpha P(u)} - \frac{1}{u - z_0} = 0.
\]
Note that (6.9) implies that saddle points associated to \( z_0 \) are independent of the choice of a branch of the logarithm. For each \( z_0 \), there are at most \( d = \deg P \) such saddle points, since the polynomial \( P'(u)(u - z_0) - \alpha P(u) \) has degree \( d \), by the assumption that \( \alpha < d \).

For all pairs \((z, u) \in \mathbb{C}^2\) except for those satisfying either \( P(u) = 0 \) or \( u = z \), define
\[
G(u, z) := \frac{1}{\alpha} \left( \log |P(u)| - \alpha \log |u - z| \right).
\]
Observe that
\[
G(u, z_0) = Re \, k(u).
\]

The integration contour for the integral (6.1) can be freely deformed as long as it does not pass through \( z_0 \). We will soon show that all the saddle points contribute to the limit of the Cauchy integral.
6.3. Properties of the saddle point curve. The saddle point curve
\[ D \subset \mathbb{CP}_u \times \mathbb{CP}_z, \]
determined by (6.9) in the affine \((u, z)\)-plane and extended by taking the closure to
\(\mathbb{CP}_u \times \mathbb{CP}_z\), is straight-forwardly related to the symbol curve \(C\) determined by (1.6)
via the explicit birational transformation \(\Theta : D \to C\) given by \(\Theta(u, z) = (C, z)\).
Namely,
\[ C = \frac{\alpha}{d - \alpha} \cdot \frac{1}{u - z} \iff u = z + \frac{\alpha}{d - \alpha} C^{-1}. \tag{6.12} \]
On the subset \(z \neq u\), \(\Theta\) is an isomorphism with the complement to \(z \neq \infty\). Clearly,
\(D\) can be investigated by using the results about \(C\) from Section 5. We will need the following properties.

Lemma 11. Under the conditions imposed on \(P\) in Section 5, the following holds:

(i) \(D\) is an irreducible rational and smooth curve, and \(\Theta\) extends to a map
\(D \to C\) which is a resolution of singularities of \(C\).

(ii) \(D\) has \(d\) branches over a neighbourhood of \(z = \infty\). One branch passes through \((\infty, \infty)\). The remaining branches pass through \(d - 1\) points of the form \((q_i, \infty)\), where \(q_1, \ldots, q_{d-1}\) are \(d - 1\) (distinct) roots of \(P'(u) = 0\).

(iii) There are no points with \(u = \infty\) on \(D\), except \((\infty, \infty)\). The intersection of
\(D\) with the diagonal \(u = z\) consists of the points \((p_i, p_i), i = 1, \ldots, d, for which \(R(p_i) = 0, together with \((\infty, \infty)\).

(iv) The meromorphic form \(Cdz\) on \(D\) given by
\[ Cdz = \frac{\alpha}{d - \alpha} \frac{dz}{u - z} \tag{6.13} \]
has the following simple poles:
(a) at the points \((u, z) = (p_i, p_i), i = 1, \ldots, d\) with residue \(\frac{1 - \alpha}{d - \alpha}\),
(b) at the points \((u, z) = (q_i, \infty), i = 1, \ldots, d - 1\) with residue \(\frac{\alpha}{d - \alpha}\),
(c) at \((\infty, \infty)\) with residue \(-1\).

Proof. (i)-(iii): For \(z\) finite, we have that \(u = \infty \iff C = 0\) by (6.12), and the last equality does not happen for finite \(z\), (comp. the proof of Theorem 9 (iv)). The local coordinate around \(z = \infty\) is \(y = 1/z\) and then (6.9) becomes
\[ P'(u)(yu - 1) = \alpha P(u) y. \tag{6.14} \]
At \(y = 0\), (6.14) simplifies to \(P'(u) = 0\), proving that there are exactly \(d - 1\) points
\((q_i, \infty), i = 1, \ldots, d - 1, with u = q_i finite. Since P(u) has only simple zeros, the gradient to (6.14) at \(y = 0, u = q_i\) equals
\[ (-P''(q_i), \alpha P(q_i)) \neq (0, 0), \]
proving that the branches of \(D\) passing through \((\infty, q_i)\) are smooth. Similar calculation gives that the branch of \(D\) passing through \((\infty, \infty)\) is smooth as well. In the
\((u, z)\)-plane, the gradient of \(P'(u)(u - z) - \alpha P(u)\) is given by
\[ (P''(u)(u - z) + P'(u) - \alpha P'(u), -P'(u)) \neq (0, 0). \]
Non-vanishing of this gradient is the consequence of the fact that otherwise \(P(u) = P'(u) = 0\) which contradicts to our assumption that \(P\) has simple roots. Hence \(D\) is a smooth curve, and \(\Theta : D \to C\) extends to a resolution of singularities of \(C\) (by sending \(d\) points of \(D\) on the line \(z = \infty\) to \((0, \infty)\), and another \(d\) points \((p_i, p_i)\) to \((\infty, p_i))\). Note, in particular, that the singularity of \(C\) at \((0, \infty)\) is resolved. \(D\)
is clearly rational and irreducible, since $C$ has these properties, see Theorem 9 i). The remaining statements follow from similar calculations.

To settle (iv), note that $\Theta(p_1, p_i) = (\infty, p_i)$, and that the $d$ branches of $C$ passing through $(0, \infty)$ correspond to the $d$ branches of $D$ which intersect $z = \infty$. Hence the calculations of the residues on $C$ in Theorem 9, v)-vi) apply. The coordinates $W$ and $C$ are related by $C = \frac{2\alpha}{d-\alpha} W$. Therefore, to get the residues of $Wdz$ in (6.13), one should multiply the residues of $Wdz$ in Theorem 9 by $\frac{\alpha}{d-\alpha}$. \hfill $\Box$

6.4. $G(w, z_0)$ distinguishes saddlepoints a.e. For a fixed $z_0$ which is not a zero of $P(w)$, let $w_1, w_2, \ldots, w_d$ be the saddle points of (6.1). Assume that $w_1$ satisfies the condition

$$G(w_1, z_0) > G(w_j, z_0), \ j = 2, \ldots, d. \quad (6.15)$$

**Lemma 12.** There exists an open dense subset $U \subset \mathbb{C}$ such that, for $z \in U$, there is a unique saddlepoint $w_1$ satisfying (6.15).

**Proof.** Argue by contradiction. Suppose that $D \subset \mathbb{C}_z$ is an arbitrary disk containing no discriminantal points; therefore there exist $d$ branches of the curve $D$ defined in $D$ which we denote by $w_i(z), \ i = 1, \ldots, d$. In a dense set two harmonic functions are either equal or distinct. VAD MENAS HR? Hence we can assume that $G(w_1, z_0) = G(w_j, z_0)$, in the disk. By analytic continuation, using the irreducibility of $D$ (Lemma 11 i)), the branch representing $w_1$ can be analytically continued to any other branch. In particular, it can be continued to the one for which $u \to \infty$. But for the corresponding continuation of the other branch, we have $u \to q_i \in \mathbb{C}$, for some $q_i : R'(q_i) = 0$ (by Lemma 11 ii)). Substituting these values into $G(u, z_0)$ gives a contradiction to the equality. I CAN NOT UNDERSTAND! Namely, for the second branch $G(u, z) \sim -\log |z|$, and for the first branch $G(u, z) \sim (\frac{2\alpha}{d-\alpha}) \log |z|$, by a calculation using Theorem 9 iii) to get that $\log |u| \sim -\log |z|$. \hfill $\Box$

6.5. A contour passing through the maximal saddlepoint. Assuming the validity of (6.15), we now show that it is possible to deform the integration contour $c$ to a contour $\gamma$ possessing the following two properties:

(*) the curve $\gamma = \Delta + \gamma_1$ passes through $w_1$ along a curve $\Delta$ of steepest descent/ascent of the function $G(w, z_0)$;

(**) there exist $\delta > 0$ such that, for all points $w \in \gamma_1$, $G(w, z_0) < G(w_1, z_0) - \delta$.

For $z_0$ fixed, consider the graph $Z$ of $w \mapsto G(w, z_0)$, as a mountainscape. Clearly all the critical points of $G(w, z_0)$ are the saddle points, since a harmonic function can not have local minima or maxima. At the pole $w = z_0$, $G(w, z_0) \to +\infty$; while $G(w, z_0) \to -\infty$ at all other poles $w$ (which are given by $P(w) = 0$). In addition, $G(w, z_0) \to +\infty$ when $|w| \to \infty$.

Intuitively, we obtain a contour satisfying the above conditions by taking a curve $\tilde{c}$ above a small circle $c$ around $z_0$ on the graph $Z$. Push $\tilde{c}$ down on $Z$, away from the peak at $z_0$, until it hits the saddles in the mountainscape. Then, additionally deform $\tilde{c}$ locally at the maximal saddle $(w_1, G(w_1, z_0))$, so that it passes through this point along a path of steepest descent/ascent, ensuring that the projection of the deformed curve in the $w$-plane satisfies (**) . Details are as follows.

Set $\delta_0 := \min\{G(w_1, z_0) - G(w_j, z_0), \ j = 2, \ldots, d\}$. Let $N$ be a small neighbourhood of $w_1$, and let $\hat{\Delta}$ be a path of steepest descent/ascent through $w_1$ in $N$. Choose points $a$ and $b$ on $\hat{\Delta}$, on each side of $w_1$, such that

$$G(a, z_0) = G(b, z_0) = G(w_1, z_0) - \tilde{\delta} > G(w_1, z_0) - \delta_0.$$
The curve $\Delta$ in (**) will be the segment of $\tilde{\Delta}$ between $a$ and $b$.

Now take $\Xi$ to be any piecewise $C^1$-path connecting $a$ to $b$ which (i) encircles $z_0$ once and (ii) avoids $N$. Set $d(\Xi) := \max\{G(w, z_0) \mid w \in \Xi\}$. If $p \in \Xi$ is not a saddle point, then there is a neighbourhood $N_p$ of $p$ divided into two open subsets by the curve $G(q, z_0) = G(p, z_0)$; one of these subsets is $N_p^- = \{q : G(q, z_0) < G(p, z_0)\}$.

If $d(\Xi) > G(w_1, z_0) - \delta$, then the compact set $M$ of $w$ such that $G(w, z_0) = d(\Xi)$ consists of a finite number of points (since $G$ is real-analytic) and possibly a finite number of segments of $\Xi$, and does not contain $a$ or $b$. Let $p$ be an isolated point in $M$. Since

$$G(p, z_0, j) \geq G(w_1, z_0) - \tilde{\delta} > G(w_j, z_0)$$

$p$ is not a saddle point. Furthermore, we can deform the part of $\Xi$ contained in a small neighbourhood $N_p$ of $p$, so that it fits in $N_p^-$, and hence $G(w, z_0, w) < d(\Xi \cap N_p)$. For a segment the argument is similar. Using compactness, we can repeat the construction for the whole $M$ and get a curve $\Xi_1$ such that $d(\Xi_1) < d(\Xi)$. Hence

$$\inf_{\Xi} d(\Xi) = G(w_1, z_0) - \delta.$$  

In particular, we can find a curve $\Xi$ such that $d(\Xi) = G(w_1, z_0) - \delta/2$. Set $\delta = \delta/2$, and let $\gamma_1$ be $\Xi$. This proves the claim.

The latter argument implies the convergence of our Cauchy integral almost everywhere which, in its turn, implies that the assumption in the following corollary is satisfied for $z_0$ in a dense open subset, see Lemma 12.

**Corollary 8.** Assume that $(w_1, z_0) \in \mathcal{D}$ is such that

$$G(w_1, z_0) > G(w_j, z_0), \; j = 2, \ldots, d.$$  

Then,

$$\lim_{m \to \infty} |I(m, s_n, c)|^{1/m} = e^{|G(z_0, w_1)|},$$

where $c$ is any contour encircling $z_0$ once counterclockwise.

**Proof.** Deform $c$ as in the first part of this section. If $M$ is the upper bound of $|P^{\gamma}(w)|$ on $\gamma_1$, then by (**) 

$$|I(m, s_n, \gamma_1)| \leq \int_{\gamma_1} e^{G(w, z_0)m}|P^{s_n}(w)|dz \leq M \text{Length}(\gamma_1)e^{m(G(w_1, z_0)-\delta)}.$$  

Comparison of (6.18) with the limit of the integral on $\Delta$ given by (6.8) proves the corollary.


6.6. **A.e. convergence of the logarithmic potentials.** By (6.1), the value at $z_0$ of the monic polynomial $\tilde{q}_n$ proportional to $q_n$ is given by

$$\tilde{q}_n(z_0) := \frac{(nd - ([\alpha n] - 1))!}{(nd)!} q_n(z_0) = \frac{([\alpha n] - 1)!}{2\pi i nd!} \int_c \frac{P^n(z)dz}{(z - z_0)^{\alpha n}}.$$  

(6.19)

The degree of $q_n$ equals $d_n = nd - (m - 1)$, where $m = [\alpha n]$. Recall that the logarithmic potential $L_{\mu_n}(z)$ of the root-counting measure $\mu_n$ of $\tilde{q}_n$ is given by

$$L_{\mu_n}(z) = \log |\tilde{q}_n|^{1/d_n}.$$  

By (6.2),

$$n = \frac{m}{\alpha} + s_n,$$
where $0 \leq s_n < 1/\alpha$. Hence

$$d_n = \left(\frac{d - \alpha}{\alpha}\right) m + (s_n d + 1) = \beta m + O(1), \quad (6.20)$$

with $\beta \equiv \frac{d - \alpha}{\alpha}$.

**Lemma 13.** In the above notation,

$$\lim_{n \to \infty} \frac{1}{d_n} \log \left(\frac{(m - 1)!(nd - (m - 1))!}{(nd)!}\right) = \frac{\beta \log \beta - (\beta + 1) \log(\beta + 1)}{\beta} \equiv B.$$

**Proof.** Straight-forward calculation using Stirling’s formula. \(\square\)

Now we can calculate the limit of the sequence \(\{1/d_n \log |\tilde{q}_n|\}\) of logarithmic potentials by taking the logarithm of (6.19) and using Lemma 13 and (6.17).

**Corollary 9.** For any point $z_0$ such that there is a unique saddle point $w_1$ satisfying $G(w_1, z_0) > G(w_j, z_0), \; 1 \neq j$, one has

$$\lim_{n \to \infty} L_{\mu_n}(z_0) = B + \frac{1}{\alpha \beta} (\log |P(w_1)|) - \alpha \log |w_1 - z_0|).$$

6.7. **Convergence in $L^1_{\text{loc}}$ and proof of the main theorem.** The first part of the following theorem is our strongest convergence result; two other claims follow from it easily. Set

$$H(u, z) := \beta^{-1} G(u, z) := \frac{1}{\alpha \beta} (\log |P(u)|) - \alpha \log |u - z|).$$

**Theorem 14.** For any polynomial $P$ of degree $d \geq 2$,

$$\lim_{n \to \infty} L_{\mu_n}(z) = B + \pi_* H(z),$$

where the limit is understood in the $L^1_{\text{loc}}$-sense. Consequently,

$$\lim_{n \to \infty} C_{\mu_n}(z) = 2 \frac{\partial \pi_* H(z)}{\partial z},$$

and,

$$\lim_{n \to \infty} \mu_n = \mu := 2 \frac{\partial^2 \pi_* H(z)}{\partial z \partial \bar{z}},$$

where the last two limits are understood in the sense of distributions.

For the proof we need the following useful corollary of Vitali’s convergence theorem. It provides a criterion for $L^1_{\text{loc}}$-convergence under the assumption that one has the pointwise a.e. convergence.

**Lemma 15.** Let \(\{p_n\}\) be a sequence of monic polynomials of strictly increasing degrees $d_n := \deg p_n \to \infty$ as $n \to \infty$. Denote by $\mu_n := \frac{1}{d_n} \sum_{i=1}^{d_n} \delta_{\zeta_i}$ the root-counting measure of $p_n$, where $\zeta_i, \; i = 1, \ldots, d_n$ are the zeros of $p_n$. Denote by $L_n(z) := \frac{1}{d_n} \log |p_n(z)|$ the logarithmic potential of $\mu_n$. Assume that

(i) there is a compact set $K \subset \mathbb{C}$ containing all the zeros of $p_n, \; n = 1, \ldots$;  
(ii) the sequence \(\{L_n(z)\}\) converges to $L(z)$ pointwise a.e. in $\mathbb{C}$.

Then, $L$ is a $L^1_{\text{loc}}$-function and $\lim_{n \to \infty} L_n = L$ in the $L^1_{\text{loc}}$-sense.
Proof. By Vitali’s theorem, we only need to check the uniform integrability of our functions on an arbitrary fixed compact set $M \supset K$. Let $E$ be a set with Lebesgue measure $\lambda(E) < \epsilon < 1$. Introduce
\[
\log_+(x) := |\log |x|| = f_{< \epsilon}(x) + f_{\geq \epsilon}(x),
\]
where $f_{< \epsilon}(x) = \log_+(x) = -\log |x|$, if $0 < x \leq \epsilon$ and $f_{< \epsilon}(x) = 0$, if $x > \epsilon$. (Thus $f_{\geq \epsilon}(x) = \log_+(x)$, if $x > \epsilon$ and 0 if $0 < x \leq \epsilon$.)
We obtain
\[
\left| \int_E L_{\nu_n}(z - \zeta)d\lambda(\zeta) \right| \leq \frac{1}{d_n} \sum_{i=1}^{d_n} \int_E \log_+(z - \zeta_i)d\lambda(\zeta)
\]
(6.21)
\[
\leq \frac{1}{d_n} \sum_{i=1}^{d_n} \int_E f_{< \epsilon}(z - \zeta_i)d\lambda(\zeta) + \frac{1}{d_n} \sum_{i=1}^{d_n} \int_E f_{\geq \epsilon}(z - \zeta_i)d\lambda(\zeta) := I_1 + I_2.
\]
(6.22)
If $D_{\epsilon}(\zeta_i)$ is a disk of radius $\epsilon$ centered at $\zeta_i$, then
\[
\int_{D_{\epsilon}(\zeta_i)} |\log |z - \zeta_i||d\lambda = -\pi \epsilon^2(\log \epsilon - 1/2).
\]
(6.23)
Hence
\[
\int_E f_{< \epsilon}(z - \zeta_i)d\lambda(\zeta) \leq \int_{D_{\epsilon}(\zeta_i)} f_{< \epsilon}(z - \zeta_i)d\lambda(\zeta) = -\pi \epsilon^2(\log \epsilon - 1/2),
\]
implying
\[
I_1 \leq -\frac{1}{d_n}(\pi \epsilon^2(\log \epsilon - 1/2)) = O(\epsilon)
\]
with a constant which only depends on $\epsilon$. Let $\delta$ be the diameter of $M$. For the second term above, let $m := \max\{-\log \epsilon, \log_+(\delta)\}$ be an upper bound of $f_{\geq \epsilon}(x - \zeta)$ for $x, \zeta \in M$. Then $I_2 \leq m\lambda(E) \leq me = o(1)$ as $\epsilon \to 0$. This proves that $L_n(z)$ are uniformly integrable. Lemma 15 is now immediate from Vitali’s convergence theorem. \qed

Proof of Theorem 14. Corollary 9 says that we have the desired pointwise convergence a.e. in $\mathbb{C}$, which together with Lemma 15 implies the first part of Theorem 14. The other parts follow from some basic properties of distributions, since $L_{\nu_n}$-convergence implies convergence as distributions. Notice that $\mu$ and its Cauchy transform are distributional derivatives of the logarithmic potential of $\mu$. \qed

We now turn to the proof of Theorem 1. By Theorem 14 (cf. Lemma 12) $\mathbb{C}$ is covered a.e. by open sets $U_i$, $i \in I$ such that in each $U_i$ there is a branch $u = u(z)$ of $D$, so that the equality:
\[
H(u(z), z) + B = L_{\mu}(z)
\]
holds in $U_i$. Consequently
\[
\frac{\partial \beta^{-1}G(u, z)}{\partial z} = C_{\mu}(z),
\]
and $u = z + (\beta C_{\mu})(z)^{-1}$ in $U_i$. The equation defining $D$ satisfied by $u$ implies that the Cauchy transform $C = C_{\mu}$ of $\mu$ satisfies, a.e. in $\mathbb{C}$, the equation
\[
(d - \alpha)C = \frac{d}{dz} \left( \log P(z + (\beta C)^{-1}) \right),
\]
(6.24)
Formula (6.24) coincides with the equation (1.3) which settles Theorem 1, except for one small detail which is the shift of the order of derivation. Above we actually prove that the sequence \{(P^n)_{[\alpha n]}^{-1}\} converges, but using e.g. the main result of [To], we get that also the sequence considered in Theorem 1 will have the same limit as \{(P^n)_{[\alpha n]}\}. \hfill \square

Proof of corollary 3. Follows immediately from Theorem 14, using the description of the tropical trace in Theorem 4 as a piecewise harmonic function and taking into account the description of the poles of \(H(w, z)\) on the saddle point curve, in Lemma 11 iv)

6.8. The saddlepoint curve \(D\) and an alternative proof that \(C\) is an aBc.
Recall the general idea for construction of affine Boutroux curves in Section 3.4. As we will now show, the curve \(C\) from the introduction is a particular instance of this construction.

The starting point is the pluriharmonic function

\[ H(u, z) := \beta^{-1}G(u, z) := \frac{1}{\alpha \beta} \left( \frac{1}{2} \left( \frac{P'(u)}{P(u)} \right) - \frac{\alpha}{u - z} \right) du + \frac{\alpha}{u - z} dz. \]

The curve \(D\) is the rational plane curve given by

\[ 2\alpha \beta \frac{\partial H}{\partial u} = \frac{P'(u)}{P(u)} - \frac{\alpha}{u - z} = 0. \]

Restricting \(H\) to \(D\), we get a simplified expression for its differential given by:

\[ dH(u, z) = \frac{1}{2\beta} \frac{1}{u - z} dz, \quad (u, z) \in D. \]

Consider the projection \(\pi_z : D \to \mathbb{C}\) induced by \(\pi_z(u, z) = z\). Locally, except for a finite number of branch points, \(z\) is a local coordinate on \(D\). For smooth points \(p = (u, z)\) on \(D\) (which all points are by Lemma 11)????, then in some neighbourhood of \(p\) in \(D\), the restriction of \(H\) to \(D\) is a real-valued harmonic function satisfying

\[ H(p) - H(p_0) = \text{Re} \int_{p_0}^p \frac{1}{\beta(u - z)} dz, \]

where \(p_0\) is another fix smooth point on \(D\). In particular, this shows that \(\omega = \frac{1}{\beta(u - z)} dz\) has imaginary periods on \(D\); this also follows from Proposition 9. Using the change of coordinates

\[ v = \frac{1}{\beta(u - z)}, \quad z = z \iff u = z + (\beta v)^{-1}, \quad z = z, \]

we see that \(D\) considered as a plane curve in coordinates \((v, z)\) is an affine Boutroux curve, since \(\omega = vdz\). The tropical trace of \(H(p)\) under the projection \(\pi_z : D \to \mathbb{C}P^1\) is given by

\[ \pi_*H(z) := \max \{ H(p) | \pi_z(p) = z \}. \]
Note that this trace does not depend on the particular choice of coordinates in \( \mathbb{C}^2 \) used to describe \( \mathcal{D} \) as a plane curve; it only depends on the projection \( \pi_z \) and the function \( H \). In terms of the coordinates \((v, z)\), \( H(u, z) = \bar{H}(v, z) \), where

\[
\bar{H}(v, z) = \beta^{-1} G(u, z) := \frac{1}{\alpha \beta} (\log |P(z + (\beta v)^{-1})| + \alpha \log |(\beta v)|),
\]

and \( \pi_z H(z) = \tilde{\pi}_z \bar{H}(z) \), where \( \tilde{\pi} : (v, z) \mapsto z \).

Since \( d\bar{H} = \frac{1}{2} vd\zeta \), this gives an alternative proof of Theorem 9 claiming that \( C \) is an aBc.

7. Final Remarks and open problems

1. Practically all the results of the present paper can be generalised to the case when \( f \) is a rational function instead of a polynomial which we plan to do in the forthcoming paper [BHS]. However poles of a rational function affect the possibility of deformation of the integration contour used in §6.2. This leads to a more delicate situation which requires special analysis.

2. The set-up of the present paper can be naturally randomised and generalised as follows. Let \( \xi \) be a probability measure compactly supported in \( \mathbb{C} \). Denote by \( P_n = \prod_{i=1}^n (x - \xi_i) \) a random polynomial of degree \( n \) whose roots are i.i.d. random variables sampled on \( \xi \). Given a sequence \( \mathcal{A} = \{\alpha_n\} \) of non-negative integers, set \( Q_n = P_n^{(\alpha_n)} \) and denote by \( \mu_n \) the root-counting measure of \( Q_n \). Results from the recent papers [PeRi, Ka] motivate the following guess. (See also results of numerical experiments using deterministic sampling in [H]).

**Conjecture 1.** In the above notation, the following two statements hold:

(i) if \( \frac{\alpha_n}{n} \to 0 \), then the sequence \( \{\mu_n\} \) converges in probability to \( \xi \);

(ii) if \( \frac{\alpha_n}{n} \to \alpha \), \( 0 < \alpha < 1 \), then the sequence \( \{\mu_n\} \) converges in probability to a measure \( \xi_\alpha \) whose support is contained in the convex hull of the support of \( \xi \);

What we have done in the present paper can be interpreted in the above terms as follows. We start with a uniform discrete measure \( \xi \) supported on the \( d \) zeros of \( P(z) \). Then we sample this measure evenly and deterministically \( nd \) times, by forming the series of polynomials \( P_n(z) := P^n(z) \) and, finally, we differentiate \( P_n(z) \) \( [na] \) times. This produces a sequence of polynomials \( \{Q_n(z)\} \) and the associated sequence of probability measures \( \{\mu_n\} \). The proportion between the number of derivations and the number of sampled points is

\[
A := \frac{\alpha}{d} n + O(1).
\]

Observe now that \( \frac{1}{\pi} \log |P(z)| \) is the logarithmic potential \( L_\xi(z) \), and hence our main result Theorem 14 can be reformulated as

\[
\mu_n \to \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{1}{1 - A} \pi_z L_\xi(z + w^{-1}) - \frac{A}{1 - A} \log |w| \right),
\]

where \( \pi_z L_\xi(z + w^{-1}) \) is taken as the maximum for the saddle point locus \( W \) of all \( w \), such that

\[
\frac{\partial G(w, z)}{\partial w} = 0,
\]

\( \partial G(w, z) \).
Figure 4. The zeros of \((P^n)^{(m)}\) for \(m \in \{0, 1, \ldots, 3n\}\), (shown by small dots), where \(P = (z - 1)(z - 6)(z - 3i)\) and \(n = 50\). The squares are the zeros of \(R(h, h')\), for \(\alpha = 1/100, 2/100, \ldots, 299/100\); the larger dots are the zeros of \(P\) and \(P'\), and the triangle is the center of mass of \(Z(P)\).

where

\[
G(w, z) := \frac{1}{1 - A}L_\xi(w) - \frac{A}{1 - A}\log |w - z|.
\]

Such interpretation of Theorem 14 makes sense if \(\xi\) is an arbitrary probability measure such that polynomials \(\{Q_n(z)\}\) are obtained by sampling independent roots according to \(\xi\). It seems plausible that the relation (7.1) holds for much more general measures than special measures with finite discrete support which we have considered in this paper.

3. Numerical experiments similar to the one shown in Fig. 4 motivate the following guess.

**Conjecture 2.** For any polynomial \(P\) of degree at least 2,

(i) the union of the supports \(\Upsilon_P = \bigcup_{\alpha \in [0, d]} S_{\alpha, P}\) is a domain inside the convex hull of the roots of \(P\). The boundary of \(\Upsilon_P\) is a subset of the curve formed by the family of branch points of \(\Gamma_{\alpha, P}\) depending on \(\alpha\);

(ii) \(\Upsilon_P\) is a concave domain.

4. Recall that the \(n\)-th Legendre polynomial is given by:

\[
L_n(z) = \frac{1}{2n!} \frac{d^n}{dz^n} [(z^2 - 1)^n].
\]

**Conjecture 3.** Let \(P := (z - 1)(z + 1)\prod_{j=1}^{d} (z - \alpha_j - \kappa \beta)\) be a polynomial, where \(d \in \mathbb{N}_0, \kappa \in \mathbb{R}\) and \(\beta \in \mathbb{C} \setminus \{0\}\). Then, for any \(n \in \mathbb{N}_0\), \(Z(L_n) \subseteq Z((P^n)^{(n)})\) when \(\kappa \to \infty\).
Conjecture 3 states that if we choose any pair \((\alpha_a, \alpha_b)\) of simple zeros of a polynomial \(P\) which are located sufficiently far from other zeros of \(P\), then (after appropriate translation, rotation, and scaling) the zeros of \((P_n)_{(n)}\) will approximately contain the zeros of \(L_n(z)\) located between the points \(\alpha_a\) and \(\alpha_b\).

In addition to the local behavior described by Conjecture 3, the global behavior of roots can be understood as follows. Consider a polynomial \(P\) with a \(d = d_1 + d_2\) zeros (counting multiplicities) which are situated in two disjoint disks \(D_1\) and \(D_2\), respectively. Then (after appropriate affine transformation) the zeros of \((P_n)_{(n)}\) lying between \(D_1\) and \(D_2\) can be approximated by the zeros of the polynomial \((W_n)_{(n)}\), where \(W = (z-1)^{d_1}(z+1)^{d_2}\). Furthermore, the error in this approximation is inversely proportional to the distance between the disks (assuming that their size is constant). 

For an isosceles triangle whose apex is far from its base, we will thus consider \(W = (z-1)^2(z+1)\), which results in the exact solution to (4.4). Consequently, the zeros on the curve segments approximate the zeros of two explicitly given analytic functions. (At least the two disjoint curves can be approximated well in this way...? The behaviour of roots is more complicated when the curves intersect.)

Similarly, for a rectangle, \(W = (z-1)^2(z+1)^2\) yields an exact solution to (4.4). Note that instead of two disks one can try to use three or more disks, but solving (4.4) exactly becomes a considerably more challenging problem.

5. Our final remark concerns Theorem 7.

Conjecture 4. Under the assumptions of Theorem 7 the signed measure whose existence is claimed in this result is unique.

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