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SINGULARITIES ON THE BOUNDARY OF THE HYPERBOLICITY REGION

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The singularities of hyperbolic polynomials (hypersurfaces) and the singularities of the boundary of the hyperbolicity region are investigated. Theorems on stabilization of these singularities in families with a fixed number of parameters and on their relationship with elliptic singularities are proved. The problems considered in this study are part of a research program focusing on singularities of boundaries of spaces of differential equations, proposed by V. I. Arnol'd.

1. INTRODUCTION

A real projective algebraic hypersurface of degree d is called hyperbolic if there exists a point in the projective space such that any real line through this point (which is called a time-like point) intersects the hypersurface at d points (counting with their multiplicities). If all these intersections are pairwise distinct for any such line, then the hypersurface is called strictly hyperbolic. A strictly hyperbolic surface of degree d in RP^n consists for even d of d/2 nested components diffeomorphic to the sphere (these components are called ovaloids) and for odd d of [d/2] nested ovaloids plus a one-sided component diffeomorphic to RP^{n-1} which is located outside the ovaloids. If for some hyperbolic hypersurface there exists a time-like point not on the hypersurface, then the hypersurface is called totally hyperbolic. The origin of the term "hyperbolic" is in the theory of hyperbolic differential equations in which a hyperbolic hypersurface is the projectivization of the zeros of the principal symbol. Various aspects of the geometry of hyperbolic hypersurfaces and their applications in mathematical physics were considered in [4, 5, 9, 10, 12, 13, 16, 19, 20].

Many problems require studying families of hyperbolic (elliptic) polynomials dependent on parameters. V. I. Arnol'd formulated a number of problems requiring the study of singularities of hyperbolic polynomials (hypersurfaces) and singularities of the boundary of the hyperbolicity and ellipticity regions (sets of parameter values of the families for which the polynomials are hyperbolic or elliptic, respectively) [15, p. 265]. These problems are a part of a research program focusing on singularities of boundaries of various spaces of differential equations, proposed by Arnol'd in [6]. In the hyperbolic case, Arnol'd conjectures can be stated as follows.

1. For families with a fixed number of parameters, the collections of typical singularities of hyperbolic hypersurfaces and boundaries of hyperbolicity regions arising in these families are stabilized up to diffeomorphism as the degree and the dimension of the hypersurface tend to infinity.

2. Similar stabilization occurs also a) for a fixed dimension of the hypersurface, when only the degree is increasing (the resulting stable collection, however, depends on the dimension of the hypersurface); b) for a fixed degree of the hypersurface, when only its dimension is increasing (the stable collection in this case depends on the degree).

3. The hyperbolicity region for families in general position in spaces of polynomials of stable dimension and degree is a topological manifold with an edge whose boundary at singular points is aligned "with the corners facing outward," i.e., the "principle of brittleness of the good" [2] is satisfied.

Similar conjectures for typical families of elliptic polynomials were proved by Matov [14].

Hyperbolicity properties have a local analog, which was considered by Atiyah, Bott, and Garding (see [10, 20] and Sec. 2 below). Arnol'd's conjecture for the local case is stated as follows.

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TABLE 1

No. of parameter	Multisingularity	Example	Curve	Boundary germ
1	A_1^-	$u^2-\sum_i v_i^2=0$	Fig. 1a	Fig. 2a
2	$A_1^-A_1^-$	$\left((u^2-u+\sum_i v_i^2)^2-\frac{1}{2}\left(u^2+\sum_i v_i^2\right)\right)\times$ $\times\left((u-10)^2-\sum_i v_i^2\right)=0$	Fig. 1b	Fig. 2b
3	A_3^-	$\left(u^2+\sum_i v_i^2-1\right)\left((u+1)^2+\sum_i v_i^2-4\right)=0$	Fig. 1c	Fig. 2c
	$A_1^-A_1^-A_1^-$	$\left(\left(\left(u-\frac{1}{2}\right)^2-\left(u-\frac{1}{2}\right)+\sum_i v_i^2\right)^2-\right.$ $-\frac{1}{2}\left(\left(u-\frac{1}{2}\right)^2+\sum_i v_i^2\right)\right)\times$ $\times\left(\left(u^2-10u+\sum_i v_i^2\right)^2-\right.$ $\left.-50\left(u^2+\sum_i v_i^2\right)\right)\times$ $\times\left((u+100)^2-\sum_i v_i^2\right)=0$	Fig. 1d	Fig. 2d

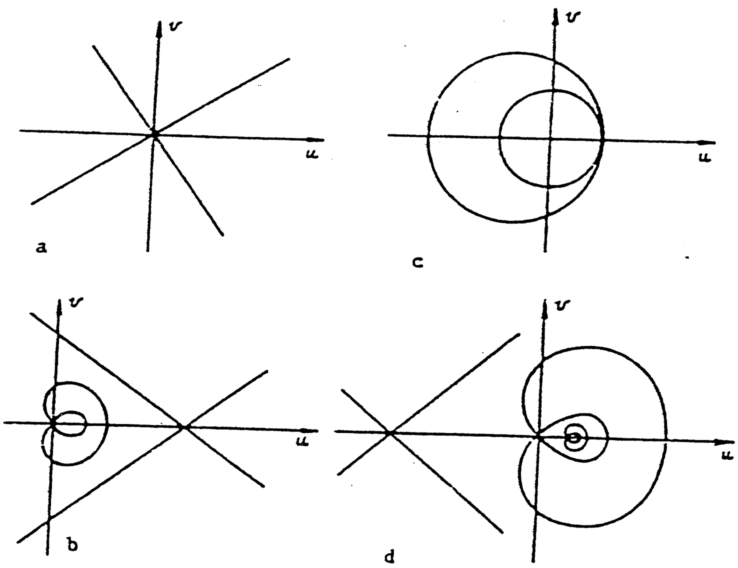

Fig. 1

4. The lists of typical hyperbolic and elliptic singularities and singularities of the boundaries of the corresponding regions in families with a fixed number of parameters stabilize as the dimension of the space increases.

In this paper, we investigate the properties of singularities of hyperbolic hypersurfaces and prove some of the conjectures. The main results can be summarized as follows.

1. Conjecture 4 is proved in Theorem 2.12. We also show that for any fixed number of parameters, the stable elliptic and hyperbolic singularities arising in the corresponding families are stably equivalent, and the corresponding singularities of the boundaries of the ellipticity and hyperbolicity regions coincide (Theorem 2.10).

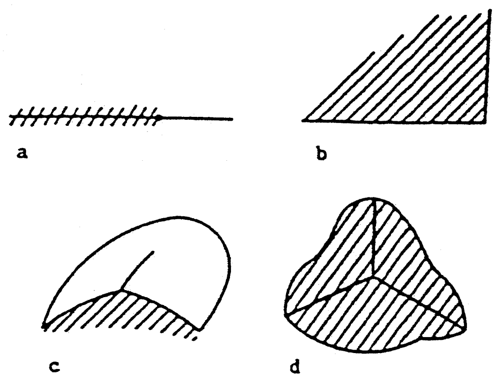

Fig. 2

2. For any natural k and any hypersurface of sufficiently high degree for which the sum of Milnor numbers of the singularities does not exceed k, the germ of its complete deformation in the space of all hypersurfaces of given degree is shown to be a versal deformation of its multisingularity (Theorem 3.3). The germ of the discriminant (the set of singular hypersurfaces) is locally diffeomorphic to a cylinder over the germ of the bifurcation diagram of the multisingularity of the given hypersurface (Corollary 3.4).

3. A complete characterization of the singularities of totally hyperbolic hypersurfaces is given. Specifically, any singularity of a totally hyperbolic hypersurface is hyperbolic in the local sense (Proposition 4.1), and any isolated hyperbolic singularity is realizable up to diffeomorphism as a unique singular point of a totally hyperbolic hypersurface (Theorem 4.2).

4. We consider the question, originally suggested by I. G. Petrovskii, of the realization of singularities of wave fronts, i.e., the orders of flatness of the ovaloids of strictly hyperbolic hypersurfaces (Theorems 4.3 and 4.4).

5. Conjecture 1 is proved for hyperbolicity in relation to a fixed time-like point (Theorems 5.2 and 5.4).

6. Stabilization of the list of typical singularities of totally hyperbolic hypersurfaces with increasing degree is proved for families with a fixed number of parameters and hypersurfaces of fixed dimension (a weaker version of conjecture 2a, see Theorem 5.3).

7. Table 1 lists the multisingularities that arise in typical one-, two-, and three-parameter families of hyperbolic hypersurfaces of sufficiently high degree with a fixed time-like point.

Figure 1 shows the hyperbolic curves on which these multisingularities are realized (hypersurfaces of arbitrary dimension are obtained by revolution of these curves). Figure 2 shows the corresponding germs of the boundary of the hyperbolicity region. The hyperbolicity region in all cases has "outward facing corners" and is a topological manifold with an edge.

Some of the results of this paper were previously reported in [11]. We would like to thank V. I. Arnol'd for suggesting the topic and for his continued support. We acknowledge the useful comments of V. A. Vasil'ev, A. M. Gabrielov, A. B. Givental', and V. M. Kharlamov.

2. STABILIZATION OF HYPERBOLIC SINGULARITIES AND THEIR RELATIONSHIP WITH ELLIPTIC SINGULARITIES

In this section, we consider the space of germs of analytic functions of n variables with a singularity at zero and prove that

1) the list of hyperbolic singularities in typical families with a fixed number of parameters stabilizes as n increases;

2) the stable list of hyperbolic singularities coincides with the stable list of elliptic singularities with the same number of parameters;

3) the stable singularities of the boundaries of ellipticity and hyperbolicity regions coincide.

2.1. *Definition.* Let h: $(R^n, 0) \to (R, 0)$ be a germ of an analytic function; the singularity at zero of the germ h is called hyperbolic if there exists a line γ through 0 (this line is called time-like) such that the germ of the hypersurface H: $\{h = 0\}$ has a constant real intersection multiplicity in the bundle of lines parallel to γ.

2.2. *Definition.* Let e: $(R^n, 0) \to (R, 0)$ be a germ of an analytic function and let 0 be an isolated real solution of the equation $e = 0$. Then the singularity at zero of the germ e is called elliptic.

Note that we will consider only singularities with a finite Milnor number. In the context of this paper, by an isolatc singularity we mean a singularity whose complexification is isolated (is of finite multiplicity), although some results are valid als for real isolated singularities (but not for isolated singularities).

The ellipticity property of a singularity is obviously invariant relative to diffeomorphisms. The corresponding assertio for isolated hyperbolic singularities is proved in a number of lemmas.

2.3. LEMMA. Let h: $(R^n, 0) \to (R, 0)$ be an isolated hyperbolic singularity, and γ a time-like line for h. Then any lin sufficiently close to γ and passing through 0 is also time-like.

Proof. For $n = 2$, the germ of the curve H is a collection of smooth branches transverse to γ [9, p. 62]. They corre spondingly make positive angles with γ. Denote the smallest of these angles by ε. Clearly, h is hyperbolic relative to an direction forming an angle less than ε with γ. Moreover, for at least one of the rays making an angle ε with γ, the real intersec tion multiplicity of the germ H is greater than for γ. We denote this ray by $\bar{\gamma}$.

For $n > 2$, consider the set $M \cong RP^{n-2}$ of all two-dimensional planes through γ. The restriction of h to each of these planes defines a hyperbolic singularity. Acting as described above, we associate to each of these planes a number $\varepsilon > 0$. Assume that there exists a sequence of planes for which the corresponding ε tend to zero. By compactness of M, this sequence may be regarded as convergent to some plane $T \in M$. Consider the line $\bar{\gamma}$ in each plane from our sequence. It follows from the above that the real intersection multiplicity of the germ H for all these lines is greater than for γ, which contradicts the upper semicontinuity of multiplicity, because γ is a limit of the sequence $\bar{\gamma}$. Q.E.D.

2.4. Let $S_\varepsilon \subset R^n$ be a small neighborhood of zero. Fix the orientation of a time-like line γ and consider the connected components of the set $\bar{H}_\varepsilon = S_\varepsilon \setminus H$. By hyperbolicity of h, we can associate to these components integers from 0 to d such that along any sufficiently close line parallel to γ these numbers are incremented by 1 as we successively traverse the components (here d is the intersection multiplicity of γ with H). For an arbitrary point $x \in \bar{H}_\varepsilon$, we denote by $\nu(x)$ the number associated to the component containing the point x.

Consider an arbitrary smooth curve ρ: $[0, 1] \to S_\varepsilon$ with endpoints lying on the surface H. The index of this curve is the difference $\nu(\rho(1)) - \nu(\rho(0))$. For curves in general position the index relative to the hypersurface H clearly does not exceed the number of intersection points of the curve with this hypersurface. Note that Arnol'd [7] introduced a similar notion of the index for the particular case of the hyperbolic hypersurface of degenerate quadratic forms in the space of all quadratic forms.

By Lemma 2.3, we can select a closed cone Γ at zero that includes γ and consists entirely of lines time-like relative to h. Denote by Γ^+ the connected component of $\Gamma \setminus 0$ such that $\nu = d$ for all its points. Define in S_ε the field of cones $\Gamma(x)$ obtained by parallel translation of the cone Γ. The smooth curve ρ: $[0, 1] \to S_\varepsilon$ is called positive if the tangent vector to ρ at an arbitrary point x belongs to $\Gamma^+(x)$.

2.5. LEMMA. Let h: $(R^n, 0) \to (R, 0)$ be an isolated hyperbolic singularity. Then in some sufficiently small neighborhood of zero the index of a positive curve in general position relative to the hypersurface H is equal to the number of intersection points of this curve with H.

Proof. It suffices to show that ν is nondecreasing along any positive curve. Let $\rho(t)$, $t \in [0, 1]$, be an arbitrary point of the positive curve ρ, $\dot{\rho}(t)$ the tangent vector to the curve at this point. Consider the bundle of lines parallel to the vector $\dot{\rho}(t)$ with a matching orientation. Since the curve ρ is positive, if the initial point $\rho(t)$ is sufficiently close to 0, then these lines are time-like for h. Therefore, ν is nondecreasing along these curves (otherwise, the number of intersection points of the line with the hypersurface H would exceed d). Thus, ν is also nondecreasing along the curve ρ in the neighborhood of the point $\rho(t)$. Q.E.D.

2.6. COROLLARY. Diffeomorphism sends an isolated hyperbolic singularity to a hyperbolic singularity. The image of the old time-like direction may be taken as the new time-like direction.

Proof. Consider the diffeomorphic image of a vector from Γ^+ tangent to a time-like line at the origin. In the new system of coordinates, construct a bundle of lines parallel to this vector. It is easy to see that the local preimage of this bundle is a bundle of positive curves leading from the connected component $\bar{H}_\varepsilon$ with $\nu = 0$ to the component with $\nu = d$. By Lemma 2.5, the intersection multiplicity with H in this bundle is d, which proves the corollary.

2.7. Now consider the space Φ_n of power series with a zero 1-jet that converge in the neighborhood of the point 0 in R^n. We will identify the hyperbolic singularities that may arise in k-parameter families in general position in Φ_n as n increases. Recall that germs of functions (of different number of variables) are called stably equivalent if they become equivalent (diffeomorphic) after nondegenerate quadratic forms of additional variables are added to them [8, p. 146].

Definition. The singularity $h \in \Phi_n$ of some class (e.g., elliptic, hyperbolic, etc.) is called stable (relative to its class) if for all $m > n$ there is a singularity of the same class in Φ_m which is stably equivalent to h.

Note that in the class of all singularities and in the class of elliptic singularities, every singularity is stable. In the class of hyperbolic singularities, the situation is different.

2.8. LEMMA. Let $h \in \Phi_n$, $n \geq 2$, be an isolated hyperbolic singularity, and γ its time-like line. Then the following assertions are equivalent:

1) the intersection multiplicity of the germ of the hypersurface H: $\{h = 0\}$ with the time-like line γ is 2;

2) the 2-jet of h is nonzero;

3) h is a stable hyperbolic singularity.

Proof. The implication $1 \Rightarrow 2$ is obvious. Let us establish the converse. Indeed, let the 2-jet of h be nonzero and let the intersection multiplicity of the germ of the hypersurface H with the time-like line γ be $d > 2$. Then by Lemma 2.3, for any line through 0 and sufficiently close to γ, the intersection multiplicity with this germ is also d. Hence, h has a zero $(d - 1)$-jet, which contradicts 2.

Let us now show that 2 implies 3. Without loss of generality, assume that the restriction of h to γ is $u_1^2 + \dots$. By Morse's lemma, h is reducible to the form $\bar{h} = \bar{u}_1^2 + F(\bar{u}_2, \dots, \bar{u}_n)$ and $\bar{h}$ is hyperbolic relative to the axis $\bar{u}_1$. Then for any $m > n$ the germ $h_m = \bar{u}_1^2 - \bar{u}_{n+1}^2 - \dots - \bar{u}_m^2 + F(\bar{u}_2, \dots, \bar{u}_n)$ has a hyperbolic singularity stably equivalent to h.

Finally, let us show that 3 implies 1. In fact, we will prove a stronger proposition: let $h \in \Phi_n$ be an isolated hyperbolic singularity for which the intersection multiplicity of the germ of the hypersurface H with the time-like line γ is $d > 2$; then h may not be stably equivalent to any hyperbolic singularity of any other number of variables.

Indeed, if $d > 2$, then, as we have shown above, the 2-jet of h is identically zero and thus h may not be stably equivalent to a germ of a function of fewer variables. Let h be stably equivalent to the singularity h' of a greater number of variables. Now, since the 2-jet of h is zero, by Proposition 6.5 of [3], h' is diffeomorphic to a singularity $h(u_1, \dots, u_n) + Q(v_1, \dots, v_q)$, where Q is a nondegenerate quadratic form (here we use the fact that h' is complex isolated). By Corollary 2.6 it remains to show that no germ of this kind may be hyperbolic. We prove this by contradiction. Assume that for some Q the germ h' is hyperbolic, then by the equivalence of 1 and 2 the time-like line γ of this germ does not lie in the subspace generated by $u_1, \dots, u_n$, and the restriction of h' to γ is αt^2, where t is a parameter on γ and α is a nonzero constant. Note that if the germ h of a function of two or more variables has an isolated hyperbolic singularity, then it necessarily changes its sign in the neighborhood of zero. Consider the line γ' obtained by a small shift of the line γ in the direction of the vector u in the subspace generated by $u_1, \dots, u_n$ and such that $\alpha h(u) > 0$. Then the restriction of h' to γ' obviously does not have roots in the neighborhood of zero and h' is thus not a hyperbolic singularity. The contradiction proves the lemma. Q.E.D.

2.9. Let us now investigate the relationship of stable hyperbolic and stable elliptic singularities, and also the relationship of the boundaries of ellipticity and hyperbolicity regions (i.e., of the sets of parameter values corresponding to elliptic and hyperbolic singularities) in k-parameter families of function germs in general position.

Definition. Let $h: (R^n, 0) \to (R, 0)$ be a function germ defining a hyperbolic singularity for which the intersection multiplicity with a time-like line is d. The hyperbolic perturbation h' is a small perturbation of the germ h such that the bundle of lines parallel to the time-like line intersects the set $\{h' = 0\}$ in the neighborhood of zero at d points (counting according to their multiplicities).

It is easy to show that hyperbolic perturbations fill the closures of some components of the complement of the bifurcation diagram of h in its versal deformation. The union of these closures will be called the hyperbolicity region of the given singularity. We similarly define the ellipticity region in the complement of the bifurcation diagram of an elliptic singularity.

2.10. THEOREM. 1) Isolated stable hyperbolic singularities of functions of n variables are stably equivalent to elliptic singularities of functions of $n - 1$ variables.

2) The hyperbolicity region in the miniversal deformation of a stable hyperbolic singularity of a function of n variables coincides with the ellipticity region of a miniversal deformation of an elliptic singularity of a function of $n - 1$ variables stably equivalent to the given hyperbolic singularity.

Proof. Consider a stable hyperbolic singularity $h \in \Phi_n$. By Lemma 2.8, it is reducible to the form $h = \alpha u_1^2 + F(u_2, \dots, u_n)$, $\alpha \neq 0$. The u_1-axis is time-like. Then in order to have intersection multiplicity 2 in the bundle of lines parallel to the u_1-axis with the germ of the hypersurface $\{h = 0\}$, it is necessary and sufficient that the germ $e = F(u_2, \dots, u_n)$ defines an elliptic singularity in the subspace generated by $u_2, \dots, u_n$, i.e., is sign-definite and its sign outside zero is the opposite of the sign of α. Hence, using Corollary 2.6, we obtain part 1.

Let us prove part 2. As before, let $h \in \Phi_n$ be a stable hyperbolic singularity reduced to the form $h = \alpha u_1^2 + F(u_2, \dots, u_n)$, where u_1 is a time-like line and $e = F(u_2, \dots, u_n)$ is an elliptic singularity stably equivalent to h. By equivalence of all

A singular hyperbolic hypersurface (in general with nonisolated singularities) is called ordinary if it has no compon that are hyperplanes intersecting the set of its time-like points.

2. Is it true that for an ordinary hyperbolic hypersurface the local number of connected components of the region H in its neighborhood is equal to the number of connected components of the set of its time-like points (Fig. 4)?

Note that in the stable case the set of time-like points is connected. This fact and a positive answer to question 2 wc imply that in the stable case the hyperbolicity region in k-parameter families is a topological manifold with corners (Arno conjecture 3).

Questions 1 and 2 apparently have simpler local analogs.

3. Is it true that for a hyperbolic singularity of three or more variables, the hyperbolicity region (Definition 2.9) occu precisely one connected component of the complement of the bifurcation diagram which is aligned with its corner fac outward?

For stable hyperbolic singularities, this follows from the results of Matov [14].

3'. Is it true that for a hyperbolic singularity of two variables, the hyperbolicity region occupies l components of complement of the bifurcation diagram, where l is the number of transversely intersecting branches of this singularity?

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