Boris Shapiro

Polynomial Solutions of Linear Differential Equations

De Gruyter Expositions in Mathematics

Volume 80

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Polynomial Solutions of Linear Differential Equations

Spectral Problems, Asymptotic and WKB-Analysis

DE GRUYTER

Mathematics Subject Classification 2020
Primary: 34M03, 30C15, 34M46; Secondary: 34M05, 34M40

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ISSN 0938-6572
ISBN 978-3-11-914710-1
ISBN 978-3-11-220774-1 (PDF)
ISBN 978-3-11-220781-9 (E-PUB)
DOI https://doi.org/10.1515/9783112207741

Library of Congress Control Number: 2026935824

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data are available on the Internet at https://dnb.dnb.de.

De Gruyter and Walter de Gruyter GmbH are part of De Gruyter Brill.
www.degruyterbrill.com

Questions about General Product Safety Regulation: productsafety@degruyterbrill.com

Printing: CPI books GmbH, Leck

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Preface

> The mathematical facts worthy of being studied are those which, by their analogy with other facts, are capable of leading us to the knowledge of a physical law. They reveal the kinship between other facts, long known, but wrongly believed to be strangers to one another.
>
> Henri Poincaré, *Science and Hypothesis*

Frankly speaking, I never expected that I would be able to write a book. Now and then I had some thoughts about a research monograph, but I was always waiting for this or that part of the next project to be completely finished. Due to a heavy teaching load I never had enough time to concentrate on putting the already obtained results in the area together.

However, when I came to Beijing Institute for Mathematical Sciences and Applications (BIMSA) for a semester and was supposed to give a lecture course, I decided, in spite of the fact that many pieces of the overall project were (and still are) unfinished, to summarize the material scattered throughout many different publications. To compensate for the incompleteness of the currently available information, I included a number of open problems at the end of each chapter hoping that the aesthetic appeal of the subject together with the abundance of concrete examples will attract other mathematicians to its study. (I strongly believe that a good mathematical theory should have a clear motivation, many examples, and connect to many previously known results, see the above epigraph by H. Poincaré.)

I put some effort in trying not to overload the text with complicated definitions and make it as understandable as I possibly could, in particular, by providing many illustrations of the root configurations for the considered polynomials. I really hope that many portions of this book are accessible to master students having some knowledge of complex analysis and potential theory.

I gladly acknowledge that this book would never have seen the light of day without the help of several of my colleagues, collaborators, and students whose collective efforts are presented below.

I want to thank my former doctoral students Tanja Bergkvist, Thomas Holst, and Hans Rullgård for their break-through contributions. Although all three have since moved on from mathematics, their impact on this work is lasting. I am also deeply grateful to my colleague Rikard Bøgvad and especially to the late Professor Jan-Erik Björk—a true Renaissance man, whose vast knowledge and interests spanned every imaginable area of human activity.

I have been fortunate to collaborate with many talented researchers across the globe, including Jorge Borrego-Morell, Emil Horozov, Andrei Martínez-Finkelshtein, Gisli Mason, Guillaume Tahar, Kouichi Takemura, Milos Tater, and many others. My sincere thanks also go to my friend and collaborator Christian Hägg, whose amazing skills and assistance with the figures were invaluable.

https://doi.org/10.1515/9783112207741-203

I am grateful to Stockholm University for granting and funding the sabbatical semester that allowed me the time to complete this project.

Last but not least, I want to thank my wife, Karin Peters, for her wholehearted support during my time in China and my children Sonya, Jacob, Elias, Benjamin, and Rebecka for being wonderful and (almost) grown-up kids.

Stora Benhamra, Sweden, January 2026 Boris Shapiro

1 Introducing the main protagonists

1.1 Motivation and short historical account

Below we discuss polynomial solutions of linear differential equations with the main emphasis on polynomial eigenfunctions of (multi-parameter) spectral problems for univariate linear ordinary differential operators with polynomial coefficients as well as their asymptotics.

This area of mathematics is by no means new. It goes back at least to the second half of the 19th century. For example, the algebraic form of the classical Lamé equation (see [197, Chapter 23]) was introduced by Gabriel Lamé in 1830s in connection with the separation of variables in the Laplace equation in $\mathbb{R}^l$ with respect to elliptic coordinates. It has the form

$$Q(z)\frac{d^2S}{dz^2} + \frac{1}{2}Q'(z)\frac{dS}{dz} + V(z)S = 0, \tag{1.1}$$

where $Q_l(z)$ is a real polynomial of degree l with all real and distinct roots, and $V(z)$ is a polynomial of degree at most $l-2$ whose choice depends on what type of solution to (1.1) we are looking for. In the second half of the 19th century, several famous mathematicians including M. Bôcher, E. Heine, F. Klein, and T. Stieltjes studied the number as well as properties of the Lamé polynomials (sometimes also referred as Lamé solutions of a given degree and a given kind).

Such solutions to (1.1) exist for some choices of $V(z)$ and are characterized by the property that their logarithmic derivative is a rational function. For a given $Q(z)$ of degree $l \geq 2$ with simple roots, there exist 2^l different kinds of Lamé polynomials depending on whether these solutions are smooth or have a square root singularity at each given root of $Q(z)$ (for details, see [153] and [197]). An excellent modern study of these questions can be found in [121].

Existence of a polynomial solution of a given linear differential equation imposes natural restrictions on the indices of its singular points and implies that its monodromy group has an invariant 1-dimensional eigenspace. Existence of a sequence of polynomial solutions for a spectral problem of a given linear differential operator imposes further restrictions on the operator. In many cases, such sequences of polynomial solutions can be studied by various classical methods of analysis and potential theory, and one can obtain detailed exact and asymptotic results about their roots.

But in spite of the relative abundance of classical results, the field is still surprisingly modern and its asymptotic aspects are closely related to several nowadays popular areas such as, for example, the WKB-expansion for linear differential equations of order exceeding 2 and the theory of differentials of high order.

It is common knowledge that very few spectral problems involving linear ordinary differential operators allow explicit calculation of their spectra and eigenfunctions. There are, however, special but important exceptions from this general rule;

https://doi.org/10.1515/9783112207741-001

the most well-known of them being the Schrödinger equation on the real line with a real quadratic potential and boundary conditions given by vanishing of eigenfunctions at $\pm\infty$.

Such potentials for which similar explicit calculations can be carried out are usually referred to as *exactly solvable*. A classical example of this kind is the Pöschl–Teller potential, see [152]. Additionally there is an interesting class of quantum mechanical systems/potentials discovered relatively recently for which only a finite part of the spectrum and the respective eigenfunctions can be explicitly obtained. Such systems are called *quasi-exactly solvable*, and they are also quite rare, see, e. g., [189].

At the same time, if we abandon the physically motivated class of second order differential equations, there exists a sufficiently large and easily characterized class of linear differential operators that possess sequences of polynomial eigenfunctions (one in each degree) and whose spectra can be explicitly found. In what follows we shall call such operators *exactly solvable* (shorthand ES-operators) and refer to their polynomial eigenfunctions as *eigenpolynomials*. Besides being closely related to the theory of orthogonal polynomials, they are much easier to analyze from the point of view of asymptotic analysis, and they are connected to interesting geometric objects such as high order differentials. Their study constitutes Part I of this book.

Another source of polynomial solutions of linear differential equations with polynomial coefficients is related to the studies of Lamé, Heine, and Stieltjes briefly mentioned above. In their approach, instead of just adding a constant term to a given operator, one is allowed to add a polynomial term of a certain degree with arbitrary coefficients, which under certain conditions guarantees the existence of finitely many polynomial solutions occurring for different choices of the added polynomial. In the classical approach one only considered operators of second order. Part II of this book is devoted to similar studies and extension of several classical results to operators of arbitrary order.

1.2 Exactly solvable operators, basic notions

The fundamental (and quite simple) definition of exactly solvable operators going back to Salomon Bochner [35], which we will use throughout this book, is as follows.

Definition 1.1. A univariate linear differential operator

$$T = \sum_{j=1}^{k} Q_j(z) \frac{d^j}{dz^j} \tag{1.2}$$

with polynomial coefficients is called *exactly solvable* if the following two conditions hold:

(i) $\deg Q_j \leq j, \forall j = 1, \ldots, k$;

(ii) $\exists j_0$ such that $\deg Q_{j_0} = j_0$.

Since $\deg Q_j \le j, j = 1, \dots, k$, we will use the notation $Q_j(z) = \sum_{i=0}^{j} q_{ji} z^i$ throughout the book.

Remark 1.1. To explain why our summation index j starts from 1 and not from 0, observe that the above condition (i) implies that the term Q_0 of any ES-operator must be constant. Since we shall mainly consider spectral problems for ES-operators, it is natural to get rid of Q_0 adding it to the spectral parameter. Another reason for our choice is that it allows us to avoid certain pathologies occurring in examples like $T = 1 + \frac{d}{dz}$, which are not ES-operators according to our definition, and although they are easy to study on their own, they have properties drastically different from those of "our" ES-operators.

Figure 1.1: Salomon Bochner, 1899–1982.

Remark 1.2. Naturally S. Bochner has not been using the term "exactly solvable" for the class of operators he introduced. This terminology appeared much later and originates from physics.

As we will show in Chapter 2, exact solvability of T is equivalent to the fact that T preserves the infinite complete flag of linear subspaces in $\mathbb{C}[z]$ given by

$$\mathbb{C}_0[z] \subset \mathbb{C}_1[z] \subset \cdots \subset \mathbb{C}_n[z] \subset \cdots$$

where $\mathbb{C}_n[z]$ is the standard $(n+1)$-dimensional linear subspace of $\mathbb{C}[z]$ containing all polynomials of degree at most n.

Choosing the standard monomial basis $\{1, z, z^2, \dots, \}$ of $\mathbb{C}[z]$, we immediately observe that, by definition, any ES-operator T acts in this basis as an infinite upper triangular matrix. Namely, for any non-negative integer n, using (1.2) we obtain

$$T(z^n) = \lambda_n z^n + \text{terms of degree less than } n$$

where the coefficients $\lambda_0, \lambda_1, \lambda_2, \dots$ (with $\lambda_0 = 0$) are given by the formula

$$\lambda_n = \sum_{j=1}^{k} (n)_j q_{j,j}, \quad n = 0, 1, 2, \ldots \tag{1.3}$$

As above $q_{j,j}$ is the coefficient of $Q_j(z)$ of (the maximal possible) degree j and

$$(n)_j = n(n-1)\ldots(n-j+1)$$

is the Pochhammer symbol. (Observe that by condition (ii) in Definition 1.1 at least one $q_{j,j}$ is non-vanishing.)

For generic ES-operators T, all λ_n, $n = 0, 1, 2, \ldots$ are distinct, and therefore $\mathbb{C}[x]$ has a basis consisting of monic eigenpolynomials $p_n(z)$ of T, $\deg p_n(z) = n$ defined for all integers $n \geq 0$.

For an arbitrary ES-operator T, we can only conclude that there always exists a minimal non-negative integer n_T such that λ_n will be distinct for all integer $n \geq n_T$. Indeed, formula (1.3) immediately implies that $|\lambda_n|$ is monotone increasing for all sufficiently large n, which guarantees that for n large, all λ_n are distinct. (For this fact to be true, we need our summation in (1.2) to start from 1 and not from 0; it does not hold for $T = 1 + \frac{d}{dz}$.) This monotonicity implies that n_T always exists. On the other hand, the restriction of T to the linear subspace $\mathbb{C}_n[z]$ for n small compared to the order k of T can be a (more or less) arbitrary triangular matrix and, in particular, can have non-trivial Jordan blocks.

Notation 1.1. For any ES-operator T, we call the latter number n_T the *least uniqueness degree* of T. Further, let us denote by $\{p_n\}_{n=n_T}^{\infty}$ the sequence of uniquely defined monic eigenpolynomials of T in degrees $n_T, n_{T+1}, n_{T+2}, \ldots$.

To build up the intuition, let us describe ES-operators of orders 1 and 2. The following facts are easy to prove.

Example 1.1. (i) Up to a scaling and affine change of z, the only ES-operator of order 1 is $T = z\frac{d}{dz}$ whose monic eigenpolynomials are $1, z, z^2, \ldots, z^n, \ldots$ and $\lambda_n = n$.

(ii) The following operators are exactly solvable of order 2:

$$T_1 = \frac{d^2}{dz^2} - 2z\frac{d}{dz}, \quad p_n(z) = H_n(z), \quad \lambda_n = -2n;$$
$$T_2 = z\frac{d^2}{dz^2} + (\alpha + 1 - z)\frac{d}{dz}, \quad p_n(z) = L_n^{\alpha}(z), \quad \lambda_n = -n;$$
$$T_3 = z^2\frac{d^2}{dz^2} + 2(z+1)\frac{d}{dz}, \quad p_n(z) = Y_n(z), \quad \lambda_n = n(n+1);$$
$$T_4 = (z^2 - 1)\frac{d^2}{dz^2} + (\beta - \alpha - (\alpha + \beta + 2)z)\frac{d}{dz}, \quad p_n(z) = P_n^{\alpha,\beta}(z), \quad \lambda_n = -n(n + \alpha + \beta + 1).$$

Here $H_n(z)$ stands for the n-th Hermite polynomial, $L_n^{\alpha}(z)$ for the n-th Laguerre polynomial with parameter α, $Y_n(z)$ for the n-th Bessel polynomial, and $P_n^{\alpha,\beta}(z)$ for the n-th

Jacobi polynomial with parameters (α, β). These four families describe all possible interesting ES-operators of order 2 up to an affine transformation and scaling. (Notice that we allow α and β to be complex.)

1.3 ES-operators and Bochner–Krall problem

Based on the observations that every ES-operator has a sequence of eigenpolynomials and that for ES-operators of order 2 one typically obtains the four classical families of orthogonal polynomials with complex-valued parameters, Salomon Bochner posed the following general problem, see [35].

Problem 1.1. Describe ES-operators whose sequences of eigenpolynomials are orthogonal with respect to some positive measure supported on $\mathbb{R}$.

In loc. cit. he solved this problem for operators of order 2 showing that the above four cases of operators T_1, T_2, T_3, T_4 with the classical values of parameters (α, β) provide the complete answer. He also made a number of important general observations. In [105] his student H. L. Krall solved Problem 1.1 for operators of order 4 obtaining a number of additional families. These families are various classical systems such as the Jacobi-type, the Laguerre-type, the Legendre-type, the Bessel, and the Hermite polynomials, see [66, Theorem 3.1].

An assortment of families corresponding to order six operators has been found in the ensuing decades, see, e. g., [106, Chapter XVI]. W. Hahn showed that the four classical families are the only orthogonal polynomial sequences $\{P_n(z)\}_{n=0}^{\infty}$ for which $\{\frac{d}{dz}P_n(z)\}_{n=0}^{\infty}$ is also an orthogonal polynomial sequence, see [81]. An analog of this was proved for the order 4 case by K. H. Kwon, L. L. Littlejohn, J. K. Lee, and B. H. Yoo [113]. The most general form of Problem 1.1 is still open for operators of order six or higher, but K. H. Kwon and J. K. Lee have found a satisfactory solution if the polynomials are required to be orthogonal with respect to a compactly supported positive measure, see [114]. (A weaker result in the same direction was somewhat earlier obtained in [28].)

Problem 1.1 is now known as the *Bochner–Krall* problem. The respective ES-operators are called *Bochner–Krall* operators, and their sequences of eigenpolynomials are called *Bochner–Krall* systems (BKS, for short). Classification of Bochner–Krall operators has attracted considerable attention, see, e. g., [66] with its 100 references and [115, 116, 117] and the references therein. But currently it is still far from being solved.

The Bochner–Krall problem and its natural generalizations provide the initial motivation for the study of ES-operators. Another motivation comes from the possibility to study in detail the asymptotic behavior of their eigenpolynomials, which leads to several interesting results similar to those obtained by the classical WKB-method for Schrödinger equations. An intriguing circumstance is that in this asymptotic study naturally appears a family of rational higher order differentials.

Figure 1.2: Eduard Heine, 1821–1881, and Thomas Stieltjes, 1856–1894.

1.4 Heine–Stieltjes theory

Obviously, any linear ordinary operator with polynomial coefficients that is not exactly solvable can have at most finitely many eigenpolynomials. However, there is a natural way to “improve the situation” by varying polynomial coefficients of the operator instead of just adding to it a constant term. In other words, instead of considering the standard spectral problem for a given linear differential operator, one should consider an appropriate multi-parameter spectral problem in which infinitely many polynomial solutions will naturally appear.

The classical example of this approach is usually referred to as the *Heine–Stieltjes theory* after E. Heine and T. J. Stieltjes who were first to develop it for operators of order 2 in the second half of the 19th century. Some old and new results in this direction are presented in Part II of our book.

Definition 1.2. Consider an arbitrary linear ordinary differential operator

$$T = \sum_{j=1}^{k} Q_j(z) \frac{d^j}{dz^j} \tag{1.4}$$

with polynomial coefficients. (Here compared to the case of (1.2), we put no restrictions on the degrees of $Q_j(z)$.) The number

$$r = \max_{i=1,\dots,k} \left(\deg Q_i(z) - i\right)$$

is called the *Fuchs index* of T. The operator T is called a *(higher) Lamé operator* if its Fuchs index r is non-negative. T is called *non-degenerate* if $\deg Q_k(z) = k + r$ and *degenerate* otherwise.

Remark 1.3. Notice that the classical Lamé operator corresponds to $k = 2$. Observe that if $r = 0$, then T is exactly solvable. Further, the non-degeneracy of T is a natural condition equivalent to the requirement that T has either a regular or a regular singular point at ∞.

Given a (higher) Lamé operator T, consider the following multi-parameter spectral problem.

Problem 1.2. For each non-negative integer n, find all polynomials $V(z)$ of degree at most r such that the equation

$$TS(z) + V(z)S(z) = 0 \tag{1.5}$$

has a polynomial solution $S(z)$ of degree n.

Following the classical terminology, we call (1.5) a *(higher) Heine–Stieltjes spectral problem*, $V(z)$ is called a *(higher) Van Vleck polynomial*, and the corresponding polynomial $S(z)$ is called a *(higher) Stieltjes polynomial*. Below, we will often skip mentioning "higher".

In Part II of the book we discuss the asymptotic root-counting measures for sequences of Heine–Stieltjes polynomials of increasing degrees when the respective sequence of Van Vleck polynomials (which all are of the same degree) has a limit. The answer is expressed in terms of this limit and the leading coefficient of the Lamé operator.

One can also ask about the asymptotic root-counting measures for the product of all Van Vleck polynomials that correspond to Stieltjes polynomials of degree $n \to \infty$. This problem seems to be considerably more difficult. Below we present its solution only in the case of Lamé operator of second order with Fuchs index 1. The answer is given in quite non-trivial terms related to the modulus of the symbol curve, which in this case is elliptic. So far our attempts to solve a similar problem for high order Lamé operators with Fuchs index 1 failed.

A promising aspect of both asymptotic theory of ES-operators and Heine–Stieltjes theory is the appearance of an interesting family of rational differentials whose order equals the order of the differential operator and which have certain properties similar to those of the famous Strebel quadratic differentials. One can hope that the material of this book will stimulate further development of the theory of higher differentials on surfaces of any genus.

1.5 Short historical notes

Heinrich Eduard Heine (Figure 1.2, left) made fundamental contributions to several areas of mathematics, mainly in real analysis, but also to special functions and mathematical logic/formalism. In particular, he introduced the notion of uniform convergence, which everybody studies in Calculus 1 classes and his name appears in Heine–Borel theorem, Heine–Cantor theorem, Heine–Stieltjes polynomials (considered below), Mahler–Heine formula, etc.

In spite of his untimely death at the age of 38, **Thomas Joannes Stieltjes** (Figure 1.2, right) has worked on almost all branches of analysis, continued fractions, and number theory. Stieltjes' work is also seen as an important first step towards the theory of Hilbert spaces. Stieltjes also contributed to the theory of divergent series and discontinuous functions, to ordinary and partial differential equations, the gamma function, interpolation, and elliptic functions.

Salomon Bochner (Figure 1.1) made substantial contributions to harmonic analysis, probability theory, differential geometry as well as history of mathematics. Several notions and results such as the Bochner integral, Bochner theorem on Fourier transforms, Bochner–Riesz means, Bochner–Martinelli formula bear at present his name. His mathematical production consists of 6 books on various subjects including history of mathematics and about 140 research papers. Among these papers 32 were published in Proc. Nat. Acad. USA, 46 in Annals of Mathematics, 3 in Acta Math., and 4 in Duke Math. J. He belonged to a sizeable group of European mathematicians of Jewish origin who moved to the USA before or during the WWII and contributed to an enormous development of mathematics in their new motherland. The list of his doctoral students looks almost scary. It contains: Richard Askey, Eugenio Calabi, Jeff Cheeger, M. T. Cheng, Charles L. Dolph, Hillel Furstenberg, Robert Gunning, Israel Halperin, Sigurdur Helgason, Carl Herz, Gilbert Hunt, Samuel Karlin, Anthony Knapp, Paco Lagerstrom, Lynn Loomis, Harry Rauch, Joseph H. Sampson, Herbert Scarf, William A. Veech, Gerard Washnitzer, Bernard Russell. The majority of these names are well known to every professional mathematician.