ROOT ASYMPTOTICS FOR THE EIGENFUNCTIONS OF UNIVARIATE DIFFERENTIAL OPERATORS

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Abstract. This paper is a brief survey of the research conducted by the author and his collaborators in the field of root asymptotics of (mostly polynomial) eigenfunctions of linear univariate differential operators with polynomial coefficients.

1. Objective

Study asymptotic properties of sequences $\{p_n(z)\}$, of polynomials/entire functions in z which

either

(1) are polynomial/entire eigenfunctions of a univariate linear ordinary differential operator with polynomial coefficients;
    or
(2) are polynomial solutions of more general pencils of such operators, e.g. homogenized spectral problems and Heine-Stieltjes spectral problems;
    or
(3) satisfy a finite recurrence relation with (in general) varying coefficients.

2. Basic notions and examples

Definition 1. *An operator $T = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$ is called* **exactly solvable** *if $\deg Q_i(z) \le i$ and there exists at least one value i such that $\deg Q_i(z) = i$.*

Obviously, $T(z^j) = a_j z^j +$ lower order terms, i.e. T acts by an (infinite) triangular matrix in the monomial basis $\{1, z, , z^2, ...\}$ of $\mathbb{C}[z]$.

Lemma 1. *For any exactly solvable T and sufficiently large n there exists a unique (up to a scalar) eigenpolynomial $p_n(z)$ of degree n.*

Typical problem. Given an exactly solvable T describe the root asymptotics for the sequence of polynomials $\{p_n(z)\}$.

2.1. **Two asymptotic measures.** Given a polynomial family $\{p_n(z)\}$ where $\deg p_n(z) = n$ we define two basic measures: (i) asymptotic root-counting measure μ; (ii) asymptotic ratio measure ν.

Definition 2. *Associate to each $p_n(x)$ a finite probability measure μ_n by placing the mass $\frac{1}{n}$ at every root of $p_n(x)$. (If some root is multiple we place at this point the mass equal to its multiplicity divided by n.) The limit $\mu = \lim_n \mu_n$ (if it exists in the sense of weak convergence) will be called the* **asymptotic root-counting measure** *of $\{p_n(z)\}$.*

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Definition 3. *Consider the ratio* $q_n(z) = \frac{p_{n-1}(z)}{p_n(z)}$. *(Assume for simplicity that* $p_n(z)$ *has no multiple roots and expand* $q_n(z) = \sum_{i=1}^{n} \frac{\kappa_{i,n}}{z-z_{i,n}}$.*) Associate to* $q_n(z)$ *the finite complex-valued measure by placing* $\kappa_{i,n}$ *at* $z_{i,n}$. *Define the* **asymptotic ratio measure** *of the sequence* $\{p_n(z)\}$ *as*

$$\nu = \lim_{n\to\infty} \nu_n.$$

Observation. Supports of μ and ν coincide but ν is often complex-valued.

2.2. **Examples.** Below we show the root distribution for $p_{55}(z)$ for 4 different exactly solvable operators

$$T_1 = z(z-1)(z-I)\frac{d^3}{dz^3}; \qquad T_2 = (z-I)(z+I)(z-2+3I)(z-3-2I)\frac{d^4}{dz^4};$$

$$T_3 = (z-I)(z+I)(z-2+3I)(z-3-2I)(z+3)\frac{d^5}{dz^5};\ T_4 = (z^2+1)(z-2+3I)(z-3-2I)(z+3)(z+1+I)\frac{d^6}{dz^6}$$

of the form $Q(z)\frac{d^k}{dz^k}$ where $Q(z)$ is a monic polynomial of degree $k+1$.

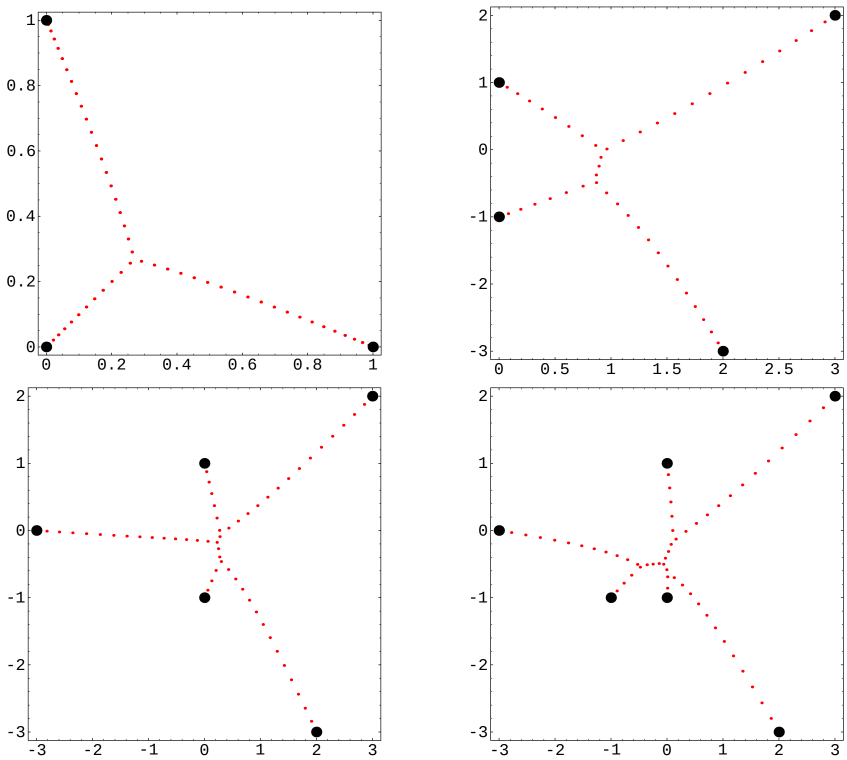

FIGURE 1. Roots of $p_{55}(z)$ for the above T's.

Explanations to Fig. 1. The larger dots show the roots of the corresponding $Q(z)$ and the smaller dots are the fifty five roots of the corresponding $p_{55}(z)$.

2.3. **Classical prototypes.**

Theorem 1 (G. Szegö). *If* $\{p_n(z)\}$ *is a family of polynomials orthogonal w.r.t a positive weight* $w(z)$ *supported on* $[-1,1]$ *such that* $\int_{-1}^{1} \ln w(z)dz < \infty$ *then the asymptotic root-counting measure has the density* $\frac{1}{\pi\sqrt{1-z^2}}$, $x \in [-1,1]$.

Theorem 2 (G. Szegö). *If $\{p_n(z)\}$ is a family of polynomials orthogonal w.r.t a weight $w(z)$ supported on $[-1,1]$ such that $\int_{-1}^{1} \frac{\ln w(z)dz}{\sqrt{1-z^2}} > -\infty$ then the asymptotic ratio measure has the density $\frac{2\sqrt{1-z^2}}{\pi}$, $z \in [-1,1]$.*

3. First results

3.1. **Non-degenerate exactly solvable operators.** The next subsection is based on [MS01], [BR02].

Definition 4. *The* **Cauchy transform** *of a (complex-valued) measure ρ satisfying $\int_{\mathbb{C}} d\rho(\xi) < \infty$ is given by*

$$C_\rho(z) = \int_{\mathbb{C}} \frac{d\rho(\xi)}{z - \xi}.$$

Example. If $\rho(z) = \frac{1}{\pi\sqrt{1-z^2}}$, $z \in [-1,1]$ then $C_\mu = \frac{1}{\sqrt{z^2-1}}$ in $\mathbb{C} \setminus [-1,1]$ and $C_\nu = \frac{2}{z+\sqrt{z^2-1}}$ in $\mathbb{C} \setminus [-1,1]$.

Definition 5. *An exactly solvable operator $T = \sum_{i=1}^{k} Q_i(z)\frac{d^i}{dz^i}$ is called* **non-degenerate** *if $\deg Q_k(z) = k$.*

Proposition 1. *Assuming that $\Psi(z) = \lim_{n\to\infty} \frac{p_n'(z)}{np_n(z)}$ exists in some open neighborhood Ω of $\mathbb{C}$ one gets that $\Psi(z)$ satisfies in Ω the algebraic equation*

$$Q_k(z)\Psi^k(z) = 1.$$

Theorem 3 (H. Rullgård). *Let $Q_k(z)$ be a monic degree k polynomial. Then there exists a unique probability measure μ_Q such that*

1) supp μ_Q is compact;

2) its Cauchy transform C_μ satisfies the equation $Q_k(z)C_\mu^k(z) = 1$ almost everywhere in $\mathbb{C}$.

Theorem 4 (Main result, see Fig. 2). *In the above notation*

1) supp μ_Q is a curvilinear tree which is straightened out by the analytic mapping

$$\xi(z) = \int_a^z \frac{dz}{\sqrt[k]{Q_k(z)}}.$$

2) supp μ_Q contains all the zeros of $Q_k(z)$ and is contained in the convex hull of those.

3) There is a natural formula for the angles between the branches, and the masses of the branches satisfy Kirchhoff law.

Below we show an example of such a measure in a proper scale and with all angles between its vertices marked, see Fig. 3.

Problem 1. *Is it true that the support of the measure μ_Q is a subset of the Stokes lines of the corresponding operator $Q\frac{d^k}{dz^k}$?*

Some partial results in this direction can be found in [STT09].

3.2. **Degenerate exactly solvable operators.** This subsection is based on [Be07].

Definition 6. *An exactly solvable T of order k is called* **degenerate** *iff $\deg Q_k < k$.*

Classical examples: $T = z\frac{d^2}{dz^2} + (az+b)\frac{d}{dz}$, $T = \frac{d^2}{dz^2} + (az+b)\frac{d}{dz}$ leading to Laguerre resp. Hermite polynomials.

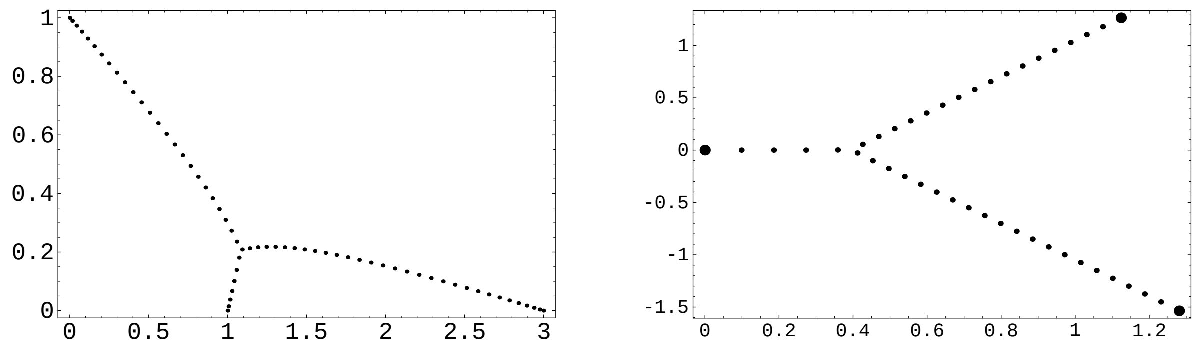

FIGURE 2. The measure μ_Q before and after the straightening transformation in the case $Q(z) = (z-1)(z-3)(z-I)$.

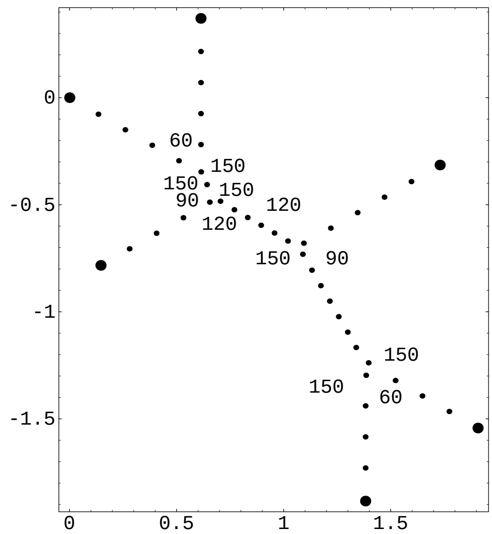

FIGURE 3. Example of μ_Q with angles.

Proposition 2. *The union of all roots of all polynomial eigenfunctions of an exactly solvable T is unbounded if and only if T is degenerate.*

Problem 2. *Given a degenerate T with the family of eigenpolynomials $\{p_n(z)\}$ how fast does the maximum r_n of the modulus of roots of $p_n(z)$ grow?*

Conjecture 1. *Given a degenerate $T = \sum_{j=1}^{k} Q_j(z) \frac{d^j}{dz^j}$ denote by j_0 the largest j for which $\deg Q_j(z) = j$. Then*

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_T$$

where $c_T > 0$ is a positive constant and

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right).$$

Corollary 1 (of the latter Conjecture)**.** *The Cauchy transform $C(z)$ of the asymptotic root measure μ of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ of a degenerate T satisfies the following algebraic equation for almost all complex z:*

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where A is the set consisting of all j for which the maximum $d := \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j} \right)$ is attained, i.e. $A = \{j : (j-j_0)/(j-\deg Q_j) = d\}$.

The latter equation for the Cauchy transform (if true) leads to very detailed information about the support of the asymptotic root-counting measure for the sequence of scaled eigenpolynomials. We illustrate this in Fig. 4.

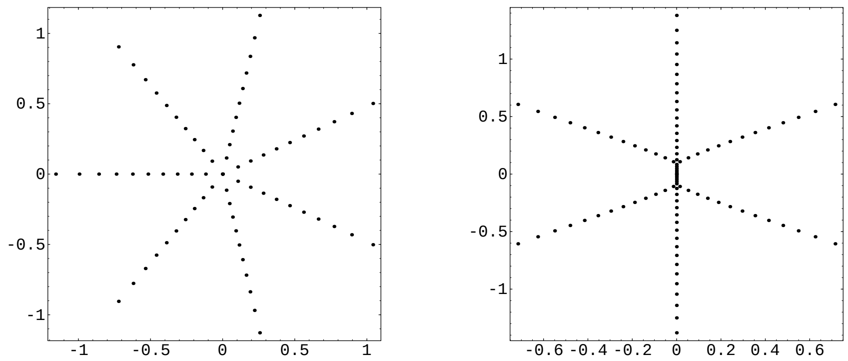

FIGURE 4. Examples of the root distributions of scaled eigenpolynomials to degenerate exactly solvable operators.

4. Homogenized spectral problem for non-degenerate T.

This section is based on [BBS09]. As an observant reader has noticed that so far only the leading coefficient of an exactly solvable operator effected the asymptotic root-counting measure, which makes the situation somewhat unsatisfactory.

To make the whole symbol of an operator important we consider (following the classical pattern of e.g. W. Wasow, M. Fedoryuk) the *homogenized spectral problem* of the form

$$T_\lambda = \sum_{i=0}^{k} Q_i(z) \lambda^{k-i} \frac{d^i}{dz^i},$$

where each $Q_i(x) = a_{ii} z^i + a_{i,i-1} z^{i-1} + \dots$ is a polynomial of degree i.

Definition 7. *A non-degenerate T is called* **of general type** *iff* $\deg Q_k(z) = k$ *and* $\sum_{i=0}^{k} a_{ii} \lambda^{k-i} = 0$ *has k distinct zeros.*

Proposition 3. *If T is of general type then*

1) for all sufficiently large n there exist exactly k distinct values $\lambda_{n,j}$, $j = 1, \dots, k$ of the spectral parameter λ such that the operator T_λ has a polynomial eigenfunction $p_{n,j}(z)$ of degree n.

2) Asymptotically $\lambda_{n,j} \sim n\lambda_j$ where $\lambda_1, \dots, \lambda_k$ is the set of roots of the algebraic equation $\sum_{i=0}^{k} a_{i,i} x^{k-i} = 0$.

Conjecture 2. *If T is of general type and all $\lambda_1, \dots, \lambda_k$ have distinct arguments then for each $j = 1, \dots, k$ $\exists!$ probability measure μ_j with compact support whose Cauchy transform $C_j(z)$ satisfies almost everywhere in $\mathbb{C}$*

$$\sum_{i=1}^{k} Q_i(z)(\lambda_j C_j(z))^i = 0.$$

Conjecture 3. $C_j(z) = \lim_{n\to\infty} \frac{p'_{n,j}(z)}{\lambda_{n,j} p_{n,j}(z)}$ *outside the support of μ_j which is the union of finitely many segments of analytic curves forming a curvilinear tree.*

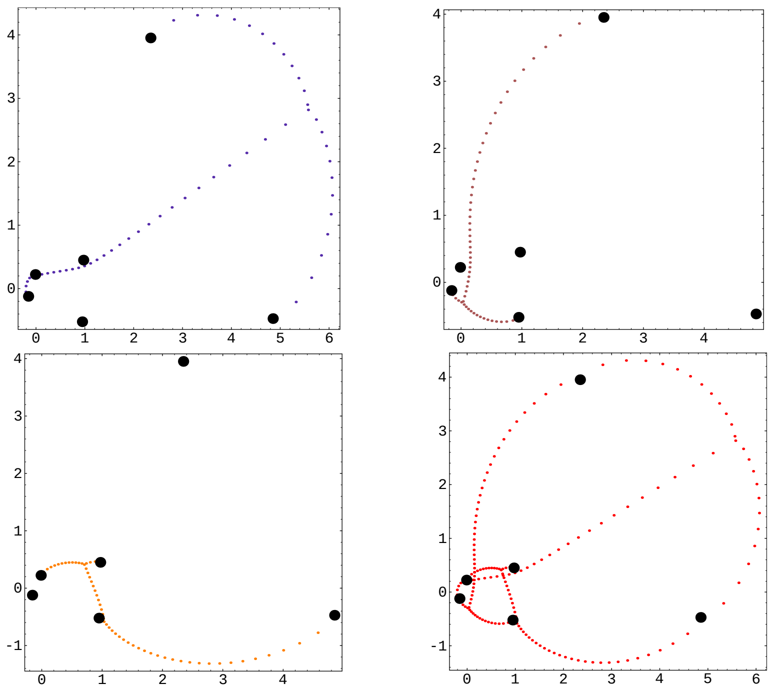

FIGURE 5. Three root-counting measures and their union for a homogenized spectral problem with an operator of order 3.

Observation. Near $\infty \in \mathbb{CP}^1$ the Cauchy transforms $\lambda_1 C_1(z), \dots, \lambda_k C_k(z)$ are independent sections of the symbol equation of T_λ considered as a branched cover over $\mathbb{CP}^1$.

Problem 3. *Find an explicit description of (the support) of the measures μ_i. Is there any relation of these measures to the periods of the plane curve $\sum_{i=1}^{k} Q_i(z) y^i = 0$?*

5. Heine-Stieltjes theory.

This section is based on [SH09]. Take an arbitrary univariate linear differential operator $T = \sum_{i=0}^{k} Q_i(z) \frac{d^i}{dz^i}$ with polynomial coefficients and set

$$r = \max_i (\deg Q_i(z) - i).$$

Definition 8. *If $r \geq 0$, $\deg Q_k(z) = k + r$ and $Q_k(z)$ has at least two distinct roots we call T a* **general Lame-type** *operator.*

Consider the following multi-parameter spectral problem. For a given non-negative integer n find all polynomials $V(z)$ of degree at most r such that the equation

$$T(p(z)) + V(z)p(z) = 0,$$

has a polynomial solution $p(z)$ of degree n. (Classically, $p(z)$ is called a *Stieltjes* polynomial and $V(z)$ is called a *Van Vleck* polynomial.)

Proposition 4. *Under the above assumptions for any sufficiently large n there exist exactly $\binom{n+r}{r}$ degree n Stieltjes polynomials $p_{n,j}(z)$ and corresponding Van Vleck polynomials $V_{n,j}(z)$.*

Proposition 5. *If a sequence $\{\widetilde{V}_{n,j_n}(z)\}$, $n = 1, \dots,$ of scaled Van Vleck polynomials converges to some polynomial $\widetilde{V}(z)$ then the sequence of finite measures $\mu_{n,j}$ of the corresponding family of eigenpolynomials $\{p_{n,j_n}(z)\}$ converges to a measure $\mu_{\widetilde{V}}$ satisfying the properties:*

a) supp $\mu_{\widetilde{V}}$ is a forest of curvilinear trees;

b) the union of the leaves of supp $\mu_{\widetilde{V}}$ coincides with the union of all zeros of $Q_k(z)$ and of $\widetilde{V}(z)$.

c) supp $\mu_{\widetilde{V}}$ is straightened out by the transformation given by

$$\int_a^z \frac{\widetilde{V}(z)dz}{Q_k(z)}.$$

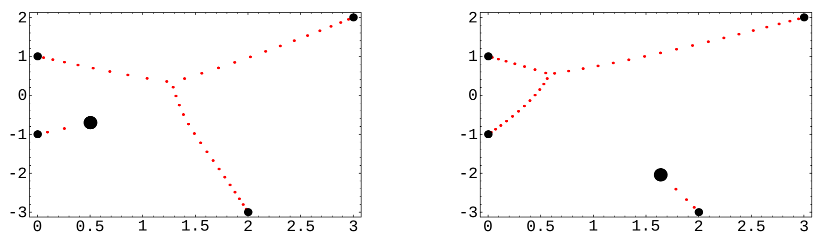

FIGURE 6. Examples of μ_Q's for $T = (z^2+1)(z+2I-3)(z-3I-2)\frac{d^3}{dz^3}$.

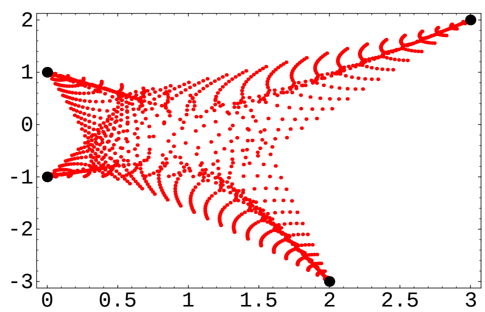

FIGURE 7. Union of μ_Q's for the above T.

Explanations to Fig.6 and 7. In Fig. 6 we give two examples of different Van Vleck polynomials $V(z)$ and the corresponding Stieltjes polynomials $p(z)$. The average size dots are the 4 roots of the polynomial $Q(z) = (z^2+1)(z+2I-3)(z-3I-2)$, the unique large dot is the only root of $V(z)$ (which is linear in this case). Small dots show the roots of $p(z)$. In Fig. 7 we show the union of all roots of $p(z)$ of degree 25 for the same problem.

6. Schrödinger operator with polynomial potential.

This section is based on [GES08], [GES208]. Consider the operator $\mathcal{H} = -\frac{d^2}{dz^2} + P(z)$ where $P(z) = z^{2l} + \sum_{i=0}^{2l-1} a_i z^i$ is a monic polynomial of even degree with real coefficients. It is well-known that the classical spectral problem

$$\mathcal{H}(y) = \lambda y \tag{6.1}$$

where y belongs to $L_2(\mathbb{R})$ has a discrete and simple spectrum $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$. Denote by $\phi_0(z), \phi_1(z), \ldots, \phi_n(z), \ldots$ the sequence of the corresponding eigenfunctions. These eigenfunctions are real entire functions of order $l+1$ and $\phi_n(z)$ has exactly n real zeros. Set $\psi_n(z) = \phi_n(\sqrt[2l]{\lambda_n} z)$ which we call the *scaled n-th eigenfunction.*

The *Stokes graph* of any complex polynomial $P(z)$ is the following object. Each root of $P(z)$ is called a *turning point.* A (local) *Stokes line* of $P(z)$ is a maximal segment of the real analytic curve containing at most two turning points (finite or infinite) which solves the equation:

$$\Re\,\xi_{z_0}(z) = 0 \quad \text{where} \quad \xi_{z_0}(z) = \int_{z_0}^{z} \sqrt{P(u)}du = 0, \tag{6.2}$$

with respect to z, where z_0 is one of the turning points of $P(z)$. The *Stokes graph* ST_P of the polynomial $P(z)$ is the union of all its local Stokes curves. A local Stokes line connecting two finite turning points, i.e. two roots of $P(z)$ is called *short.* (The Stokes graph $ST(P)$ of a generic $P(z)$ has no short Stokes lines.)

Proposition 6. *For a given positive integer l the Stokes graph $ST(z^{2l}-1)$ consists of*

1) l short Stokes lines for l odd and $l-1$ short Stokes lines for l even connecting all pairs of the roots of $z^{2l}-1$ which are symmetric w.r.t the imaginary axis;

2) for l odd each root of $z^{2l}-1$ is connected by 2 infinite Stokes lines to ∞. More exactly, the 2 infinite Stokes lines passing through the root $e^{\frac{\pi ik}{l}}$, $k = 0, \ldots, 2l-1$ are tangent at ∞ to the Stokes rays having the nearest slope to $\frac{\pi ik}{l}$;

3) for l even each root of $z^{2l}-1$ except for $\pm i$ is connected to ∞ by 2 infinite Stokes lines with the same property as above. The roots $\pm i$ have 3 infinite Stokes lines each.

Theorem 5. *For any monic polynomial $P_{\mathbb{C}}(z)$ of even degree the sequence of meromorphic functions $\{\mathcal{C}_n(z)\} = \left\{\frac{\psi_n'(z)}{n\psi_n(z)}\right\}$ converges to $\mathcal{C}(z) = -K_l\sqrt{z^{2l}-1}$ uniformly on any compact set lying in the domain $\mathbb{C} \setminus UC_l$, where $K_l = \frac{\sqrt{\pi}\Gamma\left(\frac{3l+1}{2l}\right)}{\Gamma\left(\frac{2l+1}{2l}\right)}$. (Here by $-\sqrt{z^{2l}-1}$ we mean the branch which is negative for positive $z > 1$. Also UC_l is a certain subset of local Stokes lines marked by bold on Fig. 8.)*

7. Finite recurrences.

This section is based on [BBS06]. Consider a finite recurrence of length $(k+1)$ given by

$$p_{n+1}(z) = Q_1(z)p_n(z) + \ldots + Q_k(z)p_{n-k+1},$$

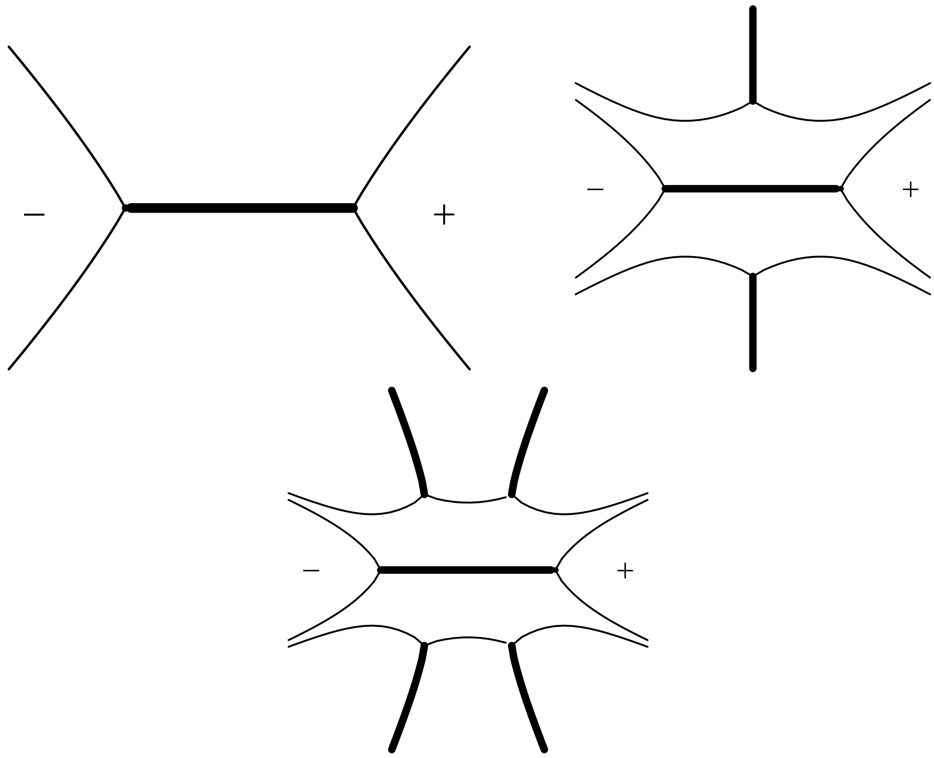

FIGURE 8. Stokes lines of $z^{2l} - 1$ for $l = 1, 2, 3$.

with polynomial or rational coefficients $\{Q_1(z), \dots, Q_k(z)\}$ uniquely determined by the initial k-tuple $\{p_0(z), \dots, p_k(z)\}$.

Theorem 6. *There exists a finite subset $\Theta \subset \mathbb{C}$ depending on the initial k-tuple and a curve Σ depending on the recurrence such that the asymptotic ratio $\Psi(z) = \lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)}$ exists and satisfies the symbol equation*

$$\Psi^k(z) = Q_1(z)\Psi^{k-1}(z) + \ldots + Q_k(z) \quad (*)$$

in $\mathbb{C} \setminus (\Sigma \cup \Theta)$. Here Σ is the so-called Stokes discriminant of $(*)$ which is the set of all z for which the equation $(*)$ has at most two roots with the same and maximal absolute value.

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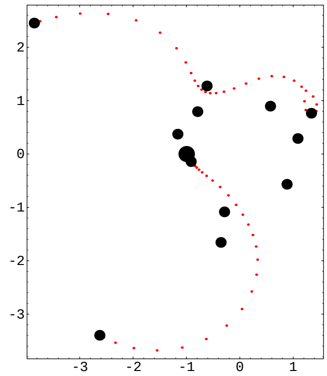

FIGURE 9. Zeros of $p_{31}(z)$ satisfying the recurrence relation $(z+1)p_n(z) = (z^2+1)p_{n-1}(z) + (z-5I)p_{n-2}(z) + (z^3-1-I)p_{n-3}(z)$.

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