Algebraic relations between harmonic and anti-harmonic moments of plane polygons

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In this paper we describe the algebraic relations satisfied by the harmonic and anti-harmonic moments of simply connected, but not necessarily convex planar polygons with a given number of vertices.

To Isaac Schoenberg and Theodore Motzkin whose insights laid the foundations of this topic.

1 Introduction and Main Results

1.1 Basic notions and background

Let μ be a a finite compactly supported Borel measure in the plane \( \mathbb{R}^2 = \mathbb{C} \). For \( j = 0, 1, \ldots \), its \( j \)-th harmonic moment is a complex number given by:

\[
m_j(\mu) \overset{\text{def}}{=} \int_{\mathbb{C}} z^j \, d\mu(z).
\]

Analogously, its \( j \)-th anti-harmonic moment is given by:

\[
m_j(\mu) \overset{\text{def}}{=} \int_{\mathbb{C}} \bar{z}^j \, d\mu(z).
\]

A function

\[
u_\mu(z) \overset{\text{def}}{=} \int_{\mathbb{C}} \ln |z - \xi| \, d\mu(\xi)
\]

is called the logarithmic potential of \( \mu \). It is harmonic outside the support of \( \mu \) and well-defined almost everywhere in \( \mathbb{C} \). The germ of \( u_\mu(z) \) at \( \infty \) is determined by the sequence of harmonic moments \( \{m_j(\mu)\}_{j=0}^\infty \): the Taylor expansion at \( \infty \) of the Cauchy transform of \( \mu \) defined as

\[
\mathcal{C}_\mu(z) \overset{\text{def}}{=} \int_{\mathbb{C}} \frac{d\mu(\xi)}{z - \xi} = \frac{\partial u_\mu(z)}{\partial z},
\]

and can be computed via the formula

\[
\mathcal{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \ldots.
\] (1)

The problem of the recovery of a measure from its logarithmic potential at \( \infty \) (alias “the inverse problem in logarithmic potential theory”) is a classical area of potential analysis, going back to the early 1920s and still quite active. One of its milestones is the fundamental paper [17], in which P.S. Novikov proved that Lebesgue

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measures of two different star-shaped domains cannot have the same logarithmic potential near $\infty$. In other words, sequences of harmonic moments of Lebesgue measures of two star-shaped plane domains cannot coincide.

For non-star-shaped domains a similar statement is false; see e.g. [4, p. 333] for examples of pairs of polygons having the same logarithmic potentials near $\infty$.

In this century, the problem reappeared in mathematical physics in connection with integrable systems and the Hele-Shaw flow, see e.g. [14, 16]. In particular, in [16] S. Natanzon and A. Zabrodin extended Novikov’s result, showing that harmonic moments can be used as “coordinates” on the set of all star-shaped domains. Their results imply that for any sequence of numbers $a_0, a_1, \ldots$, there exists a star-shaped domain whose Lebesgue measure $\mu$ satisfies the conditions $m_j(\mu) = a_j$, $j = 0, 1, \ldots$.

This claim is no longer true if one considers moments of Lebesgue measures of polygons with a fixed number $n$ of vertices. The harmonic (and anti-harmonic) moments of such polygons are algebraically dependent. The main goal of the present article is to describe these dependencies.

1.2 The main object: polygonal measures, an explicit formula, and complexity

A plane polygon $P \subseteq \mathbb{C}$ is determined by its sequence of vertices $z_1, \ldots, z_n \in \mathbb{C}$, ordered counterclockwise. However, not every sequence of $n$ points in $\mathbb{C}$ is a sequence of vertices for some simply connected polygon. It is natural to generalize the notion of the Lebesgue measure of a polygon as follows.

Let $a = (a_1, \ldots, a_n)$, $a_j \in \mathbb{C}^2$ be a sequence of points with $a_j = (x_j, y_j)$, $j = 1, \ldots, n$. Instead of $x_j$ and $y_j$, we are going to use more convenient coordinates $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$. For brevity, denote $z \equiv (z_1, \ldots, z_n)$ and $\bar{z} \equiv (\bar{z}_1, \ldots, \bar{z}_n)$.

If all $a_j \in \mathbb{R}^2 \subseteq \mathbb{C}^2$ (that is, all $x_j$ and $y_j$ are real), then every $\bar{z}_j$ is indeed the complex conjugate of $z_j$; if we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) \mapsto x + iy$, then $a_j$ becomes $z_j$. We will call such situation “the case of real vertices”. In general, though, $z$ and $\bar{z}$ are $n$-tuples of independent complex variables.

Define an orientable closed polygonal curve $\Gamma_a$ by

$$\Gamma_a \equiv [a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n] \cup [a_n, a_1].$$

Fix an auxiliary convex polygon $P_a \subset \mathbb{R}^2$ with the vertices $w_1, \ldots, w_n$, ordered counterclockwise. Let $T$ be its triangulation, i.e. a set of diagonals of $P_a$ having no common internal points and cutting $P_a$ into triangles. Let $F_{a, T} : P_a \to \mathbb{C}^2$ be the map sending every $w_j$ to $a_j$ and affine on every triangle of the triangulation. (It is easy to show that $F_{a, T}$ exists and is unique and continuous.)

The image $\Delta_{a, T} \equiv F_{a, T}(P_a) \subset \mathbb{C}^2$ is a polygonal disk in $\mathbb{C}^2$ bounded by $\Gamma_a$. The disk $\Delta_{a, T} \subset \mathbb{C}^2$ is piecewise immersed (though $F_{a, T}$ is not always an immersion). For a generic $a$, this disk is embedded.

The disk $\Delta_{a, T}$ supports the measure $\mu_{a, T} = (F_{a, T})_*, dx dy$ which is the direct image of the Lebesgue measure on $P$ under $F_{a, T}$; we call $\mu_{a, T}$ a polygonal measure.

The disk $\Delta_{a, T}$ itself depends on the triangulation $T$, but certain integrals with respect to $\mu_{a, T}$ do not.

Theorem 1.1. (i) Let $h : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function of two variables. Then the integral $\int_{\Delta_{a, T}} h \, d\mu_{a, T}$ does not depend on the triangulation $T$ and is equal to $\int_{\Gamma_a} \omega$, where $\omega$ is any $1$-form such that $d\omega = -\frac{1}{2} h \, dz \wedge d\bar{z}$.

(ii) If the vertices $a_1, \ldots, a_n$ are real, then $\mu_{a, T}$ is independent of $T$. It is supported on a compact subset of $\mathbb{R}^2$ and its density at a point $q \in \mathbb{R}^2 \setminus \Gamma_a$ equals the linking number of the $1$-cycle $\Gamma_a \subset \mathbb{R}^2$ with the $0$-cycle $q = \infty$. In particular, if all $a_1, \ldots, a_n \in \mathbb{R}^2$ are vertices of a simply connected polygon, listed counterclockwise, then $\mu_{a, T}$ is the Lebesgue measure of this polygon.

Theorem follows easily from the Stokes’ theorem; see Section 2 for a detailed proof. The last claim in assertion (ii) explains the term “polygonal measure”.

Corollary 1.2 (of assertion (i)). The harmonic moment $m_j(\mu_{a, T})$ of the polygonal measure does not depend on the triangulation $T$.

Take again $a = (a_1, \ldots, a_n)$, $a_j = (x_j, y_j) \in \mathbb{C}^2$; $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$. Obviously, $z_j$ and $\bar{z}_j$ determine $a_j$ since $x_j = (z_j + \bar{z}_j)/2$ and $y_j = (z_j - \bar{z}_j)/2i$. Set

$$\nu_k(z, \bar{z}) \equiv \left(\begin{array}{c} k \\ 2 \end{array}\right) m_{k-2}(\mu_{a, T}) \quad \text{and} \quad \bar{\nu}_k(z, \bar{z}) \equiv \left(\begin{array}{c} k \\ 2 \end{array}\right) \bar{m}_{k-2}(\mu_{a, T}).$$

In particular, $\nu_0 = \nu_1 = 0 = \bar{\nu}_0 = \bar{\nu}_1$ for all $(z, \bar{z})$. By Corollary 1.2, both sides of the equalities are independent of $T$. 

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Remark 1.3. The index shift $(k-2) \mapsto k$, used above, is convenient since the normalized moments $\nu_k$ and $\tilde{\nu}_k$ are homogeneous of degree $k$ with respect to the dilatations on $\mathbb{C}^2$. In other words, $\nu_k(tz, \tilde{z}) = t^k \nu_k(z, \tilde{z})$ where $t \overset{\text{def}}{=} (t_1, \ldots, t_n)$.

The following theorem provides explicit formulas for $\nu_k(z, \tilde{z})$ and $\tilde{\nu}_k(z, \tilde{z})$.

Theorem 1.4. For any positive integer $k \geq 2$, one has

$$\nu_k(z, \tilde{z}) = \frac{i}{4} \sum_{j=1}^{n} \left[ \frac{x_j^{k} - x_{j+1}^{k}}{x_j - x_{j+1}} \right] = \frac{i}{4} \sum_{j=1}^{n} \left[ \frac{x_j^{k-1} + x_{j+1}^{k-1} + \cdots + x_{j+1}^{k-1}}{x_j - x_{j+1}} \right].$$

The anti-harmonic moment $\tilde{\nu}_k(z, \tilde{z})$ is given by the same formula (2) with $z$ and $\tilde{z}$ interchanged.

Observe that, up to the factor $i/4$, each normalized harmonic and anti-harmonic moment of a polygonal measure is a polynomial with integer coefficients in the variables $z_1, \ldots, z_n$ and $\tilde{z}_1, \ldots, \tilde{z}_n$. Notice additionally that $\tilde{\nu}_k(z, \tilde{z}) = -\nu_k(\tilde{z}, z)$ for all $k$ and $\tilde{\nu}_2(z, \tilde{z}) = -\nu_2(z, \tilde{z})$.

1.3 Main results about the relations between the moments

In this paper, the problem of describing the algebraic relations among the moments of polygons will be understood in two different ways. We call them the algebraic and the geometric approaches, respectively. The algebraic approach amounts to finding the algebraic relations between the polynomials $\nu_j(z, \tilde{z})$, $j = 2, 3, \ldots$, while the geometric approach deals with finding the algebraic relations that include both $\nu_j(z, \tilde{z})$ and $\tilde{\nu}_j(z, \tilde{z})$, $j = 2, 3, \ldots$.

In case of the algebraic approach, our main result is relatively simple. Namely, all harmonic moments can be expressed as rational functions of the first $2n - 2$ moments $\nu_2(z, \tilde{z}), \ldots, \nu_{2n-2}(z, \tilde{z})$, and these $2n - 2$ moments are algebraically independent. More precisely, denote by $\mathbb{F}_n$ the field extension of $\mathbb{C}$, generated by the sequence of polynomials $\{\nu_j(z, \tilde{z})\}_{j=2}^{2n-2}$.

Theorem 1.5. (i) $\mathbb{F}_n = \mathbb{C}[\nu_2, \ldots, \nu_{2n-2}]$ and is isomorphic to the field of rational functions in $2n - 2$ independent complex variables. (ii) $\mathbb{F}_n \subset \mathbb{C}(z)^{\times}$, where $\mathbb{C}(z)^{\times}$ is the field of symmetric rational functions of $z_1, \ldots, z_n$.

Explicit formulas expressing harmonic moments $\nu_j(z, \tilde{z})$ with $j \geq 2$ via the first $2n - 2$ ones are given by rational functions. However, all their denominators are powers of one fixed polynomial $\mathcal{D}_n$, the determinant of the matrix (8) below. In fact, if one considers the ring extension $\mathcal{R}_n$ of $\mathbb{C}$ (as opposed to field), generated by the sequence of polynomials $\{\nu_j(z, \tilde{z})\}_{j=2}^{2n-2}$, the situation may be described in the following way.

Theorem 1.6. (i) The ring $\mathcal{R}_n = \mathbb{C}[\nu_2, \nu_3, \ldots]$ is not generated by any finite collection of harmonic moments $\nu_2, \ldots, \nu_N$. (ii) For the polynomial $\mathcal{G}_n \in \mathbb{C}[z, \tilde{z}]$, given by the determinant of (8), the localization $\mathcal{R}_n[\mathcal{G}_n]$ is isomorphic to $\mathbb{C}[\nu_2, \ldots, \nu_{2n-1}]/\left[ \mathcal{G}_n \right]$.

Notice that $\mathcal{R}_n$ does not contain the ring $\mathbb{C}(z)^{\times}$ of symmetric polynomials in the variables $z_1, \ldots, z_n$ as a subring, since the expression of the basic (e.g. elementary) symmetric polynomials via $\nu_2, \ldots, \nu_{2n-1}$ involves division by some powers of $\mathcal{G}_n$.

Further, in the geometric approach, we consider the field extension $\mathbb{F}_n$ of $\mathbb{C}$, generated by both sequences $\{\nu_j\}_{j=2}^{2n}$ and $\{\tilde{\nu}_j\}_{j=2}^{2n}$. (Recall that $\nu_2 = -\tilde{\nu}_2$.) Here the situation is more complicated.

Theorem 1.7. (i) The field $\mathbb{F}_n$ is generated by the first $4n - 5$ harmonic and anti-harmonic moments $\nu_2, \nu_3, \nu_4, \ldots, \nu_{2n-1}, \tilde{\nu}_2, \ldots, \tilde{\nu}_{2n-1}$. (ii) The field $\mathbb{F}_n$ contains a subfield $H = \mathbb{C}(z, \tilde{z})^{\times}$ of rational functions, symmetric with respect to two groups of variables $z_1, \ldots, z_n$ and $\tilde{z}_1, \ldots, \tilde{z}_n$ separately. (iii) $\mathbb{F}_n$ is an algebraic extension of $H$, generated by the single element $\nu_2$. The degree $d_n$ of this extension equals $n(n-1)!$ if $n$ is odd and equals $2((n-1)!)^2$ if $n$ is even.

Remark 1.8. Notice that any algebraic extension of a field of characteristics zero is generated by a single element. So the essence of assertion (iii) of Theorem 1.7 is that a specific element $\nu_2$ is a generator. We describe the Galois closure of this extension and its Galois group later in Section 4.
Algebraic relations between the usual moments of polygons and polytopes in several special situations were discussed in a recent (joint with K. Kohm and B. Sturmfels) paper [12] of the third author. For algebraic domains, the relations between the moments were studied in e.g. [13, Section 3].

The structure of the paper is as follows. Section 2 contains detailed proofs of Theorems 1.1, 1.4, 1.5, and 1.6, as well as some formulas related to the logarithmic potential. In Section 3, we describe the action of the group $S_n \times S_n$ on the field $\mathbb{F}_n$ and prove Theorem 1.7. In Section 4, we describe the Galois group of the (Galois closure of the) extension $\mathbb{C} \subset \mathbb{F}_n$. Section 5 contains explicit description of this extension and its Galois group in the simplest case $n = 3$ (i.e. for triangles). We finish the paper with some questions expressing our outlook at the further development of the subject, see Section 6.

2 Harmonic Moments

2.1 Formulas for the moments.

In this section we are going to prove Theorems 1.1 and 1.4. Start with the following lemma:

**Lemma 2.1.** Assume that the triangulation $\mathcal{T}$ contains the diagonal $(1, m)$ which divides $\mathcal{T}$ into two parts, $\mathcal{T}'$ and $\mathcal{T}''$. If $a = (a_1, \ldots, a_n)$, $a' = (a_1, \ldots, a_m)$ and $a'' = (a_1, a_m, a_{m+1}, \ldots, a_n)$, then $\Delta_{a, \mathcal{T}} = \Delta_{a', \mathcal{T}'} \cup \Delta_{a'', \mathcal{T}''}$, and $f_{a, \mathcal{T}} = f_{a', \mathcal{T}'} + f_{a'', \mathcal{T}''}$. 

The proof is straightforward.

**Corollary 2.2.** If $\mathcal{T}$ consists of the edges $(1, 3), (1, 4), \ldots, (1, n - 1)$, then

$$f_{a, \mathcal{T}} = \mu_{a_1, a_2, a_3} + \mu_{a_1, a_3, a_4} + \cdots + \mu_{a_1, a_{n-1}, a_n}.$$ 

(The right-hand side of the latter formula contains measures supported on triangles, so there is no need to specify a triangulation.) Notice that if the vertices are real, then, up to a sign, $\mu_{a_1, a_2, a_{k+1}}$, is the Lebesgue measure of a triangle with the vertices $a_1, a_k$ and $a_{k+1}$. If this triangle is degenerate then $\mu_{a_1, a_2, a_{k+1}}$ is zero, otherwise the sign is taken to be + if the triangle is oriented counterclockwise, and − if it is oriented clockwise.

**Proof of Theorem 1.1.** To settle assertion (i), notice first that the 2-form $h(z, \bar{z}) dz \wedge d\bar{z}$ is closed and therefore exact since $h(z, \bar{z})$ is holomorphic in both variables. So $h(z, \bar{z}) dz \wedge d\bar{z} = 2i d\omega$ for some 1-form $\omega$. If $n = 3$, then there exists only one triangulation; the equality $-\frac{1}{2} \int_{\Delta} h dz \wedge d\bar{z} = \int_{\Gamma_n} \omega$ follows from the Stokes’ theorem.

Next apply the formulas for $n = 3$ to each triangle of the triangulation $\mathcal{T}$ and add them. Observe that the integrals over the internal diagonals of the triangulation contribute two equal terms with the opposite signs and therefore cancel, while the integrals over the “sides of the polygon” (i.e., segments of $\Gamma_n$) appear once and survive in the total sum. On the other hand, all the maps $F_n, \mathcal{T}$ are the same on the sides of the polygon $P$ and map them to the same line $\Gamma_n$; thus, the integrals over $\Gamma_n$ are the same for all possible triangulations. This proves assertion (i) for any integer $n \geq 3$.

Assertion (ii) is evident for $n = 3$. Indeed, for $q$, lying outside the triangle $\Gamma_n$, the linking number of $q - \infty$ with $\Gamma_n$ is 0, and for $q$ lying inside the triangle, it equals ±1; the choice of the sign depends on the orientation of the triangle. Notice now that the map sending $\Gamma_n$ to the polygonal measure is additive, i.e., if $a = (a_1, \ldots, a_n)$, $a' = (a_1, \ldots, a_m)$, $a'' = (a_1, a_m, a_{m+1}, \ldots, a_n)$, then $\Gamma_n = \Gamma_n' + \Gamma_n''$ as 1-cycles. Therefore the linking number of $q - \infty$ with $\Gamma_n$ is equal to the sum of its linking numbers with $\Gamma_n'$ and $\Gamma_n''$. This observation, together with Lemma 2.1, allows us to finish the proof by induction on $n$.

Now we are ready to derive an explicit formula for the moments of polygonal measures.

**Proof of Theorem 1.4.** For a harmonic moment $\nu_k$, we are in the situation of Theorem 1.1 with $h(z, \bar{z}) = \frac{k(k-1)}{2} z^{k-2}$; so one can take $\omega = \frac{-i}{4} z^{k-1} dz$.

If we parameterize a segment $[p, q] \subset \mathbb{C}$ as $z = p(1 - t) + qt$, $t \in [0, 1]$, then

$$\frac{ik}{4} \int_{[p, q]} z^{k-1} d\bar{z} = \frac{ik}{4} \int_0^1 ((p(1 - t) + qt)^k - (\bar{q} - \bar{p}) dt = \frac{i}{4} \int_0^1 \frac{q - \bar{p}}{q - p} dt = \frac{i}{4} \frac{(q - \bar{p})^k - p^k}{q - p}.$$ 

Equation (2) follows now from Theorem 1.1. □
2.2 Generating functions and relations between harmonic moments.

Define now, following [18], the normalized generating function \( \Psi_\mu(w) \) for harmonic moments of a measure \( \mu \) as

\[
\Psi_\mu(w) \overset{\text{def}}{=} \sum_{j=2}^{\infty} \nu_j(\mu) w^{-j} = \sum_{j=0}^{\infty} \left( \frac{j+2}{2} \right) m_j(\mu) w^j.
\]

(3)

\( \Psi_\mu(w) \) is closely related to the Cauchy transform \( \mathcal{C}_\mu(z) \) at \( \infty \). Namely,

\[
\Psi_\mu(w) = \frac{d^2}{dw^2} \left( \sum_{j=0}^{\infty} m_j(\mu) w^{j+2} \right).
\]

At the same time, (1) for a compactly supported measure \( \mu \) and sufficiently large \( |z| \) implies \( w \mathcal{C}_\mu(z) = \sum_{j=0}^{\infty} m_j(\mu)/z^j \). Hence, if \( |w| \) is sufficiently small, then

\[
\Psi_\mu(w) = \frac{d^2}{dw^2} \left( w \mathcal{C}_\mu \left( \frac{1}{w} \right) \right).
\]

Similar multivariate generating functions were recently considered in [9].

For \( n = 3 \), sometimes we will use the notation

\[
D_{i,j,k} \overset{\text{def}}{=} \nu_2(z_i, z_j, z_k, \bar{z}_i, \bar{z}_j, \bar{z}_k),
\]

where the subscripts \( i, j, k \) may vary appropriately. Note that, for any \( n \), if the vertices are real (that is, \( z \) and \( \bar{z} \) are complex conjugate) and the points \( z_1, \ldots, z_n \) are vertices of a simply connected polygon, then the moment \( \nu_2(z, \bar{z}) \) is equal to the (signed) area of the polygon; so \( D_{i,j,k} \) is the area of a triangle with the vertices \( z_i, z_j, z_k \).

The following observation, which can be found in [18], is important in our considerations:

**Proposition 2.3.**

\[
\Psi_{\mu_z}(w) = \frac{D_{123}}{(1 - z_1 w)(1 - z_2 w)(1 - z_3 w)}.
\]

(4)

**Proof.** Substitution of (2) into (3) gives

\[
\Psi_{\mu_z}(w) = \frac{i}{4} \sum_{k=2}^{\infty} (z_1^k - z_2^k) w^{-k} + \text{cyclic} = \frac{i}{4} \left( \frac{z_1 - z_2}{z_1 - 2} \left( \frac{z_1^2}{z_1 - 2} - \frac{z_2^2}{1 - z_2 w} \right) + \text{cyclic} \right)
\]

\[
= \frac{i}{2} \frac{z_1 z_2 (z_1 - z_2) + z_2 z_3 (z_2 - z_3) + z_3 z_1 (z_3 - z_1)}{(1 - z_1 w)(1 - z_2 w)(1 - z_3 w)} = \frac{D_{123}}{(1 - z_1 w)(1 - z_2 w)(1 - z_3 w)}.
\]

(5)

(6)

(7)

(where “+cyclic” means the sum of two extra summands, cyclically shifting \( z_1 \mapsto z_2 \mapsto z_3 \mapsto z_1 \)).

Proposition 2.3 and Corollary 2.2 imply the following.

**Corollary 2.4.** For any \( z = (z_1, \ldots, z_n) \), one has

\[
\Psi_{\mu_z}(w) = \frac{D_{1,\ldots,n}}{\prod_{j=2}^{n-1} (1 - z_j w)(1 - z_{j+1} w)}.
\]

(8)

**Corollary 2.5 (of Corollary 2.4).** For every \( z = (z_1, \ldots, z_n) \), there exists a unique polynomial \( \mathcal{A}_D(w) \) of degree at most \( n - 3 \) such that

\[
\Psi_{\mu_z}(w) = \frac{\mathcal{A}_D(w)}{\prod_{j=1}^{n-1} (1 - z_j w)}.
\]

(9)
Remark 2.6. Properties of the polynomial $\mathcal{AP}_q(w)$ were studied in detail in [20] (see also [21]). For a generic $z$, define, following [20], an adjoint polynomial $A_q(u, v)$ of $z$ as the minimal degree polynomial vanishing at all points $\ell \in (\mathbb{C}^*)^n$ that correspond to lines joining $z_k$ and $z_l$ with $|k - \ell| \geq 2$. The adjoint polynomial is unique up to a multiplicative constant. It also has a multi-dimensional generalization studied in [11].

In particular, the following was proved:

Proposition ([20]).

(i) The coefficient at $w^{-3}$ in $\mathcal{AP}_q$ equals $\nu_2(z, \bar{z})$.

(ii) One has $\mathcal{AP}_q(w) = A_q(w, iw) \times \text{const.}$ where $A_q$ is the adjoint polynomial, and the constant is determined by assertion (i). \hfill \square

Next, we settle Theorem 1.5.

Proof of Theorem 1.5. First, prove assertion (ii). The field $\mathbb{C}(z)_S$ of invariant rational functions is generated by the elementary symmetric polynomials $e_1(z), \ldots, e_n(z)$. Thus it is enough to show that they belong to $\mathfrak{S}_n$.

It is a well-known fact, see, e.g., [19, Th. 4.1.1], that if

$$\sum_{j=0}^{\infty} f(j) t^j = \frac{P(t)}{Q(t)},$$

where $\deg P < \deg Q$ and $Q(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_d t^d$ with $\alpha_d \neq 0$, then for all $k \geq 0$, one has the recurrence relation:

$$f(k + d) + \alpha_1 f(k + d - 1) + \alpha_2 f(k + d - 2) + \cdots + \alpha_d f(k) = 0.$$ 

It follows from (5) that

$$\mathcal{AP}_z(w) = \prod_{j=1}^{n} (1 - z_j w) \cdot \sum_{j=0}^{\infty} \nu_{j+2}(z) w^j = (1 - e_1(z) w + \cdots + (-1)^n e_n(z) w^n) \cdot \sum_{j=0}^{\infty} \nu_{j+2}(z) w^j.$$

Since $\deg \mathcal{AP}_z \leq n - 3$, for every $k \geq -2$, one has

$$\nu_{k+2+n}(z) - e_1(z) \nu_{k+1+n}(z) + e_2(z) \nu_{k+n}(z) - \cdots + (-1)^n e_n(z) \nu_{k+2}(z) = 0. \quad (6)$$

Consider the first $n$ equations of the recurrence (6). This linear system has the form:

$$U \cdot E = V, \quad (7)$$

where $U$ is the Toeplitz $n \times n$-matrix, given by

$$U = \begin{pmatrix} \nu_{n-1}(z, \bar{z}) & \nu_{n-2}(z, \bar{z}) & \cdots & \nu_1(z, \bar{z}) & \nu_0(z, \bar{z}) \\ \nu_{n-1}(z, \bar{z}) & \nu_{n-2}(z, \bar{z}) & \cdots & \nu_2(z, \bar{z}) & \nu_1(z, \bar{z}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{2n-2}(z, \bar{z}) & \nu_{2n-3}(z, \bar{z}) & \cdots & \nu_n(z, \bar{z}) & \nu_{n-1}(z, \bar{z}) \end{pmatrix}, \quad (8)$$

and $E$ and $V$ are column vectors of length $n$, given by

$$E = (e_1(z), -e_2(z), \ldots, (-1)^{n+1} e_n(z))^T,$$

and

$$V = (\nu_{n}(z, \bar{z}), \nu_{n+1}(z, \bar{z}), \ldots, \nu_{2n-1}(z, \bar{z}))^T.$$ 

(Recall that $\nu_0 = \nu_1 = 0$). Assuming that $U$ is invertible, we get $E = U^{-1} V$. This means that every $e_j(z)$ is expressed as a rational function of the normalized moments $\nu_2(z, \bar{z}), \ldots, \nu_{2n-1}(z, \bar{z})$ with the fixed denominator, equal to the determinant of $U$. Thus assertion (ii) of Theorem 1.5 is proved.

To settle assertion (i), we argue as follows. Using the recurrence relation (6), one can express every $\nu_{k+2+n}(z, \bar{z})$, where $k \geq n - 2$, as a rational function of the first $2n - 2$ normalized harmonic moments $\nu_2, \ldots, \nu_{2n-1}, \nu_2(z, \bar{z}), \ldots, \nu_{2n-1}(z, \bar{z})$. 


Now for $s = 1, 2, \ldots$, denote by $h^s_k(x)$ the $s$-th complete symmetric polynomial of $z_1, \ldots, z_n$. The identity
\[
1 = \prod_{r=1}^n (1 - wz_r) \times \prod_{r=1}^n \frac{1}{(1 - wz_r)} = \sum_{m=0}^{\infty} (-1)^m e_m(x) w^m \sum_{k=0}^{\infty} h_k(x) w^k,
\]
implies the standard relation $\sum_{m=0}^{\infty} (-1)^m e_m(x) h_{j-m}(x) = 0$ for all $j = 1, 2, \ldots$. Corollary 2.4 implies that
\[
\sum_{j=0}^{\infty} \nu_{j+2}(x, \bar{x}) w^j = \left( \sum_{i=0}^{\infty} h_i(x) w^i \right) \sum_{\ell=1}^{n-1} D_{\ell, \ell+1}(x, \bar{x}) (1 - wz_2) \cdots (1 - wz_{\ell+1}) (1 - wz_{n-1}) \cdots (1 - wz_n)
\]
\[
= \left( \sum_{i=0}^{\infty} h_i(x) w^i \right) \sum_{\ell=1}^{n-1} D_{\ell, \ell+1}(x, \bar{x}) \times \sum_{m=0}^{n-3} (-1)^m w^m e_m(z_2, \ldots, z_\ell, z_{\ell+1}, \ldots, z_n)
\]
\[
= \left( \sum_{i=0}^{\infty} h_i(x) w^i \right) \sum_{m=0}^{n-3} w^m Q_m(x, \bar{x}),
\]
where
\[
Q_m(x, \bar{x}) \overset{\text{def}}{=} \sum_{\ell=1}^{n-1} (-1)^\ell D_{\ell, \ell+1}(x, \bar{x}) e_m(z_2, \ldots, z_\ell, z_{\ell+1}, \ldots, z_n).
\]
This equation gives
\[
\nu_{j}(x, \bar{x}) = \sum_{m=0}^{n-3} (-1)^m Q_m(x, \bar{x}) h_{j-m-2}(x),
\]
where $j = 2, 3, \ldots$, and $h_k(x) \overset{\text{def}}{=} 0$ for $k < 0$.

Multiplying (10) by $\prod_{k=1}^{n} (1 - wz_k) = \sum_{m=0}^{n} (-1)^m e_m(x) w^m$, we get
\[
\sum_{\ell=0}^{\infty} (-1)^\ell e_{\ell}(x) w^\ell \sum_{j=2}^{\infty} \nu_{j}(x, \bar{x}) w^{j-1} = \sum_{m=0}^{n-3} (-1)^m w^m Q_m(x, \bar{x}).
\]
Relation (10) leads to
\[
\mathfrak{f}_n = \mathcal{C}(e_1, \ldots, e_n, Q_0, \ldots, Q_{n-3}).
\]
We are going to show that $Q_0, \ldots, Q_{n-3}$ are algebraically independent over $\mathcal{C}(e_1, \ldots, e_n)$. Indeed, the functions $z_1, \ldots, z_n$ are the roots of
\[
t^m - e_1(x) t^{m-1} + \cdots + (-1)^n e_n(x) = 0
\]
and, therefore, belong to an algebraic extension $AE_n$ of $\mathcal{C}(e_1, \ldots, e_n)$. The same field $AE_n$ contains the functions $e_m(z_2, \ldots, z_k, z_{k+1}, \ldots, z_n)$, mentioned in (11), which implies that $D_{123, \ldots, D_{n-1}} \in AE_n$ as well. Thus, the field $\mathcal{C}(e_1, \ldots, e_n, D_{123}, \ldots, D_{1, (n-1), n})$ is an algebraic extension of $\mathfrak{f}_n$.

The polynomial $D_{123}$ depends on the variables $z_2$ and $z_3$, while $\{D_{1, k, k+1}\}$ with $k = 3, \ldots, (n - 1)$, do not. Take any $u \in \mathcal{C}$ and substitute $z_3 \mapsto z_3 + u$, leaving all other $z_j$ and all $z_j$ unchanged. This operation preserves the values of $e_1(x), \ldots, e_n(x)$, as well as the values of $D_{134}, \ldots, D_{1, (n-1), n}$. On the other hand, $D_{123}$ takes infinitely many values, as $u$ varies, and therefore it is not a root of any algebraic equation with the coefficients that depend only on $e_1(x), \ldots, e_n(x)$ and $D_{134}, \ldots, D_{1, (n-1), n}$. Now take $z_3, 3$ and consider $D_{134}$ to conclude that it is algebraically independent of $D_{145}, \ldots, D_{1, (n-1), n}$, etc.

In this way, we prove that $D_{123}, \ldots, D_{1, (n-1), n}$ are algebraically independent elements over the field $\mathcal{C}(e_1, \ldots, e_n)$. Hence, in the tower of extensions
\[
\mathcal{C}(e_1, \ldots, e_n) \subset \mathfrak{f}_n = \mathcal{C}(e_1, \ldots, e_n, Q_0, \ldots, Q_{n-3}) \subset \mathcal{C}(e_1, \ldots, e_n, D_{123}, \ldots, D_{1, (n-1), n}),
\]
the transcendence degree of the last field over the first one equals $n - 2$. The second extension is algebraic, and therefore, the first extension has the transcendence degree $n - 2$ as well. Consequently, $Q_0, \ldots, Q_{n-3}$ are algebraically independent over $\mathcal{C}(e_1, \ldots, e_n)$. Equation (12) implies that $\nu_2, \ldots, \nu_n$ are algebraically independent as well. Since $e_1(x), \ldots, e_n(x)$ are also algebraically independent, the field $\mathfrak{f}_n = \mathcal{C}(e_1, \ldots, e_n, \nu_2, \ldots, \nu_n)$ is isomorphic to the field of rational fractions in $2n - 2$ independent variables.

On the other hand, $\mathcal{C}(e_1, \ldots, e_n) \subset \mathcal{C}(\nu_2, \ldots, \nu_{2n-1}) \subset \mathfrak{f}_n$, which gives
\[
\mathfrak{f}_n = \mathcal{C}(\nu_2, \ldots, \nu_{2n-1}).
\]
This completes the proof of Theorem 1.5. \[\blacksquare\]
Remark 2.7. Equation (7) coincides with [7, equation (2.8)], if one reads the coefficient vector $E$ backwards and reflects $U$ with respect to its vertical midline.

Proof of Theorem 1.6. By (2) each polynomial $\nu_j(z, \bar{z})$ is a linear function in the variables $\bar{z} = (z_1, \ldots, z_n)$, and thus cannot be a polynomial function of the other $\nu_k$'s of degree exceeding one. On the other hand, $\nu_j$ is homogeneous of degree $j - 1$ with respect to $z = (z_1, \ldots, z_n)$, so such linear dependence is impossible as well. This proves assertion (i) of the theorem.

Further, observe that, for $j > 2n - 1$, the recurrence relation (6) allows us to express each $\nu_j$ as a rational function of $\nu_1, \ldots, \nu_{2n-1}$; its denominator is a power of $D_n \equiv \det U$, where $U$ is defined by the equation (8). This fact is equivalent to assertion (ii).

3 Action of the Group $S_n \times S_n$ and Anti-harmonic Moments

3.1 Action of the group $S_n \times S_n$.

The group $S_n \times S_n$, where $S_n$ is the symmetric group on $n$ elements, acts on the field of rational functions $\mathbb{C}(z, \bar{z})$ by permuting the variables: the first copy of $S_n$ acts on $z_1, \ldots, z_n$ while the second copy acts on $\bar{z}_1, \ldots, \bar{z}_n$. We denote the action of a pair $(\sigma, \tau)$, where $\sigma, \tau \in S_n$, on a rational function $R$, by the subscript:

$$R(z, \bar{z})_{(\sigma, \tau)} \equiv R(z_{\sigma(1)}, \ldots, z_{\sigma(n)}, \bar{z}_{\tau(1)}, \ldots, \bar{z}_{\tau(n)}).$$

In particular, $\nu_j(\sigma, \tau)$ is the result of the permutation of variables in the $j$-th normalized moment $\nu_j(z, \bar{z})$. Note that $\nu_j(\sigma, \tau)(z, \bar{z})$ is also the $j$-th normalized moment of the $a = (a_1, \ldots, a_n)$, where $a_j = (x_j, y_j)$ with $z_{\sigma(j)} = x_j + iy_j, z_{\tau(j)} = x_j - iy_j$ (in other words, $a_j = ((z_{\sigma(j)} + \bar{z}_{\tau(j)})/2 (z_{\sigma(j)} - \bar{z}_{\tau(j)})/(2i))$. If $\sigma = \tau$, then $a_1, \ldots, a_n \in \mathbb{C}^2$ are actually the same points as the vertices of the polygonal line for $\nu_j(z, \bar{z})$, but ordered differently. In particular, if all the vertices for $\nu_j(z, \bar{z})$ are real, then the vertices for $\nu_j(\sigma, \tau)(z, \bar{z})$ are real, too.

Let us compute now the stabilizer of the above $S_n \times S_n$-action. To do this, we will especially need to describe the $S_n \times S_n$-orbit of the lowest degree moment $\nu_2(z, \bar{z})$.

Observe that every $\nu_2(\sigma, \tau)(z, \bar{z})$ is a bilinear form in the variables $(z, \bar{z})$. We denote by $M(\sigma, \tau)$ its matrix in the standard basis of $\mathbb{C}^n$.

Theorem 1.4 provides that

$$M_{(id, id)} = \left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & \ldots & 0 & -1 \\ -1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & -1 & 0 \end{array}\right).$$

(15)

For an arbitrary pair $(\sigma, \tau)$, the matrix $M(\sigma, \tau)$ is obtained from (15) by permuting the rows of $M_{(id, id)}$ according to the permutation $\sigma$ and permuting the columns, according to $\tau$. (These two permutation actions commute.)

Expression for $M(\sigma, \tau)$ can be written in other terms, too. Namely, let $\mathbb{P} : [S_n] \to \text{Mat}(n, \mathbb{C})$ be the standard permutation representation of the symmetric group, i.e., for any $\sigma \in S_n$, the $(i, j)$-th entry of the $n \times n$-matrix $\mathbb{P}[\sigma]$ is 1 if $j = \sigma(i)$ and 0 otherwise. Then $M_{(id, id)} = \mathbb{P}[C - C^{-1}]$, where $C = (1, 2, \ldots, n)$ is the long cycle, sending $j$ to $j + 1$ for $j = 1, \ldots, n - 1$ and sending $n$ to 1. Therefore $M(\sigma, \tau) = \mathbb{P}[\sigma] \ast \mathbb{P}[C - C^{-1}][\tau]$. The group representation $\mathbb{P}$ is orthogonal, that is, $\mathbb{P}[\sigma] = \mathbb{P}[\sigma^{-1}](C - C^{-1})[\tau]$. For any $\sigma \in S_n$, so eventually

$$M(\sigma, \tau) = \mathbb{P}[\sigma^{-1}(C - C^{-1})[\tau].$$

(16)

Denote by $M(n)$ the set of all $n \times n$-matrices such that every row and every column of a matrix contain one entry equal to 1, another entry equal to -1, and all the remaining entries vanish. The group $S_n \times S_n$ acts on $M(n)$ by permuting of the rows and the columns. The set of all matrices $M(\sigma, \tau) \in M(n)$ for $(\sigma, \tau) \in S_n \times S_n$ is an orbit of this action, which we denote by $O_n$.

The stabilizer of $\nu_2$ under the $S_n \times S_n$-action is the stabilier of $O_n$, which we denote by $\mathcal{G}$. We want to describe $\mathcal{G}$ explicitly. Recall that $C \equiv (1, 2, \ldots, n)$ denotes the long cycle. If $n$ is even, then set $C_1 \equiv (1, 3, \ldots, 2\ell - 1), C_2 \equiv (2, 4, \ldots, 2\ell), \delta_1 \equiv (1, 2)(3, 4) \ldots (2\ell - 1, 2\ell)$, and $\delta_2 \equiv (2, 3)(4, 5) \ldots (2\ell, 1)$.

The next lemma is straightforward:

Lemma 3.1. For $n$ even, the following relations hold:
(i) \( C^2 = C_1 \cdot C_2 = C_2 \cdot C_1 \).
(ii) \( C \cdot C_1 = C_2 \cdot C \) and \( C \cdot C_2 = C_1 \cdot C \).
(iii) \( \delta_1 \cdot C_1 = C_2 \cdot \delta_1 \), \( \delta_1 \cdot C_2 = C_1 \cdot \delta_1 \), \( \delta_2 \cdot C_1 = C_2 \cdot \delta_2 \) and \( \delta_2 \cdot C_2 = C_1 \cdot \delta_2 \).
(iv) \( \delta_2 \cdot C = C \cdot \delta_1 \) and \( \delta_1 \cdot C = C \cdot \delta_2 \).

\[ \Box \]

**Proposition 3.2.**  (i) For \( n \) odd, the stabilizer \( \mathcal{G} \overset{\text{def}}{=} \text{St}(O_n) \subset S_n \times S_n \) coincides with the cyclic group \( \mathbb{Z}_n \), generated by \( (C, C) \in S_n \times S_n \).
(ii) For \( n = 2\ell \) even, the stabilizer \( \mathcal{G} \subset S_n \times S_n \) consists of all the elements
\[
(C_1^{u} \cdot C_2^{v}, C_1^{u} \cdot C_2^{v}) \in S_n \times S_n
\]
and of all elements \( (\delta_1 \cdot C_1^{u} \cdot C_2^{v}, \delta_2 \cdot C_1^{u} \cdot C_2^{v}) \in S_n \times S_n \), where \( u, v = 0, \ldots, \ell - 1 \). As an abstract group, \( \mathcal{G} \) is non-commutative, but it contains an index 2 subgroup that is isomorphic to \( \mathbb{Z}_\ell \times \mathbb{Z}_\ell \).

\[ \Box \]

**Proof.** An immediate check, using (16), shows that \( M(\sigma, \tau) = M(\sigma', \sigma', \tau') \) for all \( n \). (It also follows from the obvious fact that the polygonal line \( \Gamma_n \) does not change under the cyclic shift of the points \( a_1 \mapsto a_2 \mapsto \ldots \mapsto a_n \mapsto a_1 \).) For even \( n = 2\ell \), the same formula and Lemma 3.1 additionally imply that \( M(\sigma, \tau) = M(\sigma, \sigma, \tau) \).

Assume now that
\[
M(\sigma, \tau) = M(\sigma', \tau').
\]

Observe that relabelling the variables one can, without loss of generality, choose \( \sigma = \tau = \mathrm{id} \), where \( \mathrm{id} \) is the identity permutation. In the representation-theoretical notation, formula (17) is equivalent to \( \mathcal{P}[\sigma^{-1}(C - C^{-1})\tau] = \mathcal{P}[\sigma(C - C^{-1})] \), that is, \( \mathcal{P}[\sigma(C - C^{-1})] = \mathcal{P}[\sigma(C - C^{-1})] \).

If \( u_1, \ldots, u_n \in C^n \) is the standard basis, then the latter equation means that
\[
\mathcal{P}[\sigma(C - C^{-1})](u_i) = u_{\tau(i)+1} - u_{\tau(i)-1} = \mathcal{P}[\sigma(C - C^{-1})](u_i) = u_{\sigma(i)+1} - u_{\sigma(i)-1}
\]
for all \( i = 1, \ldots, n \). In other words, \( \tau(i) + 1 = \sigma(i) + 1 \) for all \( i \), that is, \( \tau \cdot C = C \cdot \sigma \) and \( C \cdot \tau = \sigma \cdot C \). These relations imply
\[
C^2 \cdot \tau = C \cdot \sigma \cdot C = \tau \cdot C^2.
\]

For \( n = 2\ell + 1 \), one has
\[
(C^2)^{\ell+1} \cdot \tau = \tau \cdot (C^2)^{\ell+1} \Leftrightarrow C \cdot \tau = \tau \cdot C.
\]

Since \( C \) only commutes with its own powers, one obtains \( \tau = C^k \) for some \( k = 0, \ldots, n - 1 \), implying that \( \sigma = C^k \cdot \tau \).

Consider now the case \( n = 2\ell \). Then \( C^2 = C_1 \cdot C_2 \) (a product of two independent cycles). Set \( E_1 = \{1, 3, \ldots, 2\ell - 1\} \) and \( E_2 = \{2, 4, \ldots, 2\ell\} \). Commutates with \( C_1 \cdot C_2 \), and the subgroups of \( S_n \), generated by \( C_1 \cdot C_2 \), act transitively on \( E_1 \) and \( E_2 \), respectively, one has that either \( \tau(E_1) = E_1 \) and \( \tau(E_2) = E_2 \), or \( \tau(E_1) = E_2 \) and \( \tau(E_2) = E_1 \).

In the first case, the restrictions of \( \tau \) to \( E_1 \) and \( E_2 \) commute with the cycles \( C_1 \) and \( C_2 \), respectively, and therefore \( \tau = C_1^{u} \cdot C_2^{v} \) and \( \sigma = C \cdot \tau = C^{-1} = C_1^{u} \cdot C_2^{v} \). In the second case, the same reasoning holds for the permutations \( \tau \overset{\text{def}}{=} \delta_2 \cdot \tau \). So \( \tau = \delta_2 \cdot C_1^{u} \cdot C_2^{v} \) and \( \sigma = C \cdot \tau = C^{-1} = \delta_1 \cdot C_1^{u} \cdot C_2^{v} \).

\[ \Box \]

### 3.2 Relations between harmonics and anti-harmonics moments

Let us now settle Theorem 1.7.

**Proof.** Assertions (i) and (ii) could be proved similarly to the corresponding statements in Theorem 1.5 about the field \( \delta_n \).

To prove assertion (iii), set
\[
P(t) \overset{\text{def}}{=} \prod_{(\sigma, \tau) \in (S_n \times S_n) / \mathcal{G}} (t - \nu_{2(\sigma, \tau)}).
\]

Here the index \( (\sigma, \tau) \) runs over a system of representatives of the right cosets of \( S_n \times S_n \) with respect to the stabilizer subgroup \( \mathcal{G} \). Thus \( \deg P \) is equal to the number of these right cosets, that is, to \( (n!)^2 / |\mathcal{G}| \). By Proposition 3.2, one has \( \deg P = n!(n - 1)! \) for \( n \) odd, and \( \deg P = 2((n - 1)!)^2 \) for \( n \) even.

**Lemma 3.3.** \( P(t) \) is the minimal polynomial, defining \( \nu_2 \) over the field \( \mathbb{C}(z, \bar{z})^{S_n \times S_n} \).

\[ \Box \]
Proof. It is enough to show that $P$ is irreducible.

Assume that $P(t)$ is reducible and $Q(t) = \prod_{[\sigma, \tau] \in \mathcal{U}} (t - w_{2\sigma, \tau})$ is its irreducible factor, where $U$ is some proper subset of $(S_n \times S_n)/\mathcal{G}$. Thus $\deg Q = \#U < \#(S_n \times S_n)/\mathcal{G}$. The coefficients of the polynomial $Q$ are $S_n \times S_n$-invariant, so for any $(\sigma, \tau) \in U$, the element $w_{2\sigma, \tau}$ must be a root of $Q$. By Proposition 3.2, this implies that $U$ intersects any right coset in $(S_n \times S_n)/\mathcal{G}$. Thus $\deg Q \geq \#(S_n \times S_n)/\mathcal{G}$, a contradiction. ■

Now let us show that, for generic $(z, \bar{z})$, all the roots of the polynomial $P$ defined by (19) are simple.

Lemma 3.4. For generic $(z, \bar{z})$, the values of all bilinear forms $w_{2\sigma, \tau}(z, \bar{z})$ are pairwise distinct, where $(\sigma, \tau)$ runs over all right cosets $(S_n \times S_n)/\mathcal{G}$ with respect to the stabilizer group $\mathcal{G}$. ■

Proof. Indeed, if it is not the case, then $\mathbb{C}^n \times \mathbb{C}^n$ is a union of finitely many sets $L_{\sigma, \tau, \sigma', \tau'} \overset{\text{def}}{=} \{(z, \bar{z}) \mid w_{2\sigma, \tau}(z, \bar{z}) = w_{2\sigma', \tau'}(z, \bar{z})\}$. The functions $w_{2\sigma, \tau}$ are bilinear forms, so $L_{\sigma, \tau, \sigma', \tau'}$ are quadrics. A vector space over $\mathbb{C}$ cannot be a union of finitely many nontrivial quadrics, so $L_{\sigma, \tau, \sigma', \tau'} = \mathbb{C}^n \times \mathbb{C}^n$ for some $(\sigma, \tau, \sigma', \tau')$. But then $(\sigma, \tau) \equiv (\sigma', \tau') \pmod{\mathcal{G}}$, which contradicts the choice of $(\sigma, \tau)$ and $(\sigma', \tau')$ (one element from every right coset). The lemma follows. ■

Fix some generic $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$. Then the set

$$\{(z, \bar{z}) \mid e_j(z) = c_j, e_j(\bar{z}) = d_j, j = 1, \ldots, n\}$$

is a generic $S_n \times S_n$-orbit in $\mathbb{C}^n \times \mathbb{C}^n$, where $e_j$ is the $j$-th elementary symmetric function. By Lemma 3.4, the values of $w_2$ at different points of the orbit are distinct. So the values $e_1(z), \ldots, e_n(z), e_1(\bar{z}), \ldots, e_n(\bar{z})$ and $w_2(z, \bar{z})$ determine the point $(z, \bar{z})$ completely, and therefore, determine the values $w_j(z, \bar{z})$ for all $j = 3, 4, \ldots$.

Fix some $j$, and let $Y \subset \mathbb{C}^{2n+1}$ be the closure of the set

$$\{(e_1(z), \ldots, e_n(z), e_1(\bar{z}), \ldots, e_n(\bar{z}), w_2(z, \bar{z})) \in \mathbb{C}^{2n+1} \mid (z, \bar{z}) \in \mathbb{C}^n\}.$$

For every positive integer $j$ we introduce the algebraic variety

$$\mathcal{E}_j \overset{\text{def}}{=} \{(y, c) \in Y \times \mathbb{C} \mid \exists (z, \bar{z}) \in \mathbb{C}^n : y = (e_1(z), \ldots, e_n(z), w_2(z, \bar{z})), c = w_j(z, \bar{z})\} \subset \mathbb{C}^{2n+2}.$$

By Lemma 3.4, the projection map $p : \mathcal{E}_j \to \mathbb{C}^{2n+1}$, given by $p(y, c) \overset{\text{def}}{=} y$, is generically one-to-one onto its image. Hence there exists a rational map $\tilde{R} : \mathbb{C}^{2n+1} \to \mathcal{E}_j$ such that $\tilde{R} \circ p = p^{-1}$, see [15]. If $R \overset{\text{def}}{=} q \circ \tilde{R}$, where $q : \mathbb{C}^{2n+1} \times \mathbb{C} \to \mathbb{C}$ is the standard projection, then $w_j(z, \bar{z}) = R(e_1(z), \ldots, e_n(z), e_1(\bar{z}), \ldots, e_n(\bar{z}), w_2(z, \bar{z}))$.

Thus we have shown that

$$w_j \in \mathbb{C}^{e_1(z), \ldots, e_n(z), e_1(\bar{z}), \ldots, e_n(\bar{z}), w_2(z, \bar{z})} = \mathbb{C}(z, \bar{z})^{S_n \times S_n}.$$

On the other hand, it follows from assertion (i) that

$$\tilde{\mathcal{G}}_n = \mathbb{C}^{e_1(z), \ldots, e_n(z), \bar{v}_2, \bar{v}_3, \ldots, \bar{v}_{2n-1}, \bar{v}_{2n-1}},$$

which implies $\tilde{\mathcal{G}}_n = \mathbb{C}^{e_1(z), \ldots, e_n(z), w_2} = \mathbb{C}(z, \bar{z})^{S_n \times S_n}(w_2)$. ■

Proposition 3.2 and assertion (iii) of Theorem 1.7 imply the following claim:

Corollary 3.5. $\tilde{\mathcal{G}}_n \subset \mathbb{C}(z, \bar{z})^G$, where $G \subset S_n \times S_n$ is the stabilizer group of $w_2$, described in Proposition 3.2. ■

For any nonnegative integer $k$, denote by $Y_k \subset \mathbb{C}^{2n+2k+1}$ the closure of the set

$$\{(e_1(z), \ldots, e_n(z), e_1(\bar{z}), \ldots, e_n(\bar{z}), w_2(z, \bar{z}), e_3(z, \bar{z}), \bar{e}_3(z, \bar{z}), \ldots, w_{2k+2}(z, \bar{z}), \bar{w}_{2k+2}(z, \bar{z}) \mid (z, \bar{z}) \in \mathbb{C}^n\} \subset \mathbb{C}^{2n+2k+1}.$$

Using this notation, assertion (iii) of Theorem 1.7 can be reformulated as follows.

Corollary 3.6. For every positive integer $k$, the variety $Y_k$ is birationally equivalent to $Y_0$. ■
Remark 3.7. Denote by $\hat{R}_n \overset{\text{def}}{=} \mathbb{C}[\nu_2, \nu_3, \nu_4, \ldots]$ the ring extension, generated by all harmonic and anti-harmonic moments. Although $\mathbb{C}(z, \bar{z})[S_n \times S_n] \subseteq \hat{R}_n$, it is not true that $\mathbb{C}[z, \bar{z}][S_n \times S_n] \subseteq \hat{R}_n$ because the elementary symmetric functions are only expressed as rational functions of the moments. On the other hand, the inclusion $\hat{R}_n \subseteq \mathbb{C}[z, \bar{z}]$ obviously holds.

Similarly to Theorem 1.7, the same circumstance (i.e. the presence of a denominator in formulas) does not allow us to conclude that $\nu_2, \nu_3, \ldots, \nu_{2n-1}, \nu_{2n-1}$ generate $\hat{R}_n$. Probably (though we have not yet proved this), the situation is similar to assertion (i) of Theorem 1.6: the ring cannot be generated by any proper subset of $\nu_j, \nu_{2j}, j = 2, 3, \ldots$. Also we can conjecture that an analog of assertion (ii) of the same theorem holds: all the denominators in question are powers of a single polynomial $\hat{Z}_n$.

Remark 3.8. Formulas (2) also show that $\nu_j \in \xi \nu_j$ for all $j$, where $\xi$ is an involution that reads the sequence $(12 \ldots n)$ in the opposite direction: $\xi = (1, n)(2, n-1) \ldots$. Together with a cyclic group $\mathbb{Z}_n$ with a generator $(C, C)$, the involution $\xi$ generates the dihedral group. $\blacksquare$

4 Relations between the Roots and the Galois Group of the Equation, Satisfied by $\nu_2$

Assertion (iii) of Theorem 1.7 claims that the minimal polynomial $P(t)$ for the element $\nu_2$ generates the algebraic extension of the field $\mathbb{C}[z, \bar{z}][S_n \times S_n]$ of degree

$$d_n \overset{\text{def}}{=} \begin{cases} n!(n-1)! & \text{if } n \text{ is odd,} \\ 2(n-1)!^2 & \text{if } n \text{ is even.} \end{cases}$$

This extension is not Galois; its Galois closure is the field, generated by all the roots of $P$, that is, by $\nu_{2(\sigma, \tau)}$ for all $\sigma, \tau \in S_n$. In this section, we calculate the Galois group of the closure, or, equivalently, the Galois group of the polynomial $P(t)$. To do this, we need to describe the algebraic dependencies between the polynomials $\nu_{2(\sigma, \tau)}$; by definition, the Galois group of $P(t)$ is the subgroup of $S_n \times S_n$ preserving all these dependencies.

4.1 Linear relations and dimension

Denote by $\mathfrak{M}_n$ the linear span of the set of all $n \times n$-matrices $M_{(\sigma, \tau)}$, where $(\sigma, \tau) \in S_n \times S_n$ (see equation (15) above and the text following it).

Lemma 4.1. For any $n \geq 3$, the space $\mathfrak{M}_n \subseteq \text{Mat}_n$ consists of all $n \times n$-matrices with vanishing row and column sums for each row and column. $\blacksquare$

Proof. Recall that $\mathfrak{P}$ is the standard $n$-dimensional permutation representation of the group $S_n$. By equation (16), the matrix $M_{(\sigma, \tau)}$ belongs to the image of $\mathfrak{P}$. The representation $\mathfrak{P}$ is reducible; it splits into the trivial 1-dimensional representation in the subspace $V_0 \subseteq \mathbb{C}^n$, spanned by the vector $v_0 = (1, 1, \ldots, 1)$, and the irreducible representation of dimension $(n-1)$ in the space $V \overset{\text{def}}{=} \{ (z_1, \ldots, z_n) \mid z_1 + \cdots + z_n = 0 \}$. So both $V_0$ and $V$ are invariant subspaces of all the $M_{(\sigma, \tau)}$.

The representation of $S_n$ on $V_0$ is trivial. Thus $\mathfrak{C}|_{V_0} = \text{id}$, so that $M_{(\sigma, \tau)} = 0$ on the space $V_0$. Therefore the sum of matrix elements of $M_{(\sigma, \tau)}$ in every row and column vanishes.

According to (16) the lemma is equivalent to the following statement: for every linear operator $X : V \to V$ there exist constants $a_{\sigma, \tau} \in \mathbb{C}$ such that $\sum_{\sigma, \tau \in S_n} a_{\sigma, \tau} \mathfrak{P}[\sigma^{-1} (C - C^{-1})] = X$. A standard result in representation theory says that the image of an irreducible representation of the group algebra of any finite group is the full matrix algebra of the representation space. So, for any linear operator $Y : V \to V$, there exist constants $a_{\sigma} \in S_n$ such that $\sum_{\tau \in S_n} a_{\tau} \mathfrak{P}[\tau] = Y$ on $V$, see e.g. [22]. Now Lemma 4.1 is equivalent to the following statement: for any linear operator $X : V \to V$, there exist operators $Y_{\sigma}, \sigma \in S_n$, such that

$$\sum_{\sigma \in S_n} \mathfrak{P}[\sigma^{-1} (C - C^{-1})] Y_{\sigma} = X. \quad (20)$$

To prove this claim, we observe that the operator $M_0 \overset{\text{def}}{=} \mathfrak{P}[C - C^{-1}] : V \to V$ is nonzero. So, the linear hull $W \subseteq V$ of the spaces $\mathfrak{P}[\sigma^{-1} (C - C^{-1})](V) \subseteq V, \sigma \in S_n$, is $S_n$-invariant and nonzero. The representation $V$ is irreducible which implies that $W = V$. In other words, there exist (not necessarily distinct) permutations $\sigma_1, \ldots, \sigma_{n-1} \in S_n$ and vectors $w_1, \ldots, w_{n-1} \in V$ such that the vectors $v_I = \mathfrak{P}[\sigma_i^{-1} (C - C^{-1})](w_I), i = 1, \ldots, n-1$, form a basis in $V$. 

Relations between moments of plane polygons
Let \( u_{ij} \), where \( i, j = 1, \ldots, n - 1 \), be the constants such that \( X(u_j) = \sum_{i=1}^{n-1} u_{ij} w_i \) for all \( j \). Define the operators \( Y_\sigma, \sigma \in S_n \), by the formula:

\[
Y_\sigma(e_j) = \sum_{i \in \sigma^{-1}} u_{ij} w_i
\]

(if no \( \sigma \) equals \( \sigma \), then \( Y_\sigma = 0 \) and it does not enter (20)). An immediate check shows that (20) holds, and the claim follows. Lemma 4.1 is settled.

Tomoe further, for \( 1 \leq i, j \leq n - 1 \), denote by \( \phi_{ij} \in \mathfrak{M}_n \) the matrix whose entries are equal to 1 at positions \( (i, j) \) and \( (n, i) \), and to \(-1\) at positions \((i, n)\) and \( (n, j) \), and vanish elsewhere.

**Corollary 4.2.**
(i) For any \( n \geq 3 \), \( \dim \mathfrak{M}_n = (n-1)^2 \).
(ii) For any \( n \geq 3 \), matrices \( \phi_{ij} \) with \( 1 \leq i \leq n - 1 \), \( 1 \leq j \leq n - 1 \), form a basis in \( \mathfrak{M}_n \).

\[\square\]

### 4.2 Quadratic relations

Denote by \( A_n \subseteq \mathbb{C}[z, \bar{z}] \) the subalgebra generated by the bilinear forms \( \nu_2(\sigma, \tau) \) with \( (\sigma, \tau) \in S_n \times S_n \).

**Proposition 4.3.** For any \( n \geq 3 \), one has

\[
A_n \cong \mathbb{C}[D]/(I_2)
\]

where \( D = (d_{ij}) \) is a \((n-1) \times (n-1)\) matrix with variable entries \( d_{ij} \), and \( (I_2) \) is the ideal, generated by all \( 2 \times 2 \)-minors of \( D \).

**Proof.** Observe that \( A_n \) is generated by the linear space of bilinear forms whose matrices (written in the basis \((z, \bar{z})\)) belong to \( \mathfrak{M}_n \). For simplicity, we will identify bilinear forms with their matrices and denote this space by \( \mathfrak{M}_n \) as well. By Lemma 4.2 the \((n-1)^2\) forms \( \phi_{ij}, 1 \leq i, j \leq n - 1 \), constitute a basis for \( \mathfrak{M}_n \). Explicitly,

\[
\phi_{ij} = z_i \bar{z}_j - z_i \bar{z}_n - z_n \bar{z}_j + z_n \bar{z}_n = (z_i - z_n)(\bar{z}_j - \bar{z}_n).
\]

One can easily check the equalities

\[
\phi_{i_1,j_1} \phi_{i_2,j_2} - \phi_{i_1,j_2} \phi_{i_2,j_1} = 0,
\]

coming from \( 2 \times 2 \)-minors.

So, \( A_n \) is isomorphic to the sub-algebra of \( \mathbb{C}[z, \bar{z}] \), generated by \( \phi_{ij}, 1 \leq i, j \leq n - 1 \). The substitution \( u_i = z_i - z_n, i = 1, \ldots, n - 1 \), \( u_n = z_n \) and \( v_i = \bar{z}_i - \bar{z}_n, i = 1, \ldots, n - 1 \), \( v_n = \bar{z}_n \) shows that \( A_n \) is isomorphic to \( \mathbb{C}[u_i, v_j, 1 \leq i, j \leq n - 1] \).

Now observe that \( \mathbb{C}[u_i, v_j, 1 \leq i, j \leq n - 1] \) is the coordinate ring of the Segre embedding \( \mathbb{P}^{n-2} \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{(n-1)^2-1}, \)

where

\[
([u_1 : \cdots : u_{n-1}], [v_1 : \cdots : v_{n-1}]) \mapsto [u_1 v_1 : u_1 v_2 : \cdots : u_{n-1} v_{n-1}].
\]

It is well-known (e.g. see e.g. [5, p. 14]), that the coordinate ring of \( \mathbb{P}^{(n-1)^2-1} \) is \( S = \mathbb{C}[d_{ij}]/I_2 \), where \( I_2 \) is the ideal generated by all \( 2 \times 2 \)-minors \( d_{ij} - d_{pq} \), \( 1 \leq i, j, p, q \leq n - 1 \). The image of \( (u_i, v_j) \) under Segre embedding is \( d_{ij} \). This observation completes the proof.

The Hilbert series of \( A_n \) is given in [2, p. 53] and is equal to

\[
\mathcal{H} = \sum_{d=0}^{\infty} \left( \frac{D + n - 1}{n - 1} \right)^2 t^d.
\]

It is also known that \( A_n \) is both Gorenstein and Koszul. The Gorenstein property was first proved in [8]; the Koszul property was first settled in [1]; see also [2].

Consider now the map \( \Theta_n : \mathbb{C}[x_{(\sigma, \tau)}, \sigma, \tau \in S_n] \rightarrow \mathbb{C}[z, \bar{z}] \) which sends each variable \( x_{(\sigma, \tau)} \) to \( \nu_2(\sigma, \tau) \). This map is graded and doubles the degree.

**Proposition 4.4.** Let \( n \geq 4 \), where \( 1 \leq i, j \leq n - 1 \), and let

\[
\rho_{ij} = \frac{1}{2} (x_{(1)i(1)n}, (1j)(2)n - x_{(1)i(1)n}(1j)(2)n + x_{(1)i(1)n}(1j)(2)n - x_{(1)i(1)n}(1j)(2)n)
\]

Then \( \Theta_n(\rho_{ij}) = \phi_{ij} \in \mathfrak{M}_n \).
**Proof.** For \( i = j = 1 \) the proof is an immediate check. For any other \( i \) and \( j \), one has

\[
\rho_{ij} = \mathfrak{R}[(1i), (1j)]\rho_{11},
\]

where \( \mathfrak{R} \) is a regular representation of \( S_n \times S_n \subset \mathbb{C}[x_{(\sigma, \tau)}, \sigma, \tau \in S_n] \), given by

\[
\mathfrak{R}[(\sigma', \tau')](x_{\sigma, \tau}) = x_{\sigma', \tau' \sigma, \tau}.
\]

Recall that we denote by \( \mathfrak{P} \) a \( n \)-dimensional permutation representation of \( S_n \). Then one has

\[
\Theta_n(\rho_{ij}) = \mathfrak{P}[(1i)]\Theta_n(\rho_{11})\mathfrak{P}[(1j)] = \mathfrak{P}[(1i)]\phi_{11}\mathfrak{P}[(1j)] = \phi_{ij}.
\]

The kernel \( J_n \overset{\text{def}}{=} \ker(\Theta_n) \subset \mathbb{C}[S_n \times S_n] \) is an ideal, which we call the ideal of relations. Obviously, \( J_n \) is a homogeneous ideal: \( J_n = \bigoplus_k J_{nk} \), where \( J_{nk} \overset{\text{def}}{=} \ker \Theta_n|_k \) is the kernel of \( \Theta_n \), restricted to the degree \( k \) component of the polynomial ring \( \mathbb{C}[x_{(\sigma, \tau)}, \sigma, \tau \in S_n] \).

The condition \( x = \sum_{\sigma, \tau \in S_n} u_{\sigma, \tau} x_{\sigma, \tau} \in J_n \) means that, for any \( i = 1, \ldots, n \), one has

\[
0 = \Theta(x)(e_i) = \sum_{\sigma, \tau \in S_n} u_{\sigma, \tau} \mathfrak{P}[\sigma^{-1}(C - C^{-1})e_{\tau(i)}] = \sum_{\sigma, \tau \in S_n} u_{\sigma, \tau} \mathfrak{P}[\sigma^{-1}](e_{\tau(i)+1} - e_{\tau(i)-1})
\]

(meaning addition and subtraction modulo \( n \))

\[
= \sum_{\sigma, \tau \in S_n} u_{\sigma, \tau} u_{\sigma^{-1}e_{\tau(i)+1} - e_{\tau(i)-1}}.
\]

In other words, this equality means that, for all \( i, j = 1, \ldots, n \), one has

\[
\sum_{\sigma, \tau \in S_n} u_{\sigma, \tau} = \sum_{\sigma, \tau \in S_n} u_{\sigma, \tau}.
\]

Propositions 4.3 and 4.4 imply the following.

**Corollary 4.5.** The ideal of relations is generated by all linear elements \( x = \sum_{\sigma, \tau \in S_n} u_{\sigma, \tau} x_{\sigma, \tau} \), where the coefficients \( u_{\sigma, \tau} \) satisfy equations (25), and by the quadratic elements \( \rho_{i,j}, \rho_{i,j} = \rho_{i,j}, \rho_{i,j} = \rho_{i,j} \), where the elements \( \rho_{ij} \) are defined by equation (24). \( \square \)

**Corollary 4.6 (of Corollary 4.5).** The Galois group of the Galois closure of the field extension \( \mathbb{C}[z, \bar{z}]^{S_n \times S_n} (\nu_2) : \mathbb{C}[z, \bar{z}]^{S_n \times S_n} \) consists of all maps \( \gamma : S_n \times S_n \to S_n \times S_n \) such that the linear transformation that sends \( x_{\sigma, \tau} \mapsto x_{\gamma(\sigma, \tau)} \) for all \( \sigma, \tau \in S_n \), preserves all the relations, described in Corollary 4.5. \( \square \)

5 Examples and Illustrations: Triangle

In this section, we illustrate our general results in the simplest nontrivial case \( n = 3 \), i.e. when the considered polygons are triangles.

5.1 \( S_3 \)-action and the Galois group

First of all, for \( n = 3 \), the numerator of equation (5) is a constant, so it is equal to \( \nu_2 = \nu_2(z, \bar{z}) \). Therefore, harmonic moments of a triangle are related as

\[
\frac{\nu_{j+2}}{\nu_2} = \binom{j + 2}{2} h_j(z_1, z_2, z_3),
\]

where \( h_j(z_1, z_2, z_3) \) denotes the complete symmetric function of degree \( j \) in three variables, that is, the sum of all monomials of degree \( j \) in \( z_1, z_2, z_3 \). So \( \nu_2 \) will be playing a crucial role in the following considerations. Denote \( M \overset{\text{def}}{=} \nu_2 \) for short (c.f. \( M_{d,d,d} \) in Section 3). Again, we do not distinguish bilinear forms from their matrices.
Theorem 5.1. The generators $M = \nu_2, \nu_3, \tilde{\nu}_3, \nu_4, \tilde{\nu}_4, \nu_5, \tilde{\nu}_5$ of the field $\tilde{\mathbb{F}}_3$ satisfy a sole relation $L(M, e_1, \ldots, \tilde{e}_3) = 0$, where $L \equiv \text{Res}_S(R, Q)$. Here $R = 16M^2 + \text{det} \Omega(S)$, where

$$\Omega(S) = \begin{pmatrix} 3 & e_1 & e_1 \\ e_1 & e_1^2 - 2e_2 & S \\ e_1 & S & e_1^2 - 2e_2 \end{pmatrix},$$

and

$$Q = \prod_{\sigma \in S_3} \left( S - z_1 \tilde{e}_{\sigma(1)} - z_2 \tilde{e}_{\sigma(2)} - z_3 \tilde{e}_{\sigma(3)} \right).$$

(Here $\text{Res}_S(R, Q)$ denotes the resultant of polynomials $R$ and $Q$ with respect to the variable $S$).

Remark 5.2. Explicitly, one has

$$R = -3S^2 + 2e_1 \tilde{e}_1 S + 16M^2 + e_1^2 \tilde{e}_1^2 - 4e_1^2 \tilde{e}_2 - 4e_1 \tilde{e}_1 \tilde{e}_2 + 12e_2 \tilde{e}_2.$$

$Q$ is a polynomial of degree 6 with respect to $S$; it is symmetric in the $\{z_i\}$ and the $\{\tilde{z}_i\}$ separately. Hence $Q$ can be regarded as a polynomial of degree 6 in $S$ with the coefficients being polynomials in the variables $\{e_k\}$ and $\{\tilde{e}_k\}$, $k = 1, 2, 3$. The total degree of $Q$ is 20; it contains 66 terms.

Proof of Theorem 5.1. By Theorem 1.4, $M = -\frac{1}{2} \text{det} \omega$, where

$$\omega = \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \tilde{z}_1 & \tilde{z}_2 & \tilde{z}_3 \end{pmatrix}.$$

(27)

We will follow the argument suggested by R. Bryant in [3]. One has $\omega \cdot \omega^* = \Omega(S)$, where $\omega = z_1 \tilde{z}_1 + z_2 \tilde{z}_2 + z_3 \tilde{z}_3$. Thus $16M^2 = -\text{det} \Omega(S)$ for this value of $S$. The same value of $S$ is a root of the polynomial $Q$, so $L = \text{Res}_S(R, Q) = 0$. An explicit formula for the resultant shows that $\text{Res}_S(R, Q)$ has degree 12 with respect to $M$. Theorem 1.7 implies that $L$ is the minimal polynomial for $M$.

Remark 5.3. Combining Theorem 5.1 with equations (26), one obtains a relation among $\nu_2, \nu_3, \nu_4, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\nu}_5$. We calculated it explicitly using Macaulay computer algebra system. The result is a very long polynomial with integer coefficients (of the order of several millions) which is weighted homogeneous of degree 64 with $\nu_k$ and $\tilde{\nu}_k$ having weight $k$ for $k = 2, 3, 4, 5$.

Let us now present the relations between $M_{(\sigma, \tau)}$. By equation (19), one should take one pair $(\sigma, \tau)$ for every right coset of $S_3 \times S_3$ with respect to the cyclic group generated by $(C, C)$, where $C$ is the cyclic shift $(123)$. The number of these cosets is $3!2! = 12$, and a convenient system of representatives is $\{(\sigma, \tau) \mid \sigma \in \{id, (12), \tau \in S_3\}$.

The vector space, spanned by $M_{(\sigma, \tau)}$, has dimension $(3 - 1)^2 = 4$. So, there exist $12 - 4 = 8$ independent linear relations between $M_{(\sigma, \tau)}$. Of them, 6 are two-term:

$$M_{(12), (12)\tau} + M_{id, \tau} = 0, \quad \tau \in S_3,$$

(28)

and the additional two are three-term:

$$M_{id, id} + M_{id, (123)} + M_{id, (132)} = 0,$$

$$M_{(12), id} + M_{(12), (132)} + M_{(12), (132)} = 0.$$  

(29)

The basis in the image of the map $\Theta_3$ is formed by 4 vectors, $\phi_{11}, \phi_{12}, \phi_{21},$ and $\phi_{22}$. For $n = 3$, all quadratic relations (22) reduce to only one:

$$\phi_{12} \phi_{21} = \phi_{11} \phi_{22}. $$

(30)

Direct computation shows that for $n = 3$ the forms $\phi_{ij}$ can be expressed via $M_{(\sigma, \tau)}$ as follows (recall that the general formulas (24) work only for $n \geq 4$):

$$\phi_{11} = \frac{1}{3} (M_{id, id} + 2M_{id, (123)} - M_{id, (12)} - M_{id, (23)}),$$

$$\phi_{12} = \frac{1}{3} (2M_{id, id} + M_{id, (123)} - M_{id, (12)} - 2M_{id, (23)}),$$

$$\phi_{21} = \frac{1}{3} (- M_{id, id} + M_{id, (123)} - M_{id, (12)} - 2M_{id, (23)}),$$

$$\phi_{22} = \frac{1}{3} (M_{id, id} + 2M_{id, (123)} - 2M_{id, (12)} - M_{id, (23)}).$$
Substitution of these formulas into the quadratic relation (30) gives
\[
M_{id, id}^2 + M_{id, id}M_{id, (123)} + M_{id, (123)}^2 = M_{id, (12)}^2 + M_{id, (23)}M_{id, (23)} + M_{id, (23)}^2. \tag{31}
\]

The Galois group \(G_3\) of the equation (19) permutes its 12 roots \(M_{id, \tau}, M_{(12), \tau}, \tau \in S_3\), while preserving the linear relations (28) and (29), together with the quadratic relation (31). Thus \(G_3 \subseteq S_{12}\).

For \(\gamma \in G_3\), relations (28) imply that there exists a bijection \(\gamma : S_3 \to S_3\) and a map \(\epsilon : S_3 \to \{1, -1\}\) such that \(\gamma(M_{id, \tau}) = \epsilon(\tau)M_{id, \gamma(\tau)}\) for all \(\tau \in S_3\). Then it follows from (28) that \(\gamma(M_{(12), \tau}) = -\epsilon(\tau)M_{id, (12)\gamma(\tau)} = \epsilon(\tau)M_{(12), \gamma(\tau)}\), which means that the bijection \(\gamma \in S_3\) and the map \(\epsilon\) determine \(\gamma\) uniquely. In other words, \(G_3\) is a subgroup of the Coxeter group \(B_6\) of signed permutations (which is naturally embedded into \(S_{12}\), as described above).

Further, to preserve relations (29), the map \(\gamma\) should either map the subsets \(A_3 \overset{\text{def}}{=} \{id, (123), (123)\}\) and \(S_3 \setminus A_2 = \{12, (13), (23)\}\) of \(S_3\) to themselves or to each other. In both cases, the numbers \(\epsilon(\tau)\) should remain the same, while \(\tau\) is changing within a set. The pairs \((\gamma, \epsilon) \in G_3\), where \(\gamma\) preserves the sets, form a subgroup \(G_3^+ \subseteq G_3\) of index 2.

Notice now that the quadratic form \(Q(u) = u_1^2 + u_1u_2 + u_2^2\) is \(S_3\)-invariant on the subspace \(V_3 \overset{\text{def}}{=} \{u_1e_1 + u_2e_2 + u_3e_3 | u_1 + u_2 + u_3 = 0\} \subseteq \mathbb{C}^3 = \langle e_1, e_2, e_3 \rangle\) with the permutation action of the \(S_3\). (This can be checked by an easy computation; actually, up to a factor, this form is equal to the restriction of the form \(u_1^2 + u_2^2 + u_3^2\), defined in \(\mathbb{C}^3\), to \(V_3\)). So, any mapping \(\gamma\) described above automatically preserves relation (31). Hence, the subgroup \(G_3^+\) consists of all pairs \((\gamma, \epsilon)\), where \(\gamma\) preserves the sets \(A_3\) and \(S_3 \setminus A_3\) and \(\epsilon\) is constant within either set; thus, \(G_3^+\) is isomorphic to the group \(S_3 \times S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) and contains 144 elements. The whole group \(G_3\) contains 288 elements and is a semi-direct product of \(G_3^+\) and the 2-element group \(\mathbb{Z}_2\).

5.2 Graphic presentation of the moment \(M = \nu_2\)

It follows from (26) that, in order to analyze the moments for \(n = 3\), it is enough to study the lowest moment \(\nu_2\), defined by the equation (27).

To represent the points \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{C}^2\), let us draw two triples of complex numbers: \(z = (z_1, z_2, z_3)\) and \(\bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3), \) where \(z_j = x_j + iy_j\) and \(\bar{z}_j = x_j - iy_j\), \(j = 1, 2, 3\). For generic choice of \(z_j, \bar{z}_j\), there exist unique numbers \(\alpha, \beta \in \mathbb{C}\) such that \(\alpha z_1 + \beta = z_1\) and \(\alpha z_2 + \beta = z_2\). Therefore
\[
\nu_2 = \frac{1}{2i} \det \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ 0 & 0 & z_3 - (\alpha z_3 + \beta) \end{pmatrix} = \frac{1}{2i}(z_3 - (\alpha z_3 + \beta))(z_2 - z_1). \tag{32}
\]

So if \(w = \alpha z_3 + \beta\), then the triangle \(z_1, z_2, z_3\) is similar to the triangle \(z_1, z_2, z_3\), where the similarity map sends \(z_1 \mapsto z_1, z_2 \mapsto z_2, z_2 \mapsto w\). Obviously, this condition determines \(\alpha\) uniquely. Then the vector connecting \(z_3\) and \(w\) represents the complex number \(z_3 - (\alpha z_3 + \beta)\). Thus it follows from (32) that the moment \(\nu_2\) is the product of this number by the complex number, represented by the vector that joins the vertices \(z_1\) and \(z_2\), divided by \(2i\). In particular, \(\nu_2\) is one half of the product of the lengths of two of these vectors. Thus, \(\nu_2\) can be thought as a measure of non-similarity of two triangles.

The action of the group \(S_3 \times S_3\) preserves triples \(z\) and \(\bar{z}\), but changes the numbering of these points. Identity (29) now involves moments \(\nu_2\), calculated using (32) with the same \(z_1, z_2, z_3\) in all three terms and \(\bar{z}_1, \bar{z}_2, \bar{z}_3\), changing their labels in a cycle.

If the vertices of the triangle are real, then (29) translates into a statement from the elementary Euclidean geometry. Namely, denote by \(A_j\) the point \(z_j \in \mathbb{R}^2\) and by \(C_j\), the point \(\alpha z_j + \beta\) from (32); here \(j = 1, 2, 3\). Then (29) and (32) give:

**Theorem.** Let \(A_1, A_2, A_3\) be a triangle in the plane \(\mathbb{R}^2\). Let \(C_1, C_2, C_3 \in \mathbb{R}^2\) be points such that the triangles \(A_1C_1A_2, A_2C_2A_3\) and \(A_3C_3A_1\) are similar with the similarity maps sending vertices to vertices as written (e.g. \(A_1 \mapsto A_2, A_2 \mapsto A_3, C_3 \mapsto C_1\) for the first two triangles, etc.). Then the sum of the vectors \(A_1\overrightarrow{C_1} + \overrightarrow{A_2C_2} + \overrightarrow{A_3C_3}\) vanishes. \(\Box\)
6 Further Outlook

1. According to assertion (iii) of Theorem 1.7, each moment $\nu_j(z, \bar{z})$ is a rational function of $e_1(z), \ldots, e_n(z), e_1(\bar{z}), \ldots, e_n(\bar{z})$, and $\nu_0(z, \bar{z})$. Is it possible to find these rational functions explicitly?

2. The main motivation for the present paper comes from a recent article [12] by the third author (joint with K. Kohn and B. Sturmfels), where general (not necessarily harmonic) moments for convex polytopes were considered. In particular, [12] contains a complete description of relations between the axial moments of such polytopes. A similar problem for fields and rings of general moments is still widely open and is apparently closely related to the complex questions about the ring of diagonal harmonics, defined in [10].

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