A NOTE ON REAL ZEROS OF EXPONENTIAL POLYNOMIALS

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Abstract. We discuss a real analogue of the Loxton–van der Poorten problem on zeros of linear recurrence sequences. For a linear homogeneous differential equation with constant coefficients of order k, we ask whether the number of real zeros of a non-trivial solution is bounded solely in terms of k, provided that the equation is real non-degenerate, i.e. no two distinct characteristic roots have the same real part. We prove a simple criterion for the occurrence of infinitely many real zeros. We then isolate the part of a Schanuel-type argument which gives a uniform bound for finite sets of zeros satisfying an explicit predimension-genericity condition. The remaining obstruction is formulated as a concrete independence problem for the products of the zeros with the real and imaginary parts of the characteristic roots.

1. Introduction

Let

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0, \qquad a_j \in \mathbb{C}, \tag{1}$$

be a linear homogeneous differential equation with constant coefficients, and let $\lambda_1, \dots, \lambda_s$ be its distinct characteristic roots, with multiplicities $m_1, \dots, m_s$. A solution of (1) has the form

$$f(x) = \sum_{j=1}^{s} P_j(x) e^{\lambda_j x}, \qquad \deg P_j < m_j, \tag{2}$$

where the P_j are complex polynomials. We shall call such functions *exponential polynomials* or *quasipolynomials.* The case in which all P_j are constant is also often called an exponential sum.

Zeros of exponential polynomials have been studied since the classical work of Langer, Bell, Turrittin and others; see, for example, [9, 3, 16, 6, 15, 12, 13, 17]. Our concern is not the asymptotic distribution of complex zeros, but the following elementary boundedness problem on the real axis.

Definition 1. *An equation* (1) *is called* real non-degenerate *if no two distinct characteristic roots have the same real part.*

The terminology is chosen in analogy with the usual non-degeneracy condition for linear recurrence sequences, where no quotient of two distinct characteristic roots is a root of unity. The analogy is not perfect, but it is useful: in both settings the excluded degeneracies are precisely those which allow an infinite periodic or oscillatory set of zeros.

Proposition 2. *Equation* (1) *has a non-trivial solution with infinitely many real zeros if and only if two distinct characteristic roots of* (1) *have the same real part.*

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Proof. Suppose first that $\lambda_p \neq \lambda_q$ and $\operatorname{Re}\lambda_p = \operatorname{Re}\lambda_q$. Then $\lambda_p - \lambda_q = \sqrt{-1}\beta$ with $\beta \in \mathbb{R} \setminus \{0\}$. The solution $f(x) = e^{\lambda_p x} - e^{\lambda_q x}$ has zeros at the real points $x = 2\pi m/\beta$, $m \in \mathbb{Z}$, after changing the sign of β if necessary. Thus infinitely many real zeros occur.

Conversely, assume that the real parts of the distinct characteristic roots are pairwise distinct. Order them so that $\operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_s$. If f is given by (2) and $P_s \neq 0$, then

$$e^{-\lambda_s x} f(x) = P_s(x) + \sum_{j<s} P_j(x) e^{(\lambda_j - \lambda_s)x}.$$

As $x \to +\infty$, the second term is exponentially small compared with the non-zero polynomial $P_s(x)$. Hence f has no real zeros for all sufficiently large positive x. The same argument applied to the characteristic root with minimal real part gives absence of zeros for all sufficiently large negative x. Since f is analytic and not identically zero, it has only finitely many zeros in the remaining compact interval. □

Let M_k denote the supremum, over all real non-degenerate equations of order k and all their non-trivial solutions, of the number of real zeros counted without multiplicity. Proposition 2 shows that each individual solution has only finitely many real zeros, but does not give a bound depending only on k.

Problem 3. *Is $M_k < \infty$ for every k?*

The corresponding problem for integer zeros of linear recurrence sequences is classical. The Skolem–Mahler–Lech theorem describes the zero set of a linear recurrence sequence as a union of a finite set and finitely many arithmetic progressions. Loxton and van der Poorten conjectured that, for non-degenerate recurrences of order k, the size of the finite part is bounded by a constant depending only on k [11]. This conjecture was proved quantitatively by Schmidt [14] and by Evertse–Schlickewei [7]; see also Allen [1]. Beukers proved the sharp ternary value $\mu_3 = 6$ [4], while Bavencoffe–Bezivin constructed a family giving the lower bound $\mu_k \geq \binom{k+1}{2} - 1$ [2].

We propose the following real analogue.

Conjecture 4. *For every $k \geq 1$, the number M_k is finite.*

The purpose of the present note is modest. We do not prove Conjecture 4. Instead, we isolate a natural Schanuel-predimension calculation which would give a uniform bound for those finite zero sets which satisfy an explicit genericity condition. This separates the transcendence part of the argument from the remaining, and apparently harder, linear-independence problem.

2. A conditional Schanuel bound

We shall use the following standard form of Schanuel's conjecture.

Conjecture 5 (Schanuel)**.** *For every finite set $U \subset \mathbb{C}$,*

$$\delta(U) := \operatorname{trdeg}_{\mathbb{Q}}\big(U, e^U\big) - \operatorname{ldim}_{\mathbb{Q}}(U) \geq 0,$$

where $e^U = \{e^u : u \in U\}$.

For finite sets $U, W \subset \mathbb{C}$ put

$$\delta(U/W) := \delta(U \cup W) - \delta(W).$$

Equivalently,

$$\delta(U/W) = \operatorname{trdeg}_{\mathbb{Q}}\big((U, e^U)/(W, e^W)\big) - \operatorname{ldim}_{\mathbb{Q}}(U/W). \tag{3}$$

Under Schanuel's conjecture one has

$$\delta(U/W) \geq -\delta(W). \tag{4}$$

We shall also use the elementary estimate

$$\delta(W) \leq |W|. \tag{5}$$

Indeed, if $r = \operatorname{ldim}_{\mathbb{Q}} W$, then $\operatorname{trdeg}_{\mathbb{Q}}(W, e^W) \leq 2r$, and hence $\delta(W) \leq r \leq |W|$.

For the rest of this section consider an exponential sum

$$f(x) = 1 + c_1 e^{a_1 x} + \cdots + c_n e^{a_n x}, \qquad c_j \in \mathbb{C}^*,\ a_j \in \mathbb{C}. \tag{6}$$

Write $a_j = b_j + \sqrt{-1} d_j$ with $b_j, d_j \in \mathbb{R}$, and set

$$A = \{a_1, \ldots, a_n\}, \quad \bar{A} = \{\bar{a}_1, \ldots, \bar{a}_n\}, \quad C = \{c_1, \ldots, c_n\}, \quad \bar{C} = \{\bar{c}_1, \ldots, \bar{c}_n\}.$$

Let B and D be fixed $\mathbb{Q}$-bases of the $\mathbb{Q}$-linear spans of $\{b_1, \ldots, b_n\}$ and $\{d_1, \ldots, d_n\}$, respectively. For a real number r put

$$U(r) := Ar \cup \bar{A}r, \qquad V(r) := Br \cup \sqrt{-1}Dr.$$

The sets $U(r)$ and $V(r)$ span the same $\mathbb{Q}$-vector space over $A \cup \bar{A} \cup B \cup \sqrt{-1}D$.

Definition 6. *Let $r_1, \ldots, r_\ell$ be real zeros of f. We say that this finite zero set is* Schanuel-regular *if, putting*

$$W_0 = A \cup \bar{A} \cup C \cup \bar{C}, \qquad W_j = W_0 \cup U(r_1) \cup \cdots \cup U(r_j),$$

one has

$$\delta\big(U(r_j)/W_{j-1}\big) \leq -1, \qquad j = 1, \ldots, \ell. \tag{7}$$

The condition is deliberately stated in predimension form. It is precisely the point at which the attempted proof of the full conjecture needs an additional independence input. Informally, (7) says that each new zero contributes one genuinely new algebraic relation among the corresponding exponentials, while the associated logarithms retain the expected linear independence over the previously chosen data.

Theorem 7 (Conditional Schanuel bound)**.** *Assume Schanuel's conjecture. Let f be as in* (6)*. If $r_1, \ldots, r_\ell$ is a Schanuel-regular set of real zeros of f, then*

$$\ell \leq 4n.$$

Proof. By additivity of relative predimension and by (7),

$$\delta\big(U(r_1) \cup \cdots \cup U(r_\ell)/W_0\big) = \sum_{j=1}^{\ell} \delta\big(U(r_j)/W_{j-1}\big) \leq -\ell.$$

On the other hand, Schanuel's conjecture gives, by (4),

$$\delta\big(U(r_1) \cup \cdots \cup U(r_\ell)/W_0\big) \geq -\delta(W_0).$$

Therefore $\ell \leq \delta(W_0)$. Since W_0 has at most $4n$ elements, (5) gives $\delta(W_0) \leq 4n$, and the result follows. □

Remark 8. *For a real zero r one has two algebraic equations*

$$1 + c_1 e^{a_1 r} + \cdots + c_n e^{a_n r} = 0, \qquad 1 + \bar{c}_1 e^{\bar{a}_1 r} + \cdots + \bar{c}_n e^{\bar{a}_n r} = 0.$$

Moreover the exponentials $e^{a_j r}$ and $e^{\bar{a}_j r}$ are algebraic over the field generated by the exponentials of a $\mathbb{Q}$-basis of the products $Br \cup \sqrt{-1}Dr$. Thus it is natural to expect a negative contribution to relative predimension from each sufficiently independent zero. What is not automatic is that $\mathbb{Q}$-linear independence of the zeros themselves implies the needed independence of all products $b_i r$ and $d_i r$. This is the missing step in the direct Schanuel proof of Conjecture 4.

A more concrete, but still conjectural, way to express the missing input is the following.

Conjecture 9 (Product-independence principle). *Let f be a real non-degenerate exponential sum of the form* (6)*. There is a constant $N(f)$ such that every finite set of real zeros of f with more than $N(f)$ elements contains a large subset for which the products $Br \cup \sqrt{-1}Dr$ have the expected $\mathbb{Q}$-linear dimension over the parameters and the zero equations give the predimension drop* (7)*.*

This principle is intentionally formulated as a target rather than as a theorem. It is the precise place where a proof of Conjecture 4 would have to supply new information. In particular, Schanuel's conjecture alone does not immediately turn $\mathbb{Q}$-linear independence of zeros into independence of the products with the real and imaginary parts of the characteristic roots.

3. Possible forms of the main conjecture

The discussion above suggests several levels of the same problem.

Conjecture 10 (Weak real Loxton–van der Poorten conjecture). *For every k there exists a constant M_k such that every real non-degenerate equation* (1) *of order k has at most M_k real zeros in any non-trivial solution.*

Conjecture 11 (Polynomial growth form). *The constants M_k in Conjecture 10, if finite, may be chosen to grow polynomially in k.*

The second formulation is speculative but is consistent with the known lower bounds in the discrete recurrence problem. At present even the finiteness of M_k appears open. The conditional bound in Theorem 7 indicates that the main obstacle is not the formal Schanuel-predimension calculation itself, but rather the extraction of sufficiently regular zero sets from arbitrary large zero sets.

Remark 12. *The note has been written for exponential sums, i.e. for simple characteristic roots and constant coefficients in* (6)*. General quasipolynomials with polynomial factors should be treatable by adding the coefficients of these polynomials to the parameter set. This changes the bookkeeping but not the central issue: one still needs a product-independence statement for the zeros.*

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