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Level crossing in random matrices. II. Random perturbation of a random matrix

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Abstract

In this paper we study the distribution of level crossings for the spectra of linear families $A + \lambda B$, where A and B are square matrices independently chosen from some given Gaussian ensemble and λ is a complex-valued parameter. We formulate a number of theoretical and numerical results for the classical Gaussian ensembles and some of their generalisations. Besides, we present intriguing numerical information about the monodromy distribution in case of linear families for the classical Gaussian ensembles of 3×3-matrices.

Keywords: random matrices, spectrum, level crossing

(Some figures may appear in colour only in the online journal)

1. Introduction

Given a linear operator family

$$C = A + \lambda B, \tag{1.1}$$

analysis of the dependence of its spectrum on a perturbative parameter λ is a typical problem both in fundamental natural sciences and applications, see e.g. the classical treatise [Ka]. Depending on the situation, λ is considered as a real or a complex-valued parameter.

The first time the level crossings of the spectrum (i.e. collisions of the eigenvalues) in the families (1.1) were considered in mathematics, it was in connection with the development of the linear algebra and (linear) Hamiltonian mechanics in the second half of the 19th century. One of the first mathematical observations important for the theory of level crossings is that

for any given size, the set of real symmetric matrices having a multiple eigenvalue has codimension two in the space of all symmetric matrices of that size. A similar set of Hermitian matrices has codimension three. These circumstances explain why almost all families of the form (1.1) that consist of real symmetric or Hermitian matrices have real and simple spectrum for all real values of λ. This phenomenon is usually referred to as *level crossings avoidance*. In the early 20th century level crossings reappeared in physics due to the development of quantum mechanics. Avoided level crossings have always played an important role in quantum mechanics such as nuclear physics, see e.g. [SW, DH], but also in quantum electrodynamics, see [CH], as well as in quantum chromodynamics, see e.g. [QH]. Later, the subject of quantum chaos has attracted particular interest; it is also directly linked to avoided level crossings.

On the other hand, level crossings unavoidably occur upon the analytic continuation of a real perturbation parameter λ into the complex plane, where an intricate pattern of permutations of the eigenvalues arises due to monodromy of the spectrum at each of the level crossing points. The positions of the level crossings and the monodromy of the spectrum at each of them constitute important pieces of information about the spectral properties of the linear family (1.1) and the analytic structure of its spectral surface. In particular, level crossings determine the accuracy of perturbative series in λ.

Since the late 1960s, motivated by a number of fascinating observations by Bender and Wu [BW], physicists and mathematicians started considering various cases where A and B are, for example, self-adjoint while λ is complex-valued. A very small sample of such studies can be found in e.g. [MNOP, Ro, CHM, SH, BDCP, Sm] and references therein.

Unfortunately, for particularly interesting linear families (1.1), it is usually quite difficult to exactly describe the positions of level crossings, and it is even more challenging to describe the monodromy of the spectrum when λ encircles closed curves avoiding them. As an illustration of specific examples of the physics origin, the reader might consult [ShTaQu] and [ShT], where the cases of the quasi-exactly solvable quartic and sextic are considered. The corresponding locations of level crossings are shown in figure 1. Although in both cases numerical experiments reveal very clear and intriguing patterns for the location of level crossings as well as the corresponding monodromy, mathematical proofs explaining these lattice-type patterns in figure 1 are unavailable at present.

Taking this circumstance into account, in [ShZa1] we considered the problem of finding the distribution of level crossings within the framework of the random matrix theory, and studied the case when A is a fixed matrix while B is a matrix distributed according to one of the standard Gaussian ensembles. This question and a related circle of problems about the behavior of random matrix pencils (i.e. one-dimensional families instead of individual random matrices) seem natural, important and quite new. To the best of our knowledge, the first time a similar question was considered was in [ZVW], which was published in the early 80s. At present, that paper has been cited about 60 times in the physics literature, but to the best of our understanding none of these further publications had an ambition to develop a more general theory of level crossings in random pencils. Doing so is our main intent.

One should immediately point out that compared to the study of the spectrum of random matrices, the analysis of random level crossings is a much more delicate and technical quest. Many standard tools available in the usual random matrix theory fail completely in the set-up of random matrix pencils. There is an urgent demand to develop new methods applicable to random pencils and higher dimensional linear families. For general information on the random matrix theory see e.g. [AGZ].

The present paper is a sequel to [ShZa1], and discusses level crossings in linear matrix families of the form (1.1), where A and B are independent and equally distributed matrices

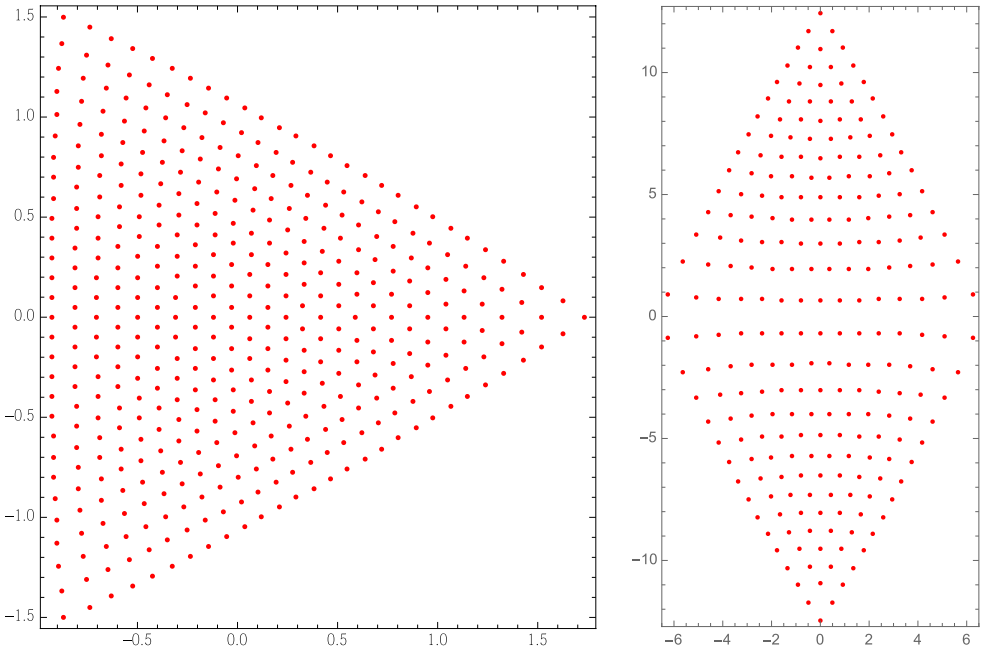

Figure 1. Level crossings for the quasi-exactly solvable quartic (left) and sextic (right) shown in the complex plane of the perturbation parameter, see [ShTaQu] and [ShT].

belonging to a certain Gaussian ensemble. To stress the equal rôle of matrices in (1.1), we denote them here by A and B as opposed to V_0 and H, which was used in [ShZa1].

It is worth noting that a somewhat similar situation, where one randomly samples coefficients of a bivariate polynomial instead of the entries of a matrix, has been earlier considered in [GP].

1.1. Summary of the main results

We will analyse level crossings in

(i) complex Gaussian ensembles $\mathrm{GE}_n^{\mathbb{C}}$ and a number of their specializations;
(ii) Gaussian orthogonal ensembles GOE_n and Gaussian unitary ensembles GUE_n;
(iii) real Gaussian ensembles $\mathrm{GE}_n^{\mathbb{R}}$.

Beside this, we present the monodromy statistics for linear families (1.1) with A and B independently sampled from the GUE_3-, GOE_3-, and $\mathrm{GE}_3^{\mathbb{C}}$-ensembles.

To properly understand what this means, we recall some notation:

(a) The complex (non-symmetric) Gaussian ensemble $\mathrm{GE}_n^{\mathbb{C}}$ is the distribution on the space $\mathrm{Mat}_n^{\mathbb{C}}$ of all complex-valued $n \times n$-matrices, where each matrix entry is an independent complex Gaussian variable distributed as $N(0, \frac{1}{2}) + \mathrm{i}N(0, \frac{1}{2})$;
(b) The Gaussian orthogonal ensemble $\mathrm{GOE}_n^{\mathbb{R}}$ is the distribution on the space $\mathrm{Sym}_n^{\mathbb{R}}$ of real-valued symmetric matrices, where each entry $e_{i,j} = e_{j,i},\ i < j$ of a matrix is an independent random variable distributed as $N(0, 1)$, and each diagonal entry $e_{i,i}$ is independently distributed as $\sqrt{2}N(0, 1)$;

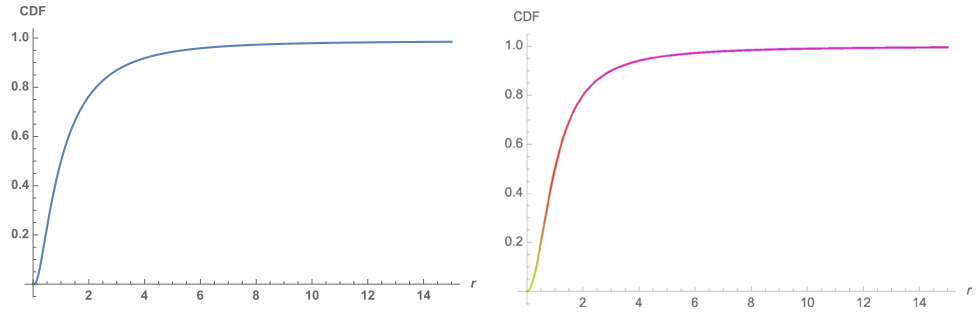

Figure 2. Smoothed empirical radial CDF of level crossings for $A + \lambda B$, where A and B are independently sampled from $\mathrm{GE}_6^{\mathbb{C}}$ (left) and the theoretical radial CDF of the uniform distribution on $\mathbb{C}P^1$ given by $\frac{r^2}{1+r^2}$ (right). (300 random pairs were used in this plot).

(c) The Gaussian unitary ensemble GUE_n-ensemble is the distribution on the space $\mathcal{H}_n$ of all Hermitian $n \times n$-matrices, where each entry $e_{i,j} = e_{j,i},\ i < j$ of a matrix is an independent random variable distributed as $N(0, \frac{1}{2}) + \mathrm{i}N(0, \frac{1}{2})$, and each diagonal entry $e_{i,i}$ is independently distributed as $N(0, 1)$;

(d) The real (non-symmetric) Gaussian ensemble $\mathrm{GE}_n^{\mathbb{R}}$ is the distribution on the space $\mathrm{Mat}_n^{\mathbb{R}}$ of real-valued $n \times n$ matrices, where each entry of a matrix is an independent real random variable distributed as $N(0, 1)$.

Let us present here a general outline of our results. For the exact formulations and details, the interested reader should consult the respective sections below. In what follows we will often use the complex projective line $\mathbb{C}P^1 \supset \mathbb{C}$ compactifying the complex plane $\mathbb{C}$ of the perturbation parameter λ. Since $\mathbb{C}P^1$ is diffeomorphic to the sphere S^2, we will consider the standard rotation invariant area element on $\mathbb{C}P^1$, normalized so that the total area of $\mathbb{C}P^1$ equals 1.

I. In the case of complex Gaussian ensembles $\mathrm{GE}_n^{\mathbb{C}}$, we prove the following claim.

Theorem 1. *For any positive integer n, if the matrices A and B are independently chosen from* $\mathrm{GE}_n^{\mathbb{C}}$, *then the distribution of level crossings in (1.1) is uniform on* $\mathbb{C}P^1$, *i.e. it coincides with the above rotation invariant area element.*

Numerical confirmation of theorem 1 is shown in figure 2. It shows a perfect match of the numerical distribution of the absolute values of level crossings obtained in our sampling with the theoretical radial CDF for the uniform distribution on $\mathbb{C}P^1$ which is given by $\frac{r^2}{1+r^2}$.

Beside theorem 1, we settle a number of its generalizations and specializations for other complex random ensembles with Gaussian entries in section 2.

II. In the case of $\mathrm{GOE}_n^{\mathbb{R}}$, we have a theoretical result for $n = 2$ and a conjecture for $n \geqslant 3$ based on extensive computer experiments.

Theorem 2. *If the matrices A and B are independently chosen from* $\mathrm{GOE}_2^{\mathbb{R}}$, *then the distribution of level crossings in (1.1) is uniform on* $\mathbb{CP}^1 \supset \mathbb{C}$, *i.e. it coincides with the above rotation invariant area element.*

Remark 1. One can easily check that the distribution of level crossings for A and B independently taken from $\mathrm{GOE}_1^{\mathbb{R}}$ is uniform on the real projective line $\mathbb{R}P^1$.

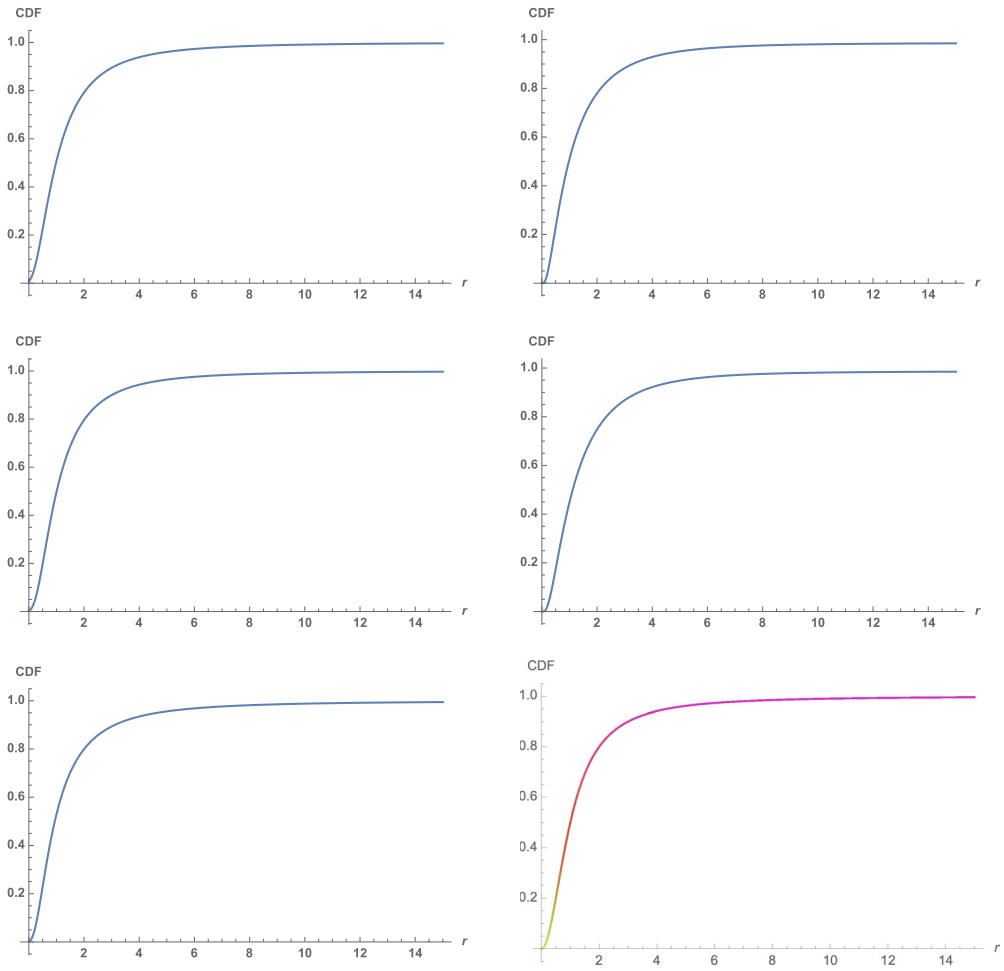

Figure 3. Smoothed empirical radial CDFs of level crossings for $n = 2, 4, 6, 8, 10$, with A and B sampled independently from GOE_n (the first 5 plots) and the theoretical CDF for the uniform distribution on $\mathbb{C}P^1$ (the last plot).

Extensive numerical experiments strongly support the following guess, which is illustrated in figure 3.

Conjecture 1. For any fixed size $n > 2$, if the matrices A and B are independently chosen from $\mathrm{GOE}_n^{\mathbb{R}}$, then the distribution of level crossings in (1.1) is uniform on $\mathbb{CP}^1 \supset \mathbb{C}$.

Remark 2. Note that the empirical radial CDFs for $n = 2, 4, 6, 8, 10$ shown in figure 3 are practically indistinguishable from the theoretical radial CDF of the uniform distribution on $\mathbb{CP}^1$. Numerics also show that the argument of the level crossings is perfectly uniformly distributed on $[0, 2\pi]$.

III. In the case of $\mathrm{GUE}_n^{\mathbb{R}}$, we again have a theoretical result for $n = 2$ and a conjecture about the existence of the asymptotic limiting distribution of level crossings when $n \to \infty$ based on computer simulations.

Remark 3. To explain our results, let us realize $\mathbb{CP}^1 \simeq S^2$ as the unit sphere in $\mathbb{R}^3$ with coordinates (X, Y, Z) and identify the complex plane of parameter $\lambda = x + \mathrm{i}y$ with the horizontal coordinate (X, Y)-plane, where X corresponds to the real axis and Y corresponds to the imaginary axis. We use the standard stereographic projection of the unit sphere $\mathbb{CP}^1 \simeq S^2 \subset \mathbb{R}^3$ from its north pole, i.e. from the point $(0, 0, 1)$ onto the (X, Y)-plane to identify the complement to the north pole in S^2 with the complex plane of parameter λ. Let us introduce the cylindrical coordinates (ρ, ψ, Y) in $\mathbb{R}^3$, where $X = \rho\cos\psi,\ Y = Y,\ Z = \rho\sin\psi$. Then (ψ, Y), $0 \leqslant \psi \leqslant 2\pi,\ -1 \leqslant Y \leqslant 1$ parametrizes the unit sphere $S^2 \simeq \mathbb{C}P^1$.

Theorem 3. *If the matrices A and B are independently chosen from* GUE_2, *then in the cylindrical coordinates (ψ, Y) on $\mathbb{C}P^1$, the distribution μ of level crossings in (1.1) is given by*

$$\mu := \mathcal{P}_{\mathrm{GUE}_2}(\psi, Y)\mathrm{d}\psi\mathrm{d}Y = \frac{|Y|\mathrm{d}\psi\mathrm{d}Y}{2\pi}, \tag{1.2}$$

where $P_{\mathrm{GUE}_2}(\psi, Y)$ denotes the density of μ in the above cylindrical coordinates.

For $n \geqslant 3$, we do not have explicit (even conjectural) formulas for the densities $\mathcal{P}_{\mathrm{GUE}_n}(x, y)$ akin to equation (1.2). However, we carried out substantial numerical experiments for matrix sizes up to six conducted as follows. For each $n \in \{2, ..., 6\}$, sampling independent pairs of GUE_n-matrices, we calculated 12 000 level crossing points for every n and plotted the values of $|Y|$ for obtained level crossings in the increasing order, see figure 4. These numerical experiments strongly suggest the following.

Conjecture 2. There exists a limiting distribution $\mathcal{P}_{\mathrm{GUE}_\infty}(Y) := \lim_{n\to\infty} \mathcal{P}_{\mathrm{GUE}_n}(Y)$.

Some discussions of conjecture 2 can be found at the end of section 4. At present we do not have a conjectural formula describing $\mathcal{P}_{\mathrm{GUE}_\infty}(Y)$ explicitly.

IV. In the case of $\mathrm{GE}_n^{\mathbb{R}}$, rather cumbersome theoretical results are available for $n = 2$ (see section 5) together with the general conjecture 3 about the asymptotic distribution of level crossings when $n \to \infty$.

Our numerical experiments illustrated in figure 5 strongly suggest the validity of the following guess to which we plan to return in [GrShZa3].

Conjecture 3. As $n \to \infty$, the sequence of level crossing distributions with A and B independently sampled from $\mathrm{GE}_n^{\mathbb{R}}$ converges to the uniform distribution on $\mathbb{CP}^1$.

It seems rather hopeless to get any explicit formulas for the distributions of level crossings of $\mathrm{GE}_n^{\mathbb{R}}$ with $n \geqslant 3$.

We have also numerically evaluated the proportion of real level crossings among the total number of level crossings. (Real level crossing are represented by the horizontal segments of the graphs in the right column of figure 5.) Our numerics suggests that for a given size n, the average number of real level crossings is close to $\sqrt{n(n-1)}$. This is the square root of the total number of level crossings, which is known to be $n(n-1)$.

Observe that in a similar situation of real random univariate polynomials of some fixed degree m, with normally distributed independent coefficients, having mean zero and the variance of the ith coefficient equal to $\binom{m}{i}$, it is known that the average number of real roots equals $\sqrt{m}$, see e.g. [Ko]. However our situation is not covered by any known theoretical results guaranteeing such behavior. We can prove that $\sqrt{2}$ is the expected average for $n = 2$, see lemma 4 below. For $n = 3, 4, 5$ with 10 000 samples, the quotient of the empirical average divided by

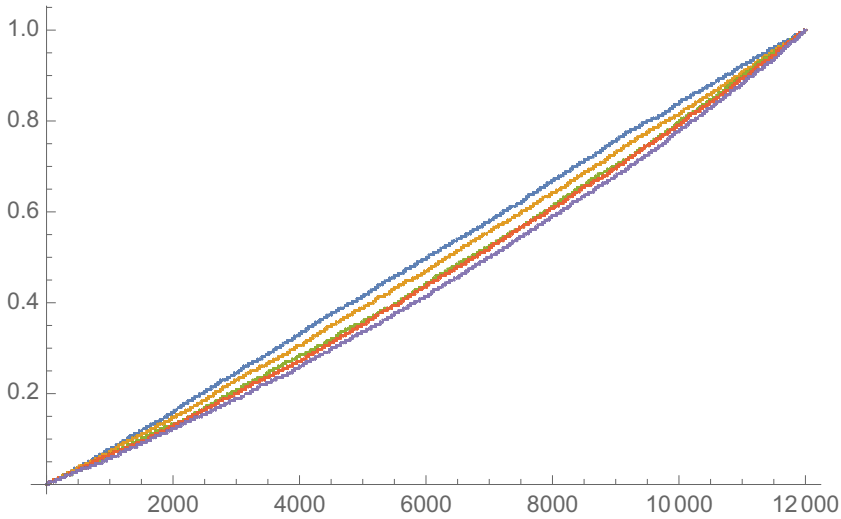

Figure 4. Empirical CDFs of $|Y|$ for level crossings with A and B independently sampled from GUE_n with $n = 2, 3, 4, 5, 6$. Curves corresponding to the increasing values of n lie one below the other; the blue straight line corresponds to $n = 2$, see (4.2).

$\sqrt{n(n-1)}$ was 1.0405, 1.0404, 1.049 57 resp. For $n = 6$ with 5000 samples, the same quotient was 1.055 86 and, finally, for $n = 10$ with 130 samples, the quotient was 1.063 82.

V. Finally, we initiate the study of the statistics of the monodromy of the random linear families (1.1). Namely, in section 5 below we present such monodromy statistics for A and B independently sampled from the GUE_3-, GOE_3-, and $\mathrm{GE}_3^{\mathbb{C}}$-ensembles. Our choice of these ensembles is justified by the fact that they provide simplest interesting examples of the monodromy groups, but at the same time, the number of possible ordered sequences of monodromy operators (i.e. transpositions of eigenvalues) in these three cases is still manageable for a detailed and reliable numerical study. Although the monodromy statistics is equally important for all matrix sizes, one can easily check that if the size of the random matrices exceeds 3, the number of possible monodromy groups increases so dramatically that using our current numerical methods it is impossible to obtain any trustworthy monodromy statistics. Numerics for the above 3×3-ensembles shows, in particular, that the monodromy sequences in which one and the same transposition consecutively repeats several times appear very seldom, if ever.

We want to emphasize that further studies in this direction are highly desirable, since the monodromy statistics carries even more interesting and valuable information about the linear family (1.1) than the distribution of its level crossings.

The structure of the paper is as follows. In section 2 we prove some general introductory results and make conclusions about the complex Gaussian ensembles. In section 3, we discuss the SO_2-action on pairs of real matrices and $\mathbb{C}P^1$, and prove a number of preliminary results. In section 4, we consider the cases of orthogonal Gaussian ensembles and Gaussian unitary ensembles. In section 5, we study linear families (1.1) for the real Gaussian ensembles. Finally, in section 6, we present our numerical results about the monodromy statistics of 3×3 linear families (1.1).

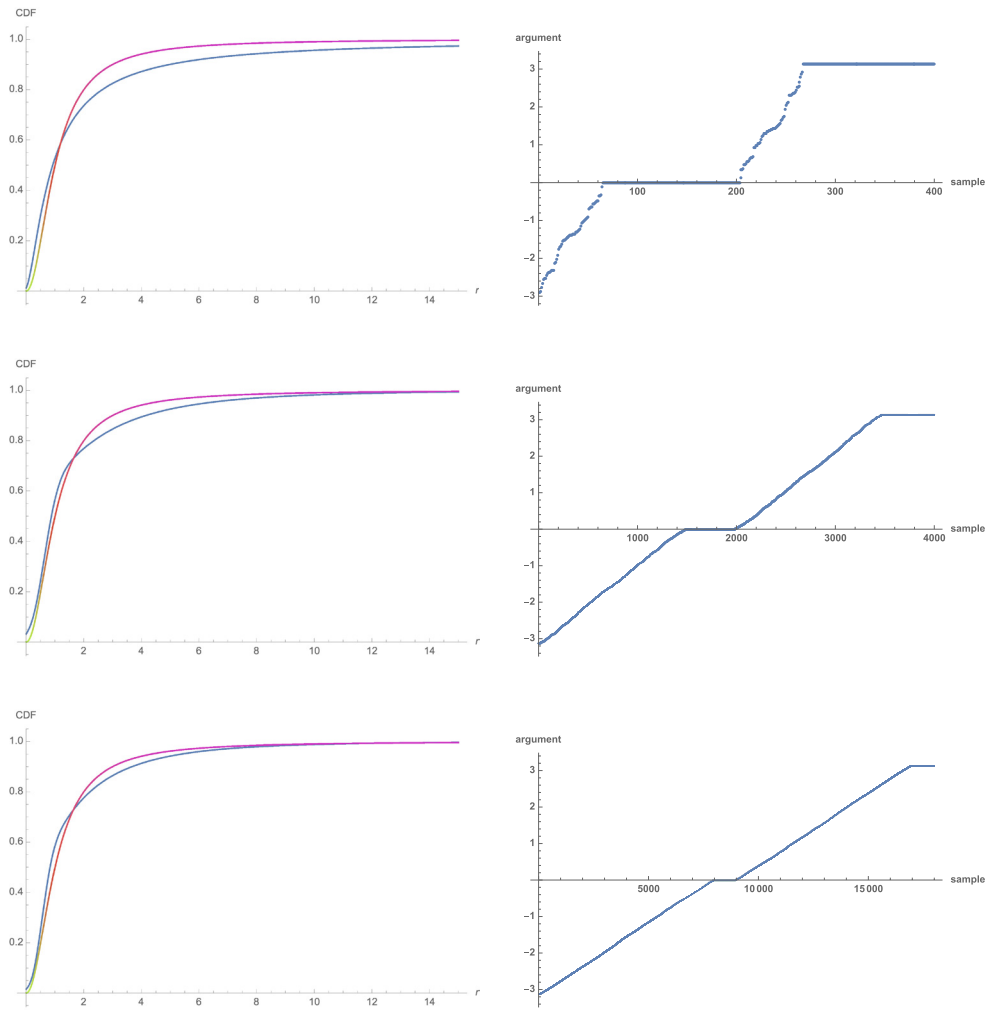

Figure 5. Smoothed empirical radial and angular CDFs of the level crossings with A and B sampled from $\mathrm{GE}_n^{\mathbb{R}}$ with $n = 2, 5, 10$, approaching that of the uniform distribution on $\mathbb{C}P^1$. The limiting theoretical radial density is shown by the rainbow curve while the experimental results are shown by the blue curves.

2. Complex Gaussian ensembles and SU_2-action on pairs of matrices

The main result of this section is as follows. (It is equivalent to theorem 1.)

Theorem 4. *For any positive integer n, if the matrices A and B are independently chosen from* $\mathrm{GE}_n^{\mathbb{C}}$, *then the distribution of level crossings in (1.1) with respect to the affine coordinate* $\lambda = x + \mathrm{i}y$ *of* $\mathbb{C}$ *is given by*

$$\mathcal{P}_{\mathrm{GE}_n^{\mathbb{C}}}(\lambda) := \mathcal{P}_{\mathrm{GE}_n^{\mathbb{C}}}(x, y)\mathrm{d}x\mathrm{d}y = \frac{\mathrm{d}x\mathrm{d}y}{\pi(1 + x^2 + y^2)^2} = \frac{\mathrm{d}x\mathrm{d}y}{\pi(1 + |\lambda|^2)^2}. \tag{2.1}$$

Remark 4. In polar coordinates (r, θ) for the complex plane of the parameter λ, the above distribution $\mathcal{P}_{\mathrm{GE}_n^{\mathbb{C}}}(\lambda)$ has the form

$$\mathcal{P}_{\mathrm{GE}_n^{\mathbb{C}}}(r,\theta)\mathrm{d}r\mathrm{d}\psi = \frac{r\mathrm{d}r\mathrm{d}\theta}{\pi(1+r^2)^2},$$

giving the radial CDF of the form

$$\Psi_{\mathrm{GE}_n^{\mathbb{C}}}(r) = \frac{r^2}{1+r^2}.$$

If we use the standard stereographic projection of the unit sphere $\mathbb{C}P^1 \simeq S^2 \subset \mathbb{R}^3$ from its north pole onto the (X, Y)-plane, then the usual area element of the sphere induced from the standard Euclidean structure in $\mathbb{R}^3$ is given by

$$\mathrm{d}A = \frac{4\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2} = \frac{4\mathrm{d}x\mathrm{d}y}{(1+|\lambda|^2)^2}.$$

The latter fact implies that the rhs of (2.1) presents the constant density $\frac{1}{4\pi}$ with respect to the standard Euclidean area measure on $S^2 \simeq \mathbb{C}P^1$, i.e. the compactified complex plane of the parameter λ. The constant density $\frac{1}{4\pi}$ provides the unit sphere with the total mass 1.

To prove theorem 4, we need the following construction. The $\mathrm{GE}_n^{\mathbb{C}}$-probability measure $\gamma := \gamma_{\mathrm{GE}_n^{\mathbb{C}}}$ on $\mathrm{Mat}_n^{\mathbb{C}}$ induces the product probability measure $\gamma^{(2)}$ on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$. Consider the *spectral determinant* $\mathfrak{D}_n \subset \mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{C}$, which is a complex algebraic hypersurface consisting of all triples (A, B, λ) such that the matrix $A + \lambda B$ has a multiple eigenvalue. Projection $\pi_n : \mathfrak{D}_n \to \mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ by forgetting the last coordinate induces a branched covering of $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ by $\mathfrak{D}_n$ of degree $n(n-1)$, whose fiber over a pair (A, B) coincides with the level crossing set of the linear family $A + \lambda B$. Taking the pullback $\pi_n^{-1}(\gamma^{(2)})$, we obtain the probability measure $\Gamma := \Gamma_{\mathrm{GE}_n^{\mathbb{C}}}$ on $\mathfrak{D}_n$. In other words, for any open subset $\mathcal{O} \subset \mathcal{D}_n$ which projects diffeomorphically on its image, $\Gamma(\mathcal{O}) = \frac{1}{n(n-1)}\gamma(\pi_n(\mathcal{O}))$. A similar construction can be used for any branched covering whose base is equipped with an arbitrary probability measure.

Now let $\kappa_n : \mathfrak{D}_N \to \mathbb{C}$ be the projection of the spectral determinant onto the last coordinate in $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{C}$, i.e. onto the λ-plane. Then the measure $\mu := \mu_{\mathrm{GE}_n^{\mathbb{C}}}$ we are looking for, coincides with the pushforward $\mu := \kappa_n(\pi_n^{-1}(\gamma^{(2)}))$. In other words, the value of measure μ on any measurable subset of $\mathbb{C}$ equals the value of measure Γ of its complete preimage in $\mathcal{D}_n$.

For our purposes, it will be more convenient to consider the space $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$, with the inclusion $\mathbb{C} \subset \mathbb{CP}^1$ given by the stereographic projection introduced in remark 3. In other words, we use $\lambda := b/a$, $(a : b)$ being the homogeneous coordinates on $\mathbb{CP}^1$. The above constructions work equally well on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$ and provide us with the measure μ supported on $\mathbb{CP}^1$. (By a slight abuse of notation we denote both measures by the same letter.)

2.1. SU_2*-action*

Consider the following action on the space $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ of pairs of complex matrices. A matrix $\mathfrak{U} \in \mathrm{SU}_2$ given by $\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$, $|u|^2 + |v|^2 = 1$ acts on the latter product space by:

$$(A, B) * \mathfrak{U} \mapsto (uA + vB, -\bar{v}A + \bar{u}B). \tag{2.2}$$

Introduce the following SU_2-action on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$ extending the above action (2.2).

A matrix $\mathfrak{U} = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$, $|u|^2 + |v|^2 = 1$ acts on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$ by:

$$(A, B, a : b) * \mathfrak{U} \mapsto (uA + vB, -\bar{v}A + \bar{u}B, \bar{u}a + \bar{v}b : -va + ub). \tag{2.3}$$

Observe that the third component of the latter action coincides with the standard SU_2-action on a point $(a : b) \in \mathbb{CP}^1$ of the conjugate matrix $\begin{pmatrix} \bar{u} & -v \\ \bar{v} & u \end{pmatrix}$.

To prove theorem 4, we will show that the above measure μ is invariant under the above SU_2-action on $\mathbb{CP}^1$. Since this action preserves the standard Fubini–Study metric on $\mathbb{C}P^1$, we can conclude that its density is constant with respect to the area form induced by the Fubini–Study metric, i.e. the one which has constant density in the cylindrical coordinates (ϕ, Z).

Our proof of theorem 4 consists of three steps. In step 1, we will show that the action (2.3) on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$ preserves the spectral determinant $\widehat{\mathfrak{D}}_n \subset \mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$. In step 2, we will prove that this action preserves the probability measure $\gamma^{(2)}$ on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$. As a consequence of steps 1 and 2, it also preserves the probability measure $\pi_n^{-1}(\gamma^{(2)})$ on $\widehat{\mathfrak{D}}_n$. In step 3, we will show the equivariance of (2.3) with respect to the projections π_n and κ_n.

Lemma 1. *The action (2.3) preserves* $\widehat{\mathfrak{D}}_n \subset \mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}} \times \mathbb{CP}^1$.

Proof. Take an arbitrary triple $(A, B, a : b)$ belonging to $\widehat{\mathfrak{D}}_n$, i.e. such that $aA + bB$ has a multiple eigenvalue, and take any $\mathfrak{U} = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in \mathrm{SU}_2$. We need to show that the triple

$$(uA + vB, -\bar{v}A + \bar{u}B, \bar{u}a + \bar{v}b : -va + ub)$$

also belongs to $\widehat{\mathfrak{D}}_n$. In other words, we need to check that if $aA + bB$ has a multiple eigenvalue, then the matrix

$$(\bar{u}a + \bar{v}b)(uA + vB) + (-va + ub)(-\bar{v}A + \bar{u}B)$$

has a multiple eigenvalue as well. The latter claim is obvious since expansion of the above expression results in $aA + bB$. □

Proof of theorem 4. To settle step 2, observe that in case of the $\mathrm{GE}_n^{\mathbb{C}}$-ensemble, the probability density to obtain a matrix $A \in \mathrm{Mat}_n^{\mathbb{C}}$ is given by:

$$\gamma(A) = \frac{1}{\pi^{n^2}} \mathrm{e}^{-\sum_{i,j=1}^{n} |A_{ij}|^2} = \frac{1}{\pi^{n^2}} \mathrm{e}^{-\mathrm{Tr}(AA^\star)},$$

where $A^\star$ stands for the conjugate-transpose of A. Therefore the density of $\gamma^{(2)}$ on $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ is given by:

$$\gamma^{(2)}(A, B) = \frac{1}{\pi^{2n^2}} \mathrm{e}^{-\mathrm{Tr}(AA^\star + BB^\star)}.$$

Setting $C = uA + vB$ and $D = -\bar{v}A + \bar{u}B$, we get the relation

$$\mathrm{Tr}(CC^\star + DD^\star) = \mathrm{Tr}(AA^\star + BB^\star) =$$

$$= \mathrm{Tr}(u\bar{u}AA^\star + v\bar{u}BA^\star + u\bar{v}AB^\star + v\bar{v}BB^\star + v\bar{v}AA^\star - \bar{u}vBA^\star - u\bar{v}AB^\star + u\bar{u}BB^\star).$$

The latter equality implies that the action (2.3) restricted to $\mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ (i.e. forgetting its action on the last coordinate $\mathbb{C}P^1$) preserves $\gamma^{(2)}$. By lemma 1, the action (2.3) preserves the hypersurface $\widehat{\mathfrak{D}}_n$ and, therefore it preserves the probability measure $\pi_n^{-1}(\gamma^{(2)})$ on it.

To settle step 3, we need to show that the measure $\mu := \kappa_n(\pi_n^{-1}(\gamma^{(2)}))$ on $\mathbb{C}P^1$ is invariant under the conjugate action of SU_2 on $\mathbb{C}P^1$, see the last component of (2.3). Take an arbitrary open set $\Omega \subset \mathbb{C}P^1$ and $g \in \mathrm{SU}_2$. Denote by $g \cdot \Omega \subset \mathbb{C}P^1$ the shift of Ω by the conjugate of g. We need to prove that $\mu(\Omega) = \mu(g \cdot \Omega)$. By definition, $\mu(\Omega) := \pi^{-1}(\gamma^{(2)})(\kappa_n^{-1}(\Omega))$ and $\mu(g \cdot \Omega) := \pi^{-1}(\gamma^{(2)})(\kappa_n^{-1}(g \cdot \Omega))$. (Observe that both $\kappa_n^{-1}(\Omega)$ and $\kappa_n^{-1}(g \cdot \Omega)$ are measurable subsets of $\widehat{\mathfrak{D}}_n$.)

Let us show that the action (2.3) by g sends $\kappa_n^{-1}(\Omega)$ to $\kappa_n^{-1}(g \cdot \Omega)$ and the action (2.3) by the inverse g^{-1} sends $\kappa_n^{-1}(g \cdot \Omega)$ to $\kappa_n^{-1}(\Omega)$, thereby implying the required coincidence of measures due to step 2. Indeed, $\kappa_n^{-1}(\Omega)$ is the set of all triples $(A, B, a : b)$ such that $aA + bB$ has a multiple eigenvalue and $(a : b) \in \Omega$. By lemma 1, acting by g on any such triple we get another triple $(\tilde{A}, \tilde{B}, \tilde{a} : \tilde{b})$ such that $\tilde{a}\tilde{A} + \tilde{b}\tilde{B}$ has a multiple eigenvalue and $(\tilde{a} : \tilde{b}) \in g \cdot \Omega$. The same argument applies to the action (2.3) by the inverse g^{-1}. □

Remark 5. Observe that an alternative way to express the fact that the rhs of (2.1) presents the constant density $\frac{1}{4\pi}$ with respect to the standard Euclidean area measure on $S^2 \simeq \mathbb{C}P^1$ is as follows. Consider the standard cylindrical coordinate system (ρ, ϕ, Z) in $\mathbb{R}^3$, where $\rho \geqslant 0, 0 \leqslant \phi \leqslant 2\pi, Z \in \mathbb{R}$. Recall that

$$X = \rho \cos\phi,\ Y = \rho \sin\phi,\ Z = Z.$$

If we consider (ϕ, Z), $0 \leqslant \phi \leqslant 2\pi, -1 \leqslant Z \leqslant 1$, as coordinates on the unit sphere $S^2 \simeq \mathbb{C}P^1$ (with both poles removed), then in these coordinates the usual area element on the sphere is given by

$$\mathrm{d}A = \mathrm{d}\phi \mathrm{d}Z.$$

Thus, in cylindrical coordinates (ϕ, Z), $0 \leqslant \phi \leqslant 2\pi$; $-1 \leqslant Z \leqslant 1$ parametrizing the unit sphere S^2, the measure $\mathcal{P}_{\mathrm{GE}_n^{\mathbb{C}}}(x, y)\mathrm{d}x\mathrm{d}y$ given by (2.1) transforms into

$$\mathcal{P}_{\mathrm{GE}_n^{\mathbb{C}}}(\phi, Z)\mathrm{d}\phi\mathrm{d}Z = \frac{\mathrm{d}\phi\mathrm{d}Z}{4\pi}. \tag{2.4}$$

In the case of 2×2-matrices, the formula

$$\mathcal{P}_{\mathrm{GE}_2^{\mathbb{C}}}(x, y)\mathrm{d}x\mathrm{d}y = \frac{\mathrm{d}x\mathrm{d}y}{\pi\left(1 + |\lambda|^2\right)^2}$$

could also be obtained by explicit calculations with the discriminantal equation, similar to what is done in sections 4–6.

Let us now present a number of generalisations of theorem 4.

Proposition 1. *The conclusion of theorem 4 holds if A and B are independently chosen from the scaled complex Gaussian ensemble* $\mathrm{GE}_{\sigma^2,n}^{\mathbb{C}}$, *i.e. the* $n \times n$ *matrix ensemble whose off-diagonal entries are i.i.d. standard normal complex variables, and whose on-diagonal entries are i.i.d. normal complex variables with an arbitrary fixed positive variance* σ^2.

(In the above notation, $\mathrm{GE}_n^{\mathbb{C}} = \mathrm{GE}_{1,n}^{\mathbb{C}}$.)

The next observation together with theorem 4 and proposition 1 allows us to substantially extend the class of complex Gaussian ensembles whose distribution of level crossings is given by (2.1), i.e. it is uniform on $\mathbb{C}P^1$.

Take any complex linear subspace $W_n \subset \mathrm{Mat}_n^{\mathbb{C}}$ such that the product space $W_n \times W_n \subset \mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ is preserved by the action (2.2). Given $\sigma > 0$, denote by $W_{\sigma^2,n}$ the space W_n with the measure induced from the scaled complex Gaussian ensemble $\mathrm{GE}_{\sigma^2,n}^{\mathbb{C}}$.

Proposition 2. *In the above notation, the level crossings of (1.1), with the random matrices A and B independently chosen from $W_{\sigma^2,n}$, are uniformly distributed on $\mathbb{C}P^1$, i.e. their probability measure is given by the right-hand side of (2.1).*

To give an example of such W_n, recall that $\mathrm{GOE}_n^{\mathbb{C}}$ is the distribution on the space $\mathrm{Sym}_n^{\mathbb{C}}$ of complex-valued symmetric matrices, where each entry $e_{i,j} = e_{j,i}$, $i < j$ of a $n \times n$-matrix has a normal distribution $N(0, 1/2) + \mathrm{i}N(0, 1/2)$, and each diagonal entry $e_{i,i}$ is distributed as $\sqrt{2}(N(0, 1/2) + \mathrm{i}N(0, 1/2))$. Observe that $\mathrm{GOE}_n^{\mathbb{C}}$ is obtained by the restriction of $\mathrm{GE}_{2,n}^{\mathbb{C}}$ to $\mathrm{Sym}_n^{\mathbb{C}}$. (Discussions of general spectral properties of complex symmetric matrices can be found in e.g. [RaGaPrPu].)

Corollary 1. *The conclusion of proposition 2 holds if A and B are independently chosen from the ensemble $\mathrm{GOE}_n^{\mathbb{C}}$, and, more generally, from the scaled ensemble $\mathrm{GOE}_{\sigma^2,n}^{\mathbb{C}}$ whose off-diagonal entries are the i.i.d. standard symmetric normal complex variables and whose diagonal entries are the i.i.d. normal complex variables with an arbitrary fixed positive variance σ^2.*

Remark 6. Further interesting examples of linear subspaces W_n covered by proposition 2 include Toeplitz matrices, band matrices, band Toeplitz matices, diagonal matrices, etc.

Proof of proposition 1. In the set-up of this proposition, the density of the probability to obtain a given matrix $A \in \mathrm{Mat}_n^{\mathbb{C}}$ with respect to the Lebesgue measure is given by the formula

$$\widetilde{\gamma}(A) = K\mathrm{e}^{-\sum_{i\neq j}|A_{ij}|^2 - W\sum_{i=1}^n |A_{ii}|^2} = K\mathrm{e}^{-\mathrm{Tr}(AA^\star) - W\sum_{i=1}^n |A_{ii}|^2},$$

where K is a normalisation constant and W is a real number. (To present a probability density in the above formula, the quadratic form $\mathrm{Tr}(AA^\star) + W\sum_{i=1}^n |A_{ii}|^2$ has to be positive-definite which implies that W can not be a large negative number.) Therefore

$$\widetilde{\gamma}^{(2)}(A, B) = K^2\mathrm{e}^{-\mathrm{Tr}(AA^\star + BB^\star) - W\sum_{i=1}^n (|A_{ii}|^2 + |B_{ii}|^2)}. \tag{2.5}$$

All we need to show is that the right-hand side of (2.5) is preserved under the action (2.3). In notation of the previous proof, we already know that $\mathrm{Tr}(CC^\star + DD^\star) = \mathrm{Tr}(AA^\star + BB^\star)$. It remains to prove that

$$\sum_{i=1}^n (|A_{ii}|^2 + |B_{ii}|^2) = \sum_{i=1}^n (|C_{ii}|^2 + |D_{ii}|^2).$$

In fact, $|A_{ii}|^2 + |B_{ii}|^2 = |C_{ii}|^2 + |D_{ii}|^2$ for each i which follows from the relation

$$|C_{ii}|^2 + |D_{ii}|^2 = (uA_{ii} + vB_{ii})(\bar{u}\bar{A}_{ii} + \bar{v}\bar{B}_{ii}) + (-\bar{v}A_{ii} + \bar{u}B_{ii})(-v\bar{A}_{ii} + u\bar{B}_{ii}) = |A_{ii}|^2 + |B_{ii}|^2.$$ □

Proof of proposition 2. Repeats the above proof of proposition 1. □

Proof of corollary 1. Both statements follow from the observation that the action (2.3) preserves the subspace $\mathrm{Sym}_n^{\mathbb{C}} \times \mathrm{Sym}_n^{\mathbb{C}} \subset \mathrm{Mat}_n^{\mathbb{C}} \times \mathrm{Mat}_n^{\mathbb{C}}$ and that, additionally, the probability measure of the ensemble $\mathrm{GOE}_{\sigma^2,n}^{\mathbb{C}}$ (supported on $\mathrm{Sym}_n^{\mathbb{C}} \times \mathrm{Sym}_n^{\mathbb{C}}$) is induced from that of $\mathrm{GE}_{(\sigma')^2,n}^{\mathbb{C}}$ for appropriate σ'. □

3. SO_2-action on pairs of real matrices

This section provides some preliminary material for our study of level crossings of (1.1) with A and B chosen from the GOE-, GUE- and $\mathrm{GE}^{\mathbb{R}}$-ensembles. A very essential feature of all these cases is that their level crossings distribution is invariant under the action of the subgroup $\mathrm{SO}_2 \subset \mathrm{SU}_2$ given by the same formula (2.2), but with real u and v satisfying $u^2 + v^2 = 1$, see lemma 2.

In the above realization of $\mathbb{C}P^1$ as the unit sphere $S^2 \subset \mathbb{R}^3$, SO_2 acts on it by rotation around the Y-axis, see figure 6 and lemma 2 below. This circumstance implies that the family of orbits of the SO_2-action on the unit sphere $S^2 \simeq \mathbb{C}P^1$ projected to the complex plane of parameter $\lambda = x + \mathrm{i}y$ will coincide with the family of circles given by

$$x^2 + (y - t)^2 = t^2 - 1, \quad |t| \geqslant 1.$$

Introduce the cylindrical coordinates (ρ, ψ, Y) in $\mathbb{R}^3$, where $X = \rho\cos\psi,\ Y = Y,\ Z = \rho\sin\psi$. Then (ψ, Y), $0 \leqslant \psi \leqslant 2\pi,\ -1 \leqslant Y \leqslant 1$ parameterises the unit sphere $S^2 \simeq \mathbb{C}P^1$. Lemma 2 implies that in the cylindrical coordinates (ψ, Y), the distributions of level crossings of the above ensembles on $\mathbb{C}P^1$ are of the form:

$$\mathrm{dens}(\psi, Y)\mathrm{d}\psi\mathrm{d}Y = \rho(Y)\mathrm{d}\psi\mathrm{d}Y,$$

for some univariate function ρ, i.e. its density depends only on Y and is independent of the angle variable ψ. (In general, $\rho(Y)\mathrm{d}Y$ can be a 1-dimensional measure which does not have a smooth density function. This happens, for example, in the case of $\mathrm{GE}_2^{\mathbb{R}}$, when $\rho(Y)\mathrm{d}Y$ has a point mass at the origin.) In the original coordinates (x, y), where $\lambda = x + \mathrm{i}y$, the distribution of level crossings for the above cases will be of the form

$$\mathrm{dens}(x, y)\mathrm{d}x\mathrm{d}y = \rho\left(\frac{2y}{x^2 + y^2 + 1}\right)\frac{4\mathrm{d}x\mathrm{d}y}{(x^2 + y^2 + 1)^2}, \tag{3.1}$$

with the same ρ as above, see proposition 3.

Therefore the problem of finding the distribution of level crossings for the Gaussian orthogonal, Gaussian unitary, and real Gaussian ensembles becomes one-dimensional which is a big advantage. In all cases under consideration, ρ has an additional property of being an even function.

We start with the following statement generalizing lemma 1.

Lemma 2. *The action of* $\mathcal{U} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in \mathrm{SO}_2 \subset \mathrm{SU}_2$ *on pairs of matrices* (A, B) *given by*

$$(A, B) * \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = (uA + vB, -vA + uB),$$

where u and v are real numbers satisfying the condition $u^2 + v^2 = 1$*, preserves the following measures on the following matrix (sub)spaces:*

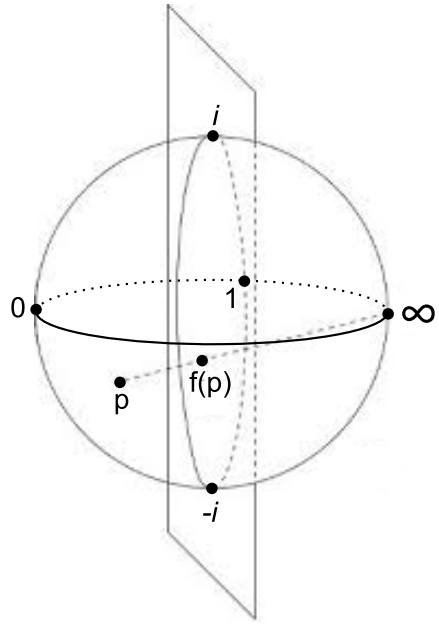

Figure 6. The SO_2-action on $\mathbb{C}P^1$ projectivising the complex plane of parameter λ.

(a) the product $\gamma^{(2)}_{\mathrm{GOE}}$ *of two* GOE_n*-measures* γ_{GOE} *on the space* $\mathrm{Sym}_n^{\mathbb{R}} \times \mathrm{Sym}_n^{\mathbb{R}}$*;*
(b) the product $\gamma^{(2)}_{\mathrm{GUE}}$ *of two* GUE_n*-measures* γ_{GUE} *on the space* $\mathcal{H}_n \times \mathcal{H}_n$*;*
(c) the product $\gamma^{(2)}_{\mathrm{GE}}$ *of two* $\mathrm{GE}_n^{\mathbb{R}}$*-measures* γ_{GE} *on the space* $\mathrm{Mat}_n^{\mathbb{R}} \times \mathrm{Mat}_n^{\mathbb{R}}$.

Proof. Similarly to lemma 1, SO_2 acts on $\widehat{\mathfrak{D}}_n \subset \mathrm{Sym}_n^{\mathbb{R}} \times \mathrm{Sym}_n^{\mathbb{R}} \times \mathbb{CP}^1$ (resp. on $\widehat{\mathfrak{D}}_n \subset \mathcal{H}_n \times \mathcal{H}_n \times \mathbb{CP}^1$ and on $\widehat{\mathfrak{D}}_n \subset \mathrm{Mat}_n^{\mathbb{R}} \times \mathrm{Mat}_n^{\mathbb{R}} \times \mathbb{CP}^1$), where $\widehat{\mathfrak{D}}_n$ is the spectral determinant, i.e. the set of all triples $(A, B, (a : b))$ such that $(a : b)$ is a level crossing point of the pair (A, B). (By a slight abuse of notation, in all cases we use the same letter for the spectral determinant.) Here SO_2 acts on $\mathbb{C}P^1$ as

$$(a : b) * \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = (ua + vb : -va + ub).$$

Note that $(ua + vb : -va + ub)$ is a level crossing point of $(uA + vB, -vA + aB)$. Indeed,

$$\begin{aligned} &(ua + vb)(uA + vB) + (-va + ub)(-vA + uB) \\ &= u^2aA + v^2bB + auvB + bvuA + v^2aA + u^2bB - auvB - bvuA \\ &= aA + bB. \end{aligned}$$

Hence SO_2 acts on $\widehat{\mathfrak{D}}_n$, and this action commutes with the projections $\pi_n : \widehat{\mathfrak{D}}_n \to \mathrm{Sym}_n^{\mathbb{R}} \times \mathrm{Sym}_n^{\mathbb{R}}$ (resp. $\pi_n : \widehat{\mathfrak{D}}_n \to \mathcal{H}_n \times \mathcal{H}_n$, and $\pi_n : \widehat{\mathfrak{D}}_n \to \mathrm{Mat}_n^{\mathbb{R}} \times \mathrm{Mat}_n^{\mathbb{R}}$), as well as with $\kappa_n : \widehat{\mathfrak{D}}_n \to \mathbb{C}P^1$. To check that the action of SO_2 on $\mathrm{Sym}_n^{\mathbb{R}} \times \mathrm{Sym}_n^{\mathbb{R}}$, $\mathcal{H}_n \times \mathcal{H}_n$, and $\mathrm{Mat}_n^{\mathbb{R}} \times \mathrm{Mat}_n^{\mathbb{R}}$, preserves the densities $\gamma^{(2)}_{\mathrm{GOE}}$, $\gamma^{(2)}_{\mathrm{GUE}}$, and $\gamma^{(2)}_{\mathrm{GE}}$, respectively, recall that these densities are given by $C_{\mathrm{GOE}_n} \mathrm{e}^{\frac{-n}{4}\mathrm{tr}(A^2+B^2)}$, $C_{\mathrm{GUE}_n} \mathrm{e}^{\frac{-n}{2}\mathrm{tr}(A^2+B^2)}$, and $C_{\mathrm{GE}_n} \mathrm{e}^{\frac{-n}{2}\mathrm{tr}(AA^T+BB^T)}$, respectively. Here $C_{\mathrm{GOE}_n}, C_{\mathrm{GUE}_n}, C_{\mathrm{GE}_n}$ are the corresponding normalising constants.

Therefore, in e.g. the orthogonal case, the density of the pair (A, B) is determined by $\mathrm{tr}(A^2 + B^2)$. At the same time

$$\begin{aligned} &\mathrm{tr}((uA + vB)^2 + (-vA + uB)^2) \\ &= \mathrm{tr}(u^2A^2 + uvAB + uvBA + v^2B^2 + v^2A^2 - uvAB - uvBA + u^2B) \\ &= \mathrm{tr}(A^2 + B^2). \end{aligned}$$

Similar calculations work in the other two cases.

The density μ of level crossing points in $\mathbb{CP}^1$ is given by $\kappa_n\left(\pi_n^{-1}(\gamma^{(2)}_{\mathrm{GOE}})\right)$ on $\mathrm{Sym}_n^{\mathbb{R}} \times \mathrm{Sym}_n^{\mathbb{R}}$, $\kappa_n\left(\pi_n^{-1}(\gamma^{(2)}_{\mathrm{GUE}})\right)$ on $\mathcal{H}_n \times \mathcal{H}_n$, and $\kappa_n\left(\pi_n^{-1}(\gamma^{(2)}_{\mathrm{GE}})\right)$ on $\mathrm{Mat}_n^{\mathbb{R}} \times \mathrm{Mat}_n^{\mathbb{R}}$ resp. That is, the measure μ of a measurable set $E \subset \mathbb{CP}^1$ is given by $\gamma^{(2)}(\pi_n(\kappa_n^{-1}(E)))$. Note that

$$\begin{aligned} \mu(g \cdot E) &= \gamma^{(2)}(\pi_n(\kappa_n^{-1}(g \cdot E))) \\ &= \gamma^{(2)}(\pi_n(g \cdot \kappa_n^{-1}(E))) \\ &= \gamma^{(2)}(g \cdot \pi_n(\kappa_n^{-1}(E))) \\ &= \gamma^{(2)}(\pi_n(\kappa_n^{-1}(E))) \\ &= \mu(E). \end{aligned}$$

Thus we can conclude that for the above three ensembles, the density of level crossing points on $\mathbb{CP}^1$ is invariant under the above action by SO_2. □

Proposition 3. *In the standard coordinates (X, Y, Z) in $\mathbb{R}^3$ introduced in remark 3, the group SO_2 acts on $\mathbb{C}P^1 \subset \mathbb{R}^3$ by rotation with respect to the Y-axis. This fact implies that in the above three cases, the distribution of level crossings in the cylindrical coordinates $(\psi; Y)$ is independent of ψ.*

Proof. We will show that for $\mathfrak{U} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}$, its action on a triple $(A, B, (\psi, Y))$ will be given by

$$(A, B, (\psi, Y)) * \mathfrak{U} = (uA + vB, -vA + uB, (\psi + 2\theta, Y))$$

implying that the action of SO_2 on $\mathbb{C}P^1$ realized as the unit sphere in $\mathbb{R}^3$ is by rotation of the sphere about the Y-axis. We only need to concentrate on the action of $\mathfrak{U}$ on the last coordinate. In the homogeneous coordinates $(a : b)$ of $\mathbb{C}P^1$, this action, by definition, is given by

$$(a : b) * \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} = (a\cos\Theta + b\sin\Theta : -a\sin\Theta + b\cos\Theta).$$

Setting $\lambda = \frac{a}{b}$ and $\lambda = x + \mathrm{i}y$, we get that

$$\lambda_\Theta := \lambda * \mathfrak{U} = \frac{\lambda\cos\Theta + \sin\Theta}{\cos\Theta - \lambda\sin\Theta}.$$

In terms of the pair (x, y), the same action is expressed as

$$(x, u) * \mathfrak{U} := (x_\Theta, y_\Theta)$$

$$= \left(\frac{(\sin\Theta + x\cos\Theta)(\cos\Theta - x\sin\Theta) - y^2\sin\Theta\cos\Theta}{(\cos\Theta - x\sin\theta)^2 + (y\sin\Theta)^2}, \frac{(\sin\Theta + x\cos\Theta)y\sin\Theta + (\cos\Theta - x\sin\Theta)y\cos\Theta}{(\cos\Theta - x\sin\theta)^2 + (y\sin\Theta)^2} \right).$$

The relations between the coordinates (x, y) in the λ-plane and the coordinates (X, Y, Z) restricted to the sphere are as follows

$$X = x(1 - Z) = \frac{2x}{x^2 + y^2 + 1}, \quad Y = y(1 - Z) = \frac{2y}{x^2 + y^2 + 1}, \quad Z = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}. \tag{3.2}$$

We have the relation

$$(\psi, Y) = \left(\arctan \frac{Z}{X}, Y \right),$$

where (X, Y, Z) are restricted to the sphere.

We need to express the above SO_2-action in the cylindrical coordinates (ψ, Y) on $S^2 \simeq \mathbb{C}P^1$. First we check that the coordinate Y is preserved. In other words, for any real pair (x, y), one forms the triple (X, Y, Z) using (3.2). Then for the above pair (x_Θ, y_Θ), one forms the triple $(X_\Theta, Y_\Theta, Z_\Theta)$ using (3.2). What we need to check is that, for any Θ, one has that $Y = Y_\Theta$. Indeed, Y_Θ is given by

$$Y_\Theta = 2\frac{((xC + S)yS + (C - xS)yC)((C - xS)^2 + (yS)^2)}{\mathrm{Exp}},$$

where $C := \cos\Theta$, $S := \sin\Theta$, and

$$\begin{aligned}\mathrm{Exp} := {} & ((xC + S)^2(C - xS)^2) - 2y^2SC(xC + S)(C - xS) + y^2S^2C^2 \\ & +(xC + S)^2y^2S^2 + 2(xC + S)(C - xS)y^2SC + (C - xS)^2y^2C^2 \\ & +(C - xS)^4 + 2(C - xS)^2y^2S^2 + y^4S^4.\end{aligned}$$

Simplifying the above formula for Y_Θ, we get

$$Y_\Theta = \frac{2y}{x^2 + y^2 + 1} = Y.$$

Now we want to find the relation between the angle ψ_Θ and the pair (ψ, Θ). Observe that

$$\tan\psi_\Theta = \frac{Z_\Theta}{X_\Theta} = \frac{x_\Theta^2 + y_\Theta^2 - 1}{2x_\Theta},$$

which using the above expressions for (x_Θ, y_Θ) gives

$$\tan\psi_\Theta = \frac{((xC + S)(C - xS) - y^2SC)^2 + ((xC + S)yS + (C - xS)yC)^2 - ((C - xS)^2 + y^2S^2)^2}{2((C - xS)^2 + y^2S^2)((S + xC)(C - xS) - y^2SC)}.$$

Simplifying, we obtain

$$\tan\psi_\Theta = \frac{(x^2 + y^2 - 1)\cos 2\Theta + 2x\sin 2\Theta}{2x\cos 2\Theta - (x^2 + y^2 - 1)\sin 2\Theta} = \frac{Z\cos 2\Theta + X\sin 2\Theta}{X\cos 2\Theta - Z\sin 2\Theta}.$$

Dividing the numerator and denominator by $X\cos 2\Theta$, we get

$$\tan\psi_\Theta = \frac{\frac{Z}{X}+\tan 2\Theta}{1-\frac{Z}{X}\tan 2\Theta} = \frac{\tan\psi+\tan 2\Theta}{1-\tan\psi\tan 2\Theta} = \tan(\psi+2\Theta),$$

which implies that $\psi_\Theta = \psi + 2\Theta$. □

Lemma 3. *If a smooth distribution which is invariant under the above* SO_2*-action is also radial in the* λ*-plane, then it is constant with respect to the spherical metric on* $\mathbb{C}P^1$.

Proof. By formula (3.1), such a distribution in the λ-plane should be of the form

$$\mathrm{dens}(x,y)\mathrm{d}x\mathrm{d}y = \rho\left(\frac{2y}{x^2+y^2+1}\right)\frac{4\mathrm{d}x\mathrm{d}y}{(x^2+y^2+1)^2}.$$

On the other hand, in the polar coordinates (r,θ) in the λ-plane, the same distribution has the form

$$\mathrm{den}(r,\theta)\mathrm{d}r\mathrm{d}\theta = R(r)\mathrm{d}r\mathrm{d}\theta,$$

implying that

$$\rho\left(\frac{2y}{x^2+y^2+1}\right)\frac{4}{(x^2+y^2+1)^2} = \frac{R(r)}{r} \Leftrightarrow \rho\left(\frac{2y}{r^2+1}\right) = F(r).$$

The lhs is a function constant on the family of circles

$$x^2+(y-t)^2 = t^2-1, \quad |t| \geqslant 1$$

while the rhs is constant on the family of circles

$$x^2+y^2 = K$$

which can only happen when both sides are constant. Since $\rho\left(\frac{2y}{r^2+1}\right) = K$, the statement follows. □

4. Gaussian orthogonal and Gaussian unitary ensembles

The next result calculates the level crossing distribution for the $\mathrm{GOE}_2^{\mathbb{R}}$- and GUE_2-ensembles.

Theorem 5.

(i) If the matrices A and B are independently chosen from $\mathrm{GOE}_2^{\mathbb{R}}$*, then the distribution of level crossings in (1.1) is uniform on* $\mathbb{CP}^1 \supset \mathbb{C}$*, i.e. its density is given by the right-hand side of (2.1).*

(ii) If the matrices A and B are independently chosen from GUE_2*, then the distribution of level crossings in* $\mathbb{C}$ *is given by*

$$\mathcal{P}_{\mathrm{GUE}_2}(x,y)\mathrm{d}x\mathrm{d}y = \frac{4|y|\mathrm{d}x\mathrm{d}y}{\pi(1+x^2+y^2)^3} = \frac{1}{\pi}\left|\frac{y}{1+x^2+y^2}\right|\frac{4\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2}, \tag{4.1}$$

which matches the general formula (3.1).

Remark 7. In the cylindrical coordinates (ψ, Y) on $\mathbb{C}P^1$, where $0 \leqslant \psi \leqslant 2\pi$ and $-1 \leqslant Y \leqslant 1$, one has

$$\mathcal{P}_{\mathrm{GUE}_2}(\psi, Y)\mathrm{d}\psi\mathrm{d}Y = \frac{|Y|\mathrm{d}\psi\mathrm{d}Y}{2\pi}. \tag{4.2}$$

This fact establishes the equivalence of Part (ii) of theorem 5 with theorem 3.

The main argument of our proof is similar to our other proofs dealing with the case $n = 2$, comp. [ShZa1]; it has an advantage that one obtains more detailed information. Notice that the ensemble GOE_n (resp. GUE_n) is invariant under the conjugation by orthogonal (resp. unitary) matrices implying that for any pair of GOE_n-matrices (A, B), (resp. GUE_n-matrices) we can conjugate $A + \lambda B$ by an orthogonal (resp. unitary) matrix to make A diagonal.

Proof of theorem 5. We start with Part (i). By the above, we assume without loss of generality that A is a diagonal matrix, i.e. $A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, where α_1 and α_2 are the eigenvalues of A satisfying the condition $\alpha_1 \leqslant \alpha_2$. Moreover, we can shift our matrix family so that $A = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}$, where $\Delta = \alpha_2 - \alpha_1 \geqslant 0$.

We know that level crossing points of the linear family $A + \lambda B$ are exactly the zeroes of the discriminant $\mathrm{Dsc}(\lambda)$ of the characteristic polynomial $\chi(\lambda, t)$ with respect to the variable t, where

$$\chi(\lambda, t) = \det(A + \lambda B + tI) = t^2 + t(\lambda\,\mathrm{Tr}\,(B) + \Delta) + \lambda^2 \det(B) + \lambda b_{11}\Delta. \tag{4.3}$$

The latter discriminant equals

$$\mathrm{Dsc}(\lambda) = \lambda^2((b_{22} - b_{11})^2 + 4|b_{12}|^2) + 2\lambda\Delta(b_{22} - b_{11}) + \Delta^2. \tag{4.4}$$

Therefore, since all coefficients of the latter equation are real and the discriminant of $\mathrm{Dsc}(\lambda)$ considered as a quadratic equation in λ is given by

$$D = -4\Delta^2|b_{12}|^2 \leqslant 0,$$

level crossing points of a generic pair (A, B) form a complex conjugate pair $(\lambda, \bar{\lambda})$, where

$$\lambda = \Delta\frac{b_{11} - b_{22} + 2i|b_{12}|}{(b_{22} - b_{11})^2 + 4|b_{12}|^2} \quad \text{and} \quad \bar{\lambda} = \Delta\frac{b_{11} - b_{22} - 2i|b_{12}|}{(b_{22} - b_{11})^2 + 4|b_{12}|^2}. \tag{4.5}$$

In order to find the distribution of λ, we will first find its conditional distribution assuming that Δ is constant. Set $\Sigma := \frac{b_{11} - b_{22}}{\Delta}$ and $\Theta := \frac{2|b_{12}|}{\Delta}$ giving $\lambda = \frac{1}{\Sigma - \mathrm{i}\Theta}$.

Since $b_{11}, b_{22} \sim N(0, 2)$ and are independent, we get that $\Sigma \sim N(0, \frac{4}{\Delta^2})$. Further, $\Theta \sim \frac{2}{\Delta}|N(0, 1)|$, which can be expressed using χ_1-distribution, see e.g. Therefore, the conditional PDFs of Σ and Θ are given by

$$\mathcal{P}^{\Delta}_{\Sigma}(u) = \frac{\Delta}{2\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{u^2\Delta^2}{8}}$$

and

$$\mathcal{P}^{\Delta}_{\Theta}(v) = \begin{cases} \frac{\Delta}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{v^2\Delta^2}{8}}, & \text{for } v \geqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since Σ depends on b_{11} and b_{22}, while Θ depends of b_{12}, we get that Σ and Θ are independent random variables. Therefore, their joint distribution is given by

$$\mathcal{P}^{\Delta}_{(\Sigma,\Delta)}(u,v) = \mathcal{P}^{\Delta}_{\Sigma}(u) \cdot \mathcal{P}^{\Delta}_{\Theta}(v) = \begin{cases} \frac{\Delta^2}{4\pi}\mathrm{e}^{-\frac{\Delta^2(u^2+v^2)}{8}}, & \text{for } v \geqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$

Introduce $\mathfrak{X} := \frac{\Sigma}{\Sigma^2+\Theta^2}$ and $\mathfrak{Y} := \frac{\Theta}{\Sigma^2+\Theta^2}$ implying that $\lambda = \frac{1}{\Sigma - \mathrm{i}\Theta} = \mathfrak{X} + \mathrm{i}\mathfrak{Y}$. Since the Jacobian of the variable change is given by

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{(u^2+v^2)^2} = (x^2+y^2)^2,$$

the joint distribution of $\mathfrak{X}$ and $\mathfrak{Y}$ coincides with

$$\mathcal{P}^{\Delta}_{(\mathfrak{X},\mathfrak{Y})}(x,y) = \begin{cases} \frac{\Delta^2}{4\pi(x^2+y^2)^2} \cdot \mathrm{e}^{-\frac{\Delta^2}{8(x^2+y^2)}}, & \text{for } y \geqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the conditional distribution of λ with Δ fixed equals

$$\mathcal{P}^{\Delta}(\lambda) = \frac{\Delta^2}{4\pi \mid \lambda \mid^4} \cdot \mathrm{e}^{-\frac{\Delta^2}{8|\lambda|^2}}.$$

The distribution of pairs of eigenvalues (α_1, α_2) with $\alpha_1 \leqslant \alpha_2$ of a GOE_2-matrix is given by

$$\mathcal{P}(\alpha_1,\alpha_2) = \frac{(\alpha_2-\alpha_1)}{4\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{\alpha_1^2+\alpha_2^2}{4}},$$

where $-\infty < \alpha_1 \leqslant \alpha_2 < \infty$.

Thus, the distribution of λ with $\mathrm{Im}\,\lambda > 0$ is given by

$$\begin{aligned} \mathcal{P}_{>0}(\lambda) &= \iint_{-\infty<\alpha_1\leqslant\alpha_2<\infty} \frac{(\alpha_2-\alpha_1)}{4\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{\alpha_1^2+\alpha_2^2}{4}} \cdot \frac{(\alpha_2-\alpha_1)^2}{4\pi \mid \lambda \mid^4} \cdot \mathrm{e}^{-\frac{(\alpha_2-\alpha_1)^2}{8|\lambda|^2}}\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_1 \\ &= \iint_{-\infty<\alpha_1\leqslant\alpha_2<\infty} \frac{(\alpha_2-\alpha_1)^3}{16\sqrt{2}\cdot\pi^{3/2}} \cdot \frac{1}{\mid \lambda \mid^4} \cdot \mathrm{e}^{-\frac{\alpha_1^2+\alpha_2^2}{4}} \cdot \mathrm{e}^{-\frac{(\alpha_2-\alpha_1)^2}{8|\lambda|^2}}\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_1 \\ &= \frac{2}{\pi(1+\mid \lambda \mid^2)^2}. \end{aligned}$$

To get the actual PDF of λ, we must divide the previous answer by 2, getting

$$\mathcal{P}_{\mathrm{GOE}_2}(\lambda) = \frac{1}{\pi(1+\mid \lambda \mid^2)^2}$$

which settles Part (i).

Now we prove Part (ii) by using the same methods as for the GOE_2-ensemble to calculate the distribution of level crossings for the GUE_2-case. As in the previous case, level crossing point λ with nonnegative imaginary part is given by

$$\lambda = \Delta \frac{b_{11} - b_{22} + 2ib_{12}}{(b_{22} - b_{11})^2 + 4 \mid b_{12} \mid^2} = \frac{1}{\Sigma - \mathrm{i}\Theta},$$

where $\Sigma := \frac{b_{11}-b_{22}}{\Delta}$ and $\Theta := \frac{2|b_{12}|}{\Delta}$.

Since $b_{11}, b_{22} \sim N(0,1)$ and are independent, we obtain $b_{22} - b_{11} \sim N(0,2)$, and hence, $\Sigma \sim N\left(0, \frac{2}{\Delta^2}\right)$. Therefore, the conditional PDF of Σ is given by

$$\mathcal{P}^{\Delta}_{\Sigma}(u) = \frac{1}{\frac{\sqrt{2}}{\Delta}\sqrt{2}\sqrt{\pi}} \mathrm{e}^{-u^2\Delta^2/4} = \frac{\Delta}{2\sqrt{\pi}} \mathrm{e}^{-u^2\Delta^2/4}.$$

Since $\mathrm{Re}(b_{12}), \mathrm{Im}(b_{12}) \sim N(0, \frac{1}{2})$, then $\frac{1}{1/\sqrt{2}}|b_{12}| \sim \chi_2$. Thus, the conditional PDF of Θ is given by

$$\mathcal{P}^{\Delta}_{\Theta}(v) = \begin{cases} \frac{\Delta^2 y}{2} \mathrm{e}^{-v^2\Delta^2/4}, & \text{for } v \geqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$

The joint distribution of Σ and Θ gives us the conditional distribution of $\frac{1}{\lambda}$. Since b_{12} is independent of b_{11} and b_{22}, then Σ and Θ are also independent random variables which implies that the conditional PDF of $\frac{1}{\lambda}$ is the product of the PDFs of Σ and Θ, i.e.

$$\mathcal{P}^{\Delta}_{1/\lambda}(u,v) = \mathcal{P}^{\Delta}_{\Sigma}(u) \cdot \mathcal{P}^{\Delta}_{\Theta}(v) = \frac{\Delta^3}{4\sqrt{\pi}} v \mathrm{e}^{-(u^2+v^2)\Delta^2/4}.$$

Introducing as in the previous case $\mathfrak{X} := \frac{\Sigma}{\Sigma^2+\Theta^2}$ and $\mathfrak{Y} := \frac{\Theta}{\Sigma^2+\Theta^2}$, we get $\lambda = \frac{1}{\Sigma - \mathrm{i}\Theta} = \mathfrak{X} + \mathrm{i}\mathfrak{Y}$. As above the Jacobian of the variable change is given by

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{(u^2+v^2)^2} = (x^2+y^2)^2,$$

and the joint distribution of $\mathfrak{X}$ and $\mathfrak{Y}$ coincides with

$$\mathcal{P}^{\Delta}_{(\mathfrak{X},\mathfrak{Y})}(x,y) = \frac{\mathcal{P}^{\Delta}_{\Sigma}(u) \cdot \mathcal{P}^{\Delta}_{\Theta}(v)}{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|} = \frac{y\Delta^3 \mathrm{e}^{-\Delta^2/4(x^2+y^2)}}{4\sqrt{\pi}(x^2+y^2)^3}.$$

As $\mathfrak{X} = \mathrm{Re}(\lambda)$ and $\mathfrak{Y} = \mathrm{Im}(\lambda)$, then for a given value of Δ, the conditional distribution of λ is given by

$$\mathcal{P}^{\Delta}(\lambda) = \frac{\mathrm{Im}(\lambda)\Delta^3 \mathrm{e}^{-\Delta^2/4|\lambda|^2}}{4\sqrt{\pi}|\lambda|^6}.$$

Finally, in order to find the (unconditional) distribution of λ, we recall that the PDF of the joint distribution for pairs of eigenvalues $\alpha_1 \leqslant \alpha_2$ of a random matrix belonging to GUE_2 is given by

$$\mathcal{P}(\alpha_1, \alpha_2) = \frac{1}{2\pi}(\alpha_2 - \alpha_1)^2 \mathrm{e}^{-(\alpha_1^2+\alpha_2^2)/2} = \frac{\Delta^2}{2\pi} \mathrm{e}^{-(\alpha_1^2+\alpha_2^2)/2}.$$

Therefore, since $\Delta = a_2 - a_1$, the distribution for level crossing point λ with $\operatorname{Im}\lambda \geqslant 0$ is given by

$$\begin{aligned}\mathcal{P}_{>0}(\lambda) &= \iint_{-\infty<a_1\leqslant a_2<+\infty} \mathcal{P}^{\Delta}(\lambda)\cdot\mathcal{P}(\alpha_1,\alpha_2)\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1\\ &= \iint_{-\infty<a_1\leqslant a_2<+\infty} \frac{(\alpha_2-\alpha_1)^5}{8\pi^{3/2}}\cdot\frac{\operatorname{Im}(\lambda)}{|\lambda|^6}\cdot \mathrm{e}^{\frac{-(\alpha_2-\alpha_2)^2}{4|\lambda|^2}-\frac{(\alpha_1^2+\alpha_2^2)}{2}}\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1\\ &= \frac{8\operatorname{Im}(\lambda)}{\pi(1+|\lambda|^2)^3}.\end{aligned}$$

Therefore, the actual distribution for level crossing point $\lambda \in \mathbb{C}$ equals

$$\mathcal{P}_{\mathrm{GUE}_2}(\lambda) = \frac{4|\operatorname{Im}(\lambda)|}{\pi(1+|\lambda|^2)^3}$$

which settles Part (ii). □

To conclude this section, let us briefly discuss conjectures 1 and 2 that were formulated in the introduction.

Conjecture 1 states that, for every $n \geqslant 2$, Part (i) of theorem 5 actually holds if matrices A and B are independently sampled from GOE_n. This is supported by notable numerical evidence. Since the conjectural distribution of level crossings in this case is uniform on $\mathbb{C}P^1$, one can hope that as in the case of the complex Gaussian ensembles, there exists some additional symmetry of such pencils (beside the SO_2-action presented in section 3), which might explain this behavior. We were unfortunately unable to find any group larger than SO_2 preserving the required distributions. At the moment, the extension of Part (i) of theorem 5 is purely computational and remains conceptually unexplained. On the other hand, for $n > 2$, it is impossible to imitate the proof of Part (i) since discriminants of a random polynomials of degree exceeding 2 are extremely difficult to study analytically.

Compared to conjecture 1, conjecture 2 is a much weaker statement, claiming only the existence of the limiting distribution of level crossings when $n \to \infty$. Its numerical evidence is also somewhat weaker since it is difficult to compare results for different values of n. Still, figure 4 shows the monotone sequence of convex empirical CDFs of the random variable $|Y|$ for $n = 2, \ldots, 6$. (We restricted our experiments to $n \leqslant 6$, as when n increases the calculation of the level crossing points quickly becomes numerically challenging and unstable.) Due to the action of SO_2, if one can prove that for any given n, the CDF of $|Y|$ is convex, then this would probably imply conjecture 2. At the moment we lack an insight which might help to guess the explicit form of the asymptotic CDF of $|Y|$.

5. Real gaussian ensembles

Our next two results are analogs of the theorem 5 in the case of the $\mathrm{GE}_2^{\mathbb{R}}$-ensemble. However, the formulas are much more cumbersome.

Theorem 6.

(i) The density of the average of the two level crossing points (λ_+, λ_-) with respect to the Lebesgue measure on the real axis is given by the following single integral

$$\rho_{\frac{\lambda_+ + \lambda_-}{2}}(x) = \int_{-1}^{1} \frac{|t|}{\pi\sqrt{2-2t}\,(x^2t^2+1)^2}\mathrm{d}t, \tag{5.1}$$

where $x \in \mathbb{R}$.

(ii) The density of the product of the two level crossing points (λ_+, λ_-) with respect to the Lebesgue measure on the real axis is given by

$$\rho_{\frac{D_A}{D_B}}(x) = \Theta(x)\left[\frac{1}{2(x+1)^2} - \int_{-\infty}^{0} \frac{y\mathrm{e}^{-y(1+x)/2}}{8}\mathrm{erfc}(\sqrt{-y})\mathrm{erfc}(\sqrt{-xy})\mathrm{d}y\right] + \Theta(-x)\left[\frac{1}{(x+1)^2}\left(1 + \frac{3x-1+(x-3)\sqrt{-x}}{\sqrt{8}(1-x)^{3/2}}\right)\right], \tag{5.2}$$

where $x \in \mathbb{R}$ and $\mathrm{erfc}(t)$ stands for the standard complementary error function given by $\mathrm{erfc}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty \mathrm{e}^{-\tau^2}\mathrm{d}\tau$.

For the actual distribution of the level crossings on the complex λ-plane we were only able to obtain the following complicated claim.

Proposition 4.

(i) For $\lambda = x + \mathrm{i}y$ and $y \neq 0$, the distribution of level crossings of (1.1) with A and B independently taken from the $\mathrm{GE}_2^{\mathbb{R}}$-ensemble is given by the triple integral:

$$\mathcal{P}_{\mathrm{GE}_2^{\mathbb{R}}}(x,y)\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\mathrm{d}a\int_0^\infty \mathrm{d}r\int_{-\infty}^{\infty}\mathrm{d}b\cdot \mathrm{e}^{-\frac{r^2+b^2+(r^2-b^2)(x^2+y^2)+\left(\frac{ar+x(r^2-b^2)}{b}\right)^2}{2}} \cdot\left|\frac{yr}{2\pi^2 b}(r^2-b^2)^2\right|\cdot\frac{\Theta\left((r^2-b^2)(x^2+y^2)+\left(\frac{ar+x(r^2-b^2)}{b}\right)^2 - a^2\right)}{\sqrt{(r^2-b^2)(x^2+y^2)+\left(\frac{ar+x(r^2-b^2)}{b}\right)^2 - a^2}}\mathrm{d}x\mathrm{d}y, \tag{5.3}$$

where Θ is the Heaviside Θ-function, i.e. $\Theta(t) = 0$ for $t < 0$ and $\Theta(t) = 1$ for $t > 0$.

(ii)

$$\mathcal{P}_{\mathrm{GE}_2^{\mathbb{R}}}(x,0)\mathrm{d}x\mathrm{d}y = \frac{\sqrt{2}}{\pi}\frac{\mathrm{d}x\delta y}{(1+x^2)^2}. \tag{5.4}$$

Here δy means the delta-function in the y-direction concentrated on the x-axis. In other words, the distribution of the level crossings consists of its regular part given by (i) which is absolutely continuous with respect to the Lebesgue measure in the plane and its singular part given by (ii) which is concentrated on the x-axis.

In order to prove theorem 6 and proposition 4, we will use the standard presentation of real 2×2-matrices as linear combinations of Pauli matrices which was extensively applied in [ShZa1]. Namely, let $A = (a_+, \mathrm{i}a_-, a_\Delta)\cdot\vec{\sigma}$ be a real 2-by-2 matrix with normal variables, generic up to additional multiples of identity. Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the standard triple of Pauli matrices. Denote the coefficient vector (a_+, a_-, a_Δ) by $\vec{A}$ and consider the inner product on such triples using a Minkowski metric:

$$\vec{A}\cdot\vec{B} :=_{\mathrm{def}} \vec{A}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\vec{B}. \tag{5.5}$$

Notice that the discriminant of A, i.e. the expression which vanishes if and only A has a multiple eigenvalue, is given by

$$D_A = \vec{A}\cdot\vec{A} = a_+^2 - a_-^2 + a_\Delta^2. \tag{5.6}$$

Similarly construct $B = (b_+, \mathrm{i}b_-, b_\Delta)\cdot\vec{\sigma}$ and consider the linear family

$$C = A + \lambda B. \tag{5.7}$$

We get

$$D_C = D_A + \lambda^2 D_B + 2\lambda\vec{A}\cdot\vec{B} \tag{5.8}$$

with zeroes at

$$\lambda = -\frac{\vec{A}\cdot\vec{B}}{D_B} \pm \sqrt{\left(\frac{\vec{A}\cdot\vec{B}}{D_B}\right)^2 - \frac{D_A}{D_B}}. \tag{5.9}$$

Firstly, let us prove theorem 6.

Proof. To settle Part (i), observe that $\frac{\lambda_+ + \lambda_-}{2} = -\frac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}}$ which amounts to computing a single delta. In this case we make the isometric transformation $b_- \mapsto -b_-$ and let a be the component of $\vec{A}$ along $\vec{B}$, which allows us to work in a Euclidean space for the purposes of computing $\vec{A}\cdot\vec{B}$. We also use spherical coordinates for $\vec{B}$ given by:

$$b_+ = R_B \sin\phi_B \cos\theta_B; \tag{5.10}$$

$$b_\Delta = R_B \sin\phi_B \sin\theta_B; \tag{5.11}$$

$$b_- = -R_B \cos\phi_B; \tag{5.12}$$

$$-\frac{\vec{A}\cdot\vec{B}}{D_B} = -\frac{aR_B}{R_B^2(1-2\cos^2\phi_B)} = \frac{a}{R_B\cos(2\phi_B)}. \tag{5.13}$$

Note that here R_B^2 is χ_3^2-distributed whereas r_B^2 is χ_2^2-distributed. Next observe that $c \equiv \cos\phi_B$ is uniformly distributed for any spherically symmetric distribution, which means that if $c = \pm\sqrt{\frac{1-t}{2}}$, then

$$\rho_{\cos\phi_B}(c)\mathrm{d}c = \frac{\mathrm{d}c}{2}; \tag{5.14}$$

$$\rho_{1-2\cos^2\phi_B}(t)\mathrm{d}t = \frac{2\left|\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\frac{1-t}{2}}\right|\mathrm{d}t}{2}\mathrm{d}t = \frac{\mathrm{d}t}{2\sqrt{2-2t}}; \tag{5.15}$$

$$R_B \propto \rho_{\sqrt{\chi_3^2}}(R) = \sqrt{\frac{2}{\pi}}R^2\mathrm{e}^{-R^2/2}. \tag{5.16}$$

So the distribution of the average of two level crossings simply becomes

$$\rho_{\frac{\lambda_+ + \lambda_-}{2}}(x) = \iiint \delta\left(x + \frac{a}{Rt}\right)\rho(a,R,t)\mathrm{d}a\mathrm{d}R\mathrm{d}t. \tag{5.17}$$

Resolving the delta with respect to a gives $\left|\frac{\mathrm{d}a}{\mathrm{d}x}\right| = |Rt|$ which implies that

$$\begin{aligned}\rho_{\frac{\lambda_+ + \lambda_-}{2}}(x) &= \int_{-1}^{1}\mathrm{d}t\int_0^\infty \mathrm{d}R\left|R^3 t\right|\frac{\mathrm{e}^{-\frac{R^2+(xtR)^2}{2}}}{2\pi\sqrt{2-2t}}\\ &= \int_{-1}^{1}\mathrm{d}t\int_0^\infty \mathrm{d}R\frac{|t|}{2\pi\sqrt{2-2t}}R^3\mathrm{e}^{-(1+x^2t^2)\frac{R^2}{2}}\\ &= \int_{-1}^{1}\frac{|t|}{\pi\sqrt{2-2t}\left(x^2t^2+1\right)^2}\mathrm{d}t.\end{aligned} \tag{5.18}$$

To settle Part (ii), compute the distribution of $D_B = (b_+^2 + b_\Delta^2) - b_-^2$:

$$\begin{aligned}\rho_D(D) &= \iint_0^\infty \delta(D+y-x)\rho_{\chi_2^2}(x)\frac{\mathrm{e}^{-y/2}}{\sqrt{8\pi y}}\mathrm{d}x\mathrm{d}y\\ &= \frac{\mathrm{e}^{-D/2}}{\sqrt{8}}\left(1-\Theta(-D)\mathrm{erf}(\sqrt{-D})\right).\end{aligned} \tag{5.19}$$

It is worth noting that in the positive range this is just $\frac{\rho_{\chi_2^2}(x)}{\sqrt{2}}$, so the probability that the discriminant is positive is $\frac{1}{\sqrt{2}}$.

The distribution of the product $\lambda_+\lambda_- = \frac{D_A}{D_B}$ is the D-ratio distribution:

$$\rho_{\lambda_+\lambda_-}(x) = \int_{-\infty}^{\infty}|y|\rho_D(y)\rho_D(xy)\mathrm{d}y. \tag{5.20}$$

We split the latter integral into four parts depending on the signs of x and y:

$$\rho_{++} = \int_0^\infty \frac{y\mathrm{e}^{-y(1+x)/2}}{8} = \frac{1}{2(x+1)^2}; \tag{5.21}$$

$$\rho_{+-} = \int_{-\infty}^{0} -\frac{y\mathrm{e}^{-y(1+x)/2}}{8}\mathrm{erfc}(\sqrt{-y})\mathrm{erfc}(\sqrt{-xy}); \tag{5.22}$$

$$\rho_{--} = \int_{-\infty}^{0} -\frac{y\mathrm{e}^{-y(1+x)/2}}{8}\mathrm{erfc}(\sqrt{-y}) = \frac{1}{2(x+1)^2}\left(1+\frac{3x-1}{\sqrt{2}(1-x)^{3/2}}\right); \tag{5.23}$$

$$\rho_{-+} = \int_0^\infty \frac{y\mathrm{e}^{-y(1+x)/2}}{8}\mathrm{erfc}(\sqrt{-xy}) = \frac{1}{2(x+1)^2}\left(1+\frac{(x-3)\sqrt{-x}}{\sqrt{2}(1-x)^{3/2}}\right). \tag{5.24}$$

Observe that only one integral out of four can not be computed in a closed form, but it can be computed numerically using e.g. Mathematica. Combining terms, we get

$$\begin{aligned}\rho_{\frac{D_A}{D_B}}(x) = \Theta(x)&\left[\frac{1}{2(x+1)^2} - \int_{-\infty}^{0}\frac{y\mathrm{e}^{-y(1+x)/2}}{8}\mathrm{erfc}(\sqrt{-y})\mathrm{erfc}(\sqrt{-xy})\mathrm{d}y\right]\\ &+\Theta(-x)\left[\frac{1}{(x+1)^2}\left(1+\frac{3x-1+(x-3)\sqrt{-x}}{\sqrt{8}(1-x)^{3/2}}\right)\right]\end{aligned} \tag{5.25}$$

which is the required expression. □

In order to settle proposition 4 we need the following lemma.

Lemma 4. *If A and B are independently chosen from the* $\mathrm{GE}_2^{\mathbb{R}}$*-ensemble, then the probability of attaining a real pair of level crossing points* $\lambda_\pm$ *in the family* $C = A + \lambda B$ *equals* $\frac{1}{\sqrt{2}}$.

Proof. We use a result from [ShZa1] saying that the proportion of real eigenvalues for a fixed A is given by

$$\kappa(a_+, a_-, a_\Delta) = \begin{cases} 1 & \text{if } D_A < 0 \\ 1 - \frac{1}{\pi}\arccos\frac{a_-^2}{a_+^2 + a_\Delta^2} & \text{if } D_A \geqslant 0, \end{cases} \tag{5.26}$$

see formula (5.43) in [ShZa1]. The expectation value over the set of matrices with positive discriminant is given by

$$\langle\kappa\rangle_{D_A\geqslant 0} = \iiint_{D_A\geqslant 0}\left(1 - \frac{1}{\pi}\arccos\frac{a_-^2}{a_+^2 + a_\Delta^2}\right)\rho(a_+, a_-, a_\Delta)\mathrm{d}a_+\mathrm{d}a_-\mathrm{d}a_\Delta. \tag{5.27}$$

Using spherical coordinates relative to the a_--axis we can simplify the integral as:

$$\int_0^\infty 2\pi r^2\frac{\mathrm{e}^{-r^2/2}}{(2\pi)^{3/2}}\mathrm{d}r\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\left(1 - \frac{1}{\pi}\arccos\frac{\cos^2\phi}{1-\cos^2\phi}\right)d(\cos\phi) = \sqrt{2} - 1. \tag{5.28}$$

On the other hand, the contribution of the set of matrices with $D_A < 0$ is just

$$\langle\kappa\rangle_{D_A<0} = \iiint_{D_A<0}\rho(a_+, a_-, a_\Delta)\mathrm{d}a_+\mathrm{d}a_-\mathrm{d}a_\Delta = P(D_A < 0) = 1 - \frac{1}{\sqrt{2}} \tag{5.29}$$

where the last step follows from equation (5.19). Thus the total probability of getting a real crossing value is

$$\langle\kappa\rangle = \sqrt{2} - 1 + 1 - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \tag{5.30}$$

□

Proof of proposition 4. Due to the isotropy of a normally distributed vector, we are free to rotate the coordinate system in the (b_+, b_Δ)-plane such that $\vec{B} = (r_B, b_-, 0)$. This B-dependent choice of a basis has no impact on the distribution of $\vec{A}$ which has the normally distributed entries (a_1, a_-, a_2) in this basis.

To settle Part (i) of the proposition, assume that the level crossing points $\lambda_\pm = x \pm \mathrm{i}y$ are complex conjugate, in which case we get

$$x = -\frac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}} = \frac{a_1 r_B - a_- b_-}{r_B^2 - b_-^2}; \tag{5.31}$$

$$y = \sqrt{\frac{D_A}{D_B} - \left(\frac{\vec{A}\cdot\vec{B}}{D_B}\right)^2} = \sqrt{\frac{a_1^2 - a_-^2 + a_2^2}{r_B^2 - b_-^2} - x^2}. \tag{5.32}$$

Therefore the density of the joint distribution with respect to the Lebesgue measure in the plane takes the form

$$\rho(x,y) = \iiint_A \iiint_B \delta\left(x + \frac{a_1 r_B - a_- b_-}{r_B^2 - b_-^2}\right) \delta\left(y - \sqrt{\frac{a_1^2 - a_-^2 + a_2^2}{r_B^2 - b_-^2} - x^2}\right) \rho(\vec{A}, \vec{B}). \tag{5.33}$$

Resolving the first delta with respect to a_-, we get

$$a_- = \frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}; \tag{5.34}$$

$$\left|\frac{\mathrm{d}a_-}{\mathrm{d}x}\right| = \left|\frac{r_B^2 - b_-^2}{b_-}\right|. \tag{5.35}$$

Then resolving the second delta with respect to a_2^2, we obtain

$$a_2^2 = (r_B^2 - b_-^2)(x^2 + y^2) + \left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2 - a_1^2; \tag{5.36}$$

$$\left|\frac{\mathrm{d}(a_2^2)}{\mathrm{d}y}\right| = \left|2y(r_B^2 - b_-^2)\right|. \tag{5.37}$$

Inserting, we get

$$\begin{aligned}\rho(x,y) = \int_{a_1} \iiint_B \left|\frac{2y}{b_-}(r_B^2 - b_-^2)^2\right| \rho_{a_-}\left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right) \\ \cdot \rho_{a_2^2}\left((r_B^2 - b_-^2)(x^2 + y^2) + \left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2 - a_1^2\right) \\ \cdot \rho_{a_1}(a_1)\rho_{r_B}(r_B)\rho_{\theta_B}(\theta_B)\rho_{b_-}(b_-).\end{aligned} \tag{5.38}$$

Expanding the expression and integrating out θ_B gives us:

$$\begin{aligned}\rho(x,y) = \int_{-\infty}^{\infty} \mathrm{d}a_1 \int_0^{\infty} \mathrm{d}r_B \int_{-\infty}^{\infty} \mathrm{d}b_- \left|\frac{2y}{b_-}(r_B^2 - b_-^2)^2\right| \frac{\mathrm{e}^{-\left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2/2}}{\sqrt{2\pi}} \\ \cdot \frac{\mathrm{e}^{-\left((r_B^2 - b_-^2)(x^2+y^2) + \left(\frac{a_1 r_B + x(r_b^2 - b_-^2)}{b_-}\right)^2 - a_1^2\right)/2}}{\sqrt{2\pi}\sqrt{(r_B^2 - b_-^2)(x^2 + y^2) + \left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2 - a_1^2}} \\ \cdot \Theta\left[(r_B^2 - b_-^2)(x^2 + y^2) + \left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2 - a_1^2\right] \\ \cdot \frac{\mathrm{e}^{-a_1^2/2}}{\sqrt{2\pi}} r_B \mathrm{e}^{-r_B^2/2} \frac{\mathrm{e}^{-b_-^2/2}}{\sqrt{2\pi}}.\end{aligned} \tag{5.39}$$

After some extra simplifications, we get

$$\rho(x,y) = \int_{-\infty}^{\infty} \mathrm{d}a_1 \int_0^{\infty} \mathrm{d}r_B \int_{-\infty}^{\infty} \mathrm{d}b_- \left| \frac{y r_B}{2\pi^2 b_-} (r_B^2 - b_-^2)^2 \right| \mathrm{e}^{-\frac{a_1^2 + r_B^2 + b_-^2}{2}}$$
$$\cdot \frac{\mathrm{e}^{-((r_B^2 - b_-^2)(x^2+y^2) + \left(\frac{a_1 r_B + x(r_b^2 - b_-^2)}{b_-}\right)^2 - a_1^2)/2}}{\sqrt{(r_B^2 - b_-^2)(x^2+y^2) + \left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2 - a_1^2}}$$
$$\cdot \Theta\left[(r_B^2 - b_-^2)(x^2+y^2) + \left(\frac{a_1 r_B + x(r_B^2 - b_-^2)}{b_-}\right)^2 - a_1^2\right]. \quad (5.40)$$

Suppressing the superfluous subscripts from the integration variables, we obtain the triple integral from the formulation of proposition 4.

To settle Part (ii), observe that by formula (3.1), the density of a distribution the level crossings invariant under the SO_2-action on the real axis should be proportional to $\frac{1}{(1+x^2)^2}$. By lemma 4 the total mass of the measure of level crossings concentrated on the real axis equals $\frac{1}{\sqrt{2}}$. Using this normalization, we arrive at the expression (5.4). □

Let us now comment on conjecture 3, which claims that the limit as $n \to \infty$ of the sequence of the level crossing distributions with A and B independently taken from $\mathrm{GE}_n^{\mathbb{R}}$ is given by the uniform distribution on $\mathbb{C}P^1$. Figure 5 indicates that as n increases, the proportion of real level crossings decreases, the angle distribution becomes closer to the uniform, and the radial distribution approaches the theoretical one given by $\frac{r^2}{1+r^2}$. At the moment, it seems in principle possible to rigorously prove that the proportion of real level crossings decreases when n increases and that the angular distribution becomes more uniformly spread out. On the other hand, we have no idea how to approach the behavior of the asymptotic radial distribution. As the present section shows, already the analytic study of the case $n = 2$ is highly technical. We plan to return to this problem in the future.

6. Monodromy distribution for 3 × 3 Gaussian ensembles

In this final section, we initiate the study of a very important problem about the distribution of the monodromy groups in random linear families (1.1). Below, we present numerical results about the monodromy in random 3×3 linear matrix families (1.1) for the cases of the GUE_3-, GOE_3-, and $\mathrm{GE}_3^{\mathbb{C}}$-ensembles. Some of these numerical results are rather unexpected, see remark 9.

General observations. Observe that, for generic pairs of matrices A and B from GUE_n or GOE_n, all level crossings are simple and arise in complex conjugate pairs; $\binom{n}{2}$ of them lying in the upper half-plane and $\binom{n}{2}$ lying symmetrically in the lower half-plane. We can additionally assume that all level crossings in the upper half-plane have distinct real parts since the coincidence of the real parts happens with probability 0. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_{\binom{n}{2}}$ level crossing points in the upper half-plane ordered by the increase of their real parts. Since generically level crossing points are simple, let $\sigma_1, \sigma_2, \ldots, \sigma_{\binom{n}{2}}$ be the associated sequence of transpositions obtained as follows, see figure 7. Under our assumptions, for every real λ, the spectrum of $A + \lambda B$ is real and simple, which means that no monodromy of the spectrum occurs when λ belongs to the real axis $\mathbb{R} \subset \mathbb{C}$.

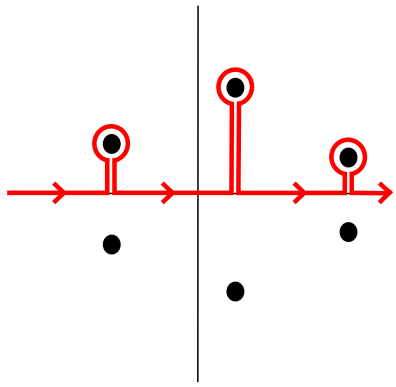

Figure 7. Creating the monodromy sequence.

If λ_i is the ith level crossing point in the upper half-plane in the order of increasing real parts, consider the path in the λ-plane starting on the real axis at $\tau = \mathrm{Re}(\lambda_i)$, going straight up to λ_i, making a small loop encircling λ_i counterclockwise, and returning back to τ_i. As a result, one gets a transposition σ_i of two real eigenvalues corresponding to $\tau_i = \mathrm{Re}(\lambda_i)$. Doing this for each λ_i, $i = 1, \ldots, \binom{n}{2}$, we obtain a sequence of $\binom{n}{2}$ transpositions $\left(\sigma_1, \sigma_2, \ldots, \sigma_{\binom{n}{2}}\right)$, $\sigma_i \in S_n$.

One can verify that the obtained sequence $\left(\sigma_1, \sigma_2, \ldots, \sigma_{\binom{n}{2}}\right)$ of transpositions satisfies the following two conditions:

(i) for general A and B, they generate the symmetric group S_n;
(ii) the product $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{\binom{n}{2}}$ coincides with the inverse permutation $(n, n-1, \ldots, 1)$.

Indeed, for general A and B, the characteristic polynomial $\chi(\lambda, t) = \det(A + \lambda B + tI)$ is an irreducible bivariate polynomial, which means that the above mentioned monodromy group acts transitively on the spectrum of $A + \lambda B$, i.e. on set of roots of $\chi(\lambda, t)$ w.r.t. the variable t when λ is some fixed generic number. Additionally, for general A and B, the spectrum of $A + \lambda B$ is real and distinct for every real λ, which implies (i). To explain (ii), observe that the product $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{\binom{n}{2}}$ acts on the spectrum of $A + \lambda B$, where λ is a large negative number, by interchanging every possible pair of eigenvalues. This results in the complete reversion of the order of eigenvalues, corresponding to a large negative number λ when it moves along the real axis to large positive values. (This phenomenon can be easily understood when considering a generic pair of A and B very close to a non-generic pair $(\tilde{A}, \tilde{B})$ consisting of two real diagonal matrices.)

Notice that the statistics of the monodromy sequences of transpositions for GUE_n and GOE_n are invariant under conjugation by the inverse permutation $(n, n-1, \ldots, 1)$, as well as under reversing the order of the transpositions. These symmetries can be explained as consequences of the symmetries of the ensembles.

Namely, if the matrix $A + \lambda B$ has eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the matrix $-A - \lambda B$ has eigenvalues $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$. These matrix pencils share the same level crossing points, and if a loop in $\mathbb{CP}^1$ permutes the eigenvalues of $A + \lambda B$, then it applies the same permutation to the eigenvalues of $-A - \lambda B$. However, when we compute the monodromy associated to a pair of matrices in our ensembles, we order the (real) eigenvalues for real λ, and the transpositions associated to each level crossing point are written with respect to this ordering.

Since the eigenvalues of $-A-\lambda B$ will have the ordering opposite to those of $A+\lambda B$, the monodromy associated to the pair $(-A,-B)$ will be the monodromy of (A,B), conjugated by $(n,n-1,\ldots,1)$. Since the pairs (A,B) and $(-A,-B)$ have the same probability density, each of the admissible sequences of transpositions will appear with the same frequency as its conjugate.

The other symmetry of our data is its invariance under reversing the order of the transpositions. It can be similarly explained by the equal probability density for the pairs (A,B) and $(A,-B)$. If level crossing points of $A+\lambda B$ are $\lambda_1,\lambda_2,\ldots,\lambda_{n(n-1)}$, then level crossing points of $A-\lambda B$ are $-\lambda_1,-\lambda_2,\ldots,-\lambda_{n(n-1)}$. Level crossing points come in conjugate pairs, and the same transpositions are associated to these pairs, so if $\lambda_1,\lambda_2,\ldots,\lambda_{\binom{n}{2}}$ are level crossing points of $A+\lambda B$ in the upper half-plane, then $-\overline{\lambda}_1,-\overline{\lambda}_2,\ldots,-\overline{\lambda}_{\binom{n}{2}}$ are level crossing points of $A-\lambda B$ in the upper half-plane. Since we order them according to the increase of their real parts—which have been inverted—it now remains to show that the transposition associated to (A,B,λ_i) is the same as that associated to $(A,-B,-\overline{\lambda}_i)$. Since the transposition associated to the level crossing point is the same as that associated to its conjugate, we can instead consider $(A,-B,-\lambda_i)$. Observe that the transposition associated to (A,B,λ_i) is determined by the eigenvalues of

$$A+(\mathrm{Re}(\lambda_i)+\epsilon\mathrm{Im}(\lambda_i))B$$

for $0\leqslant\epsilon\leqslant 1$, and in the same way the transposition associated to $(A,-B,-\lambda_i)$ is determined by

$$A+(\mathrm{Re}(-\lambda_i)+\epsilon\mathrm{Im}(-\lambda_i))(-B)=A+(\mathrm{Re}(\lambda_i)+\epsilon\mathrm{Im}(\lambda_i))B.$$

These coincide, and we conclude that the monodromy sequence associated to $(A,-B)$ is the reverse of that associated to (A,B).

Statistical results for GUE_3*- and* GOE_3*-ensembles.*

For $n=3$, it is easy to check that there are only 8 triples of transpositions in S_3 satisfying conditions (i) and (ii). These triples are: $(12)(12)(13)$; $(12)(13)(23)$; $(12)(23)(12)$; $(13)(12)(12)$; $(13)(23)(23)$; $(23)(12)(23)$; $(23)(13)(12)$; $(23)(23)(13)$. (For comparison, for $n=4$, there are already 3840 6-tuples of transpositions in S_4 satisfying (i) and (ii).)

Remark 8. Observe that, in general, for a given positive integer n, the number of different sequences of $\binom{n}{2}$ transpositions satisfying the above conditions (i) and (ii) is closely related to a specific instance of the so-called simple Hurwitz numbers which in our case counts the number of branched coverings of $\mathbb{C}P^1$ of degree n by Riemann surfaces of genus $\binom{n-1}{2}$ with $\binom{n}{2}$ simple affine ramification points and one complicated ramification point at infinity, see e.g. [ShShV]. Not all such coverings can be realized by plane curves, which additionally explains why the monodromy problem is difficult both in the deterministic and the random set-ups, see also section 7. (The authors want to thank an anonymous referee for bringing up this point.)

Numerical experiments were carried out in MATLAB. Namely, the MATLAB-code computed the transposition associated to level crossing point λ of a pair of matrices (A,B). More exactly, the program calculated the eigenvalues of $A+(\mathrm{Re}(\lambda)+\epsilon\mathrm{Im}(\lambda))B$ as ϵ runs from 0 to 1 in steps of 0.01. A typical plot of the eigenvalues during this process is shown in figure 8. At $\epsilon=0$ all of the eigenvalues are real, so we can number them in the increasing order. For each new ϵ, the new eigenvalues are assigned the same numbers as the closest eigenvalues obtained for the previous value of ϵ. Then, when two eigenvalues collide at $\epsilon=1$, the numbers assigned to these two colliding eigenvalues give the transposition corresponding to level crossing point

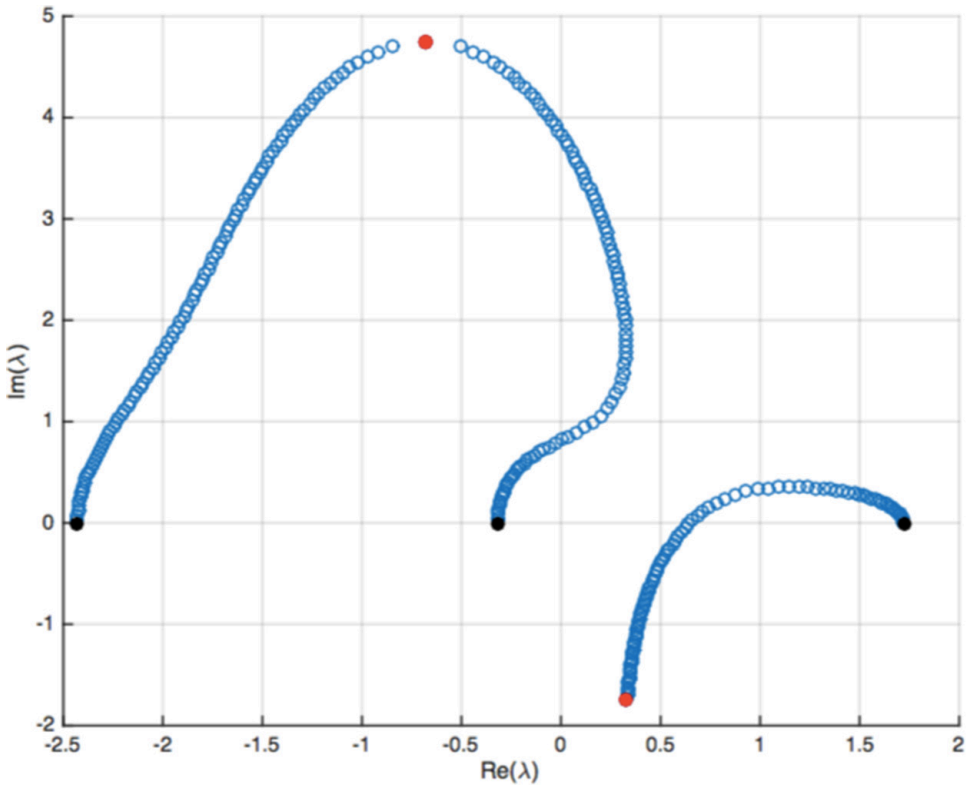

Figure 8. The first and second eigenvalues (of totally 3) collide as the parameter approaches a level crossing point. This results in the monodromy trasformation encoded by the transposition (12).

λ. By following this procedure shown in figure 8 for each of level crossing points in the upper half-plane in order of increasing real part, one obtains triples of transpositions associated to (A, B). This triple of transpositions completely determines the monodromy of the linear family (1.1). Because errors can occur if the real parts of different level crossing points are very close, we discarded such pairs of matrices when gathering monodromy statistics. This procedure was carried out in case of the GUE_3- and GOE_3-ensembles. The resulting statistics for the GUE_3- (top) and the GOE_3-ensembles (bottom) are shown in figure 9.

Statistical results for $\mathrm{GE}_3^{\mathbb{C}}$*-ensemble.*

In this case, in order to calculate the monodromy sequence for a general matrix family of the form (1.1), we must first choose a base point for the system of closed paths in the λ-plane which (generically) is not a level crossing point. We choose $\lambda = 0$, since typically the origin is not a level crossing point for a general pair of matrices, and the preimages of 0 are precisely the eigenvalues of A. Using $\lambda = 0$ as a base point, we need to order our level crossing points with respect to the origin and to choose a system of paths such that

(i) each path begins and ends at 0;
(ii) each path goes around exactly one level crossing point;
(iii) each path does not intersect any other path except at the origin.

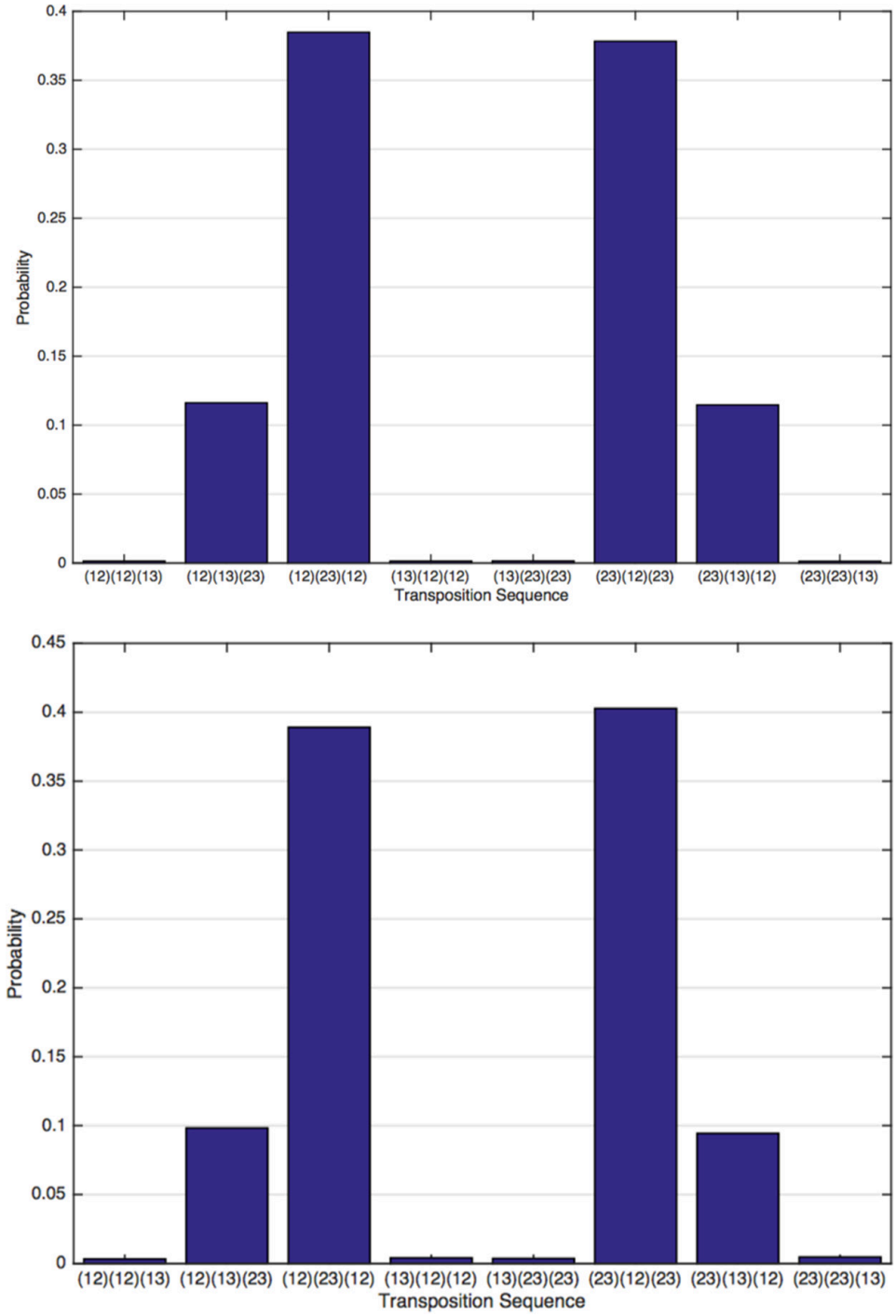

Figure 9. The probabilities of the monodromy triples of transpositions for the GUE_3- and GOE_3-ensembles.

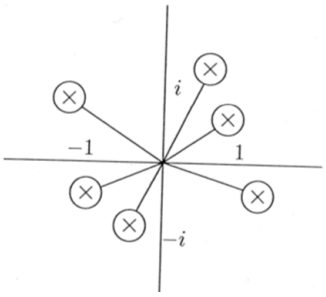

Figure 10. An example of paths in the λ-plane chosen to determine the monodromy for pairs (A, B) from the $\mathrm{GE}_3^{\mathbb{C}}$-ensemble.

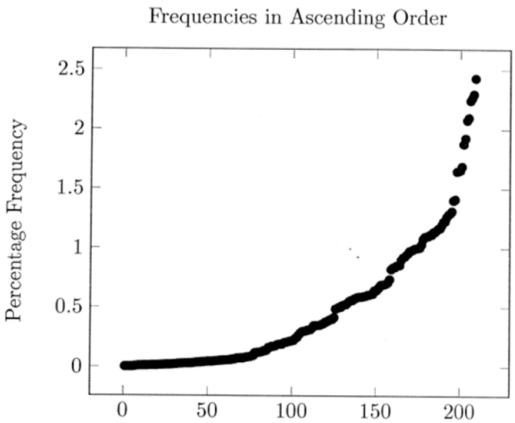

Figure 11. Frequencies of 240 possible 6-tuples of transpositions from S_3 in the ascending order.

As already mentioned, these level crossing points are all generically simple; so as λ traverses a path around one level crossing point and returns to the origin, exactly two of the eigenvalues of A will interchange. Thus we obtain a transposition in the symmetric group S_n. To do this, we have to order the preimages of our starting point (i.e. the eigenvalues of A) and keep track of how these preimages change as we follow each path. This procedure gives us an $n(n-1)$-tuple of transpositions in S_n. Since the concatenation of all paths encompasses all of our level crossing points, the product of all transpositions in the chosen order equals to the identity permutation. When A and B are independently chosen from $\mathrm{GE}_n^{\mathbb{C}}$, the arguments of our level crossing points are uniformly distributed, and so we may order our level crossing points by the argument. However, the choice of which level crossing point is the first one, and whether the level crossing points are ordered clockwise or counterclockwise, is in our hands. The paths we choose will start and end at 0, and go around these level crossing points in a natural way. An example of how we choose such paths is shown in figure 10.

For A and B chosen from $\mathrm{GE}_3^{\mathbb{C}}$, there are 240 admissible sequences of the 6-tuples of transpositions $(\sigma_1, \sigma_2, \ldots, \sigma_6)$ from S_3 satisfying the conditions:

(i) they generate the symmetric group S_3;
(ii) the product $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_6$ coincides with the identity permutation $(1, 2, 3)$.

Using a similar MATLAB-code to determine the monodromy transpositions, we generated 150 000 random matrix pairs in $\mathrm{GE}_3^{\mathbb{C}}$ and calculated their monodromy sequences. Our numerical results show the following, see figure 11.

(i) Of the 240 possible cases, only 209 were realized and only 204 were realized more than once.
(ii) The most common monodromy sequences were $(23)(12)(23)(12)(23)(12)$, which occurred with the frequency 2.43 % and $(12)(13)(13)(23)(23)(12)$ which occurred with the frequency 2.29 %.
(iii) Monodromy sequences in which one permutation occurs four times in a row followed by two occurrences of another permutation and their cyclic permutations (for example, $(12)(12)(12)(12)(13)(13)$ or $(12)(23)(23)(23)(23)(12)$) were the most rare, occurring only once or not at all.

Remark 9. One particularly strange and interesting result is that the labelling of the eigenvalues seems to affect the frequencies with which certain monodromy sequences appear. In the case of $\mathrm{GE}_3^{\mathbb{C}}$-matrices, one can relabel the three preimages of $\lambda = 0$, i.e. the eigenvalues of A, by using the action of S_3. Usually, about half of these six group elements change the frequency by either doubling or halving the original one. The other half of the group tends to keep the frequency the same, but exactly which members of S_3 do what varies from case to case. We have not been able to find a pattern of, or an explanation to, why relabelling changes the frequencies in this peculiar way.

7. Final remarks

In connection with our topic, one can naturally ask why we only restrict ourselves to consideration of the distributions of a single level crossing point on $\mathbb{C}$ and are not trying to obtain information about the joint distribution of all $n(n-1)$ level crossing points which obviously exists in all the above cases. It turns out that for $n > 3$, not all $n(n-1)$-tuples of complex numbers can be realized as level crossings and even the description of the loci of realizable $n(n-1)$-tuples is very complicated. This fact definitely means that at least for $n > 3$, to get the joint distribution of level crossings on such loci will be a formidable (if not completely impossible) task, comp. e.g. [OnSh]. On the other hand, in the simplest case $n = 2$, we calculate and use such joint distributions below.

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