19. R. F. Bikbaev and A. I. Bobenko, "On finite-gap integration of the Landau—Lifshitz equa tion. XYZ case," Preprint LOMI. E-8-83. Leningrad. LOMI (1983).
20. B. A. Dubrovin and S. M. Natanzon, "Real two-zoned solutions of the sine-Gordon equation," Funkts. Anal., 16, No. 1, 27-43 (1982).
21. A. P. Veselov, "Landau—Lifshits equation and integrable systems of classical mechanics, Dokl. Akad. Nauk SSSR, 270, No. 5, 1094-1097 (1983).
22. H. Bateman and A. Erdelyi, Higher Transcendental Functions, Vol. 3, McGraw-Hill.
23. V. A. Steklov, Motion of a Rigid Body in a Fluid [in Russian], Kharkov (1893).
24. F. Kötter, "Über die Bewegung eines festen Körpers in einer Flussigkeit. I, II," J. Re Angew. Math., 109, 51-81, 89-111 (1892).
25. M. M. Bogdan, "Nonlinear self-localized excitations in one-dimensional crystals," Autho Abstract of Candidate's Dissertation, Kharkov, FTINT (1983).
26. I. V. Cherednik, "Integrability of equations of a two-dimensional asymmetric chiral $O(3)$-field and its quantum analogue," Yad. Fiz., 33, No. 1, 278-282 (1981).

HIGHER-DIMENSIONAL ANALOGS OF THE THEOREMS OF NEWTON AND IVORY

A. D. Vainshtein and B. Z. Shapiro

UDC 514.

Well-known theorems of Newton and Ivory [1, 2] assert that the potential of a charged metallic ellipsoid equals a constant in the interior of the ellipsoid and is constant on the confocal ambient ellipsoids. In this paper we prove the analogs of these theorems for hyperboloids of arbitrary signature in Euclidean space of arbitrary dimension.

The authors thank V. I. Arnol'd for formulating this problem and for his constant suppo and A. B. Givental' and B. V. Yusin for useful discussions.

1. Three-Dimensional Case. First, we state the result for the case of a one-sheeted hyperboloid in three-dimensional space. To this end, we include the given hyperboloid in a family of confocal surfaces. This family traces a net of orthogonal lines on the surface of the hyperboloid. The closed (open) lines of this net will be referred to as the parallels (respectively, meridians) of the ellipsoid. The family of meridians extends to a fibering c the simply connected domain bounded by the hyperboloid into open curves; these will be referred to as the meridians of the inner domain. Similarly, the family of parallels extends to a fibering of the outer, nonsimply connected domain bounded by the hyperboloid on closed curves, termed the parallels of the outer domain.

THEOREM 1. There exists a unique (modulo a constant factor) surface current, flowing along the meridians of the hyperboloid, which produces a magnetic field that vanishes in the inner domain and is directed along parallels in the outer domain of the hyperboloid. Similarly, there exists a unique (modulo a constant factor) surface current, flowing along the parallels of the hyperboloid, which produces a magnetic field that vanishes in the outer domain and is directed along meridians in the inner domain of the hyperboloid.

The magnetic field in the inner domain, outside of a charged conducting ellipsoid confocal to the given hyperboloid, coincides, up to its sign, with the electric field of the ellipsoid. Also, the magnetic field in the region of the outer domain between the sheets of a confocal two-sheeted hyperboloid coincides, up to its sign, to the electric field produced by two charges equal in magnitude, distributed on the sheets of the conducting two-sheeted hyperboloid.

The fields constructed in Theorem 1 also yield exact solutions of the problems of potential flow of an incompressible fluid through the inner domain of a triaxial hyperboloid and of the vortex-free flow around this hyperboloid.

Theorem 1 was communicated to the authors by Arnold (see [3, 4]), who asked whether thi theorem may be generalized to higher dimensions and signatures. The theorems of Newton and

All-Union Biotechnical Scientific-Research Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 1, pp. 20-24, January-March, 1985. Original article submitted July 11, 1983.

of Ivory were extended to the case of a two-sheeted hyperboloid in a space of arbitrary dimension by Arnold [5, 6] and Shcherbak, respectively.

2. General Case. Consider in n-dimensional Euclidean space a central hypersurface of second degree (quadric). We include our surface in a family of confocal surface

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i + \lambda} = 1, \tag{1}$$

$a_n < a_{n-1} < \ldots < a_{k+2} < 0 < a_{k+1} < \ldots < a_1$, as the surface corresponding to the value $\lambda = 0$, and define the elliptic coordinates of an arbitrary point in space as the values of λ for which a hypersurface of the family (1) passes through this point. Our hypersurface is diffeomorphic to the direct product $S^l \times \mathbf{R}^k$, $k + l = n - 1$. It distinguishes two regions in space: the inner one, diffeomorphic to $D^{l+1} \times \mathbf{R}^k$, and the outer one, diffeomorphic to $S^l \times \mathbf{R}^k \times \mathbf{R}_+$, where $\mathbf{R}_+$ designates the real positive half axis.

There are two fiberings of the inner region:

a) one into "meridians," diffeomorphic to $\mathbf{R}^k$, and given by the condition of constancy of the elliptic coordinates that are negative (in the inner region);

b) the second, into domains orthogonal to the "meridians," diffeomorphic to D^{l+1} and given by the condition of constancy of the elliptic coordinates that are positive (in the inner region).

Similarly, there are two fiberings of the outer region:

a) one into "parallels," diffeomorphic to S^l and given by the condition of constancy of the elliptic coordinates that are positive (in the outer region);

b) the second, into domains orthogonal to the "parallels," diffeomorphic to $\mathbf{R}^k \times \mathbf{R}_+$ and given by the condition of constancy of the elliptic coordinates that are negative (in the outer region).

Let ω be an arbitrary current, regarded as a form with distribution coefficients (see [7]). We shall say that form ω is harmonic off our hypersurface if it is continuous off this hypersurface, coclosed, and if its exterior derivative is a form supported on the hypersurface.

THEOREM 2. 1) There exists a unique (modulo a constant factor) differential form ω^+, which is nontrivial in the inner region of the space, is harmonic off the hypersurface, is decomposable in elliptic coordinates, and has no real poles.

2) There exists a unique (modulo a constant factor) differential form ω^-, which is nontrivial in the outer region of the space, is harmonic off the hypersurface, is decomposable in elliptic coordinates, and has no real poles.

The forms ω^+ and ω^- can be computed explicitly.

THEOREM 3. 1) The form ω^+ vanishes identically in the outer region of the space, and in the inner region is representable as $\omega^+ = \Phi_l(\xi)d\xi$, where $d\xi = d\xi_1 \wedge \cdots \wedge d\xi_l$, and $\xi_l \leqslant \xi_{l-1} \leqslant \ldots \leqslant \xi_1$ are positive (in the entire space) elliptic coordinates.

2) The form ω^- vanishes identically in the inner region of the space, and in the outer region is representable as $\omega^- = \Phi_k(\eta)d\eta$, where $d\eta = d\eta_1 \wedge \cdots \wedge d\eta_k$, and $\eta_k \leqslant \eta_{k-1} \leqslant \ldots \leqslant \eta_1$ are negative (in the entire space) elliptic coordinates.

THEOREM 4. Function $\Phi_m(x)$ is given by the formula

$$\Phi_m(x) = \Phi_m(x_1, \ldots, x_m) = \prod_{1 \leqslant i < j \leqslant m} (x_i - x_j) \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + a_j)^{-1/2}. \tag{2}$$

Formula (2) has the following geometrical meaning.

Given our hypersurface, we construct two sets. The first lies in the inner region and is obtained by fixing the coordinates $\eta_k = -a_2 < \eta_{k-1} = -a_3 < \ldots < \eta_1 = -a_{k+1} = \zeta$. The second lies in the outer region and is obtained by fixing the coordinates $\zeta = -a_{k+2} = \xi_l < \ldots < -a_{n-1} = \xi_2 < -a_n = \xi_1$. These sets will be referred to as the focal sets of our hypersurface.

Proposition. The focal sets are smooth manifolds. The first of them is a two-sheeted hyperboloid diffeomorphic to the disconnected union of two copies of $\mathbf{R}^l$, whereas the second is an ellipsoid diffeomorphic to S^k.

We call homeoid density on the nondegenerate second degree central surface V the form of highest degree which is obtained from the Euclidean volume form of the layer enclosed be tween V and a hypersurface homothetic to V relative to its center when the width of this la is made infinitesimally small. The homeoid density on an intersection of nondegenerate cen tral hypersurfaces is defined similarly. On the focal sets the homeoid density is defined a the limit of the corresponding forms on nondegenerate intersection-manifolds for a fixed va of the integral of these forms on the manifold.

THEOREM 5. Form ω^+ in the inner region and form ω^- in the outer region are induced by the homeoid densities on the focal sets by translation along the hypersurfaces of family (1 transverse to these sets.

Remark. The assertion of Theorem 5 remains true if, instead of the first focal set, o considers the manifold given by the condition of constancy of the coordinates $-a_1 < \eta_k < -a_2 < \eta_{k-1} < \ldots < \eta_1 = -a_{k+1} = \zeta$ (which is diffeomorphic to $\mathbf{R}^l$), and, instead of the second focal set, the manifold given by the condition of constancy of the coordinates $\zeta = -a_{k+1} = \xi_l < \ldots < \xi_2 < -a_n < \xi_1$ (which is diffeomorphic to a disconnected union of 2^{l-1} copies of S^l

3. Proofs of the Theorems. Proof of Theorems 2-4. Let ω be the sought-for form. The

$$d * \omega = 0, \quad d\omega = \delta(\zeta)\, d\zeta \wedge \Omega, \qquad (3$$

where $\delta(\zeta)$ is the delta-function and Ω is a form on the hypersurface. In agreement with the assumption of Theorem 2, we seek ω in the form $\omega = a(\xi, \eta, \zeta)\, d\xi' \wedge d\eta'$, where $d\xi' = d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_q}$, $d\eta' = d\eta_{j_1} \wedge \cdots \wedge d\eta_{j_r}$. Then from (3) we obtain

$$d * \omega = d\left(a h_{\xi''} h_{\eta''} h_\zeta h_{\xi'}^{-1} h_{\eta'}^{-1}\, d\xi'' \wedge d\zeta \wedge d\eta''\right) = 0, \qquad (4$$

$$d\omega = \sum_i \frac{\partial a}{\partial \xi_i''} d\xi_i'' \wedge d\xi' \wedge d\eta' + \sum_j \frac{\partial a}{\partial \eta_j''} d\eta_j'' \wedge d\xi' \wedge d\eta' + \frac{\partial a}{\partial \zeta} d\zeta \wedge d\xi' \wedge d\eta' = \delta(\zeta)\, d\zeta \wedge \Omega.$$

Here $d\xi''$ is a decomposable form, which completes $d\xi'$ up to $d\xi$, $h_{\xi'}$ is the product of the La coefficients appearing in $d\xi'$, and $d\eta''$, $h_{\xi''}$, $h_{\eta'}$, $h_{\eta''}$ are defined similarly. From (4) we ob tain the system of equations

$$\frac{\partial}{\partial \xi_i'}\left(\frac{a h_{\xi''} h_{\eta''} h_\zeta}{h_{\xi'} h_{\eta'}}\right) = \frac{\partial}{\partial \eta_j'}\left(\frac{a h_{\xi''} h_{\eta''} h_\zeta}{h_{\xi'} h_{\eta'}}\right) = 0,$$
$$\frac{\partial a}{\partial \xi_i''} = \frac{\partial a}{\partial \eta_j''} = 0, \qquad (5$$
$$\frac{\partial a}{\partial \zeta} = \delta(\zeta)\, b(\xi, \eta),$$

and $\Omega = b(\xi, \eta)\, d\xi' \wedge d\eta'$.

Taking account of the explicit expressions of the Lamé coefficients for elliptic coordinates [8] we get

$$\frac{h_{\xi''} h_{\eta''} h_\zeta}{h_{\xi'} h_{\eta'}} = \frac{\Phi_{l-q}(\xi'')\, \Phi_{k-r}(\eta'')\, \Phi_1(\zeta)}{\Phi_q(\xi')\, \Phi_r(\eta')} \cdot \frac{\prod_i (\xi_i'' - \zeta) \prod_j (\eta_j'' - \zeta) \prod_{t,u} (\xi_t'' - \eta_u'')}{\prod_{v,w} (\xi_v' - \eta_w')},$$

where functions $\Phi_m(x)$ are defined by formula (2). Substituting in (5) and solving the equations in the order in which they are written, we get

$$b(\xi, \eta) = B \Phi_q(\xi')\, \Phi_r(\eta') \prod_{i,j} (\xi_i' - \eta_j'),$$
$$a(\xi, \eta, \zeta) = (\chi(\zeta) + C)\, b(\xi, \eta), \qquad (6$$

where $\chi(\zeta)$ is the Heaviside function, and B and C are constants.

Now let us impose the condition that ω have no poles. Recalling that the elliptic coordinates themselves degenerate at some points of the space, we are led to the investigation of the zeros of the expression

$$\prod_i(\xi_i'-\zeta)\prod_j(\eta_j'-\zeta)\prod_{m,p}(\xi_m'-\eta_p'')\prod_{s,t}(\xi_s'-\xi_t'')\prod_{u,v}(\eta_u'-\xi_v'')\prod_{w,z}(\eta_w'-\eta_z'').$$

Since always $-a_1 \leqslant \eta_k \leqslant -a_2 \leqslant \ldots \leqslant -a_{k+1} \leqslant \zeta \leqslant -a_{k+2} \leqslant \xi_l \leqslant \ldots \leqslant -a_n \leqslant \xi_1$, we conclude that $\prod_{m,p}(\xi_m'-\eta_p'')$ and $\prod_{u,v}(\eta_u'-\xi_v'')$ do not vanish in E^n. Further, the expressions $\prod_{s,t}-(\xi_s'\xi_t'')$ and $\prod_{w,z}(\eta_w'-\eta_z'')$ must have roots for $\zeta > 0$ as well as for $\zeta < 0$, provided that the number of components in these products is different from zero. This implies that either $d\xi' = d\xi$ or $d\xi'' = d\xi$ and, similarly, either $d\eta' = d\eta$ or $d\eta'' = d\eta$. Finally, the expression $\prod_i(\xi_i'-\zeta)$ has a zero for $\zeta > 0$, and expression $\prod_j(\eta_j'-\zeta)$ — a zero for $\zeta < 0$. Consequently, if $d\xi' = d\xi$, then $d\eta'' = d\eta$ and $C = 0$, whereas $d\xi'' = d\xi$ implies $d\eta' = d\eta$ and $C = -1$. In conjunction with (6) this proves the stated results.

Theorem 1 follows from Theorems 2 and 3; the current is interpreted as the form Ω, and the magnetic field as the vector field corresponding to the form $*\omega$.

Proof of Theorem 5. Since the forms we are interested in are defined modulo a constant factor, it suffices to show that for every pair of points $x' = (\xi', \eta, \zeta)$ and $x'' = (\xi'', \eta, \zeta)$ lying in the inner region of the space the ratio $\omega^+(x')/\omega^+(x'')$ equals the ratio of the homeoid densities at the points ξ' and ξ'' on the first focal set (there is no point in taking the coordinates η, ζ of the points x' and x'' distinct, because neither ω^+ nor the homeoid density depend on these coordinates).

It is clear that the homeoid density on the focal set is inversely proportional to the product of the Lamé coefficients corresponding to the fixed elliptic coordinates (see [9]). Therefore, the ratio of the homeoid densities at the points ξ' and ξ'' equals

$$\frac{h_\eta(x'')h_\xi(x'')}{h_\eta(x')h_\xi(x')}=\left(\prod_{i,j}\frac{\xi_i''-\eta_j}{\xi_i'-\eta_j}\prod_m\frac{\xi_m''-\zeta}{\xi_m'-\zeta}\right)^{1/2}.$$

On the other hand, Theorems 3 and 4 yield

$$\frac{\omega^+(x')}{\omega^+(x'')}=\frac{\Phi_l(\xi')\,d\xi(x')}{\Phi_l(\xi'')\,d\xi(x'')}=\frac{\Phi_l(\xi')\,h_\xi(x'')}{\Phi_l(\xi'')\,h_\xi(x')}=\left(\prod_{i,j}\frac{\xi_i''-\eta_j}{\xi_i'-\eta_j}\prod_m\frac{\xi_m''-\zeta}{\xi_m'-\zeta}\right)^{1/2}.$$

Theorem 5 for the form ω^+ is thus proved. The proof for the form ω^- is analogous.

LITERATURE CITED

1. I. Newton, Mathematical Principles of Natural Philosophy, F. Cajori (ed.), Cambridge (1934).
2. J. Ivory, "On the attraction of homogeneous ellipsoids," Philos. Trans., 99, 345-372 (1809).
3. V. I. Arnold, "Magnetic analogs of the theorems of Newton and Ivory," Usp. Mat. Nauk, 38, No. 5, 145-146 (1983).
4. V. I. Arnold, "Some algebrogeometrical aspects of the Newton attraction theory," in: Arithmetic and Geometry, Vol. 2: Geometry, Birhäuser, Basel—Boston (1983).
5. V. I. Arnold, "On the Newtonian potential of hyperbolic layers," Tr. Tbiliskogo Univ., Ser. Mat. Mekh. Astron., 232-233, Nos. 13-14, 23-29 (1982).
6. V. I. Arnold, "On the Newtonian attraction of accumulations of dustlike particles," Usp. Mat. Nauk, 37, No. 4, 125 (1982).
7. G. de Rham, Varieties Differentiables, Hermann, Paris (1955).
8. C. G. J. Jacobi, Vorlesungen über Dynamik, G. Reimer Verlag, Berlin (1884).
9. R. Z. Muratov, Potentials of the Ellipsoid [in Russian], Atomizdat, Moscow (1976).