

LEVEL CROSSING IN RANDOM MATRICES. III. ANALOGS OF GIRKO'S CIRCULAR AND WIGNER'S SEMICIRCLE LAWS

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ABSTRACT. We study the asymptotic distribution of level crossings for random matrix pencils $A_n + \lambda B_n$ in several ensembles, including complex and real i.i.d. matrices and Gaussian/Hermitian settings. We derive a representation of the normalized log-discriminant in terms of pairwise eigenvalue interactions and formulate conditions under which its limit is governed by a deterministic potential.

Under assumptions combining a uniform circular law, logarithmic tail control, and small-spacing (repulsion) estimates, we prove convergence of the empirical measure of level crossings to an explicit deterministic limit. In the complex Gaussian case these assumptions are verified (modulo a uniformity step), while in the general i.i.d. setting the results are conditional and motivated by universality theory.

We further analyze the real case, showing that any limiting measure does not concentrate on the real projective line under suitable hypotheses, and discuss analogous phenomena for elliptic/Hermitian ensembles. Our results highlight the role of logarithmic energy and universality in governing spectral degeneracies of random matrix pencils.

1. INTRODUCTION

Classical random matrix theory is largely concerned with the asymptotic distribution of eigenvalues of a single large random matrix. Wigner's semicircle law and Girko's circular law are the two basic universality statements of this kind; see e.g. [Gi1, Gi2, Me, AGZ]. The present paper studies a closely related, but genuinely different, object: the distribution of degeneracies in a random one-parameter spectral family

$$A + \lambda B, \quad \lambda \in \mathbb{C}. \quad (1.1)$$

A level crossing is a value of λ for which $A + \lambda B$ has a multiple eigenvalue. Thus the problem is not to describe where the eigenvalues of one random matrix lie, but rather where the branch points of the spectral cover of the pencil are located in the parameter sphere $\mathbb{C}P^1$.

Such pencils occur naturally in perturbation theory: A is an unperturbed operator, B is a perturbation, and λ is the perturbation parameter; see [Ka]. They also arise in questions about monodromy of spectra and random Riemann surfaces, cf. [GP].

The present work continues the study initiated in [ShZa, GrShZa], where several exact formulas were obtained for Gaussian pencils. Several results in this paper are conditional on hypotheses such as uniform local circular law (UCL) and small-repulsion (SR). Outside the Gaussian setting, these should be viewed as conjectural inputs motivated by universality results in non-Hermitian random matrix theory.

For complex Gaussian matrices, the picture is especially simple. If A and B are independent complex Ginibre matrices, then for every matrix size the level crossings

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are uniformly distributed on $\mathbb{C}P^1$; in the affine coordinate $\lambda = x + iy$ this means the density

$$\frac{dx dy}{\pi(1 + |\lambda|^2)^2}. \quad (1.2)$$

This exact finite-dimensional identity, recalled below from [GrShZa], is the starting point of the paper. Our main aim is to understand to what extent (1.2) survives beyond the Gaussian situation.

The first part of the paper treats non-Hermitian complex pencils with independent entries. We formulate a conditional asymptotic uniform law: if the normalized eigenvalue process of

$$\frac{A_n + \lambda B_n}{\sqrt{n(1 + |\lambda|^2)}}$$

satisfies a uniform circular-law statement and if close eigenvalue pairs do not contribute to the leading logarithmic energy, then the empirical measure of level crossings converges in probability to (1.2). The proof is potential-theoretic. The discriminant of the characteristic polynomial records precisely the level crossings, and the Poincaré–Lelong formula converts the asymptotics of the normalized log-discriminant into the limiting level-crossing measure. In this sense the result is the level-crossing counterpart of Girko’s circular law.

The real case contains an additional phenomenon absent from complex Ginibre pencils. Since real pencils are invariant under conjugation, level crossings may lie on the real projective line, and such real crossings could in principle produce a singular component in the limiting measure. We show that a singular component on $\mathbb{R}P^1$ is excluded under a stronger non-concentration assumption near $\mathbb{R}P^1$. The weaker condition that the expected number of exactly real crossings is $o(n^2)$ only implies that the finite- n mass on $\mathbb{R}P^1$ vanishes. Under such a non-concentration assumption, the possible singular contribution on $\mathbb{R}P^1$ is ruled out, and the remaining problem concerns the two-dimensional distribution in the complex plane. We then formulate a real i.i.d. conjecture which refines the naive uniform-law expectation by allowing a correction depending on the real symmetry class. For GOE pencils we go further: after normalization, the entire dependence on the parameter λ is encoded by the pseudocovariance

$$\tau(\lambda) = \frac{1 + \lambda^2}{1 + |\lambda|^2}.$$

At the level of expected logarithmic potentials, this reduces the possible limiting densities to expressions obtained by applying $\frac{1}{2\pi}\Delta_\lambda$ to a potential of the form

$$\frac{1}{2} \log(1 + |\lambda|^2) + G\left(\frac{|1 + \lambda^2|^2}{(1 + |\lambda|^2)^2}\right).$$

In particular, the GOE uniformity conjecture of [GrShZa] is reduced to the single analytic assertion that the pseudocovariance does not affect the leading n^2 -term of the log-discriminant. This identifies precisely where the difficulty of the real symmetric case lies.

The Hermitian case is the level-crossing analogue of the Wigner regime, and it is qualitatively different from the Ginibre case. The limiting distribution is not expected to be uniform on $\mathbb{C}P^1$. Already for GUE_2 one has the explicit density from [GrShZa]

$$\frac{4 |\Im \lambda| dx dy}{\pi (1 + |\lambda|^2)^3},$$

which vanishes on the real axis and is invariant under the natural $SO(2)$ -symmetry of the pencil. For general GUE_n we prove that the large- n problem can be expressed

through the Gaussian elliptic ensemble. The normalized matrix

$$\frac{A_n + \lambda B_n}{\sqrt{1 + |\lambda|^2}}$$

has elliptic parameter determined by

$$|\tau(\lambda)|^2 = \frac{|1 + \lambda^2|^2}{(1 + |\lambda|^2)^2} = 1 - Y^2,$$

where Y is the height coordinate on the sphere. Assuming a uniform logarithmic-energy theorem for the elliptic ensemble in the bulk, we obtain a conditional version of the existence of the limiting GUE_n level-crossing density on every compact subset away from the equator $Y = 0$. The limiting density is given potential-theoretically by

$$\frac{1}{2\pi} \Delta_\lambda \left[\frac{1}{2} \log(1 + |\lambda|^2) + G(1 - Y^2) \right],$$

where G is the logarithmic energy of the corresponding elliptic law. This proves the proposed GUE limiting picture off the real axis and gives the natural candidate for the universal Wigner-type law for Hermitian pencils.

2. COMPLEX MATRICES

Recall that the $GE_n^{\mathbb{C}}$ -ensemble on $Mat_n^{\mathbb{C}}$ is the distribution where each entry of a $n \times n$ -matrix is an independent complex random variable distributed as $N(0, \frac{1}{2}) + iN(0, \frac{1}{2})$, and $Mat_n^{\mathbb{C}}$ is the space of all complex-valued $n \times n$ -matrices. The $GE_{\sigma^2, n}^{\mathbb{C}}$ -ensemble on $Mat_n^{\mathbb{C}}$ is the distribution where each off-diagonal entry of a $n \times n$ -matrix is an independent complex random variable distributed as $N(0, \frac{1}{2}) + iN(0, \frac{1}{2})$ and each diagonal entry is distributed as $N(0, \sigma^2) + iN(0, \sigma^2)$. (Obviously, the $GE_n^{\mathbb{C}}$ -ensemble is a special case of the $GE_{\sigma^2, n}^{\mathbb{C}}$ -ensemble for $\sigma = 1$.) Recall the following result of [GrShZa] dealing with complex Gaussian ensembles.

Theorem A (cf. Theorem 1 and Proposition 1 of [GrShZa]). *For any given size $n \geq 1$, if A and B are independently chosen from the $GE_{\sigma^2, n}^{\mathbb{C}}$ -ensemble, then the distribution of the level crossings in the affine coordinate $\lambda = x + iy$ of \mathbb{C} is given by*

$$\mathcal{P}_{GE_{\sigma^2, n}^{\mathbb{C}}}(x, y) dx dy = \frac{dx dy}{\pi(1 + x^2 + y^2)^2} = \frac{dx dy}{\pi(1 + |\lambda|^2)^2}. \quad (2.1)$$

Remark 1. An alternative way to express this fact is as follows. Consider the standard cylindrical coordinate system (ρ, ϕ, Z) in \mathbb{R}^3 , where $\rho \geq 0, 0 \leq \phi \leq 2\pi, Z \in \mathbb{R}$. Recall that

$$X = \rho \cos \phi, Y = \rho \sin \phi, Z = Z.$$

If we consider (ϕ, Z) , $0 \leq \phi \leq 2\pi, -1 \leq Z \leq 1$, as coordinates on the unit sphere $S^2 \simeq \mathbb{C}P^1$ (with both poles removed), then in these coordinates the usual area element on the sphere is given by

$$dA = d\phi dZ.$$

Thus, in cylindrical coordinates (ϕ, Z) , $0 \leq \phi \leq 2\pi; -1 \leq z \leq 1$ parameterising the unit sphere S^2 , the measure $\mathcal{P}_{GE_{\sigma^2, n}^{\mathbb{C}}}(x, y) dx dy$ given by (2.1) transforms into

$$\mathcal{P}_{GE_{\sigma^2, n}^{\mathbb{C}}}(\phi, Z) d\phi dZ = \frac{d\phi dZ}{4\pi}. \quad (2.2)$$

Remark 2. Observe that formula (2.1) is independent of the size n and also of the variance σ^2 . For $n = 1$, formula (2.1) gives the distribution of the quotient of two independent complex Gaussian random variables.

In fact, the right-hand side of (2.1) gives the distribution of level crossings for a much broader class of complex Gaussian ensembles. Namely, consider the following $SU(2)$ -action on $Mat_n^{\mathbb{C}} \times Mat_n^{\mathbb{C}}$. A matrix $\mathfrak{U} = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$, $|u|^2 + |v|^2 = 1$ from $SU(2)$ acts on the latter product as:

$$\mathfrak{U} : (A, B) \mapsto (uA + vB, -\bar{v}A + \bar{u}B). \quad (2.3)$$

Definition 1. We say that a linear subspace $W \subset Mat_n^{\mathbb{C}}$ is *adjusted to the $SU(2)$ -action* (2.3) if the product space $W \times W \subset Mat_n^{\mathbb{C}} \times Mat_n^{\mathbb{C}}$ is preserved by (2.3).

Theorem B (See Proposition 2, [GrShZa]). *Any complex linear subspace $W \subset Mat_n^{\mathbb{C}}$ adjusted to the above $SU(2)$ -action with the measure induced from $GE_{\sigma^2, n}^{\mathbb{C}}$, $\sigma^2 > 0$ has the property that the distribution of the level crossings of (1.1) with random A and B independently taken from W according to the latter induced measure is given by (2.1), i.e., uniformly distributed on $\mathbb{C}\mathbb{P}^1$.*

Remark 3. Interesting examples of linear subspaces W adjusted to the $SU(2)$ -action include complex symmetric matrices, complex Toeplitz matrices, complex band matrices, complex band Toeplitz matrices, complex diagonal matrices etc.

Since (2.1) appear as the distribution of level crossings for complex Gaussian ensembles, we will call it the *uniform law* for level crossings. Let us now formulate the *asymptotic uniform law* for complex non-Gaussian ensembles which we consider as a natural analog of Girko's circular law for the spectrum of random complex matrices.

Theorem 1 (Conditional asymptotic uniform law for full complex pencils). *Let $A_n = (a_{ij}^{(n)})$ and $B_n = (b_{ij}^{(n)})$ be independent $n \times n$ random matrices with i.i.d. complex entries. Assume that*

$$\mathbb{E} a_{11}^{(n)} = \mathbb{E} b_{11}^{(n)} = 0, \quad \mathbb{E} |a_{11}^{(n)}|^2 = \mathbb{E} |b_{11}^{(n)}|^2 = \sigma^2 > 0.$$

For $\lambda \in \mathbb{C}$, set

$$C_n(\lambda) := A_n + \lambda B_n, \quad \widehat{C}_n(\lambda) := \frac{C_n(\lambda)}{\sigma \sqrt{n(1 + |\lambda|^2)}}.$$

Let $\zeta_1^{(n)}(\lambda), \dots, \zeta_n^{(n)}(\lambda)$ be the eigenvalues of $\widehat{C}_n(\lambda)$, and let

$$L_{n,\lambda} := \frac{1}{n} \sum_{k=1}^n \delta_{\zeta_k^{(n)}(\lambda)}.$$

Also set

$$M_{n,\lambda} := \frac{1}{n(n-1)} \sum_{i \neq j} \delta_{(\zeta_i^{(n)}(\lambda), \zeta_j^{(n)}(\lambda))}.$$

Assume in addition that the following three conditions hold.

(UCL) For every compact set $K \subset \mathbb{C}$ and every bounded continuous $f : \mathbb{C} \rightarrow \mathbb{C}$,

$$\sup_{\lambda \in K} \left| \int_{\mathbb{C}} f dL_{n,\lambda} - \int_{\mathbb{C}} f d\sigma_{\text{circ}} \right| \rightarrow 0 \quad \text{in probability,}$$

where σ_{circ} is the uniform probability measure on the unit disk.

(SR) For every compact set $K \subset \mathbb{C}$ and every $\eta > 0$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\lambda \in K} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbf{1}_{\{|\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)| \leq \varepsilon\}} \left| \log |\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)| \right| > \eta \right) = 0.$$

(LT) For every compact set $K \subset \mathbb{C}$ and every $\eta > 0$,

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\lambda \in K} \iint_{\max\{|z|, |w|\} > R} \log(2 + |z| + |w|) dM_{n,\lambda}(z, w) > \eta \right) = 0.$$

Then the empirical level-crossing measure of the pencil

$$A_n + \lambda B_n$$

converges weakly in probability to the uniform measure on $\mathbb{C}\mathbb{P}^1$, i.e. to the probability measure with density

$$\frac{dx dy}{\pi(1 + |\lambda|^2)^2}$$

in the affine coordinate $\lambda = x + iy$.

Proposition 1 (A sufficient uniform circular-law input). *Let $K \subset \mathbb{C}$ be compact. Suppose that the normalized entries*

$$X_\lambda := \frac{a_{11} + \lambda b_{11}}{\sigma \sqrt{1 + |\lambda|^2}}, \quad \lambda \in K,$$

form a compact family of centered, unit-variance, non-degenerate laws for which Girko's hermitization estimates hold uniformly in the following sense: for every compact $D \subset \mathbb{C}$ and every $\eta > 0$,

$$\sup_{\lambda \in K} \mathbb{P} \left(\sup_{z \in D} \left| \frac{1}{n} \log |\det(\widehat{C}_n(\lambda) - zI)| - \int_{\mathbb{C}} \log |w - z| d\sigma_{\text{circ}}(w) \right| > \eta \right) \rightarrow 0,$$

after the usual logarithmic regularization at the singularity. Then condition (UCL) in Theorem 1 holds on K .

Remark 4. The above uniform hermitization input is known in the standard sub-gaussian bounded-density setting from local circular-law estimates, with constants uniform on compact families of entry laws; see Girko's hermitization method [Gi2] and the local circular law of Bourgade–Yau–Yin [BYY]. In finite-moment settings the same conclusion is expected after truncation and replacement, but this requires the corresponding uniform least-singular-value bounds.

Proof. By Girko's hermitization formula, convergence of the logarithmic potentials

$$z \mapsto \frac{1}{n} \log |\det(\widehat{C}_n(\lambda) - zI)|$$

to the logarithmic potential of σ_{circ} , uniformly for $\lambda \in K$ and locally uniformly in the regularized variable z , implies weak convergence of the empirical eigenvalue measures $L_{n,\lambda}$ to σ_{circ} , uniformly in $\lambda \in K$. More explicitly, let $f \in C_c^2(\mathbb{C})$. Integration by parts gives

$$\int_{\mathbb{C}} f dL_{n,\lambda} = \frac{1}{2\pi} \int_{\mathbb{C}} (\Delta f)(z) \frac{1}{n} \log |\det(\widehat{C}_n(\lambda) - zI)| dx dy,$$

and the same identity holds with $L_{n,\lambda}$ replaced by σ_{circ} . Taking the difference, then taking the supremum over $\lambda \in K$, the asserted uniform convergence of the regularized logarithmic potentials implies

$$\sup_{\lambda \in K} \left| \int f dL_{n,\lambda} - \int f d\sigma_{\text{circ}} \right| \rightarrow 0 \quad \text{in probability}$$

for all $f \in C_c^2(\mathbb{C})$. Approximation of bounded continuous test functions by compactly supported smooth functions, together with the tightness which follows from the same logarithmic-potential estimates, gives the stated form of (UCL). \square

Lemma 1 (Uniform two-point Wegner estimate implies (SR)). *Let $K \subset \mathbb{C}$ be compact. Suppose that for every $R > 0$ there is a constant $C_{K,R}$ such that, uniformly in n and uniformly in $0 < r < 1$,*

$$\mathbb{E} \left[\sup_{\lambda \in K} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbf{1}_{\{|\zeta_i^{(n)}(\lambda)| \leq R, |\zeta_j^{(n)}(\lambda)| \leq R, |\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)| \leq r\}} \right] \leq C_{K,R} r^2.$$

Assume also that, for every $\eta > 0$,

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{\lambda \in K} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbf{1}_{\{\max(|\zeta_i^{(n)}(\lambda)|, |\zeta_j^{(n)}(\lambda)|) > R\}} > \eta \right) = 0.$$

Then condition (SR) holds.

Proof. Decompose the small-distance logarithm dyadically:

$$\mathbf{1}_{\{|z-w| \leq \varepsilon\}} |\log |z-w|| \leq \sum_{m \geq 0} |\log(2^{-m} \varepsilon)| \mathbf{1}_{\{|z-w| \leq 2^{-m} \varepsilon\}}.$$

Restrict first to $|z|, |w| \leq R$, take the supremum over $\lambda \in K$, and then take expectations. The assumed uniform two-point estimate gives

$$\mathbb{E} \left[\sup_{\lambda \in K} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbf{1}_{\{|\zeta_i - \zeta_j| \leq \varepsilon, |\zeta_i|, |\zeta_j| \leq R\}} |\log |\zeta_i - \zeta_j|| \right] \leq C_{K,R} \sum_{m \geq 0} |\log(2^{-m} \varepsilon)| 2^{-2m} \varepsilon^2 = O_{K,R}(\varepsilon^2 |\log \varepsilon|).$$

The contribution from pairs with at least one eigenvalue outside $R\mathbb{D}$ is negligible by the second assumption, after first choosing R large. Markov's inequality now gives (SR). \square

Proposition 2 (Fixed-parameter small repulsion for complex Ginibre pencils). *If A_n and B_n are complex Ginibre matrices, then the fixed-parameter version of (SR) holds: for every fixed $\lambda \in \mathbb{C}$ and every $\eta > 0$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n(n-1)} \sum_{i \neq j} \mathbf{1}_{\{|\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)| \leq \varepsilon\}} |\log |\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)|| > \eta \right) = 0.$$

The compact-uniform version required in Theorem 1 follows if, in addition, one has a modulus-of-continuity estimate in λ strong enough to pass from fixed parameters to finite nets.

Proof. For every fixed λ , the normalized matrix $\widehat{C}_n(\lambda)$ is again a standard complex Ginibre matrix. Its eigenvalues form a determinantal point process with two-point correlation

$$\rho_2^{(n)}(z, w) = K_n(z, z)K_n(w, w) - |K_n(z, w)|^2.$$

The standard Ginibre kernel bound (see, e.g., [Me, Ch. 15]) gives, uniformly in n and locally uniformly in the bulk,

$$\rho_2^{(n)}(z, w) \leq Cn^2 |z-w|^2.$$

after the global normalization used above. Integrating this estimate over $|z-w| \leq r$ gives the fixed-parameter two-point bound. The exponentially small tail of the Ginibre spectral radius controls the contribution from outside a large disk. The dyadic argument in Lemma 1, without the supremum over K , then proves the stated fixed-parameter small-repulsion estimate. \square

Remark 5 (Status of the hypotheses beyond the Gaussian case). Proposition 1 shows that the global input (UCL) is not expected to be a new obstacle: it is a uniform version of the circular law, obtained by Girko hermitization and the standard local circular-law estimates for i.i.d. non-Hermitian matrices; see, for example, [Gi2, BYY]. The additional genuinely local input is (SR). Lemma 1

reduces (SR) to a two-point small-gap estimate, i.e. to local eigenvalue repulsion for the pencil.

For each fixed parameter in the complex Gaussian case this repulsion follows from the determinantal Ginibre kernel; see [Me, Ch. 15]. The compact-uniform form of (SR) additionally requires a modulus-of-continuity estimate in the pencil parameter. For general complex i.i.d. entries, we do not claim that (SR) is presently proved in the form stated above. Rather, it is the natural estimate predicted by bulk universality of local non-Hermitian spectral statistics, as in the universality results of Tao and Vu [TV], together with the uniform local circular-law technology of Bourgade–Yau–Yin [BYY] and a uniformity argument in the parameter λ . Thus, outside the Gaussian case, Theorem 1 should be read as conditional on this small-repulsion input. For discrete entries, or for real ensembles near \mathbb{RP}^1 , this two-point input is much more delicate and should be kept as an explicit assumption.

Proof of Theorem 1. Let

$$p_n(\lambda, t) := \det(C_n(\lambda) - tI), \quad \Delta_n(\lambda) := \text{Disc}_t p_n(\lambda, t).$$

The zeros of $\Delta_n(\lambda)$, counted with multiplicity, coincide with the level-crossing points of the pencil $A_n + \lambda B_n$. Since

$$\deg_\lambda \Delta_n = n(n-1),$$

the empirical level-crossing measure is

$$\mu_n = \frac{1}{n(n-1)} \sum_{\Delta_n(\lambda)=0} \delta_\lambda.$$

By the Poincaré–Lelong formula,

$$\mu_n = \frac{1}{2\pi n(n-1)} \Delta_\lambda \log |\Delta_n(\lambda)|.$$

Let $\xi_1^{(n)}(\lambda), \dots, \xi_n^{(n)}(\lambda)$ be the eigenvalues of $C_n(\lambda)$. Then

$$\Delta_n(\lambda) = \prod_{i < j} (\xi_i^{(n)}(\lambda) - \xi_j^{(n)}(\lambda))^2.$$

Since

$$\xi_k^{(n)}(\lambda) = \sigma \sqrt{n(1+|\lambda|^2)} \zeta_k^{(n)}(\lambda),$$

we get

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \log(\sigma \sqrt{n(1+|\lambda|^2)}) + \frac{2}{n(n-1)} \sum_{i < j} \log |\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)|.$$

Equivalently,

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \log(\sigma \sqrt{n(1+|\lambda|^2)}) + \iint_{\mathbb{C}^2 \setminus \text{Diag}} \log |z - w| dM_{n,\lambda}(z, w),$$

where

$$M_{n,\lambda} := \frac{1}{n(n-1)} \sum_{i \neq j} \delta_{(\zeta_i^{(n)}(\lambda), \zeta_j^{(n)}(\lambda))}.$$

Fix a compact set $K \subset \mathbb{C}$. For $0 < \varepsilon < 1 < R$, define the two-sided truncation

$$\Phi_{\varepsilon,R}(z, w) := \max\{\log \varepsilon, \min(\log |z - w|, \log R)\}.$$

This is a bounded continuous function on \mathbb{C}^2 . Moreover, for bounded continuous functions one has

$$\iint \Phi_{\varepsilon,R} dM_{n,\lambda} = \iint \Phi_{\varepsilon,R} dL_{n,\lambda}(z) dL_{n,\lambda}(w) + O_{\varepsilon,R}(1/n),$$

because $M_{n,\lambda}$ differs from $L_{n,\lambda} \otimes L_{n,\lambda}$ only by the diagonal correction. Hence (UCL) implies

$$\sup_{\lambda \in K} \left| \iint \Phi_{\varepsilon,R}(z,w) dM_{n,\lambda}(z,w) - \iint \Phi_{\varepsilon,R}(z,w) d\sigma_{\text{circ}}(z) d\sigma_{\text{circ}}(w) \right| \rightarrow 0$$

in probability.

We now remove the truncations. The lower truncation error is supported on $\{|z-w| < \varepsilon\}$, and is bounded by

$$\iint_{\{|z-w| < \varepsilon\}} |\log|z-w|| dM_{n,\lambda} + |\log \varepsilon| M_{n,\lambda}\{|z-w| < \varepsilon\},$$

which is negligible uniformly on K by (SR). The upper truncation error is supported on $\{|z-w| > R\}$ and is bounded by a constant multiple of

$$\iint_{\max\{|z|,|w|\} > R/2} \log(2+|z|+|w|) dM_{n,\lambda}(z,w),$$

which is negligible uniformly on K by (LT) as $R \rightarrow \infty$. The same upper truncation error is absent for the circular law once $R > 2$, since σ_{circ} is supported on the unit disk, and the lower truncation converges by monotone convergence. Consequently

$$\sup_{\lambda \in K} \left| \iint \log|z-w| dM_{n,\lambda} - \iint \log|z-w| d\sigma_{\text{circ}}(z) d\sigma_{\text{circ}}(w) \right| \rightarrow 0$$

in probability, after taking first $n \rightarrow \infty$, then $\varepsilon \downarrow 0$, and finally $R \rightarrow \infty$. Denote the limiting energy by

$$I_{\text{circ}} := \iint \log|z-w| d\sigma_{\text{circ}}(z) d\sigma_{\text{circ}}(w).$$

Thus

$$\sup_{\lambda \in K} \left| \frac{2}{n(n-1)} \sum_{i < j} \log |\zeta_i^{(n)}(\lambda) - \zeta_j^{(n)}(\lambda)| - I_{\text{circ}} \right| \rightarrow 0$$

in probability.

Therefore,

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \frac{1}{2} \log(1+|\lambda|^2) + c_n + o_{\mathbb{P}}(1)$$

in probability in $L_{\text{loc}}^1(\mathbb{C})$, where

$$c_n := \log(\sigma\sqrt{n}) + I_{\text{circ}}.$$

This is the natural mode of convergence, since $\log |\Delta_n|$ has logarithmic singularities at level crossings and hence cannot converge locally uniformly on sets meeting its zero set. Applying $\frac{1}{2\pi} \Delta_\lambda$ in the sense of distributions, the constants c_n disappear and we get

$$\mu_n \implies \frac{1}{2\pi} \Delta_\lambda \left(\frac{1}{2} \log(1+|\lambda|^2) \right) dx dy = \frac{dx dy}{\pi(1+|\lambda|^2)^2}.$$

This is exactly the uniform distribution on $\mathbb{C}\mathbb{P}^1$. \square

3. REAL MATRICES

3.1. Real orthogonal case. The uniform law seems to be valid for a number of real Gaussian ensembles on the spaces of real and real symmetric matrices. Recall that the $GOE_n^{\mathbb{R}}$ -distribution on $Sym_n^{\mathbb{R}}$ is the distribution where each entry $e_{i,j} = e_{j,i}$, $i < j$ of a $n \times n$ -matrix has a normal (and independent of the other entries) distribution $N(0,1)$, and each diagonal entry $e_{i,i}$ is independently distributed as $\sqrt{2}N(0,1)$. Here $Sym_n^{\mathbb{R}}$ is the space of real-valued symmetric matrices. Let us formulate the main result and conjecture from [GrShZa] concerning random pencils of matrices with real coefficients.

Theorem C. *If A and B are independently chosen from $GOE_2^{\mathbb{R}}$, then the distribution of the branch points is uniform on $\mathbb{C}\mathbb{P}^1 \supset \mathbb{C}$, i.e., its density in the affine coordinate $\lambda = x + iy$ is given by (2.1).*

Conjecture 1. *Conclusion of Theorem C is valid if A and B are independently chosen from any $GOE_n^{\mathbb{R}}$, $n \geq 3$.*

Remark 6. Observe that Theorem 1 is, in general, wrong for real symmetric matrices, i.e. scaling of the main diagonal of real symmetric matrices by a large number leads to a distribution of the branch points different from (2.1). In particular, this distribution is not radial in the complex λ -plane. On the other hand, numerical experiments suggest that if one scales the main diagonal by a number in $[0, 1]$, formula (2.1) is still valid! At the moment it is unclear for which value of the scaling parameter, the “phase transition” happens.

3.2. Reduction of the GOE conjecture to pseudocovariance-independence.

In this subsection we refine Conjecture 1 by exploiting the special Gaussian structure of the GOE ensemble. We show that the conjecture reduces to a single analytic question about the dependence of the log-discriminant on the pseudocovariance parameter.

Let A_n, B_n be independent GOE matrices, and consider the pencil

$$C_n(\lambda) := A_n + \lambda B_n, \quad \lambda \in \mathbb{C}.$$

For $i < j$, the entries of $C_n(\lambda)$ are of the form

$$(C_n(\lambda))_{ij} = a_{ij} + \lambda b_{ij},$$

where $a_{ij}, b_{ij} \sim N(0, 1)$ are independent real Gaussians. On the diagonal,

$$(C_n(\lambda))_{ii} = a_{ii} + \lambda b_{ii}, \quad a_{ii}, b_{ii} \sim N(0, 2).$$

Define the normalized matrix

$$\widehat{C}_n(\lambda) := \frac{C_n(\lambda)}{\sqrt{1 + |\lambda|^2}}.$$

Then the entries of $\widehat{C}_n(\lambda)$ are centered complex Gaussian random variables with

$$\mathbb{E}|\widehat{C}_n(\lambda)_{ij}|^2 = 1, \quad \mathbb{E}(\widehat{C}_n(\lambda)_{ij}^2) = \tau(\lambda),$$

where

$$\tau(\lambda) := \frac{1 + \lambda^2}{1 + |\lambda|^2}.$$

Thus the law of $\widehat{C}_n(\lambda)$ depends on λ only through the single complex parameter $\tau(\lambda)$, and hence through

$$q(\lambda) := |\tau(\lambda)|^2 = \frac{|1 + \lambda^2|^2}{(1 + |\lambda|^2)^2}.$$

Let $\Delta_n(\lambda)$ denote the discriminant of the characteristic polynomial of $C_n(\lambda)$, so that the empirical level-crossing measure is given by

$$\mu_n = \frac{1}{2\pi n(n-1)} \Delta_\lambda \log |\Delta_n(\lambda)|.$$

Remark 7 (Status of the real-case assumptions). The main technical input in both the real symmetric and Hermitian cases is control of the logarithmic energy uniformly in the elliptic parameter. This means two separate estimates: a global elliptic-law input and a local small-spacing/no-clustering input which makes the logarithmic singularity negligible. For Gaussian elliptic ensembles these estimates are supported by explicit Pfaffian or determinantal correlation kernels; for general

real i.i.d. entries they should follow only after combining local circular/elliptic-law estimates with bulk universality. We therefore state the real results below as conditional statements, except in the special finite-dimensional cases quoted from [GrShZa].

The mode of convergence for normalized log-discriminants should be L_{loc}^1 in probability, or equivalently distributional convergence after applying $(2\pi)^{-1}\Delta_\lambda$, rather than locally uniform convergence. Indeed $\log|\Delta_n|$ has logarithmic poles at the level crossings.

Proposition 3 (GOE structural reduction for expected potentials). *Let A_n, B_n be independent GOE matrices and set*

$$U_n(\lambda) := \mathbb{E} \left[\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| \right].$$

Then there exists a function $H_n : [0, 1] \rightarrow \mathbb{R}$ such that

$$U_n(\lambda) = \frac{1}{2} \log(1 + |\lambda|^2) + H_n \left(\frac{|1 + \lambda^2|^2}{(1 + |\lambda|^2)^2} \right).$$

Consequently any locally uniform limit of U_n has this form.

Proof. Write

$$C_n(\lambda) = A_n + \lambda B_n = \sqrt{1 + |\lambda|^2} \widehat{C}_n(\lambda).$$

Since the discriminant is homogeneous of degree $n(n-1)$ in the matrix entries,

$$\log |\Delta_n(C_n(\lambda))| = n(n-1) \frac{1}{2} \log(1 + |\lambda|^2) + \log |\Delta_n(\widehat{C}_n(\lambda))|.$$

It remains to understand the law of the second term. The entries of $\widehat{C}_n(\lambda)$ are centered complex Gaussian variables. Their variance is normalized to one, and their pseudocovariance equals

$$\tau(\lambda) = \mathbb{E} \widehat{c}_{ij}(\lambda)^2 = \frac{1 + \lambda^2}{1 + |\lambda|^2}.$$

For a centered complex Gaussian vector, the joint law is determined by its covariance and pseudocovariance matrices. Hence the law of $\widehat{C}_n(\lambda)$ depends on λ only through $\tau(\lambda)$.

If $\tau' = e^{2i\theta}\tau$, then multiplication of the whole matrix by $e^{i\theta}$ sends the Gaussian ensemble with pseudocovariance τ to the one with pseudocovariance τ' . This multiplication rotates all eigenvalues by $e^{i\theta}$ and multiplies the discriminant by a complex number of modulus one times the deterministic homogeneous factor; therefore $\log |\Delta_n(\widehat{C}_n(\lambda))|$ depends on $\tau(\lambda)$ only through $|\tau(\lambda)|$. Taking expectations gives

$$U_n(\lambda) = \frac{1}{2} \log(1 + |\lambda|^2) + H_n(|\tau(\lambda)|^2)$$

for a deterministic function H_n on $[0, 1]$, which is the claimed formula. Any locally uniform subsequential limit has the same structural form by passing to the limit in this identity. \square

Proposition 3 shows that Conjecture 1 reduces to understanding whether the pseudocovariance parameter contributes to the leading n^2 -asymptotics of the log-discriminant.

Theorem 2 (Conditional GOE asymptotic law). *Let A_n, B_n be independent GOE matrices, and let μ_n be the empirical level-crossing measure of $A_n + \lambda B_n$.*

Assume that there exists a deterministic function $G \in L_{\text{loc}}^1([0, 1])$ such that

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| - \frac{1}{2} \log(1 + |\lambda|^2) - G(q(\lambda)) - c_n \rightarrow 0$$

in probability in $L_{\text{loc}}^1(\mathbb{C})$, where c_n is independent of λ .

Then

$$\mu_n \implies \frac{1}{2\pi} \Delta_\lambda \left[\frac{1}{2} \log(1 + |\lambda|^2) + G(q(\lambda)) \right] dx dy.$$

In particular, if G is constant, then μ_n converges to the uniform measure on \mathbb{CP}^1 .

Proof. The proof is identical to that of Theorem 1 once the stated L_{loc}^1 asymptotic for the log-discriminant is available. The Poincaré–Lelong formula identifies μ_n with the distributional Laplacian of the normalized log-discriminant, and distributional convergence of Laplacians follows from L_{loc}^1 convergence of the potentials. Since $\Delta_\lambda c_n = 0$, the constants disappear. \square

Conjecture 2 (Pseudocovariance-independence). *Under the assumptions above, the pseudocovariance parameter does not contribute to the leading asymptotics:*

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \frac{1}{2} \log(1 + |\lambda|^2) + c_n + o(1)$$

in probability in $L_{\text{loc}}^1(\mathbb{C})$.

Conjecture 2 implies Conjecture 1. It isolates the core analytic problem: to show that the pseudocovariance

$$\tau(\lambda) = \frac{1 + \lambda^2}{1 + |\lambda|^2}$$

does not affect the leading-order logarithmic energy of the eigenvalues of $A_n + \lambda B_n$.

Remark 8. At the special value $\lambda = i$, one has $\tau(i) = 0$, and the entries of $\widehat{C}_n(i)$ are circularly symmetric complex Gaussians (subject only to the symmetry constraint). This suggests comparing the general case to $\lambda = i$ via a Gaussian interpolation in τ , which may provide a route to proving Conjecture 2.

Remark 9. Conjecture 3 would follow from convergence of the normalized log-discriminant to the logarithmic potential whose Laplacian is the stated density.

3.3. General real case. Recall that the $GE_n^{\mathbb{R}}$ -ensemble on $Mat_n^{\mathbb{R}}$ is the distribution where each entry of a $n \times n$ -matrix is an independent real random variable distributed as $N(0, 1)$, and $Mat_n^{\mathbb{R}}$ is the space of all real-valued $n \times n$ -matrices.

In [GrShZa] we were only able to obtain integral representation for the level crossing distribution for $GE_2^{\mathbb{R}}$, see Theorem 6 in loc.cit.

We now formulate the large-size problem for real i.i.d. entries. This is the real analogue of the conditional Girko-type theorem above, but the real symmetry introduces an additional issue: a priori a positive proportion of level crossings could accumulate near the real projective line. Thus the natural statement has two parts, an absolutely continuous two-dimensional part and a possible singular part supported on \mathbb{RP}^1 .

Conjecture 3 (General real i.i.d. level-crossing law). *Let $A_n = (a_{ij})$ and $B_n = (b_{ij})$ be independent real $n \times n$ matrices with i.i.d. entries satisfying*

$$\mathbb{E}a_{11} = \mathbb{E}b_{11} = 0, \quad \mathbb{E}a_{11}^2 = \mathbb{E}b_{11}^2 = 1,$$

and assume a standard moment condition strong enough to imply the circular law and the corresponding local no-clustering estimates. Let μ_n^{LC} be the empirical measure of the $n(n-1)$ level crossings of $A_n + \lambda B_n$, counted with multiplicity and normalized to have total mass one. Then, as $n \rightarrow \infty$,

$$\mu_n^{LC} \implies \frac{dx dy}{\pi(1 + |\lambda|^2)^2}$$

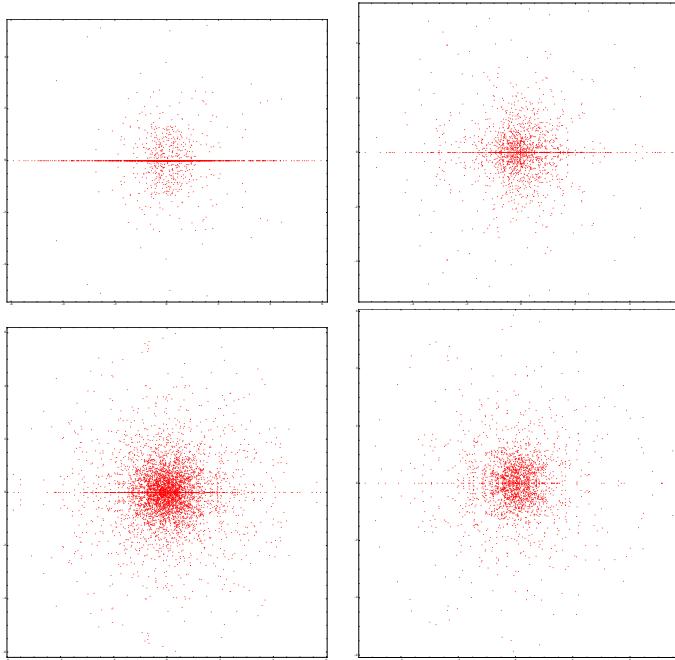


FIGURE 1. $GE_n^{\mathbb{R}}$ -distributions of the branch points, for $n = 2, 5, 15, 25$ approaching the uniform distribution on $\mathbb{C}P^1$.

in probability on $\mathbb{C}P^1$. Equivalently, the limiting law is the uniform spherical measure.

Remark 10. The conjecture is meant for genuinely full real i.i.d. matrices, not for real symmetric matrices. It is consistent with the numerical evidence in [GrShZa], where the real Gaussian ensemble $GE_n^{\mathbb{R}}$ appears to approach the same uniform measure on $\mathbb{C}P^1$. The expected number of exactly real level crossings may grow like $O(n)$, but this is negligible on the $n(n-1)$ scale. What is needed for the limiting statement is the stronger absence of an $O(n^2)$ cloud of almost real crossings.

Remark 11 (Assumptions in the full real i.i.d. case). Away from $\mathbb{R}P^1$, the normalized pencil $(A_n + \lambda B_n)/\sqrt{n(1 + |\lambda|^2)}$ is a non-Hermitian matrix with complex entries whose real and imaginary parts are correlated. The expected inputs are therefore the real analogue of the complex assumptions in Theorem 1: a uniform circular or elliptic law and a uniform small-spacing estimate. The global part is in the scope of circular-law and local circular-law results such as [BYY]; the small-spacing part is predicted by local universality of non-Hermitian eigenvalue statistics, as in [TV], but should not be presented here as proved in this full uniformity.

The additional assumption excluding concentration near $\mathbb{R}P^1$ is a separate real phenomenon. In the Gaussian generalized eigenvalue problem the number of real generalized eigenvalues is only of order $n^{1/2}$ in expectation [EKS], and the real Ginibre correlation functions are explicitly described by Pfaffian formulas [FN]. These results support the expectation that real or almost-real crossings are negligible on the $n(n-1)$ scale, but they do not by themselves prove the stated no-concentration hypothesis for general real i.i.d. pencils.

Theorem 3 (Conditional real i.i.d. theorem). *Let A_n, B_n be as in Conjecture 3. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, set*

$$\widehat{C}_n(\lambda) = \frac{A_n + \lambda B_n}{\sqrt{n(1 + |\lambda|^2)}}.$$

Assume that, uniformly for λ in compact subsets of $\mathbb{C} \setminus \mathbb{R}$, the empirical eigenvalue measure of $\widehat{C}_n(\lambda)$ converges to the circular law and that the small-spacing condition (SR) and logarithmic-tail condition (LT) appearing in Theorem 1 hold. Assume moreover that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{E}N_n(\varepsilon)}{n(n-1)} = 0,$$

where $N_n(\varepsilon)$ denotes the number of level crossings lying within spherical distance ε from \mathbb{RP}^1 . Then μ_n^{LC} converges in probability to the uniform measure (2.1) on \mathbb{CP}^1 .

Proof. On compact subsets of $\mathbb{C} \setminus \mathbb{R}$ the argument is identical to the proof of Theorem 1: the normalized log-discriminant satisfies

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \frac{1}{2} \log(1 + |\lambda|^2) + c_n + o_{\mathbb{P}}(1),$$

in probability in L_{loc}^1 on such compact subsets, where c_n is independent of λ . Applying $(2\pi)^{-1} \Delta_\lambda$ in the sense of distributions gives the density (2.1) away from \mathbb{RP}^1 . The additional no-concentration assumption and Theorem 4 rule out any missing singular mass on \mathbb{RP}^1 , completing the proof. \square

Conjecture 4 (Possible non-universal real corrections). *For real ensembles with additional linear constraints, such as real symmetric or structured real matrices, the limiting density need not be uniform. If the limit is absolutely continuous and invariant under the natural $SO(2)$ -action, then it should have the form*

$$\rho(\lambda) dx dy = h \left(\frac{|1 + \lambda^2|^2}{(1 + |\lambda|^2)^2} \right) \frac{dx dy}{\pi(1 + |\lambda|^2)^2},$$

for a non-negative function h normalized by total mass one. In the full real i.i.d. case we expect $h \equiv 1$.

Theorem 4 (No concentration near the real projective line). *Let A_n, B_n be independent $n \times n$ random real matrices, and let μ_n be the empirical level-crossing measure of $A_n + \lambda B_n$.*

For $\varepsilon > 0$, let

$$N_n(\varepsilon) := \#\{\lambda \in \text{supp } \mu_n : \text{dist}_{\mathbb{CP}^1}(\lambda, \mathbb{RP}^1) < \varepsilon\},$$

counted with multiplicity. Assume that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\mathbb{E}N_n(\varepsilon)}{n(n-1)} = 0.$$

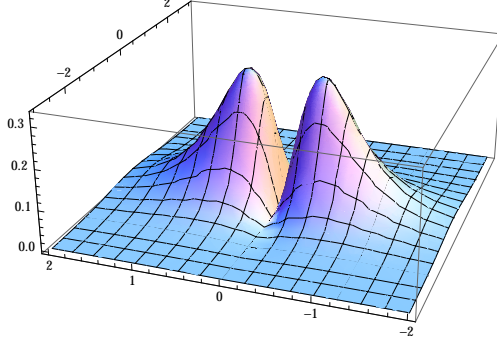
Then every subsequential weak limit in probability of μ_n gives zero mass to \mathbb{RP}^1 . Equivalently, along every almost-sure weakly convergent subsequence extracted from a subsequence converging in probability, the limiting measure assigns no mass to \mathbb{RP}^1 .

Proof. For every $\varepsilon > 0$,

$$\mu_n((\mathbb{RP}^1)_\varepsilon) = \frac{N_n(\varepsilon)}{n(n-1)}.$$

Hence Markov's inequality and the assumption imply

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\mu_n((\mathbb{RP}^1)_\varepsilon) > \eta) = 0$$

FIGURE 2. Density of GUE_2 in \mathbb{C} .

for every $\eta > 0$.

Let $\mu_{n_k} \Rightarrow \mu$ in probability. Passing to a further subsequence if necessary, assume the convergence is almost sure. Choose continuous cutoffs $0 \leq \chi_j \leq 1$ on $\mathbb{C}\mathbb{P}^1$ such that $\chi_j = 1$ on $\mathbb{R}\mathbb{P}^1$ and $\text{supp } \chi_j \subset (\mathbb{R}\mathbb{P}^1)_{\varepsilon_j}$, where $\varepsilon_j \downarrow 0$. By another diagonal extraction, using the preceding probability estimate, we may assume that

$$\limsup_{k \rightarrow \infty} \mu_{n_k}((\mathbb{R}\mathbb{P}^1)_{\varepsilon_j}) = 0$$

almost surely for every fixed j . Therefore

$$\mu(\mathbb{R}\mathbb{P}^1) \leq \int \chi_j d\mu = \lim_{k \rightarrow \infty} \int \chi_j d\mu_{n_k} \leq \limsup_{k \rightarrow \infty} \mu_{n_k}((\mathbb{R}\mathbb{P}^1)_{\varepsilon_j}) = 0.$$

Thus $\mu(\mathbb{R}\mathbb{P}^1) = 0$. □

Corollary 1. *If $\mathbb{E}N_n^{\mathbb{R}} = o(n^2)$, then*

$$\mu_n(\mathbb{R}\mathbb{P}^1) \rightarrow 0$$

in probability.

Remark 12. Under the assumption that the expected number of level crossings in any ε -neighborhood of $\mathbb{R}\mathbb{P}^1$ is $o(n^2)$ uniformly as $\varepsilon \rightarrow 0$, every subsequential limit assigns zero mass to $\mathbb{R}\mathbb{P}^1$.

We do not currently verify this condition for general real ensembles, and leave it as an assumption.

4. HERMITIAN MATRICES

To describe an analog of Wigner's semicircle law, we now turn to level crossings for linear families of random Hermitian matrices. Recall that the GUE_n -ensemble is the distribution on the space \mathcal{H}_n of all Hermitian $n \times n$ -matrices given by the density

$$\frac{1}{Z_n} e^{-\frac{n}{2} \text{tr} H^2}, \quad H \in \mathcal{H}_n,$$

where $Z_n = 2^{n/2} \pi^{n^2/2}$.

Theorem D (See Theorem 3 of [GrShZa]). *If A and B are independently chosen from GUE_2 , then the distribution of the level crossings in \mathbb{C} is given by*

$$\mathcal{P}_{GUE_2}(x, y) dx dy = \frac{4}{\pi} \frac{|y| dx dy}{(1 + x^2 + y^2)^3}. \quad (4.1)$$

To formulate our further results we need to impose certain assumptions.

Assumption 1 (Uniform log-energy convergence in the elliptic bulk). For every $\varepsilon > 0$, uniformly for $\tau \in [0, 1 - \varepsilon]$,

$$\frac{1}{n(n-1)} \sum_{i \neq j} \log |\lambda_i - \lambda_j| = G(\tau^2) + o_{\mathbb{P}}(1),$$

where

$$G(\tau^2) = \iint_{\mathbb{C}^2} \log |z - w| \mu_{\tau}(dz) \mu_{\tau}(dw),$$

and μ_{τ} is the elliptic law.

Remark 13 (Motivation for Assumption 1). Let $\log_{\delta} r := \max\{\log r, \log \delta\}$ and split

$$\frac{1}{n(n-1)} \sum_{i \neq j} \log |\lambda_i - \lambda_j| = L_{n,\delta}^{\text{far}}(\tau) + L_{n,\delta}^{\text{near}}(\tau).$$

For fixed $0 < \delta < 1 < R$, replace $\log |z - w|$ by the two-sided truncation

$$\Phi_{\delta,R}(z, w) := \max\{\log \delta, \min(\log |z - w|, \log R)\}.$$

This kernel is bounded and continuous. Since the complex elliptic Ginibre ensemble is determinantal with an explicit kernel, the one- and two-point correlation functions admit bulk asymptotics uniform in $\tau \in [0, 1 - \varepsilon]$. Hence

$$L_{n,\delta,R}^{\text{trunc}}(\tau) = \iint \Phi_{\delta,R}(z, w) \mu_{\tau}(dz) \mu_{\tau}(dw) + o_{\mathbb{P}}(1)$$

uniformly in τ .

The remaining step is precisely the logarithmic-uniform-integrability input: one lets first $n \rightarrow \infty$, then $\delta \downarrow 0$, and finally $R \rightarrow \infty$, using the local bulk two-point estimate near the diagonal and the uniform logarithmic tail bound at infinity. This is why Assumption 1 is stated as an assumption rather than as a consequence of weak convergence alone.

Theorem 5 (GUE level-crossing limit away from the real axis). *Assume Assumption 1 for the Gaussian elliptic ensemble on compact subsets of $|\tau| < 1$. Then the empirical level-crossing measures for the pencil*

$$A_n + \lambda B_n, \quad A_n, B_n \sim GUE_n,$$

converge locally on $\mathbb{C}\mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$ to the measure

$$\frac{1}{2\pi} \Delta_{\lambda} \left[\frac{1}{2} \log(1 + |\lambda|^2) + G(1 - Y^2) \right] dx dy,$$

where

$$Y = \frac{2\Im \lambda}{1 + |\lambda|^2}$$

and G is the logarithmic energy of the elliptic law with parameter

$$|\tau| = \sqrt{1 - Y^2}.$$

Proof of Theorem 5. Let

$$C_n(\lambda) = A_n + \lambda B_n, \quad A_n, B_n \sim GUE_n,$$

and define the normalized matrix

$$\widehat{C}_n(\lambda) := \frac{C_n(\lambda)}{\sqrt{1 + |\lambda|^2}}.$$

Let $\xi_1(\lambda), \dots, \xi_n(\lambda)$ and $\hat{\xi}_1(\lambda), \dots, \hat{\xi}_n(\lambda)$ denote the eigenvalues of $C_n(\lambda)$ and $\widehat{C}_n(\lambda)$ respectively. Then

$$\xi_i(\lambda) = \sqrt{1 + |\lambda|^2} \hat{\xi}_i(\lambda),$$

and hence

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \frac{1}{2} \log(1 + |\lambda|^2) + \frac{1}{n(n-1)} \sum_{i \neq j} \log |\hat{\xi}_i(\lambda) - \hat{\xi}_j(\lambda)|.$$

By the Gaussian covariance computation, the law of $\widehat{C}_n(\lambda)$ depends on λ only through the parameter

$$\tau(\lambda) := \frac{1 + \lambda^2}{1 + |\lambda|^2}.$$

Moreover, since multiplication by a complex phase rotates the spectrum, the quantity

$$\frac{1}{n(n-1)} \sum_{i \neq j} \log |\hat{\xi}_i - \hat{\xi}_j|$$

depends only on $|\tau(\lambda)|$.

Define

$$q(\lambda) := |\tau(\lambda)|^2 = \frac{|1 + \lambda^2|^2}{(1 + |\lambda|^2)^2}.$$

In stereographic coordinates on S^2 , one checks that

$$q(\lambda) = 1 - Y^2, \quad Y = \frac{2\Im\lambda}{1 + |\lambda|^2}.$$

By Assumption 1, for every $\varepsilon > 0$, there exists a continuous function $\mathcal{I}_\varepsilon : [0, 1 - \varepsilon] \rightarrow \mathbb{R}$ such that

$$\frac{1}{n(n-1)} \sum_{i \neq j} \log |\hat{\xi}_i - \hat{\xi}_j| = \mathcal{I}_\varepsilon(|\tau|) + o_{\mathbb{P}}(1),$$

uniformly for $|\tau| \leq 1 - \varepsilon$.

It follows that for all λ such that $|\tau(\lambda)| \leq 1 - \varepsilon$,

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| = \frac{1}{2} \log(1 + |\lambda|^2) + G_\varepsilon(q(\lambda)) + o_{\mathbb{P}}(1),$$

where $G_\varepsilon(q) := \mathcal{I}_\varepsilon(\sqrt{q})$.

Set

$$F_\varepsilon(\lambda) := \frac{1}{2} \log(1 + |\lambda|^2) + G_\varepsilon(1 - Y^2).$$

Then F_ε is continuous and the convergence above is understood in probability in L_{loc}^1 on compact subsets of the region $|\tau(\lambda)| \leq 1 - \varepsilon$. This is the appropriate potential-theoretic mode of convergence because the left-hand side has logarithmic singularities at level crossings. Thus

$$\frac{1}{n(n-1)} \log |\Delta_n(\lambda)| \rightarrow F_\varepsilon(\lambda)$$

in probability in L_{loc}^1 on this region.

By standard potential-theoretic arguments, this implies convergence of Laplacians in the sense of measures:

$$\mu_n^{LC} = \frac{1}{2\pi n(n-1)} \Delta_\lambda \log |\Delta_n(\lambda)| \implies \frac{1}{2\pi} \Delta_\lambda F_\varepsilon(\lambda) dx dy,$$

locally on the region $|\tau(\lambda)| \leq 1 - \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, this proves the existence of a limiting density away from the set $|\tau(\lambda)| = 1$, i.e. away from the equator $Y = 0$ on S^2 . The density is given by

$$\Psi(\lambda) = \frac{1}{2\pi} \Delta_\lambda \left[\frac{1}{2} \log(1 + |\lambda|^2) + G(1 - Y^2) \right],$$

where $G(q)$ denotes the logarithmic energy of the elliptic law with parameter \sqrt{q} .

This proves the theorem. \square

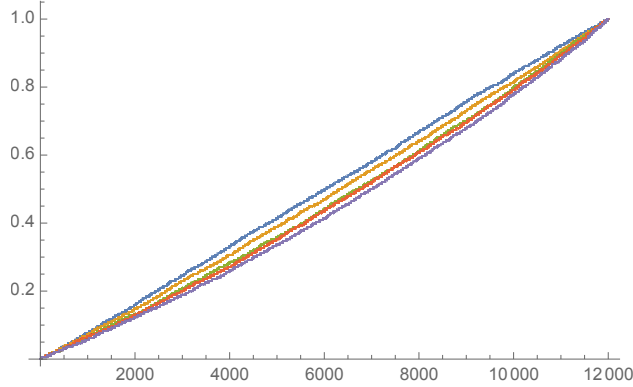


FIGURE 3. Empirical distributions of $|Y|$ for (1.1) taken from GUE_n with $n = 2, 3, 4, 5, 6$. (Curves corresponding to the increasing values of n lie one below the other; the blue straight line corresponds to $n = 2$.)

We do not have an exact formula for the density $\mathcal{P}_n^{GUE}(x, y)$, for $n > 3$. However these densities have simpler formulas in cylindrical or spherical coordinates on \mathbb{CP}^1 . In particular density $\frac{1}{\pi} \frac{1}{(1+x^2+y^2)^2}$ transforms in the constant density $\frac{1}{4\pi}$ in the cylindrical coordinates and the density $\frac{4}{\pi} \frac{|y|}{(1+x^2+y^2)^3}$ becomes $\mathcal{P}_2^{GUE}(\theta, \phi) = \sin^2\theta$. One can show that each $\mathcal{P}_n^{GUE}(\theta, \phi)$ is independent of the azimuthal angle ϕ .

Conjecture 5. *There exists $\Psi(\theta, \phi) := \lim_{n \rightarrow \infty} \mathcal{P}_n^{GUE}(\theta, \phi)$, see Fig. 3.*

Remark 14. The preceding theorem treats the Gaussian Hermitian case through the elliptic ensemble. The expected universality statement is that the same limiting level-crossing law holds for general Wigner matrices. We spell out the corresponding ensemble and conjecture, since this is the Hermitian analogue of Wigner's semicircle law in the present level-crossing problem.

Definition 2. Fix a real-valued probability distribution μ with mean zero and finite moments, and a complex-valued probability distribution ν with mean zero, unit variance, finite moments, and satisfying the Hermitian symmetry convention $h_{ji} = \overline{h_{ij}}$. We denote by $\mathcal{HE}_{\mu, \nu, n}$ the Wigner ensemble of Hermitian $n \times n$ matrices whose diagonal entries are i.i.d. with law μ and whose upper-triangular off-diagonal entries are i.i.d. with law ν .

Conjecture 6 (Hermitian Wigner universality). *Let A_n and B_n be independent matrices from $\mathcal{HE}_{\mu, \nu, n}$, normalized in the usual Wigner scaling. Then the empirical level-crossing measures of $A_n + \lambda B_n$ converge, as $n \rightarrow \infty$, to the same probability measure Ψ as in Conjecture 5, i.e. the limiting law is universal inside the Hermitian symmetry class and does not depend on the particular laws μ and ν .*

Conjecture 7 (Potential-theoretic form of the Hermitian limit). *The universal Hermitian limiting measure is the $SO(2)$ -invariant measure on \mathbb{CP}^1 given away from \mathbb{RP}^1 by*

$$\frac{1}{2\pi} \Delta_\lambda \left[\frac{1}{2} \log(1 + |\lambda|^2) + G(1 - Y^2) \right] dx dy, \quad Y = \frac{2\Im\lambda}{1 + |\lambda|^2},$$

where $G(q)$ is the logarithmic energy of the elliptic law with elliptic parameter \sqrt{q} . The measure has no atom or singular component on \mathbb{RP}^1 .

Theorem 6 (Conditional Hermitian Wigner theorem). *Assume the following two uniform inputs for the Wigner-type elliptic ensembles arising from*

$$\frac{A_n + \lambda B_n}{\sqrt{1 + |\lambda|^2}} :$$

(i) *the empirical eigenvalue distribution converges to the elliptic law with parameter $|\tau(\lambda)|$, uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$; and (ii) the normalized logarithmic pair-energy converges uniformly to the elliptic logarithmic energy $G(|\tau(\lambda)|^2)$, with negligible contribution from eigenvalue spacings tending to zero. Then the conclusion of Conjecture 7 holds locally on $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. If, in addition, the empirical level-crossing measures do not concentrate in an $o(1)$ -neighborhood of \mathbb{RP}^1 , then the same convergence holds on all of \mathbb{CP}^1 .*

Proof. The proof repeats the proof of Theorem 5. The normalization separates the deterministic factor $\frac{1}{2} \log(1 + |\lambda|^2)$ from the logarithmic pair-energy of the normalized eigenvalues. By the assumed elliptic-law and log-energy convergence, the normalized log-discriminant converges to

$$\frac{1}{2} \log(1 + |\lambda|^2) + G(1 - Y^2).$$

Applying the Poincaré–Lelong formula gives the stated Laplacian. The final no-concentration assumption removes the only possible missing singular mass on \mathbb{RP}^1 . \square

Proposition 4 (Gaussian Hermitian pencils). *For A_n and B_n independent GUE matrices, the two uniform inputs in Theorem 6 are satisfied on every compact subset of $\mathbb{C} \setminus \mathbb{R}$. Consequently the Hermitian limiting measure is rigorously obtained there from the elliptic logarithmic energy.*

Proof. Fix a compact set $K \subset \mathbb{C} \setminus \mathbb{R}$. For $\lambda \in K$ the normalized matrix

$$\widehat{C}_n(\lambda) = \frac{A_n + \lambda B_n}{\sqrt{1 + |\lambda|^2}}$$

is a Gaussian elliptic ensemble. Its correlation parameter is

$$\tau(\lambda) = \frac{1 + \lambda^2}{1 + |\lambda|^2}, \quad 1 - |\tau(\lambda)|^2 = \frac{4(\Im \lambda)^2}{(1 + |\lambda|^2)^2}.$$

Since K is separated from the real axis, there exists $\delta > 0$ such that $|\tau(\lambda)| \leq 1 - \delta$ for every $\lambda \in K$.

For Gaussian elliptic ensembles with $|\tau| \leq 1 - \delta$, the explicit determinantal correlation kernels imply the global elliptic law uniformly in τ ; see the elliptic ensemble computations of [SCS, FZ, Fo]. Thus the empirical eigenvalue measures of $\widehat{C}_n(\lambda)$ converge, uniformly for $\lambda \in K$, to the elliptic law with parameter $|\tau(\lambda)|$.

It remains to justify convergence of the logarithmic pair-energy. Split

$$\log |z - w| = \log |z - w| \mathbf{1}_{|z-w|>\varepsilon} + \log |z - w| \mathbf{1}_{|z-w|\leq\varepsilon}.$$

For the first term the kernel is continuous after an outer truncation and hence the uniform elliptic law gives convergence to the corresponding truncated elliptic energy. For the second term, the determinantal two-point correlation function is locally bounded, uniformly for $|\tau| \leq 1 - \delta$, and has the usual quadratic repulsion at the diagonal. Therefore, for compact spectral windows,

$$\mathbb{E} \frac{1}{n(n-1)} \sum_{i \neq j} \mathbf{1}_{|\hat{\xi}_i - \hat{\xi}_j| \leq \varepsilon} |\log |\hat{\xi}_i - \hat{\xi}_j|| \leq C_\delta \varepsilon^2 |\log \varepsilon| + o(1),$$

uniformly in $\lambda \in K$. Letting first $n \rightarrow \infty$ and then $\varepsilon \downarrow 0$ proves that the small-spacing contribution is negligible. Removing the outer truncation is justified by the

compact support and logarithmic-tail bounds for the elliptic law, again uniformly for $|\tau| \leq 1 - \delta$. Hence the normalized logarithmic pair-energy converges uniformly on K to the logarithmic energy $G(|\tau(\lambda)|^2)$ of the elliptic law, which is precisely the pair-energy input required in Theorem 6. \square

Remark 15 (Non-Gaussian Hermitian/Wigner entries). For Wigner entries with smooth bounded densities, subexponential tails, and matching first two moments, the same verification is expected to follow from the local elliptic law and bulk universality for elliptic random matrices, uniformly for $|\tau| \leq 1 - \delta$. Thus away from \mathbb{RP}^1 the Hermitian conditional theorem is essentially reduced to known elliptic-law and level-repulsion technology. What remains outside the present verification is uniform control as $\Im\lambda \rightarrow 0$, where the elliptic ensemble approaches the Hermitian regime, and the possible concentration of level crossings near \mathbb{RP}^1 .

Remark 16. Observe that the probability density (4.1) is invariant under the change of variables $t \leftrightarrow t^{-1}$ which corresponds to considering the family $tA + B$ instead of $A + tB$; the latter change having no effect on the spectral distributions.

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REFERENCES

- [AGZ] Anderson, G. W., Guionnet, A., Zeitouni, O., An Introduction to Random Matrices. Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010. xiv+492 pp.
- [GP] Galligo, A., Poteaux, A., Computing monodromy via continuation methods on random Riemann surfaces, *Theor. Comp. Sci.*, 412, 16 (2011), 1492–1507.
- [Gi1] Girko, V. L., The V -transformation, *Dokl. Akad. Nauk Ukr.SSR, Ser. Mat.* 3, 5–6 (1982). (in Russian).
- [Gi2] Girko, V. L., The Circle Law, *Theory of Probabilities and their Applications*, 29, 4, 669–679 (1984).
- [Ka] Kato, T., Perturbation theory for linear operators. Reprint of the 1980 edition. *Classics in Mathematics*. Springer-Verlag, Berlin, 1995. xxii+619 pp.
- [Me] Mehta, M. L., Random matrices. Third edition. *Pure and Applied Mathematics (Amsterdam)*, 142. Elsevier/Academic Press, Amsterdam, 2004. xviii+688 pp.
- [BYY] Bourgade, P., Yau, H.-T., Yin, J., Local circular law for random matrices, *Probab. Theory Related Fields* 159 (2014), no. 3–4, 545–595. doi:10.1007/s00440-013-0514-z.
- [TV] Tao, T., Vu, V., Random matrices: universality of local spectral statistics of non-Hermitian matrices, *Ann. Probab.* 43 (2015), no. 2, 782–874. doi:10.1214/13-AOP876.
- [EKS] Edelman, A., Kostlan, E., Shub, M., How many eigenvalues of a random matrix are real?, *J. Amer. Math. Soc.* 7 (1994), no. 1, 247–267. doi:10.1090/S0894-0347-1994-1231689-0.
- [FN] Forrester, P. J., Nagao, T., Eigenvalue statistics of the real Ginibre ensemble, *Phys. Rev. Lett.* 99 (2007), 050603. doi:10.1103/PhysRevLett.99.050603.
- [SCS] Sommers, H.-J., Crisanti, A., Sompolinsky, H., Stein, Y., Spectrum of large random asymmetric matrices, *Phys. Rev. Lett.* 60 (1988), 1895–1898.
- [FZ] Feinberg, J., Zee, A., Non-Hermitian random matrix theory: method of Hermitian reduction, *Nuclear Phys. B* 504 (1997), no. 3, 579–608.
- [Fo] Forrester, P. J., *Log-Gases and Random Matrices*, London Mathematical Society Monographs Series, 34. Princeton University Press, Princeton, NJ, 2010.
- [ShZa] Shapiro, B., Zarembo, K., Level Crossing in Random Matrices. I. Random perturbation of a fixed matrix, *Journal of Physics A: Mathematical and Theoretical*, Volume 50(4).
- [GrShZa] Grøsfjeld, T., Shapiro, B., Zarembo, K., On level crossing in random matrix pencils. II. Random perturbation of a random matrix, *J. Phys. A* 52 (2019), no. 21 <https://doi.org/10.1088/1751-8121/ab1733>
- [ZVW] Zirnbauer, M. R., Verbaarschot, J. J. M., Weidenmüller, H. A., Destruction of order in nuclear spectra by a residual GOE interaction, *Nuclear Physics A* 411 (1983) 161–180.

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