COUNTING REAL ZEROS OF EXPONENTIAL SUMS

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Abstract. Below we show that for any positive integer k, there exists a positive integer $N(k)$ such that any exponential sum of order (at most) k either has at most $N(k)$ real zeros or infinitely many. Moreover, we show that the set of exponential sums of order (at most) k with more than $2k-3$ real zeros has real codimension 1 in the set of all all exponential sums of order (at most) k.

1. Introduction

Consider a linear homogeneous differential equation

$$y^{(k)} + c_{k-1}y^{(k-1)} + ... + c_0 y = 0, \qquad (1)$$

with arbitrary complex constant coefficients $c_{k-1}, \dots, c_0$.

The main motivation of the present study is the conjecture of J. H. Loxton and A. J. van der Poorten from 1977 (Conjecture $1'$ of [17]) claiming that for any positive integer k, there exists a constant μ_k such that any non-trivial solution of any equation (1) either has at most μ_k integer zeros or it has infinitely many integer zeros.

This conjecture was first settled by W. M. Schmidt in 1999, see [23] and also by J. H. Evertse and H. P. Schlickewei, see [7]. Apparently the best known at the moment upper bound for μ_k was obtained in [2] and is given by

$$\mu_k < e^{e^{k^{\sqrt{11k}}}}.$$

In Problem 7 of [8], the second author jointly with R. Fröberg asked whether there exists an analog of the latter result of Schmidt-Evertse-Schlickewei for the set of real roots of non-trivial solutions of (1). The main purpose of this paper is to provide a positive answer to the latter question together with some additional information.

Given a function $y(x) = p_1(x)e^{\lambda_1 x} + \dots + p_\ell(x)e^{\lambda_\ell x}$, where all λ_i's are distinct complex numbers, and $p_i(x)$'s are univariate polynomials, we call $y(x)$ an *exponential sum of order* $\ell + \deg p_1(x) + \dots + \deg p_\ell(x)$. Obviously, any non-trivial solution $y(x)$ of (1) is an exponential sum of order at most k and viceversa.

Zero distributions of exponential sums have been studied at least since the 1920's, see e.g. [3], [6], [18], [25], [26], [27], [29] and references therein. The case of exponential sums with real exponents is closely related to the case of almost periodic functions classically studied by H. Weyl [28], B. Ya. Levin [15], B. M. Levitan [16], M. G. Krein [11] etc. It is also related to the fewnomial theory as developed by A. Khovanskii in [10].

Denote by Eq_k the affine space of all linear equations of the form (1) with coordinates $(c_{k-1}, \dots, c_0)$ and denote by $Eq_k^{\leq n} \subset Eq_k$ the (closed) subset of all equations

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such that any non-trivial solution of these equations has at most n real zeros counting multiplicities. (Obviously, for $n < k-1$, the set $Eq_k^{\leq n}$ is empty.) Additionally, denote by $Eq_k^{\infty} \subset Eq_k$ the set of all equations (1) possessing a non-trivial solution with infinitely many real zeros. (The set Eq_k^{∞} is explicitly characterised in Lemma 4 below.) Observe that for any n, $Eq_k^{\leq n} \cap Eq_k^{\infty} = \emptyset$, and

$$Eq_k^{\leq k-1} \subseteq Eq_k^{\leq k} \subseteq \cdots \subseteq \ldots.$$

The main qualitative result of this note is as follows.

Theorem 1. (i) *For all $n = k-1, \ldots, 2k-3$,*

$$\dim_{\mathbb{R}} \left(Eq_k^{\leq n} \setminus Eq_k^{\leq n-1} \right) = \dim_{\mathbb{R}} Eq_k = 2k.$$

(ii) *The closure of the set $Eq_k^{\leq 2k-3}$ coincides with Eq_k. In other words, any non-trivial solution of a generic equation* (1) *has a most $2k-3$ real zeros and this bound is sharp.*

Our main result is as follows.

Theorem 2. *For any positive integer $k \geq 2$, there exists a positive integer $N(k) \geq 2k-3$ such that $Eq_k = Eq_k^{\leq N(k)} \cup Eq^{\infty}(k)$. In other words, there exists an upper bound of the number of real zeros taken over the set of all solutions of all equations* (1) *under the additional assumption that we only consider solutions with a finite number of real zeros, i.e. we disregard all solutions with infinitely many real roots.*

Remark 3. Observe that the set of solutions considered in Theorem 2 is an open subset of $\mathbb{C}P^{k-1} \times \mathbb{C}^k$ which implies that the existence of $N(k)$ is by no means obvious.

The structure of the paper is as follows. In § 2, we present our proofs of the results formulated above. In Appendix 3, we provide some additional information about the integer roots of exponential sums. Finally, in § 4 we present a number of open problems related to this topic.

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2. Proofs

It is a classical result that for any exponential sum

$$E(x) = \sum_{j=1}^{\ell} p_j(x)_j e^{\lambda_j x}, \tag{2}$$

with pairwise distinct exponents λ_j, its zeros concentrate on the finite union of a straight lines whose directions are perpendicular to possible differences $\lambda_{j'} - \lambda_{j''}$, $j' \neq j''$; the exact location of these lines in $\mathbb{C}$ depends on a particular choice of the coefficients $p_j(x)$, see e.g. [6, 18, 21]. The directions perpendicular to differences $\lambda_{j'} - \lambda_{j''}$, $j' \neq j''$ are classically referred to as *Stokes directions*. In particular, as soon as $k > 1$, the number of complex zeros of $E(z)$ is always infinite.

Lemma 4. *For a given equation* (1)*, a direction $\alpha \in \mathbb{R}P^1$ is Stokes for* (1) *if and only if for any straight line L with slope α, there exists a non-trivial solution of* (1) *with infinitely many zeros belonging to L.*

Proof. By definition of a Stokes direction, there exist at least two characteristic roots $\lambda_{j'}$ and $\lambda_{j''}$ such that $\alpha = i\rho(\lambda_{j'} - \lambda_{j''})$, where $\rho \in \mathbb{R}$. Then the linear combination

$$ae^{\lambda_{j'}x} + be^{\lambda_{j''}x}$$

for appropriate a and b has infinitely many zeros on a line with slope α. On the other hand, if α is not a Stokes direction, then there is a finite upper bound on the number of zeros of any solution of (1), see e.g. Proposition 14 of [21]. □

Our proof of Theorem 1 consists of two parts. Firstly, we show that a generic equation (1) with a Stokes line sufficiently close to the real axis possesses a non-trivial solution with as least $2k-3$ real roots. Secondly, we show that the set of equations (1) possessing a non-trivial solution with at least $2k-2$ real zeros can not contain an open subset in Eq_k.

The first part of the proof is settled by the following statement.

Proposition 5. *For a generic equation* (1) *and any its Stokes direction* $\alpha \in \mathbb{R}P^1$, *there exists a small neighborhood* $U_\alpha \subset \mathbb{R}P^1$ *of* α *such that for any* $\beta \in U_\alpha$, *there exists a non-trivial solution of* (1) *with at least* $2k-3$ *roots on* L_β, *where* L_β *is any straight line with slope* β.

Proof. By rescaling and shift we can assume that $\lambda_1 = 0$ and $\lambda_k = 1$. Denote $a = (a_2, \dots, a_{k-1})$ and $r = (r_2, \dots, r_{k-1})$. Set $a_k = -1 - \sum_{i=2}^{k-1} a_i$ and define

$$F(a, x) := 1 + \sum_{i=2}^{k} a_i e^{\lambda_i x}.$$

Observe that for all $a = (a_2, \dots, a_{k-1})$, $F(a, x)$ vanishes at $x = r_1 = 0$. Define the mapping

$$\Phi : \mathbb{C}^{k-2} \times \mathbb{C}^{k-2} \to \mathbb{C}^{k-2}, \quad \text{given by} \quad \Phi(a, r) = (F(a, r_2), ..., F(a, r_{k-1})).$$

Consider the function $a = \rho(r)$ implicitly defined by the relation

$$F(\rho(r), r) = 0.$$

Since for any fixed $r = (r_2, \dots, r_{k-1})$, $F(a, r)$ is an affine function of a which implies that $\rho(r)$ is well-defined as soon as the differential

$$\frac{\partial \Phi(a, r)}{\partial a} = \left\{ e^{\lambda_j r_i} - e^{\lambda_k r_i} \right\}_{i,j=2}^{k-1} \tag{3}$$

is non-degenerate. An easy computation shows that

$$\det \left(\frac{\partial \Phi(a, r)}{\partial a} \right) = \det \left\{ e^{\lambda_j r_i} \right\}_{i,j=1,2}^{k-1,k}. \tag{4}$$

Lemma 6. *For a generic* (1), *the matrix* $\frac{\partial \Phi(a,r)}{\partial a}$ *is non-degenerate for* $r = u(0) = (2\pi i, \dots, 2\pi i(k-2)) \in \mathbb{C}^{k-2}$, *with* $\rho(u(0)) = 0$. *The corresponding* $F(a, x)$ *is given by* $F(a, x) = -e^x + 1$.

Proof. For the tuple $u(0)$, the right-hand side of (4) is the Vandermonde determinant equal to $\prod_{1 \le j' < j'' \le k-1} (e^{2\pi i \lambda_{j'}} - e^{2\pi i \lambda_{j''}})$. Under the genericity condition

$$\lambda_{j'} - \lambda_{j''} \notin \mathbb{Z}, \quad 1 \le j', j'' \le k-1, \tag{5}$$

the latter is non-zero. As $1 - e^x$ vanishes at all $2\pi i j, j \in \mathbb{Z}$, it has to be equal to the uniquely defined function $F(\rho(u(0)), x)$. □

By continuity, $\det\left(\frac{\partial\Phi(a,r)}{\partial a}\right) \neq 0$ for all $u \in \mathbb{C}^{k-1}$ sufficiently close to $u(0)$, thus defining a as a function of u in some neighborhood of $u(0)$. The function ρ restricted to this neighborhood will be denoted by ρ_+.

From the equation $\Phi(\rho_+(u), u) = 0$ we find that

$$\frac{\partial\rho_+}{\partial u} = -\left(\frac{\partial\Phi(a,u)}{\partial a}\right)^{-1} \cdot \frac{\partial\Phi(a,u)}{\partial u},$$

and

$$\frac{\partial\Phi(a,r)}{\partial r}(a(0), r(0)) = \operatorname{diag}\left\{\sum_{l=2}^{k} \lambda_l a_l e^{\lambda_l r_j}\right\}_{j=2}^{k-1}\Bigg|_{(0,u(0))} = -Id,$$

so

$$\frac{\partial\rho_+}{\partial r}(u(0)) = \left(\frac{\partial\Phi(a,u)}{\partial a}(0, u(0))\right)^{-1}.$$

Now, replace $u(0)$ with $w(0) = -u(0) = (-2\pi i, -4\pi i, \ldots, -2(k-2)\pi i) \in \mathbb{C}^{k-2}$, i.e., with the $(k-2)$-tuple of roots of $e^x - 1$ opposite to $u_j(0)$. The same computations as above hold, and the formulas are the same up to replacing λ_j by $-\lambda_j$. So, in some neighborhood of $w(0)$ there is a uniquely defined mapping $a = \rho_-(w)$ satisfying the condition $\Phi(\rho_-(w), w) = 0$ with

$$\frac{\partial\rho_-}{\partial w}(w(0)) = \left(\frac{\partial\Phi(a,w)}{\partial a}\right)^{-1}(0, w(0)).$$

Taking the composition, we obtain the holomorphic mapping $w = \xi(u)$ with $\xi = \rho_-^{-1} \circ \rho_+$ which is defined in some neighborhood of $u(0)$ and such that

$$M(\lambda) = \frac{\partial\xi}{\partial u}(u(0)) = \frac{\partial\Phi(a,w)}{\partial a}(0, w(0)) \cdot \left(\frac{\partial\Phi(a,u)}{\partial a}(0, u(0))\right)^{-1}$$

is non-degenerate.

We need to find pairs $(u, w = \xi(u))$ such that all u_j, w_l lie on the same line $L \subset \mathbb{C}$ close to $i\mathbb{R}$ and passing through $r_0 = 0$. First, consider the case of $L = i\mathbb{R}$, i.e. $(u, w) \in (i\mathbb{R})^{2k-4} \subset \mathbb{C}^{2k-4} = \mathbb{C}_r^{k-2} \times \mathbb{C}_w^{k-2}$.

Lemma 7. *Assume that* $\det(Im\, M(\lambda)) \neq 0$. *Then* $(i\mathbb{R})^{2k-4}$ *intersects the graph* Γ_ξ *of* ξ *transversally.*

Proof. Standard computation. □

Corollary 8. *If* $\det(Im\, M(\lambda)) \neq 0$, *then for any line* L *close to imaginary axis, there exists a solution having* $2k-3$ *zeros on* L *close to* $2\pi ij, j = -k+2, ..., k-2$.

Indeed, the set $\{(r, w), r_j, w_l \in \ell\} \subset \mathbb{C}^{2k-2}$ is a deformation of $(i\mathbb{R})^{2k-2}$, so it intersects Γ_ρ at a point close to $(r(0), w(0))$.

□

Example 9. *For* $k = 3$, *we obtain*

$$\frac{\partial\Phi(a,w)}{\partial a}(0, w(0)) = e^{-2\pi i\lambda_2} - 1, \quad \textit{and} \quad \frac{\partial\Phi(a,r)}{\partial a}(0, r(0)) = e^{2\pi i\lambda_2} - 1,$$

which implies that

$$M(\lambda) = \frac{e^{-2\pi i\lambda_2} - 1}{e^{2\pi i\lambda_2} - 1} = -e^{-2\pi i\lambda_2} \notin \mathbb{R}$$

if and only if $Re\,\lambda_2 \notin 1/2\mathbb{Z}$.

Let us now compute $Im\, M(\lambda)$. We have

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & \\ & \dots & \dots & \\ 0 & \dots & -1 & 1 \end{pmatrix} \frac{\partial\Phi(a,w)}{\partial a}(0, w(0)) =$$

$$= \begin{pmatrix} 1 & \dots & 1 \\ e^{-2\pi i\lambda_1} & \dots & e^{-2\pi i\lambda_{k-1}} \\ & \dots & \\ e^{-2\pi i(k-2)\lambda_1} & \dots & e^{-2\pi i(k-2)\lambda_{k-1}} \end{pmatrix} \operatorname{diag}\left\{e^{-2\pi i\lambda_j} - 1\right\}_{j=2}^{k-1}. \qquad (6)$$

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & \\ & \dots & \dots & \\ 0 & \dots & -1 & 1 \end{pmatrix} \frac{\partial\Phi(a,u)}{\partial a}(0, u(0)) =$$

$$= \begin{pmatrix} 1 & \dots & 1 \\ e^{2\pi i\lambda_2} & \dots & e^{2\pi i\lambda_{k-1}} \\ & \dots & \\ e^{2\pi i(k-2)\lambda_2} & \dots & e^{2\pi i(k-2)\lambda_{k-1}} \end{pmatrix} \operatorname{diag}\left\{e^{2\pi i\lambda_j} - 1\right\}_{j=2}^{k-1}. \qquad (7)$$

Therefore, up to a conjugation by a real matrix (which doesn't affect the non-degeneracy of $Im\, M(\lambda)$), we get

$$M(\lambda)^T = \begin{pmatrix} 1 & e^{2\pi i\lambda_2} & \dots & e^{2\pi i(k-2)\lambda_2} \\ & \dots & \dots & \\ 1 & e^{2\pi i\lambda_k} & \dots & e^{2\pi i(k-2)\lambda_k} \end{pmatrix}^{-1} \operatorname{diag}\left\{-e^{-2\pi i\lambda_j}\right\}_{j=2}^{k-1}$$
$$\begin{pmatrix} 1 & e^{-2\pi i\lambda_2} & \dots & e^{-2\pi i(k-2)\lambda_2} \\ & \dots & \dots & \\ 1 & e^{-2\pi i\lambda_k} & \dots & e^{-2\pi i(k-2)\lambda_k} \end{pmatrix}. \qquad (8)$$

$M(\lambda)^T$ is the operator which takes a vector $a = (a_2, \dots, a_{k-1})$, produces a vector of evaluations of the sum $f(a,\lambda) = \sum_{j=2}^{k-1} a_j e^{2\pi i j\lambda}$ at the points $\lambda = -\lambda_2, \dots, -\lambda_{k-1}$, then multiplies these values by the values of $-e^{2\pi i\lambda}$ at these points, and finally finds a function $g(z) = \sum_{j=2}^{k-1} b_j e^{2\pi i j\lambda}$ taking these values at points λ_l.

Set $t = e^{2\pi i z}$, $t_l = e^{2\pi i\lambda_l}$. Using this notation, $M(\lambda)^T$ is the matrix of the operator $\mathcal{M}$ which maps a polynomial f to a polynomial g such that $g(1/t) = -tf(t)$ at points $t = t_l$ in the basis $\{1, t, \dots, t^{k-3}\}$. In other words, the rows of $M(\lambda)$ are the coefficients of the polynomials $g_j(t)$ such that $g_j(t_l^{-1}) = -t_l^j$, $j = 1, \dots, k-2$.

Lemma 10. *Let $D(t) = \prod(t - t_l^{-1}) = t^{k-2} + \sum_{j=1}^{k-2} d_j t^{k-3-j}$. Then*

$$g_1 = -d_{k-2}^{-1}\left(t^{k-4} + \sum_{j=1}^{k-3} d_j t^{k-3-j}\right), \quad g_j = (-1)^{j-1} g_1^j \mod D(t).$$

In order to prove that $\det Im\, M(\lambda) \neq 0$ generically, one should show this for just one tuple of λ_l. Take $t_l^{-1} = exp\left(2\pi i\frac{l-1}{k-2} + i\epsilon\right)$. Then $D(t) = t^{k-2} - e^{i(k-2)\epsilon}$, and $g_j = -e^{i(k-2)\epsilon} t^{k-2-j}$. Therefore $Im\, M(\lambda) = -\sin((k-2)\epsilon)\Delta^c$, where $\Delta^c_{ij} = \delta_{i,k-1-j}$. Hence $\det Im\, M(\lambda) \neq 0$.

Now we start proving the second part of Theorem 1. Let $\Gamma_{n,k} \subset \mathbb{C}P_a^{k-1} \times \mathbb{C}_\lambda^{k-1} \times \mathbb{C}_z^n$ be the closure of the set of solutions of the system

$$\bigcap_{i=1}^{n} \left\{\sum_{j=1}^{k} a_j e^{\lambda_j z_i} = 0.\right\},$$

where $\lambda_1 + \cdots + \lambda_k = 0$.

Lemma 11. *For any $k \geq 1$ and $n \geq 1$, $\Gamma_{n,k}$ is a complex analytic subvariety of dimension $\dim_{\mathbb{C}} \Gamma_{n,k} = 2k-2$.*

Proof. Indeed, for any non-trivial solution of any equation, the set of its zeros is discrete. Moreover, typically the number of zeros is infinite which implies that the set $\Gamma_{n,k}$ has the same dimension for any $n \geq 1$. □

REMARK, Show that $\Gamma_{n,k}$ is irreducible???

Consider the projection $\pi_z : \mathbb{C}P_a^{k-1} \times \mathbb{C}_\lambda^{k-1} \times \mathbb{C}_z^n \to \mathbb{C}_z^n$ and restrict it to $\Gamma_{n,k}$. Set $I_{n,k} := \pi_z(\Gamma_{n,k}) \subseteq \mathbb{C}_z^n$.

Lemma 12. *For $n \geq 2k-2$, the map $\pi_z : \Gamma_{n,k} \to I_{n,k}$ is discrete over a generic point in $I_{n,k}$. In particular, for $n \geq 2k-2$, $I_{n,k}$ is a complex analytic variety of dimension $2k-2$.*

Maybe $\pi_z : \Gamma_{n,k} \to I_{n,k}$ is $1-1$ over a generic point in the image. Is it true that $I_{n,k}$ is equidimensional?

Proof. There is a natural mapping $I_{n,k} \to I_{2k-2,k}$, sending a tuple $(z_1, \ldots, z_n)$ to $(z_1, \ldots, z_{2k-2})$, which commutes with π_z. Therefore, it is enough to prove it for $n = 2k-2$.

As the set $\Gamma_{n,k}$ is locally parameterized by (a, λ), it is enough to provide one point $(a, \lambda, z) \in \Gamma_{n,k}$ where the mapping $\phi : (a, \lambda) \to z$ has a non-degenerate differential $d\phi$. For $n = 2k-2$ this means that ϕ is a local diffeo.

Let us compute $d\phi$. Let $F : \mathbb{C}P_a^{k-1} \times \mathbb{C}_\lambda^{k-1} \times \mathbb{C}_z^{2k-2} \to \mathbb{C}P^{2k-2}$ be the mapping $F(a, \lambda, z) = (\sum_{j=1}^k a_j e^{\lambda_j z_i})_{i=1}^{2k-2}$, so $\Gamma_{2k-2,k} = \{F = 0\}$. Then $F(a, \lambda, \phi(a, la)) \equiv 0$, and

$$d\phi = -\left(\frac{\partial F}{\partial z}\right)^{-1} \left(\frac{\partial F}{\partial(a, \lambda)}\right).$$

The matrix $\frac{\partial F}{\partial z} = \operatorname{diag}\{\sum_{j=1}^k a_j \lambda_j e^{\lambda_j z_i}\}$ is generically non-degenerate.

To compute the second matrix, we take a chart $a_k = 1$. Also, we assume $\lambda_k = 1$. Then

$$\frac{\partial F}{\partial(a, \lambda)} = \left(e^{\lambda_j z_i} | a_j z_i e^{\lambda_j z_i}\right)_{i,j=1}^{2k-2,k-1}, \tag{9}$$

and

$$\det \frac{\partial F}{\partial(a, \lambda)} = \prod_{j=1}^{k-1} a_j \cdot \det M, \text{where } M = \left(e^{\lambda_j z_i} | z_i e^{\lambda_j z_i}\right)_{i,j=1}^{2k-2,k-1} \tag{10}$$

For $a = (1, 1, \ldots, 1)$, $\lambda = (\frac{1}{k}, \ldots, \frac{k-1}{k}, 1)$ the function

$$f(z) = \sum a_j e^{\lambda_j z} = \sum_{j=1}^k e^{\frac{j}{k} z} = \frac{e^z - 1}{1 - e^{-z/k}}$$

has zeros $z_i = 2\pi I l(i)$, where $l(i) = i$ for $i = 1, \ldots, k-1$, and $l(i) = i+1$ for $i = k, \ldots, 2k-2$ (where $I = \sqrt{-1}$).

For these values of (a, λ) we have

$$\sum_{j=1}^k a_j \lambda_j e^{\lambda_j z} = \sum_{j=1}^k \frac{j}{k} e^{\frac{j}{k} z} = f'(z) = \frac{e^z(1 - e^{-z/k}) - \frac{1}{k}(e^z - 1)e^{-z/k}}{(1 - e^{-z/k})^2} = \frac{ke^z - (k+1)\, e^{z - z/k} + e^{-z/k}}{k(1 - e^{-z/k})^2}.$$

For the values of z_i above, $e^{z_i} = 1 \neq e^{z_i/k}$, so $f'(z_i) = \frac{1}{1 - e^{-z_i/k}} \neq 0$ for all $i = 1, \ldots, 2k-2$. Thus the matrix $\frac{\partial F}{\partial z}$ is non-degenerate for these values of (a, λ, z).

It remains to show that $\det M \neq 0$ for these values of (a, λ, z). Note that $z_{i+k-1} = z_i + 2\pi I k$ for $i = 1, \ldots, k-1$. Therefore $e^{\lambda_j z_i} = e^{\lambda_j z_{i+k-1}}$, $i = 1, \ldots, k-1$, and $z_{i+k-1} e^{\lambda_j z_{i+k-1}} = z_i e^{\lambda_j z_i} + 2\pi I k e^{\lambda_j z_i}$. Therefore the matrix M has the form

$$M = \begin{pmatrix} M_1 & M_2 \\ M_1 & M_2 + 2\pi I k M_1 \end{pmatrix},$$

where $M_1 = \{e^{\lambda_j z_i}\}_{i,j=1}^{k-1}$. Subtracting the first $(k-1)$ rows from the last $(k-1)$ rows, we see that $\det M = (2\pi I k)^{k-1} (\det M_1)^2$. Finally,

$$\det M_1 = \det\{e^{2\pi I \frac{j}{k} i}\}_{i,j=1}^{k-1} = \prod_{1 \le j \le k-1} e^{2\pi I \frac{j}{k}} \cdot \prod_{1 \le j < j' \le k-1} \left(e^{2\pi I \frac{j}{k}} - e^{2\pi I \frac{j'}{k}}\right) \neq 0. \tag{11}$$

□

Remark. Set $I_{n,k}^{\mathbb{R}} := I_{n,k} \cap \mathbb{R}^n \subset \mathbb{C}_z^n$. Then, for $n \ge 2k-2$, $\dim_{\mathbb{R}} I_{n,k}^{\mathbb{R}} \le 2k-2$. (More generally, $\mathbb{R}^n$ can be substitutes by a totally real subvariety of dimension n.) Set $\Delta_{n,k} := \pi_z^{-1}(I_{n,k}^{\mathbb{R}}) \subseteq \Gamma_{n,k}$.

Let Δ_2 be the critical locus of the projection π_z. Then

$$\Delta_{n,k} = (\Delta_{n,k} \setminus \Delta_2) \cup (\Delta_{n,k} \cap \Delta_2)$$

Lemma 13. *For any k and $n \ge 2k-2$, $\dim_{\mathbb{R}} \Delta_{n,k} \setminus \Delta_2 \le 2k-2$.*

Proof. On $\Delta_{n,k} \setminus \Delta_2$ the projection π_z is a local biholomorphism. □

Introduce $\pi_\lambda : \mathbb{C}P_a^{k-1} \times \mathbb{C}_\lambda^{k-1} \times \mathbb{C}_z^n \to \mathbb{C}_\lambda^{k-1}$. The fiber of the projection $\pi_\lambda : \Delta_{n,k} \to J_{n,k}$ are real 1-dimensional since the parallel translation of the roots on the real axis preserves the equation.

Corollary 14. *For $n \ge 2k-2$, $\dim_{\mathbb{R}} \pi_\lambda(\Delta_{n,k} \setminus \Delta_2) \le 2k-3 < \dim \mathbb{C}_\lambda^{k-1}$.*

Proposition 15. *For $n \ge 2k-2$, $\dim_{\mathbb{C}} \Delta_2 < k-1$.*

Proof of Theorem 2. Let us first observe that each $Eq_k \le n \subseteq Eq_k$ can be fibered over the

□

3. Appendix. Integer roots of exponential polynomials

This material is included here for the sake of completeness and is mainly borrowed from [8]. Take the first non-trivial case $m = 2k-1$ and assume that $(0 = t_1 < t_2 < ... < t_{2k-1})$ all are rational numbers. Multiply them by their least common denominator and assume, therefore, that they are positive integers. We have the following intriguing problem.

Question 4. Which sequences $I = (0 = i_1 < i_2 < ... < i_{2k-1})$ are bad and which are good meaning that there exist pairwise distinct $(x_1, ..., x_k)$ forming the $(2k-1) \times k$-matrix of incomplete rank.

We get the following reduction of Question 4. For any sequence of non-negative integers $J = (0 \le j_1 < j_2 < ... < j_k)$ consider the associated Schur polynomial $S_J(x_1, ..., x_k)$ given by

$$S_J(x_1, ..., x_k) = \begin{vmatrix} x_1^{j_1} & x_2^{j_1} & ... & x_k^{j_1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{j_k} & x_2^{j_k} & ... & x_k^{j_k} \end{vmatrix} / W(x_1, ..., x_k),$$

where $W(x_1, ..., x_k)$ is the van der Monde determinant.

Now given a sequence $I = (0 = i_1 < i_2 < ... < i_{2k-1})$ cover it by k subsequences $J_1, ..., J_k$ of length k each. For example, take $J_1 = (i_1, ..., i_{k-1}, i_k)$;

$J_2 = (i_1, ..., i_{k-1}, i_{k+1})$; $J_3 = (i_1, ..., i_{k-1}, i_{k+2})$; ...; $J_k = (i_1, ..., i_{k-1}, i_{2k-1})$. Take the corresponding Schur polynomials $S_{J_1}(x_1, ..., x_k), ..., S_{J_k}(x_1, ..., x_k)$ and form the ideal $< I_{J_1,...,J_k} >$ within the ring $\mathcal{S}[x_1, ..., x_k]$ of all symmetric functions in $x_1, ..., x_k$ generated by $S_{J_1}(x_1, ..., x_k)$, ..., $S_{J_k}(x_1, ..., x_k)$. Finally, take the quotient ring

$$\mathcal{R}_{J_1,...,J_k} = \mathcal{S}[x_1, ..., x_k]/ < I_{J_1,...,J_k} > .$$

Proposition 16. *The sequence I of length $2k-1$ is good (resp. bad) if for any its covering by subsequences $J_1, ..., J_k$ each of length k the quotient ring $\mathcal{R}_{J_1,...,J_k}$ is a finite-dimensional vector space over $\mathbb{C}$, i.e. is a 0-dimensional ring (resp. is a ring of positive dimension).*

Remark 17. *The same statement holds for sequences I of an arbitrary length bigger than $2k-1$ and their coverings.*

4. Final remarks

Given a slope $\alpha \in \mathbb{R}P^1$, let L_α be a straight line in $\mathbb{C}$ with slope α. (We use the real and the imaginary parts of z as standard coordinates in $\mathbb{C} \simeq \mathbb{R}^2$ and L_α is defined up to a parallel translation.)

Definition 18. For a given equation (1) and a slope $\alpha \in \mathbb{R}P^1$, define the *oscillation number* of (1) in the direction α as:

$$O(\alpha) := \sup_{y \in Sol} \sharp(y, L_\alpha), \tag{12}$$

where Sol denotes the projective space of all non-trivial solutions of (1) considered up to non-vanishing factor and $\sharp(y, L_\alpha)$ is the counted with multiplicities number of zeros of a nontrivial solution $y \in Sol$ lying on the straight line L_α. The number $O(\alpha)$ considered as an integer-valued function of $\alpha \in \mathbb{R}P^1$ is called the *oscillation function* of (1).

Remark 19. (i) by translation invariance, the right-hand side of (12) is independent of a particular choice of L_α;
(ii) the number $O(\alpha)$ is allowed to attain value $+\infty$;
(iii) for any equation (1) of order k and any $\alpha \in \mathbb{R}P^1$, $O(\alpha) \geq k-1$.

Further, for a given slope $\alpha \in \mathbb{R}P^1$ and a non-negative number $w \geq 0$, let $S_{\alpha,w}$ be an infinite open strip of width w bounded by two straight lines with slope α. By definition, $S_{\alpha,0}$ is a straight line with slope α. (Notice that $S_{\alpha,w}$ is defined up to a parallell translation.)

Definition 20. For a given equation (1) and a given $\alpha \in \mathbb{R}P^1$, define the *width* of (1) in the direction α as

$$W(\alpha) := \sup_{w} \{\sharp(y, S_{\alpha,w}) \leq O(\alpha),\ \forall y \in Sol\}, \tag{13}$$

where $\sharp(y, S_{w,\alpha})$ is the number of counted with multiplicities zeros of a nontrivial solution $y \in Sol$ lying in the strip $S_{\alpha,w}$. The number $W(\alpha)$ considered as an non-negative function of $\alpha \in \mathbb{R}P^1$ is called the *width function* of (1).

Remark 21. (i) by translation invariance, the right-hand side of (13) is independent of a particular choice of $S_{\alpha,w}$;
(ii) $W(\alpha)$ is allowed to attain value $+\infty$; if $W(\alpha) = +\infty$, then $S_{\alpha,+\infty} \simeq \mathbb{C}$.

The following question is the main topic of the present paper.

Problem 22. *For a given equation* (1)*, find/estimate its oscillation function $O(\alpha)$ and its width function $W(\alpha)$.*

Example 23. (1) For any equation (1) of the first order, $O(\alpha) \equiv 0$ and $W(\alpha) \equiv +\infty$ for any $\alpha \in \mathbb{R}P^1$.
(2) For equations (1) of order 2, the situation is as follows. If (1) has coinciding characteristic numbers $\lambda_1 = \lambda_2$, then $O(\alpha) \equiv 1$ and $W(\alpha) \equiv +\infty$. If $\lambda_1 \neq \lambda_2$, then $O(\alpha) = 1$ for all α not perpendicular to $\lambda_1 - \lambda_2$ and $O(\alpha^\star) = +\infty$ for the unique value of $\alpha^\star$ perpendicular to $\lambda_1 - \lambda_2$. $W(\alpha^\star) = +\infty$ while for $\alpha \neq \alpha^\star$, $W(\alpha) = ...$
(3) For an equation of any order k with all coinciding characteristic numbers, $O(\alpha) \equiv k - 1$ and $W(\alpha) \equiv +\infty$ for any $\alpha \in \mathbb{R}P^1$.

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