SHADOWS OF POLYNOMIALS AND GAUSS-LUCAS THEOREM

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To the memory of Blagovest Hristov Sendov

ABSTRACT. Geometry of polynomials whose classical achievements are beautifully summarized in Marden's treatise [Ma] deals with the location of roots and critical points of univariate polynomials. Below, given a univariate polynomial $P(z)$, we consider the closure of the union of all roots of all polynomials of the form $\frac{\mathrm{d}^m}{\mathrm{d}z^m}(P^n(z))$ where n and m are arbitrary positive integers; we call this closure the *shadow* of $P(z)$. Under some mild non-degeneracy assumptions, the shadow of $P(z)$ is a domain inside the convex hull of its zero locus. Imposing some extra assumptions, we describe the boundary and study some properties of the shadow. Many open questions are formulated.

1. Introduction

The area of mathematics called the *geometry of polynomials* was initiated by C. F. Gauss after his discovery of what is now called *Gauss-Lucas theorem* and his interpretation of the critical points of a polynomial as the points of equilibria for the logarithmic electrostatic field created by the configuration of charges placed at the roots of the polynomial where the charge placed at a given root equals its multiplicity. Since then numerous generalizations of both Gauss-Lucas theorem and the study of points of equilibria for configurations of point charges in the plane and higher dimensions have been suggested. A number of fascinating (and still open) conjectures relating the roots and the critical points of a complex univariate polynomials have been formulated over the years; among them the most notorious being the Sendov and the Smale conjectures. In addition to the critical points of a given polynomial $P(z)$, i.e., the roots of $P'(z)$, many papers discuss the roots of higher derivatives of $P(z)$, its polar derivatives etc. ADD MANY REFERENCES HERE INCLUDING SENDOV-SENDOV...

In this paper, for any given polynomial P of degree $\mathfrak{d} \geq 1$, we consider a double-indexed family $\{\mathcal{P}_{m,n,P}\}$ of polynomials given by

$$\mathcal{P}_{m,n,P}(z) := \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left(P^n(z)\right), \ n = 1, 2, 3, \dots \ \text{and} \ m = 0, 1, \dots, n\mathfrak{d} - 1.$$

Following [BoHaSh], we call polynomials $\mathcal{P}_{m,n,P}(z)$ *Rodrigues' descendants* of P. We will refer to the roots of $\mathcal{P}_{m,n,P}(z)$ as *generalized critical points* of $P(z)$. (Since a polynomial P will be fixed, in what follows, we sometimes skip the index P and use the notation $\mathcal{P}_{m,n}(z)$.)

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Further, let us define the sequence $\{Q_{n,P}\}_{n=1}^{\infty}$ as

$$Q_{n,P}(z) := \prod_{m=0}^{n\mathfrak{d}-1} \mathcal{P}_{m,n,P}(z). \tag{1.1}$$

Example 1. An interesting example of polynomials $\mathcal{P}_{n,n}(z)$ has been introduced in 1816 by (Benjamin) Olinde Rodrigues who discovered his famous formula

$$P_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left((z^2-1)^n\right) \tag{1.2}$$

for the Legendre polynomials which since then became a standard tool in the theory of classical orthogonal polynomials and special functions, see e.g. [AbSt]. (Later the same formula was rediscovered by J. Ivory and C. G. Jacobi, see [As]). For many classical families of orthogonal polynomials there is an analog of Rodrigues formula.

The main object of our study is the asymptotics as $n \to \infty$ of the sequence $\{\mu_n\}$ of the root-counting measures for the polynomial sequence $\{Q_n\}$ and show that it weakly converges to a certain probability measure μ_P supported on a domain Υ_P which we call the *shadow* of the polynomial P and which lies inside the convex hull of its roots, see illustrations in Fig. 1. We will show that Υ_P is the closure of the set of all generalized critical points of $P(z)$ and we shall study its properties.

The shadow of a polynomial has been earlier introduced in our manuscript [BoHaSh] where the following conjecture was formulated. We say that a polynomial P of degree at least 3 has roots *in convex position* if they are not aligned and each of them is a vertex of their convex hull. In particular, every cubic polynomial P whose roots are not aligned has this property.

Conjecture 1. *For any polynomial P of degree at least* 2 *whose roots are in convex position but do not form a regular polygon,*
(i) *Υ_P is a concave domain.*
(ii) *The boundary of Υ_P is contained in the union of all critical values (w.r.t. z) of the rational function*

$$F_\alpha(z) = z - \alpha\frac{P(z)}{P'(z)}. \tag{1.3}$$

where the parameter α runs over the interval $[0,\mathfrak{d}]$. In other words, Υ_P consists of all u for which the family $\tilde{\Phi}(\alpha, z, u) = \alpha P(z) + (u-z)P'(z)$ has a multiple root w.r.t. z when $\alpha \in [0,\mathfrak{d}]$.

Remark 1. *In the case when the roots of $P(z)$ form a regular polygon, Υ_P is the union of straight intervals connecting its vertices with its center.*

Recall that a polar derivative of a polynomial $P(z)$ of degree $\mathfrak{d}$ w.r.t. a point $u \in \mathbb{C}$ is defined by

$$D_u P(z) = \mathfrak{d}P(z) + (u-z)P'(z),$$

see [Ma]. Thus if we introduce the *generalized polar derivative* of $P(z)$ w.r.t a point $u \in \mathbb{C}$ with parameter α as given by

$$D_u^{(\alpha)} P(z) = \alpha P(z) + (u-z)P'(z)$$

then Conjecture 1 claims that the boundary of Υ_P is contained in the union of all u for which there exists $\alpha \in [0,\mathfrak{d}]$ such that $D_u^{(\alpha)} P(z)$ has a multiple root w.r.t. z.

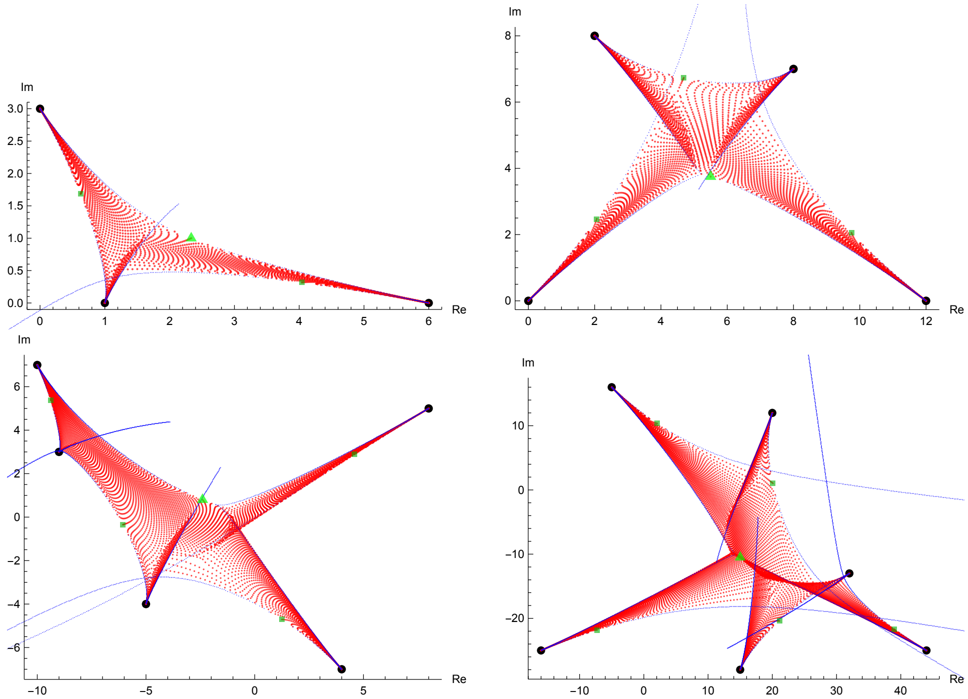

FIGURE 1. The union of all zeros of $\mathcal{P}_{m,30,P}(z)$ for $m = 0, 1, \dots, 30 \deg P - 1$ shown by small red dots. The large dots are the zeros of P, the large squares are the critical points of P, the triangle is the center of mass of the zero locus of P, and the small squares are the branch points of (5.1) and (5.3) in the z-plane.

The goal of this paper is to settle Conjecture 1. In case when the roots of $P(z)$ are not in convex position Conjecture 1 is no longer true as it stands. We are currently looking for its generalization.

Corollary 1. *The usual critical points of P and its mass center lie on the boundary of Υ_P.*

The structure of the paper is as follows......

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2. PROPERTIES OF THE SHAPE CURVE Θ_P AND THE SHAPE CONTOUR $\mathcal{B}_P$

Notation 1. *Given a polynomial $P(z)$ as above, consider the trivariate polynomial $\Phi_P(\alpha, z, y)$ given by*

$$\Phi_P(\alpha, z, y) := \sum_{k=0}^{\eth} \frac{\alpha - k}{k!} P^{(k)}(z) y^k. \tag{2.1}$$

Denote by $\Gamma_P \subset \mathbb{C}_\alpha \times \mathbb{C}_z \times \mathbb{C}_y$ the affine surface in the space with coordinates (α, z, y) given by the equation $\Phi_P(\alpha, z, y) = 0$ and denote by $\Gamma_P(\alpha) \subset \mathbb{C}_z \times \mathbb{C}_y$

the affine curve obtained as the plane section of Γ_P if one fixes α. Finally, for α fixed, let $Br_P(\alpha) \subset \mathbb{C}_z$ be the branching locus of the projection of the curve $\Gamma_P(\alpha)$ on the z-coordinate. Define the **discriminant of critical values** *(shorthand, the* **c.v-discriminant***) of the polynomial P as given by the formula*

$$Dsc_P(\alpha,z) := Discriminant_y(\Phi_P(\alpha,z,y)) = Resultant(\Phi_P(\alpha,z,y), \frac{d}{dy}\Phi_P(\alpha,z,y), y)$$

and define the **c.v-curve** *$\mathfrak{C}_P \subset \mathbb{C}_\alpha \times \mathbb{C}_z$ of P as the zero locus of its c.v-discriminant. (The explanation for the terminology can be found in § 2.) We will call $\Phi_P(\alpha,z,y)$ the* **generating family** *of the c.v-curve $\mathfrak{C}_P$.*

Observe that $Dsc_P(\alpha,z)$ is a bivariate polynomial in the variables (α,z) and that $Br_P(\alpha)$ is given by the condition $Dsc_P(\alpha,z)=0$ where α is fixed.

Notation 2. *Given a univariate polynomial P, define its* **shape curve** *Θ_P as*

$$\Theta_P := \cup_{\alpha\in\mathbb{R}} Br_P(\alpha) \subset \mathbb{C}_z \simeq \mathbb{R}^2$$

and define its **shape contour** *$\mathcal{B}_P$ as*

$$\mathcal{B}_P = \cup_{\alpha\in[0,\mathfrak{d}]} Br_P(\alpha) \subset \Theta_P.$$

The set $\mathcal{B}_P$ is shown by the blue curves in Figure 1.

Remark 2. *Obviously, Θ_P is obtained by projecting to the coordinate $\mathbb{C}_z$ the restriction $\mathfrak{C}_P^{\mathbb{R}}$ of the c.v-curve $\mathfrak{C}_P \subset \mathbb{C}_\alpha \times \mathbb{C}_z$ to the 3-dimensional space $\mathbb{R}_\alpha \times \mathbb{C}_z$.*

Theorem 1. *If the roots of P are in convex position then the boundary of Υ_P is contained in the shape contour $\mathcal{B}_P$.*

Lemma 2. *In the above notation,*

$$\text{(i)}\quad \Phi_P(\alpha,z,y) = \alpha P(z+y) - y\frac{d}{dy}P(z+y);$$

(ii) *$Dsc_P(\alpha,z) = b(\alpha-\mathfrak{d})\hat{D}_P(\alpha,z)$, where $\hat{D}_P(\alpha,z)$ is a bivariate polynomial in (α,z), b is the leading coefficient of P, and $\mathfrak{d} = \deg P$.*
(iii) *The c.v-curve $\mathfrak{C}_P \subset \mathbb{C}_\alpha \times \mathbb{C}_z$ can be obtained as the zero locus of the discriminant with respect to the variable u of the trivariate polynomial*

$$\widehat{\Phi}(\alpha,z,u) = \alpha P(u) - (u-z)P'(u). \tag{2.2}$$

(iv) *For any fixed α (real or complex), the divisor $\mathcal{B}_P(\alpha)$ coincides with the divisor of all critical values (in the variable u) of the rational function*

$$F_\alpha(u) = u - \alpha\frac{P(u)}{P'(u)}. \tag{2.3}$$

Therefore the shape curve Θ_P is the union of all critical values of $F_\alpha(u)$ taken over all $\alpha\in\mathbb{R}$.

Proof. To prove (i), observe that

$$\Phi_P(\alpha,z,y) := \sum_{k=0}^{\mathfrak{d}} \frac{\alpha-k}{k!}P^{(k)}(z)y^k = \alpha\sum_{k=0}^{\mathfrak{d}}\frac{P^{(k)}(z)}{k!}y^k - \sum_{k=1}^{\mathfrak{d}}\frac{P^{(k)}(z)}{(k-1)!}y^k = \alpha P(z+y) - y\frac{d}{dy}P(z+y).$$

The last equality is valid since P is a univariate polynomial of degree $\mathfrak{d}$ to which we apply the Taylor expansion in the variable y at the point z. □

Lemma 3. *Consider the expansion of $\hat{D}_P(\alpha, z)$ in powers of z:*

$$\hat{D}_P(\alpha, z) = \kappa_P^{(0)} z^{2\mathfrak{d}-2} + \kappa_P^{(1)} z^{2\mathfrak{d}-3} + \kappa_P^{(2)} z^{2\mathfrak{d}-4} + \cdots + \kappa_P^{(2\mathfrak{d}-3)} z + \kappa_P^{(2\mathfrak{d}-2)}.$$

Then, for every $j = 1, \ldots, 2\mathfrak{d}-2$, $\kappa_P^{(j)}$ is a polynomial in α and coefficients of P. Moreover except for $j = 2\mathfrak{d}-3$ the degree of $\kappa_P^{(j)}$ in the variable α equals j. In the remaning case the degree of $\kappa_P^{(2\mathfrak{d}-3)}$ in the variable α equals $2\mathfrak{d}-4$.

For α fixed, define the positive divisor $\Theta_P(\alpha)$ in the complex plane $\mathbb{C}_z$ obtained as the locus of solutions $\hat{D}_P(\alpha, z) = 0$. If $\kappa_P \neq 0$ which is equivalent to $P'(z)$ having simple roots, then $\Theta_P(\alpha)$ is a divisor of degree $2d-2$ for every (complex) value of α.

Lemma 4. (i) The shape curve $\Theta_P \subset \mathbb{C}_z \simeq \mathbb{R}^2$ is real semi-algebraic in the coordinates (x, y) where x and y are the real and imaginary parts of z respectively.
(ii) If P' has only simple roots, then $\Theta_P(0) = 2 \cdot \mathbf{div}(P')$.
(iii) $\Theta_P(1) \supset \mathbf{div}(P)$.
(iv) $\Theta_P(\mathfrak{d}) \ni$ mass center.
(v) For generic P, Θ_P is smooth except for the cusps at all roots of P.
(vi) $\lim_{\alpha\to\pm\infty} \Theta_P(\alpha) =$???

Proof. To settle (i) notice that as we mentioned in Remark 2, the shape curve is the projection of the curve $\mathfrak{C}_P^{\mathbb{R}} \subset \mathbb{R}^3$ to the last two coordinates. $\mathfrak{C}_P^{\mathbb{R}}$ is obviously real algebraic. So its projection is real semialgebraic. Figure 2 below shows that semialgebraicity can actually occur.

To settle (ii), first observe that the family $\widetilde{\Phi}(\alpha, z, y)$ can be rewritten as

$$\widehat{\Phi}(\alpha, z, u) = \alpha P(u) - (u - z)P'(u)$$

where $u = z + y$. Thus $\widehat{\Phi}(0, z, u) = -(u - z)P'(u)$. If P' has only simple roots then $\widehat{\Phi}(0, z, u)$ has a multiple root in the variable u if and only if z coincides with one of the roots of $P'(u)$. The degree of $\Theta_P(1)$ equals $2\mathfrak{d}-2$. Thus it is more or less clear that $\Theta_P(0) = 2 \cdot \mathbf{div}(P')$.

To settle (iii), observe that $\widehat{\Phi}(1, z, u) = P(u) - (u - z)P'(u)$. If $P(z) = (u - u_1)(u - u_2)\ldots(u - u_{\mathfrak{d}})$, then $P'(u) = P(u)\left(\frac{1}{u-u_1} + \frac{1}{u-u_2} + \cdots + \frac{1}{u-u_{\mathfrak{d}}}\right)$. Thus if $z = u_j$ then $\widehat{\Phi}(1, z, u)$ has a double root at u_j. Computer experiments show that for a generic $P(u)$, $\widehat{\Phi}(1, z, u)$ has additional $\mathfrak{d}-2$ distinct roots away from $\mathbf{div}(P)$.

To settle (iv), observe that $\widehat{\Phi}(\mathfrak{d}, z, u) = \mathfrak{d}P(u) - (u - z)P'(u)$. Assume that $P(u) = (u-u_1)(u-u_2)\ldots(u-u_{\mathfrak{d}})$. Thus the mass center $\mathfrak{m}$ is given by $\frac{u_1+u_2+\cdots+u_{\mathfrak{d}}}{\mathfrak{d}}$. Further $\widehat{\Phi}(\mathfrak{d}, \frac{\beta_1}{\mathfrak{d}}, u)$ will become a polynomial of degree at most $\mathfrak{d}-2$, i.e. its two leading coefficieints will cancel out. Thus it will gain a multiple root at ∞. □

Lemma 5. *Given a monic polynomial P, the α-curve $\mathfrak{C}_P \subset \mathbb{C}_\alpha \times \mathbb{C}_z$ is birationally equivalent to the curve $\Lambda_P \subset \mathbb{C}_\nu \times \mathbb{C}_\mu$ which is induced from the standard discriminant in the space of all monic polynomials of degree d by the map given by the 2-dimensional affine subspace*

$$\Psi_P(\nu, \mu, u) = P(u) + \nu(\mathfrak{d}P(u) - uP'(u)) + \mu P'(u), \tag{2.4}$$

under the birational variable change $\nu = \frac{1}{\alpha-\mathfrak{d}}$, $\mu = \frac{z}{\alpha-\mathfrak{d}} \Leftrightarrow \alpha = \frac{1}{\nu} + \mathfrak{d}$, $z = \frac{\mu}{\nu}$.

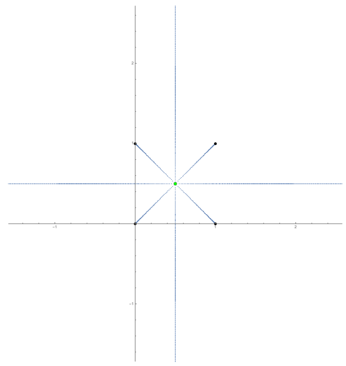

FIGURE 2. The shape curve of the quartic polynomial $P(z) = z(z-1)(z-I)(z-1-I)$.

Proof. Assume that $P(u) = u^{\mathfrak{d}} + \beta_1 u^{\mathfrak{d}-1} + \beta_2 u^{\mathfrak{d}-2} + \cdots + \beta_{\mathfrak{d}}$. Then using (2.2) we get

$$\widehat{\Phi}_P(\alpha, z, u) = \alpha u^{\mathfrak{d}} + \alpha\beta_1 u^{\mathfrak{d}-1} + \ldots \alpha\beta_{\mathfrak{d}} + z(\mathfrak{d}u^{\mathfrak{d}-1} + (\mathfrak{d}-1)\beta_1 u^{\mathfrak{d}-2} + \cdots + \beta_{\mathfrak{d}-1})$$
$$-(\mathfrak{d}u^{\mathfrak{d}} + (\mathfrak{d}-1)\beta_1 u^{\mathfrak{d}-1} + \cdots + \beta_{\mathfrak{d}-1}u) = (\alpha - \mathfrak{d})u^{\mathfrak{d}} + ((\alpha - \mathfrak{d} + 1)\beta_1 + z\mathfrak{d})u^{\mathfrak{d}-1} +$$
$$+((\alpha - \mathfrak{d} + 2)\beta_2 + z(\mathfrak{d}-1)\beta_1)u^{\mathfrak{d}-2} + ((\alpha - \mathfrak{d} + 3)\beta_3 + z(\mathfrak{d}-2)\beta_2)u^{\mathfrak{d}-3} + \ldots.$$

Diving $\widehat{\Phi}(\alpha, z, u)$ by $(\alpha - \mathfrak{d})$ and using $\nu = \frac{1}{\alpha - \mathfrak{d}}$ we get the family

$$u^{\mathfrak{d}} + ((1+\nu)\beta_1 + z\mathfrak{d}\nu)u^{\mathfrak{d}-1} + ((1+2\nu)\beta_2 + z(\mathfrak{d}-1)\beta_1\nu)u^{\mathfrak{d}-2} + \cdots + (1+\mathfrak{d}\nu)\beta_{\mathfrak{d}} + z\beta_{\mathfrak{d}-1}\nu.$$

Denoting $\mu = z\nu$, we get the family

$$\Psi_P(\nu, \mu, u) = u^{\mathfrak{d}} + ((1+\nu)\beta_1 + \mu\mathfrak{d})u^{\mathfrak{d}-1} + ((1+2\nu)\beta_2 + \mu(\mathfrak{d}-1)\beta_1)u^{\mathfrak{d}-2} + ((1+3\nu)\beta_3 + \mu(\mathfrak{d}-2)\beta_2)u^{\mathfrak{d}-3}$$
$$\cdots + (1+\mathfrak{d}\nu)\beta_{\mathfrak{d}} + \mu\beta_{\mathfrak{d}-1}.$$

Regrouping the terms we get

$$\Psi_P(\nu, \mu, u) = P(u) + \nu(\beta_1 u^{\mathfrak{d}-1} + 2\beta_2 u^{\mathfrak{d}-2} + \cdots + \mathfrak{d}\beta_{\mathfrak{d}}) + \mu(\mathfrak{d}u^{\mathfrak{d}-1} + (\mathfrak{d}-1)u^{\mathfrak{d}-2} + \cdots + \beta_{\mathfrak{d}-1})$$
$$= P(u) + \nu(\mathfrak{d}P(u) - uP'(u)) + \mu P'(u).$$

The curve $\Lambda_P \subset \mathbb{C}_\nu \times \mathbb{C}_\mu$ consisting of those pairs (ν, μ) for which (2.4) has a multiple root in the variable u is birationally equivalent to the α-curve $\mathfrak{C}_P$. The isomorphism is induced by the birational variable change $\nu = \frac{1}{\alpha - \mathfrak{d}}$, $\mu = \frac{z}{\alpha - \mathfrak{d}} = z\nu$. □

Remark 3. *We can also use the generating family*

$$\Xi_P(\nu, z, u) = P(u) + \nu(\mathfrak{d}P(u) - uP'(u)) + z\nu P'(u),$$

where $\nu = \frac{1}{\alpha - \mathfrak{d}}$. *This family gives the same shape curve* Θ_P *as the initial family when* ν *runs over* $\mathbb{R}$ *and the same shape contour* $\mathcal{B}_P$ *when* ν *runs over the interval* $(-\infty, -\frac{1}{\mathfrak{d}}]$.

Remark 4. Using (2.2), we can determine the branching locus $Br_P(\alpha) \subset \mathbb{C}_z$ as the locus of all z such that the hyperbola $\frac{\alpha}{u-z}$ has a non-transversal intersection with the graph of the logarithmic derivative $\frac{P'(u)}{P(u)}$. Indeed, $Br_P(\alpha)$ consists of those

z for which $\widehat{\Phi}(\alpha,z,u)$ has a multiple root in the variable u. Zeros of $\widehat{\Phi}(\alpha,z,u)$ are given by the equation

$$\frac{P'(u)}{P(u)}=\frac{\alpha}{u-z}$$

which implies the claim.

3. Examples: Simplifying the discriminant

In this section, we will simplify $\mathcal{D}^{\alpha}_{P(z)}:=\text{Discriminant}_u(f(u))$, where $f(u):=\alpha P(u)+(z-u)P'(u)$ and $P(z):=z^{\mathfrak{d}}+a_{\mathfrak{d}-1}z^{\mathfrak{d}-1}+\ldots+a_1z+a_0$.

3.1. **Cubic polynomials.** We begin with the polynomial $P(z):=z^3+bz^2+cz+\kappa$. In this situation we have $\mathfrak{d}=3$, and

$$\begin{aligned}\mathcal{D}^{\alpha}_{P(z)}&=36\left(b^2-3c\right)z^4+\left[48b(b^2-3c)-4(2b^3-9bc+27\kappa)\alpha\right]z^3\\&+\left[8(b^2-3c)(2b^2+3c)-4(4b^4-21b^2c+18c^2+27b\kappa)\alpha+4(b^2-3c)^2\alpha^2\right]z^2\\&+\left[16bc(b^2-3c)+4(-4b^3c+15bc^2+18b^2\kappa-81c\kappa)\alpha+4(b^2-3c)(bc-9\kappa)\alpha^2\right]z\\&+4c^2(b^2-3c)+4(-3b^2c^2+10c^3+8b^3\kappa-27bc\kappa)\alpha\\&+(13b^2c^2-48c^3-48b^3\kappa+198bc\kappa-243\kappa^2)\alpha^2-6\left(b^2c^2-4c^3-4b^3\kappa+18bc\kappa-27\kappa^2\right)\alpha^3\\&+\left(b^2c^2-4c^3-4b^3\kappa+18bc\kappa-27\kappa^2\right)\alpha^4.\end{aligned}\tag{3.1}$$

Notice that

$$\begin{aligned}\mathcal{D}^{0}_{P(z)}&=\text{Discriminant}_u(0\cdot P(u)+(z-u)P'(u))=(P'(z))^2\cdot\text{Discriminant}_z(P'(z))\\&=36\left(b^2-3c\right)z^4+48b\left(b^2-3c\right)z^3+8\left(b^2-3c\right)\left(2b^2+3c\right)z^2\\&+16bc\left(b^2-3c\right)z+4c^2\left(b^2-3c\right).\end{aligned}$$

By comparing the latter expression to (3.1), we see that

$$\begin{aligned}\mathcal{D}^{\alpha}_{P(z)}&=\mathcal{D}^{0}_{P(z)}-\left[4(2b^3-9bc+27\kappa)\alpha\right]z^3-\left[4(4b^4-21b^2c+18c^2+27b\kappa)\alpha-4(b^2-3c)^2\alpha^2\right]z^2\\&+\left[4(-4b^3c+15bc^2+18b^2\kappa-81c\kappa)\alpha+4(b^2-3c)(bc-9\kappa)\alpha^2\right]z\\&+4(-3b^2c^2+10c^3+8b^3\kappa-27bc\kappa)\alpha+(13b^2c^2-48c^3-48b^3\kappa+198bc\kappa-243\kappa^2)\alpha^2\\&-6\left(b^2c^2-4c^3-4b^3\kappa+18bc\kappa-27\kappa^2\right)\alpha^3+\left(b^2c^2-4c^3-4b^3\kappa+18bc\kappa-27\kappa^2\right)\alpha^4.\end{aligned}\tag{3.2}$$

Furthermore, note that

$$\begin{aligned}\alpha\left(\mathcal{D}^{0}_{P(z)}-\mathcal{D}^{-1}_{P(z)}\right)&=-4(2b^3-9bc+27\kappa)\alpha z^3-4(5b^4-27b^2c+27c^2+27b\kappa)\alpha z^2\\&+4(-5b^3c+18bc^2+27b^2\kappa-108c\kappa)\alpha z+4(-8b^2c^2+29c^3+27b^3\kappa-108bc\kappa+108\kappa^2)\alpha.\end{aligned}$$

Adding and subtracting the above expression from (3.2) and simplifying the resulting expression, we obtain

$$\begin{aligned}\mathcal{D}^{\alpha}_{P(z)}&=\mathcal{D}^{0}_{P(z)}+\alpha\left(\mathcal{D}^{0}_{P(z)}-\mathcal{D}^{-1}_{P(z)}\right)+4\alpha(\alpha+1)(b^2-3c)^2z^2+4\alpha(\alpha+1)(b^2-3c)(bc-9\kappa)z\\&+4(5b^2c^2-19c^3-19b^3\kappa+81bc\kappa-108\kappa^2)\alpha+(13b^2c^2-48c^3-48b^3\kappa+198bc\kappa-243\kappa^2)\alpha^2\\&-6(b^2c^2-4c^3-4b^3\kappa+18bc\kappa-27\kappa^2)\alpha^3+(b^2c^2-4c^3-4b^3\kappa+18bc\kappa-27\kappa^2)\alpha^4.\end{aligned}\tag{3.3}$$

Clearly, the last four terms in (3.3) resemble

$$\text{Discriminant}_z(P(z)) = b^2c^2 - 4c^3 - 4b^3\kappa + 18bc\kappa - 27\kappa^2, \tag{3.4}$$

while the z and z^2 terms are identical to those found in

$$\frac{\alpha(\alpha+1)}{4}\,\text{Discriminant}_z(P'(z))\,\mathcal{D}^{\mathfrak{d}}_{P(z)} = 4\alpha(\alpha+1)(b^2-3c)^2z^2$$
$$+ 4\alpha(\alpha+1)(b^2-3c)(bc-9\kappa)z + 4\alpha(\alpha+1)(b^2-3c)(c^2-3b\kappa). \tag{3.5}$$

By adding and subtracting (3.5) from (3.3) and using (3.4), this yields

$$\mathcal{D}^{\alpha}_{P(z)} = \mathcal{D}^{0}_{P(z)} + \alpha\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right)$$
$$+ \alpha(\alpha+1)\left(\frac{\text{Discriminant}_z(P'(z))\,\mathcal{D}^{\mathfrak{d}}_{P(z)}}{4} + (\alpha(\alpha-7)+16)\,\text{Discriminant}_z(P(z))\right), \tag{3.6}$$

or, apparently,

$$\mathcal{D}^{\alpha}_{P(z)} = \mathcal{D}^{0}_{P(z)} + \alpha\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right) + \frac{\alpha(\alpha+1)}{2!}\left(\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right) - \left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right)\right)$$
$$+ \alpha(\alpha+1)(\alpha+2)(\alpha-9)\cdot\text{Discriminant}_z(P(z)). \tag{3.7}$$

3.2. **Quartic and higher polynomials.** We will now consider the polynomial $P(z) := z^4+bz^3+cz^2+ez+f$. In this situation we have $\mathfrak{d}=4$, and as in the previous subsection we want to simplify $\mathcal{D}^{\alpha}_{P(z)} = \text{Discriminant}_u(\alpha\cdot P(u) + (z-u)P'(u))$.

By considering the pattern in equation (3.7) above, we quickly find that

$$\mathcal{D}^{\alpha}_{P(z)} = \mathcal{D}^{0}_{P(z)} + \alpha\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right) + \frac{\alpha(\alpha+1)}{2!}\left(\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right) - \left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right)\right)$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{3!}\left[\left(\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right) - \left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right)\right) - \left(\left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right) - \left(\mathcal{D}^{-2}_{P(z)} - \mathcal{D}^{-3}_{P(z)}\right)\right)\right]$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{4!}\left[\left[\left(\left(\mathcal{D}^{0}_{P(z)} - \mathcal{D}^{-1}_{P(z)}\right) - \left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right)\right) - \left(\left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right) - \left(\mathcal{D}^{-2}_{P(z)} - \mathcal{D}^{-3}_{P(z)}\right)\right)\right]\right.$$
$$\left. - \left[\left(\left(\mathcal{D}^{-1}_{P(z)} - \mathcal{D}^{-2}_{P(z)}\right) - \left(\mathcal{D}^{-2}_{P(z)} - \mathcal{D}^{-3}_{P(z)}\right)\right) - \left(\left(\mathcal{D}^{-2}_{P(z)} - \mathcal{D}^{-3}_{P(z)}\right) - \left(\mathcal{D}^{-3}_{P(z)} - \mathcal{D}^{-4}_{P(z)}\right)\right)\right]\right]$$
$$+ \alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha-22)\cdot\text{Discriminant}_z(P(z)). \tag{3.8}$$

Note that if Y_k denotes the coefficient of $\prod_{j=1}^{k}(\alpha+j-1)/j$ in equation (3.8), $k=0,\dots,4$ (with $Y_0 = \mathcal{D}^0_{P(z)}$), then $Y_{k+1} = Y_k - Y'_k$, $k=0,\dots,3$. Here Y'_k is equal to Y_k with the indices of all generalized polar derivatives contained therein shifted down by 1. This "sum of finite differences" hints at a differential operator being at work. It also hints at a sum of binomial coefficients with alternating signs. In fact, we find that

$$\mathcal{D}^{\alpha}_{P(z)} = \binom{0}{0}\mathcal{D}^{0}_{P(z)} + \alpha\left(\binom{1}{0}\mathcal{D}^{0}_{P(z)} - \binom{1}{1}\mathcal{D}^{-1}_{P(z)}\right) + \frac{\alpha(\alpha+1)}{2!}\left(\binom{2}{0}\mathcal{D}^{0}_{P(z)} - \binom{2}{1}\mathcal{D}^{-1}_{P(z)} + \binom{2}{2}\mathcal{D}^{-2}_{P(z)}\right)$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{3!}\left(\binom{3}{0}\mathcal{D}^{0}_{P(z)} - \binom{3}{1}\mathcal{D}^{-1}_{P(z)} + \binom{3}{2}\mathcal{D}^{-2}_{P(z)} - \binom{3}{3}\mathcal{D}^{-3}_{P(z)}\right)$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{4!}\left(\binom{4}{0}\mathcal{D}^{0}_{P(z)} - \binom{4}{1}\mathcal{D}^{-1}_{P(z)} + \binom{4}{2}\mathcal{D}^{-2}_{P(z)} - \binom{4}{3}\mathcal{D}^{-3}_{P(z)} + \binom{4}{4}\mathcal{D}^{-4}_{P(z)}\right)$$
$$+ \alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha-22)\cdot\text{Discriminant}_z(P(z)). \tag{3.9}$$

Equation (3.9) can be written more succinctly as

$$\mathcal{D}^{\alpha}_{P(z)} = \sum_{k=0}^{2\mathfrak{d}-4} \left(\prod_{j=1}^{k} \frac{\alpha+j-1}{j} \right) \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathcal{D}^{-j}_{P(z)} \right) + \text{Discriminant}_z(P(z))(\alpha-22) \prod_{j=0}^{2\mathfrak{d}-4} (\alpha+j). \tag{3.10}$$

Note that (3.10) is identical to equation (3.7) for cubic polynomials, i.e. $\mathfrak{d} = 3$, if the 22 is replaced by 9. If P is a quintic polynomial, i.e. $\mathfrak{d} = 5$, then equation (3.10) holds if the 22 is replaced by 41. For sextic P, the 22 should be replaced by 66. For general $\mathfrak{d} \geq 2$, this pattern indicates that

$$\begin{aligned} \mathcal{D}^{\alpha}_{P(z)} = &\sum_{k=0}^{2\mathfrak{d}-4} \left(\left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathcal{D}^{-j}_{P(z)} \right) \left(\prod_{j=1}^{k} \frac{\alpha+j-1}{j} \right) \right) \\ &+ \text{Discriminant}_z(P(z))(\alpha - 3\mathfrak{d}^2 + 8\mathfrak{d} - 6) \prod_{j=0}^{2\mathfrak{d}-4} (\alpha+j), \end{aligned} \tag{3.11}$$

where the sequence of numbers given by $3\mathfrak{d}^2 - 8\mathfrak{d} + 6$, $\mathfrak{d} = 1, 2, 3, \ldots$ corresponds to the first spoke of a hexagonal spiral; see sequence A056105 in the OEIS. By changing the upper limit of summation in (3.11) from $2\mathfrak{d} - 4$ to $2\mathfrak{d} - 2$, equation (3.11) can be written as

$$\mathcal{D}^{\alpha}_{P(z)} = \sum_{k=0}^{2\mathfrak{d}-2} \left(\left(\prod_{j=1}^{k} \frac{\alpha+j-1}{j} \right) \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathcal{D}^{-j}_{P(z)} \right) \right). \tag{3.12}$$

Now let E be the forward shift operator that changes $\mathcal{D}^{-j}_{P(z)}$ to $\mathcal{D}^{-j-1}_{P(z)}$, and let $\Delta = E - 1$ be the forward difference. Using this notation, and the fact that the inner sum in (3.12) is 0 if $k > 2\mathfrak{d} - 2$, it follows from (3.12) that

$$\begin{aligned} \mathcal{D}^{\alpha}_{P(z)} &= \sum_{k=0}^{\infty} \left(\left(\prod_{j=1}^{k} \frac{\alpha+j-1}{j} \right) \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathcal{D}^{-j}_{P(z)} \right) \right) \\ &= \sum_{k=0}^{\infty} \left(\left(\prod_{j=1}^{k} \frac{\alpha+j-1}{j} \right) (-\Delta)^k \, \mathcal{D}^{0}_{P(z)} \right) = (\Delta+1)^{-\alpha} \, \mathcal{D}^{0}_{P(z)}. \end{aligned} \tag{3.13}$$

The last equality in (3.13) follows when $(\Delta + 1)^{-\alpha}$ is interpreted as a power series acting on the vector space of polynomials in one variable. Furthermore, by applying the inverse Euler transform to the first series in (3.13) (see [Ko]), this yields the following result:

Proposition 6. *Let $P(z)$ be a polynomial of degree $\mathfrak{d} \geq 2$ and let $\mathcal{D}^{\alpha}_{P(z)} :=$ $\text{Discriminant}_u(\alpha P(u) + (z-u)P'(u))$. Then*

$$\mathcal{D}^{\alpha}_{P(z)} = E^{-\alpha} \, \mathcal{D}^{0}_{P(z)}, \tag{3.14}$$

where $E^{-\alpha} \, \mathcal{D}^{0}_{P(z)}$ denotes the application of the forward shift operators in the Taylor series for $E^{-\alpha}$ at $E = 1$ to $\mathcal{D}^{0}_{P(z)}$.

Proof. Note that an Euler sum of $E^{-\alpha} \, \mathcal{D}^{0}_{P(z)}$ is the series after the first equality in (3.13), that all series involved converge, and that, consequently, equations (3.12) and (3.14) are equivalent. Thus, to prove (3.14), it is sufficient to show that

$$\sum_{k=0}^{2p-2}\left(\left(\prod_{j=1}^{k}\frac{\lambda+j-1}{j}\right)\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j(-j)^p\right)\right)=\begin{cases}0 & \text{if } p<2,\\ \lambda^p & \text{if } p\geq 2.\end{cases} \tag{3.15}$$

for any fixed numbers $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$. (*** Some details about fibers here? ***) The cases $p = 0$ and $p = 1$ in equation (3.15) follow easily from direct calculations. When $p \geq 2$, we see that

$$\prod_{j=1}^{k}\frac{\lambda+j-1}{j}=\frac{(\lambda)_k}{k!}$$

and

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j(-j)^p=(-1)^p\sum_{j=0}^{k}\binom{k}{j}(-1)^j j^p=(-1)^{k+p}\,k!\left\{{p \atop k}\right\},$$

where $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ is the Pochhammer symbol and $\left\{{p \atop k}\right\}$ is a Stirling number of the second kind. Consequently, since $\left\{{p \atop k}\right\} = 0$ for all $k > p$, the left-hand side of equation (3.15) can be written as

$$(-1)^p\sum_{k=0}^{2p-2}(-1)^k\left\{{p \atop k}\right\}(\lambda)_k=(-1)^p\sum_{k=0}^{p}(-1)^k\left\{{p \atop k}\right\}(\lambda)_k=(-1)^p(-\lambda)^p=\lambda^p.$$

□

Remark 5. *Note that we can write* $E^{-\alpha}$ *as* $\sum_{k=0}^{\infty}\binom{-\alpha}{k}(E-1)^k$ *or* $\sum_{k=0}^{\infty}\binom{-\alpha}{k}\Delta^k$.

Remark 6. *Note that* $\mathcal{D}^{0}_{P(z)} = (P'(z))^2 \cdot \mathrm{Discriminant}_z(P'(z))$*, but no similarly "simple forms" of* $\mathcal{D}^{-k}_{P(z)}$ *are currently known for* $k = 1, 2, 3, \ldots$

By evaluating the inner sum in equation (3.12) for $k = 2\mathfrak{d} - 2$ and making use of the forward difference and forward shift operators, we obtain the following conjecture.

Conjecture 2. *Let* $P(z)$ *be a polynomial of degree* $\mathfrak{d} \geq 2$*. Then*

$$\mathrm{Discriminant}_z(P(z))=\frac{1}{(2\mathfrak{d}-2)!}\sum_{j=0}^{2\mathfrak{d}-2}\binom{2\mathfrak{d}-2}{j}(-1)^j\mathcal{D}^{-j}_{P(z)}=\frac{(E-1)^{2\mathfrak{d}-2}}{(2\mathfrak{d}-2)!}\mathcal{D}^{0}_{P(z)}, \tag{3.16}$$

and

$$\mathrm{Discriminant}_z(P(z))=\frac{1}{-3(\mathfrak{d}-1)^2\,(2\mathfrak{d}-3)!}\sum_{j=0}^{2\mathfrak{d}-3}\binom{2\mathfrak{d}-3}{j}(-1)^j\mathcal{D}^{-j}_{P(z)}. \tag{3.17}$$

Remark 7. *By averaging the sums in* (3.16) *and* (3.17)*, we get*

$$\mathrm{Discriminant}_z(P(z)) = \frac{1}{12(\mathfrak{d}-1)^2\,(2\mathfrak{d}-3)!}\sum_{j=0}^{2\mathfrak{d}-2}\left(3(\mathfrak{d}-1)\binom{2\mathfrak{d}-2}{j}-2\binom{2\mathfrak{d}-3}{j}\right)(-1)^j\mathcal{D}^{-j}_{P(z)}. \tag{3.18}$$

Remark 8. *It follows from* (3.16) *and* (3.17) *that*

$$\mathcal{D}^{-(2\mathfrak{d}-2)}_{P(z)}=\sum_{j=0}^{2\mathfrak{d}-3}\left(\frac{2}{3-3\mathfrak{d}}\binom{2\mathfrak{d}-3}{j}-\binom{2\mathfrak{d}-2}{j}\right)(-1)^j\mathcal{D}^{-j}_{P(z)}. \tag{3.19}$$

4. Final Remarks and open problems

1. Practically all the results of the present paper can be generalised to the case when f is a rational function instead of a polynomial which we plan to do in the future. However poles of a rational function restrict the possibility of deformation of the integration contour used in § 5. This leads to a more delicate situation which requires special analysis.

2. The set-up of the present paper can be randomised and generalized as follows. Let ξ be a probability measure compactly supported in $\mathbb{C}$. Denote by $P_n = \prod_{i=1}^{n}(x-\xi_i)$ a random polynomial of degree n whose roots are i.i.d. random variables sampled on ξ. Given a sequence $\mathcal{A} = \{\alpha_n\}$ of non-negative integers, set $Q_n = P_n^{(\alpha_n)}$ and denote by μ_n the root-counting measure of Q_n. Results from the recent papers [PeRi, Ka] motivate the following guess.

Conjecture 3. *In the above notation, the following two statements hold:*

(i) if $\frac{\alpha_n}{n} \to 0$, then the sequence $\{\mu_n\}$ converges in probability to ξ;

(ii) if $\frac{\alpha_n}{n} \to \alpha,\ 0 < \alpha < 1$, then the sequence $\{\mu_n\}$ converges in probability to a measure ξ_α whose support is contained in the convex hull of the support of ξ;

What we have done in the present paper can be interpreted in the above terms as follows. We start with a uniform discrete measure ξ supported on the d zeros of $P(z)$. Then we sample this measure evenly and deterministically nd times, by forming the series of polynomials $P_n(z) := P^n(z)$ and, finally, we differentiate $P_n(z)$ $[n\alpha]$ times. This produces a sequence of polynomials $\{Q_n(z)\}$ and the associated sequence of probability measures $\{\mu_n\}$. The proportion between the number of derivations and the number of sampled points is

$$A := \frac{\alpha}{d}n + O(1).$$

5. Appendix 1. Necessary results from [BoHaSh]

In what follows, we will always assume that a polynomial $P(z)$ under consideration satisfies the condition $\mathfrak{d} := \deg P \geq 2$. The remaining case $\mathfrak{d} = 1$ is trivial.

For any polynomial P and its Rodrigues' descendant $\mathcal{R}_{m,n,P}(z)$, denote by $\mu_{m,n,P}$ the *root-counting measure* of $\mathcal{R}_{m,n,P}(z)$ and by

$$\mathcal{C}_{m,n,P}(z) := \frac{\mathcal{R}'_{m,n,P}(z)}{(\mathfrak{d}n - m)\cdot \mathcal{R}_{m,n,P}(z)}$$

its *Cauchy transform*. Notice that $\mathfrak{d}n - m = \deg \mathcal{R}_{m,n,P}$. (For the used basic definitions from potential theory consult § 6 and [Ra].)

Theorem 7. *For any polynomial P and a given positive number $\alpha < \mathfrak{d}$, there exists a weak limit*

$$\mu_{\alpha,P} = \lim_{n\to\infty} \mu_{[\alpha n],n,P}.$$

Moreover, its Cauchy transform $\mathcal{C}_{\alpha,P}$ defined as the pointwise limit

$$\mathcal{C} := \mathcal{C}_{\alpha,P}(z) := \lim_{n\to\infty} \mathcal{C}_{[\alpha n],n,P}(z)$$

exists almost everywhere (a.e.) in $\mathbb{C}$ and satisfies the algebraic equation:

$$\sum_{k=0}^{d} \frac{\alpha^{k-1}\,(\alpha-k)(\mathfrak{d}-\alpha)^{\mathfrak{d}-k}}{k!}\, P^{(k)}\mathcal{C}^{\mathfrak{d}-k} = 0. \tag{5.1}$$

Corollary 2. *The scaled Cauchy transform $\mathcal{W}$ defined by*

$$\mathcal{W} := \mathcal{W}_{\alpha,P} := \frac{\mathfrak{d}-\alpha}{\alpha}\mathcal{C}_{\alpha,P} \tag{5.2}$$

satisfies a simpler algebraic equation:

$$\sum_{k=0}^{\mathfrak{d}} \frac{\alpha-k}{k!}\, P^{(k)}\mathcal{W}^{\mathfrak{d}-k} = 0. \tag{5.3}$$

Corollary 3. *Equations* (5.1) *and* (5.3) *define rational affine curves which are irreducible if and only if all roots of P are simple. If P has roots of multiplicity at least 2, then* (5.3) *admits a finite number of "trivial" factors of the form $\mathcal{W} = (b-z)^{-1}$, where b is such a root. The remaining factor of* (5.3) *is irreducible and can be written as*

$$\alpha\mathcal{W} = \frac{P'\left(z+\mathcal{W}^{-1}\right)}{P\left(z+\mathcal{W}^{-1}\right)} = \frac{\mathrm{d}\log P(z+\mathcal{W}^{-1})}{\mathrm{d}z}. \tag{5.4}$$

Remark 9. Equation (5.4) implies the following relation satisfied by the original Cauchy transform

$$(\mathfrak{d}-\alpha)\mathcal{C} = \frac{\mathrm{d}\log P(z+\frac{\alpha}{\mathfrak{d}-\alpha}\mathcal{C}^{-1})}{\mathrm{d}z}. \tag{5.5}$$

Remark 10. Observe that, for any $0 < \alpha < \mathfrak{d}$, the support $S_{\alpha,P}$ of $\mu_{\alpha,P}$ is contained in the convex hull of the zero locus of the polynomial P.

Corollary 4. *For any polynomial P and $0 < \alpha < \mathfrak{d}$, the support $S_{\alpha;P}$ of $\mu_{\alpha;P}$ consists of finitely many compact semi-analytic curves and points. The measure $\mu_{\alpha;P}$ has point masses if and only if $\alpha < 1$.*

6. Appendix 2. Basics of logarithmic potential theory

For the convenience of our readers, let us briefly recall some notions and facts used throughout the text. Let μ be a finite compactly supported complex measure in the complex plane $\mathbb{C}$. Define the *logarithmic potential* of μ as

$$L_\mu(z) := \int_{\mathbb{C}} \ln|z-\xi|\,\mathrm{d}\mu(\xi)$$

and the *Cauchy transform* of μ as

$$\mathcal{C}_\mu(z) := \int_{\mathbb{C}} \frac{\mathrm{d}\mu(\xi)}{z-\xi}.$$

Standard facts about the logarithmic potential and the Cauchy transform include:

- $\mathcal{C}_\mu$ and L_μ are locally integrable; in particular they define distributions on $\mathbb{C}$ and therefore can be acted upon by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.

- $\mathcal{C}_\mu$ is analytic in the complement in $\mathbb{C}P^1 \simeq \mathbb{C} \cup \{\infty\}$ to the support of μ. For example, if μ is supported on the unit circle (which is the most classical case), then $\mathcal{C}_\mu$ is analytic both inside the open unit disc and outside the closed unit disc.
- the relations between μ, $\mathcal{C}_\mu$ and L_μ are as follows:
$$\mathcal{C}_\mu = 2\frac{\partial L_\mu}{\partial z} \quad \text{and} \quad \mu = \frac{1}{\pi}\frac{\partial \mathcal{C}_\mu}{\partial \bar z} = \frac{2}{\pi}\frac{\partial^2 L_\mu}{\partial z \partial \bar z} = \frac{1}{2\pi}\left(\frac{\partial^2 L_\mu}{\partial x^2} + \frac{\partial^2 L_\mu}{\partial y^2}\right).$$
They should be understood as equalities of distributions.
- the Laurent series of $\mathcal{C}_\mu$ in a neighborhood of ∞ is given by
$$\mathcal{C}_\mu(z) = \frac{m_0(\mu)}{z} + \frac{m_1(\mu)}{z^2} + \frac{m_2(\mu)}{z^3} + \dots,$$
where
$$m_k(\mu) = \int_{\mathbb{C}} z^k \, \mathrm{d}\mu(z), \; k = 0, 1, \dots$$
are the harmonic moments of measure μ.

Given a polynomial p, we associate to p its standard *root-counting measure*
$$\mu_p = \frac{1}{\deg p} \sum_i m_i \delta(z - z_i),$$
where the sum is taken over all distinct roots z_i of p and m_i is the multiplicity of z_i.

One can easily check that the Cauchy transform of μ_p is given by
$$\mathcal{C}_{\mu_p} = \frac{1}{\deg p} \cdot \frac{p'}{p}.$$

For more relevant information on the Cauchy transform we will probably recommend a short and well-written treatise [Ga].

The above notions of a complex measure μ compactly supported in $\mathbb{C}$, its logarithmic potential L_μ, and its Cauchy transform $\mathcal{C}_\mu$ have natural extensions to similar objects $\bar\mu, \bar L_{\bar\mu}, \bar{\mathcal{C}}_{\bar\mu}$ defined on $\mathbb{C}P^1 \supset \mathbb{C}$ and such that the main relations between these objects are preserved. They are constructed as follows.

(i) For a finite complex measure μ compactly supported in $\mathbb{C}$, we introduce the signed measure $\bar\mu$ of total mass 0 defined on $\mathbb{C}P^1$ by adding to μ the point measure $-\mathfrak{m} \cdot \delta(\infty)$ placed at ∞, where $\mathfrak{m} = \int_{\mathbb{C}} \mathrm{d}\mu$. (It is natural to think of $\bar\mu$ as an exact 2-current on $\mathbb{C}P^1$.)

(ii) The logarithmic potential L_μ is defined as a function on $\mathbb{C} \subset \mathbb{C}P^1$ with a logarithmic singularity at ∞. In terms of a local coordinate $w = 1/z$ at ∞ the logarithmic potential is L^1_{loc} near ∞, and this implies that we may talk about its derivatives. In the following when we think of L_μ as an object on $\mathbb{C}P^1$ we denote it by $\bar L_{\bar\mu}$.

Recall that on any complex manifold the exterior differential d (acting on currents) is standardly decomposed as $d = d' + d''$, where d' is the holomorphic part and d'' is the anti-holomorphic part. For a function f on a Riemann surface with a local coordinate z, we get
$$\mathrm{d}'f = \frac{\partial f}{\partial z}\,\mathrm{d}z \quad \text{and} \quad \mathrm{d}''f = \frac{\partial f}{\partial \bar z}\,\mathrm{d}\bar z.$$

The above quantities $\bar{\mu}$ and $\bar{L}_{\bar{\mu}}$ satisfy the relation

$$\bar{\mu}\ \mathrm{d}x \wedge \mathrm{d}y = \frac{i}{\pi} d' d'' \bar{L}_{\bar{\mu}}.$$

More explicitly, we have that

$$\bar{\mu}\,\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2\pi}\left(\frac{\partial^2 \bar{L}_{\bar{\mu}}}{\partial x^2} + \frac{\partial^2 \bar{L}_{\bar{\mu}}}{\partial y^2}\right)\mathrm{d}x \wedge \mathrm{d}y = \frac{2}{\pi}\frac{\partial^2 \bar{L}_{\bar{\mu}}}{\partial z \partial \bar{z}}\ \mathrm{d}x \wedge \mathrm{d}y = \frac{i}{\pi}\frac{\partial^2 \bar{L}_{\bar{\mu}}}{\partial z \partial \bar{z}}\ \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where $\frac{\partial^2 \bar{L}_{\bar{\mu}}}{\partial z \partial \bar{z}}$ is understood as a distribution on $\mathbb{C}P^1$.

(iii) Finally, the Cauchy transform $\bar{\mathcal{C}}_{\bar{\mu}}$ should be understood as an 1-current given by the relation

$$\bar{\mathcal{C}}_{\bar{\mu}} = 2\,\mathrm{d}' \bar{L}_{\bar{\mu}} = 2\frac{\partial \bar{L}_{\bar{\mu}}}{\partial z}\ \mathrm{d}z.$$

Then

$$\bar{\mu}\,\mathrm{d}x \wedge \mathrm{d}y = \frac{i}{\pi}\,\mathrm{d}'\,\mathrm{d}'' \bar{L}_{\bar{\mu}} = -\frac{i}{2\pi}\,\mathrm{d}'' \bar{\mathcal{C}}_{\bar{\mu}} = \frac{i}{2\pi}\frac{\partial \bar{\mathcal{C}}_{\bar{\mu}}}{\partial \bar{z}}\ \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

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