

# GENERALIZED LAGUERRE INEQUALITIES AND NON-NEGATIVE SYMMETRIC POLYNOMIALS

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*To the late Professor George Csordas with admiration*

ABSTRACT. We discuss the convex cone of generalized Laguerre-type inequalities posed in [4]. The originally expected finite simplicial description of this cone appears to be false. We give instead a precise reduction to a cone of non-negative symmetric forms, describe the latter by duality as the polar of a conical hull naturally associated with real-rooted polynomials, and recall the half-degree principle of Timofte–Riener which reduces the required testing family to points with few distinct coordinates. This gives a workable finite-dimensional semialgebraic description and raises the question whether the cone is non-polyhedral in higher degrees.

## 1. INTRODUCTION

The classical Laguerre inequality says that any real polynomial  $p(x)$  with all real and simple roots satisfies, for all real  $x$ , the inequality

$$L_1(p(x)) = (p'(x))^2 - p(x)p''(x) > 0.$$

Higher Laguerre inequalities were discovered by J. L. W. V. Jensen [5] already in 1913. In [3] it was proved that they give necessary and sufficient conditions for a polynomial to have all real roots. Namely, define the  $n$ -th Laguerre expression  $L_n$  by the formula:

$$L_n(p(x)) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} p^{(j)}(x) p^{(2n-j)}(x).$$

One of the main results of [3] claims that a polynomial  $p$  of degree  $n$  has all real and simple zeros if and only if  $L_k(p(x)) > 0$  for all  $k = 1, 2, \dots, n$ . A decade ago the first author jointly with W. H. Foster [4] initiated the study of the convex cone  $\mathcal{C}_n$  of bilinear operators of the form

$$U(p(x)) = \sum_{j=0}^n a_j p^{(j)}(x) p^{(2n-j)}(x), \quad a_j \in \mathbb{R} \tag{1.1}$$

such that for all real  $x$  and any real polynomial  $p(x)$  with all real roots,

$$U(p(x)) \geq 0. \tag{1.2}$$

They conjectured that for any positive integer  $n$  the cone  $\mathcal{C}_n$  is polytopal and also conjectured the explicit form of its generators in the first non-trivial case  $n = 2$ , see Conjectures 1 and 2 on p. 104 of [4]. A natural subsequent guess was that, for arbitrary  $n$ , the corresponding cone should be simplicial and generated by a small explicit list of non-negative symmetric forms. This guess is too optimistic

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and appears to be false in general. The purpose of the present note is therefore more modest: we record the exact reduction to a cone of non-negative symmetric forms and give a corrected dual description of this cone. This description is not a finite list of generators, but it reduces the problem to a natural semialgebraic family related to real-rooted polynomials.

We first translate the problem in the language of non-negative symmetric polynomials. Translation-invariance of (1.1) implies the following.

**Lemma 1.** *Operator (1.1) satisfies (1.2) if and only if*

$$\Sigma_U = \sum_{j=0}^n a_j j! (2n-j)! \sigma_j \sigma_{2n-j} \quad (1.3)$$

*is a non-negative polynomial (for any number of variables). Here  $\sigma_j$  is the  $j$ -th elementary symmetric function in some number of variables.*

*Proof of Lemma 1.* Let

$$p(x) = \prod_{i=1}^m (x - r_i)$$

be real-rooted. At a point  $x$  which is not a zero of  $p$  one has

$$\frac{p^{(j)}(x)}{j! p(x)} = \sigma_j \left( \frac{1}{x - r_1}, \dots, \frac{1}{x - r_m} \right). \quad (1.4)$$

Therefore

$$\frac{U(p(x))}{p(x)^2} = \sum_{j=0}^n a_j j! (2n-j)! \sigma_j(\mathbf{y}) \sigma_{2n-j}(\mathbf{y}), \quad \mathbf{y} = \left( \frac{1}{x - r_1}, \dots, \frac{1}{x - r_m} \right).$$

Thus non-negativity of (1.3) implies (1.2). Conversely, given any vector  $\mathbf{y} = (y_1, \dots, y_m)$  with non-zero real coordinates, choose  $x = 0$  and  $r_i = -1/y_i$ . Then (1.4) realizes this vector. Points with some zero coordinates follow by continuity. Hence (1.2) for all real-rooted polynomials is equivalent to non-negativity of (1.3) on all real vectors.  $\square$

**Lemma 2.** *The linear space  $\mathcal{L}_n$  spanned by symmetric functions  $\sigma_j \sigma_{2n-j}$ ,  $j = 0, \dots, n$  coincides with the space of all symmetric functions of degree  $2n$  such that each variable has degree at most 2. Another natural basis of  $\mathcal{L}_n$  consists of all monomial symmetric functions corresponding to the Young diagrams of area  $2n$  and at most 2 rows.*

*Proof of Lemma 2.* Each product  $\sigma_j \sigma_{2n-j}$  is symmetric, homogeneous of degree  $2n$ , and no variable occurs with exponent larger than 2. Hence  $\mathcal{L}_n$  is contained in the stated space. Conversely, the latter space is spanned by monomial symmetric functions corresponding to partitions of  $2n$  with all parts at most 2, equivalently to Young diagrams with at most 2 columns. After conjugation this is the same indexing set as partitions with at most 2 rows. The transition matrix between the products  $\sigma_j \sigma_{2n-j}$ ,  $0 \leq j \leq n$ , and these monomial symmetric functions is triangular with non-zero diagonal after ordering by the number of squared variables. Hence the two spaces coincide.  $\square$

In view of Lemma 1 the problem of description of  $\mathcal{C}_n$  reduces to the description of the cone  $\mathcal{K}_n$  consisting of all non-negative polynomials of the form  $\sum_{j=0}^n \alpha_j \sigma_j \sigma_{2n-j}$ . (By definition  $\mathcal{K}_n$  is a convex cone in  $\mathcal{L}_n$ .) The main result of this note is as follows.

**Theorem 1** (Dual description of  $\mathcal{K}_n$ ). *Let  $\mathcal{K}_n$  denote the cone of all coefficient vectors  $(\beta_0, \beta_1, \dots, \beta_n) \in \mathbb{R}^{n+1}$  such that*

$$\sum_{j=0}^n \beta_j \sigma_j(\mathbf{x}) \sigma_{2n-j}(\mathbf{x}) \geq 0 \quad (1.5)$$

for every real vector  $\mathbf{x} = (x_1, \dots, x_m)$  and every  $m$ . Then  $\mathcal{K}_n$  is the dual cone

$$\mathcal{K}_n = \left( \text{cone} \left\{ (\sigma_0 \sigma_{2n}, \sigma_1 \sigma_{2n-1}, \dots, \sigma_n^2)(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m, m \geq 0 \right\} \right)^\vee, \quad (1.6)$$

where the dual is taken with respect to the standard pairing in  $\mathbb{R}^{n+1}$ . Equivalently, if

$$q_{\mathbf{x}}(t) = \prod_{i=1}^m (1 + x_i t) = \sum_{k=0}^m e_k(\mathbf{x}) t^k,$$

then  $\mathcal{K}_n$  is the dual of the conical hull of the vectors

$$(e_0 e_{2n}, e_1 e_{2n-1}, \dots, e_n^2)$$

obtained from all real-rooted polynomials  $q_{\mathbf{x}}$  normalized by  $q_{\mathbf{x}}(0) = 1$ .

*Proof.* The inequalities defining  $\mathcal{K}_n$  are linear in  $(\beta_0, \dots, \beta_n)$ , hence  $\mathcal{K}_n$  is a closed convex cone. For a fixed real vector  $\mathbf{x}$ , put

$$v(\mathbf{x}) = (\sigma_0 \sigma_{2n}, \sigma_1 \sigma_{2n-1}, \dots, \sigma_n^2)(\mathbf{x}).$$

Then  $(\beta_0, \dots, \beta_n) \in \mathcal{K}_n$  if and only if

$$\langle (\beta_0, \dots, \beta_n), v(\mathbf{x}) \rangle \geq 0$$

for every  $\mathbf{x}$ . This is exactly the definition of the dual cone to the conical hull generated by the vectors  $v(\mathbf{x})$ . The formulation in terms of  $q_{\mathbf{x}}$  is the same statement, since the coefficients of  $q_{\mathbf{x}}$  are the elementary symmetric functions  $e_k(\mathbf{x}) = \sigma_k(\mathbf{x})$ .  $\square$

**Theorem 2** (Reduction to few distinct coordinates). *A vector  $(\beta_0, \dots, \beta_n)$  belongs to  $\mathcal{K}_n$  if and only if*

$$\sum_{j=0}^n \beta_j \sigma_j(\mathbf{x}) \sigma_{2n-j}(\mathbf{x}) \geq 0 \quad (1.7)$$

for all real vectors  $\mathbf{x}$  having at most  $\max\{n, 2\}$  distinct coordinates. In particular, for  $n \geq 2$  it is enough to test points with at most  $n$  distinct coordinates. Consequently the cone dual to  $\mathcal{K}_n$  is generated by the vectors corresponding to polynomials of the form

$$q(t) = \prod_{\nu=1}^r (1 + t_{\nu} t)^{m_{\nu}}, \quad r \leq n, \quad (1.8)$$

where  $t_{\nu} \in \mathbb{R}$ ,  $m_{\nu}$  are non-negative integers, and  $r \leq \max\{n, 2\}$ .

*Proof.* The left-hand side of (1.7) is a symmetric form of degree  $2n$ . The half-degree principle for symmetric polynomials, due to Timofte [7] and in the form used here to Riener [6], says that non-negativity of a symmetric form of degree  $2n$  on real space is equivalent to its non-negativity on the subset of points having at most  $\max\{\lfloor (2n)/2 \rfloor, 2\} = \max\{n, 2\}$  distinct coordinates. This proves the first assertion. Substituting a point with at most  $n$  distinct coordinates into  $q_{\mathbf{x}}(t) = \prod_i (1 + x_i t)$  gives precisely the corresponding family (1.8), and hence gives the stated reduced generating family for the dual cone.  $\square$

## 2. SMALL DEGREES: EVIDENCE AND POLYTOPAL SUBCONES

We now spell out the first non-trivial cases explicitly. The vectors below should not be interpreted as giving the whole cone. Rather, they give natural polytopal subcones inside the actual cone. Numerical experiments and the dual semialgebraic description above suggest that the true cone is larger and, in higher degrees, typically non-polyhedral.

**2.1. Forms of degree four.** In degree four one has  $n = 2$  and considers operators

$$U(p; x) = a_0 p''(x)^2 + a_1 p'(x) p'''(x) + a_2 p(x) p^{(4)}(x). \quad (2.1)$$

The corresponding symmetric form is

$$24a_0\sigma_4 + 6a_1\sigma_1\sigma_3 + 4a_2\sigma_2^2. \quad (2.2)$$

Besides the square  $V_{2,0} = p''(x)^2$ , two natural degree-four forms are

$$V_{2,1}(p; x) = 3p''(x)^2 - 4p'(x)p'''(x) + p(x)p^{(4)}(x), \quad (2.3)$$

$$V_{2,2}(p; x) = 3p''(x)^2 - 2p'(x)p'''(x) - p(x)p^{(4)}(x). \quad (2.4)$$

Under the reduction of Lemma 1 these become, up to the common positive factor 4,

$$\Phi_{2,1} = \sigma_2^2 - 6\sigma_1\sigma_3 + 18\sigma_4, \quad (2.5)$$

$$\Phi_{2,2} = -\sigma_2^2 - 3\sigma_1\sigma_3 + 18\sigma_4. \quad (2.6)$$

Thus the conjectural degree-four basis from [4] is

$$B_2 = \{V_{2,0}, V_{2,1}, V_{2,2}\}. \quad (2.7)$$

Equivalently, in coefficient-vector notation for  $(p''^2, p'p''', pp^{(4)})$ , the three rays are

$$(1, 0, 0), \quad (3, -4, 1), \quad (3, -2, -1). \quad (2.8)$$

For completeness we also record the relation with the classical Laguerre inequality. The first non-square form is the second Jensen–Laguerre form:

$$L_2(p; x) = \frac{1}{12} (3p''(x)^2 - 4p'(x)p'''(x) + p(x)p^{(4)}(x)),$$

so its non-negativity on real-rooted polynomials follows from Jensen’s higher Laguerre inequalities. The second ray  $V_{2,2}$  is the additional degree-four ray isolated in [4]. In the present notation its non-negativity is equivalent to the non-negativity of (2.6). By the half-degree principle, this can be tested on vectors with at most two distinct coordinates.

*Remark 1.* The three forms in (2.7) give a useful polygonal subcone, but this subcone should not be confused with the whole cone. In the affine normalization  $a_0 = 1$  the numerical picture suggested by the dual description is a convex curvilinear region: two sides appear to be straight, while the remaining side appears to be a non-linear convex arc. Thus even in degree four the geometry is better viewed as “a polygonal subcone inside a larger cone” rather than as a simplicial finite-ray description of the full cone.

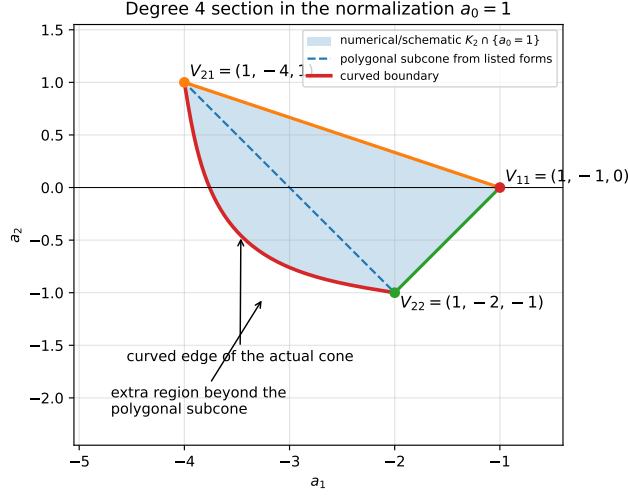


FIGURE 1. Schematic numerical picture of the degree-4 section in the normalization  $a_0 = 1$ . The dashed triangle is the polytopal subcone generated by the listed forms  $V_{21}, V_{11}, V_{22}$ . The shaded curvilinear region represents the larger cone suggested by the dual/half-degree description; the most important feature is the curved boundary edge, which is not captured by the polygonal subcone.

**2.2. Conjectural patterns in degrees six and eight.** The following experimentally observed vectors are useful as test cases and as historical motivation. They are best understood as generators of natural polytopal subcones of the corresponding true cones. They should be treated as conjectural beyond the parts covered by known generalized Laguerre inequalities.

For degree six, i.e.  $n = 3$ , the proposed non-square generators of a polytopal subcone in the coefficient basis

$$(p^{(3)}p^{(3)}, p^{(2)}p^{(4)}, p'p^{(5)}, pp^{(6)})$$

are

$$(10, -15, 6, -1), \quad (5, -5, -1, 1), \quad (10, -5, -4, -1). \quad (2.9)$$

They satisfy the formal relations

$$V_{3,1} + V_{3,2} = 5V_{2,1}, \quad V_{3,2} + V_{3,3} = 5V_{2,2}, \quad (2.10)$$

and hence

$$V_{3,1} + 2V_{3,2} + V_{3,3} = 30V_{1,1}.$$

Here  $V_{3,1}$  is the usual generalized Laguerre ray, whereas the positivity of  $V_{3,2}$  and  $V_{3,3}$  is a separate question.

The same source suggests the recursive pattern

$$V_{n,i} + V_{n,i+1} = c_n V_{n-1,i}, \quad c_n = \frac{4n-2}{n-1}, \quad (2.11)$$

with

$$v_{n,1} = \left( \frac{1}{2} \binom{2n}{n}, \dots, (-1)^{n+j} \binom{2n}{j}, \dots, (-1)^n \right), \quad (2.12)$$

which is the generalized Laguerre ray. For  $n = 4$  this gives the tentative coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 35 & -56 & 28 & -8 & 1 \\ 35/3 & -14 & 0 & 10/3 & -1 \\ 35/3 & -28/3 & -14/3 & 4/3 & 1 \\ 35 & -14 & -14 & -6 & -1 \end{pmatrix}, \quad (2.13)$$

where the first row is the square  $(p^{(4)})^2$  and the other rows are the putative non-square rays.

Finally, the Gaussian test gives a necessary but not sufficient condition for non-negativity. If

$$U(p; x) = \sum_{j=0}^m a_j p^{(m-j)}(x) p^{(m+j)}(x),$$

then, applying the operator to  $p(x) = e^{-x^2}$  and using Rodrigues' formula, one obtains the necessary condition

$$\sum_{j=0}^m a_j H_{m-j}(x) H_{m+j}(x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \quad (2.14)$$

where  $H_k$  are the Hermite polynomials in the corresponding normalization. This test is useful for excluding candidate rays, but it is not sufficient: for example the coefficient vector  $(10, -15, 8, -3)$  passes the Gaussian/Hermite test but fails on some real-rooted polynomials.

**2.3. Relation with the work of Blekherman–Riener.** The paper [1] studies cones of symmetric non-negative forms and symmetric sums of squares for a fixed number of variables and fixed degree. Their results are directly relevant background, especially in degree four. However, they do not by themselves give the finite-ray description originally expected here. The reason is that our cone is the intersection, over all numbers of variables, of the non-negativity cones with the special subspace

$$\text{span}\{\sigma_j \sigma_{2n-j} : 0 \leq j \leq n\},$$

and the number of variables is allowed to vary. Thus their theory supports the use of symmetric-form methods and gives valuable structural information, but the specific cone  $\mathcal{K}_n$  still requires the dual description and half-degree reduction given above. In particular, the quartic classification is compatible with the degree-four example, whereas for  $n \geq 3$  it does not imply that  $\mathcal{K}_n$  is polyhedral or simplicial.

### 3. FINAL REMARKS

The original finite-generator statement should be regarded as a conjectural picture which fails in the present generality. The dual description above is canonical and stable under increasing the number of variables, while the half-degree principle gives a practical testing family. A natural remaining problem is to determine for which values of  $n$  the cone  $\mathcal{K}_n$  is polyhedral. The evidence suggests that non-polyhedrality may occur in higher degree, but this remains to be proved.

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