PARAMETRIC POINCARÉ-PERRON THEOREM WITH APPLICATIONS

By

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Abstract. We prove a parametric generalization of the classical Poincaré-Perron theorem on stabilizing recurrence relations, where we assume that the varying coefficients of a recurrence depend on auxiliary parameters and converge uniformly in these parameters to their limiting values. As an application, we study convergence of the ratios of families of functions satisfying finite recurrence relations with varying functional coefficients. For example, we explicitly describe the asymptotic ratio for two classes of biorthogonal polynomials introduced by Ismail and Masson.

1 Introduction

Consider a usual linear recurrence relation of length $k+1$, with constant coefficients

$$u_{n+1} + \alpha_1 u_n + \alpha_2 u_{n-1} + \ldots + \alpha_k u_{n-k+1} = 0, \tag{1.1}$$

with $\alpha_k \neq 0$.

Definition 1. The left-hand side of the equation

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \ldots + \alpha_k = 0 \tag{1.2}$$

is called the **characteristic polynomial** of recurrence (1.1). Denote the roots of (1.2) by $\lambda_1, \ldots, \lambda_k$ and call them the **spectral numbers** of the recurrence.

The following simple theorem can be found, e.g., in [20, Ch. 4].

Theorem 1. *Let $k \in \mathbb{N}$ and consider a k-tuple $(\alpha_1, \ldots, \alpha_k)$ of complex numbers, with $\alpha_k \neq 0$. For any function $u : \mathbb{Z}_+ \to \mathbb{C}$, the following conditions are equivalent.*

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*J. B. passed away unexpectedly on April 8, 2009 at the age of 40. We dedicate this paper (started jointly with J. B. in Spring 2004) to the memory of this talented and tragic human being. Rest in peace, Julius.

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(i) $\sum_{n\geq 0} u_n t^n = P(t)/Q(t)$, *where* $Q(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k$ *and* $P(t)$ *is a polynomial in t whose degree is smaller than k.*

(i) *For all* $n \geq k-1$, *the function* u_n *satisfies relation* (1.1).

(ii) *For all* $n \geq 0$,

$$u_n = \sum_{i=1}^{r} P_i(n)\lambda_i^n, \tag{1.3}$$

where $\lambda_1, \ldots, \lambda_r$ *are the distinct spectral numbers of* (1.1) *with multiplicities* $m_1, \ldots, m_r$, *respectively, and* $P_i(n)$ *is a polynomial in the variable n of degree at most* $m_i - 1$, *for* $1 \leq i \leq r$.

The k-tuple $(u_0, \ldots u_{k-1})$ that can be chosen arbitrarily is called the **initial k-tuple**. Denote the k-dimensional space of all initial k-tuples by $\mathbb{C}^k$.

Definition 2. Recurrence relation (1.1) and its characteristic polynomial (1.2) are called **maxmod-generic** if there exists a unique and simple spectral number λ_{max} of this recurrence satisfying $|\lambda_{max}| = \max_{1\leq i\leq k} |\lambda_i|$. Otherwise, (1.1) and (1.2) are called **maxmod-nongeneric**. The number λ_{max} is referred to as the **leading spectral number** of (1.1) or the **leading root** of (1.2).

Definition 3. An initial k-tuple of complex numbers $(u_0, u_1, \ldots, u_{k-1}) \in \mathbb{C}^k$ is called **fast growing** with respect to a given maxmod-generic recurrence (1.1) if the coefficient κ_{max} of λ_{max}^n in (1.3) is nonvanishing; that is, $u_n = \kappa_{max}\lambda_{max}^n + \ldots$ with $\kappa_{max} \neq 0$. Otherwise, the k-tuple $(u_0, u_1, \ldots, u_{k-1})$ is said to be **slow growing**.

Remark 1. Note that by Definition 2, the leading spectral number λ_{max} of any maxmod-generic recurrence has multiplicity one. An alternative characterization of fast growing initial k-tuples is that they have the property $\lim_{n\to\infty} u_{n+1}/u_n = \lambda_{max}$. One easily sees that the set of all slowly growing initial k-tuples is a (complex) hyperplane in $\mathbb{C}^k$, its complement being the set of all fast growing k-tuples. The latter hyperplane of slow growing k-tuples can be found explicitly using linear algebra.

A famous and frequently used generalization of Theorem 1 in the case of variable coefficients was obtained by H. Poincaré [18], in 1885, and later extended by O. Perron [17].

Theorem 2 (Poincaré-Perron)**.** *If the coefficients* $\alpha_{i,n}$, $i = 1, \ldots, k$ *of a linear homogeneous difference equation*

$$u_{n+k} + \alpha_{1,n}u_{n+k-1} + \alpha_{2,n}u_{n+k-2} + \cdots + \alpha_{k,n}u_n = 0 \tag{1.4}$$

have limits $\lim_{n\to\infty} \alpha_{i,n} = \alpha_i, \quad i = 1, \ldots, k$, *and if the roots* $\lambda_1, \ldots, \lambda_k$ *of the characteristic equation* $t^k + \alpha_1 t^{k-1} + \cdots + \alpha_k = 0$ *have distinct absolute values, then*

(i) *for any solution* u *of* (1.4)*, either* $u(n) = 0$ *for all sufficiently large* n *or* $\lim_{n\to\infty} u(n+1)/u(n)$ *is one of the roots of the characteristic equation.*

(ii) *If additionally* $\alpha_{k,n} \neq 0$ *for all* n*, then for every* λ_i *there exists a solution* u *of* (1.4) *with* $\lim_{n\to\infty} u(n+1)/u(n) = \lambda_i$.

Remark 2. If, as above, λ_{max} denotes the root of the limiting characteristic equation, with the maximal absolute value, then under the assumptions of Theorem 2 (ii), the set of solutions of (1.4) for which $\lim_{x\to\infty} u(n+1)/u(n) \neq \lambda_{max}$ is a complex hyperplane in the space of all solutions. For the latter fact to hold, the assumption that all λ_i's have distinct absolute values can be replaced with the weaker assumption of maxmod-genericity of the limiting recurrence; see details in Lemma 3. But, in general, there seems to be no easy way to determine this hyperplane explicitly.

A number of generalizations and applications of the Poincaré-Perron Theorem can be found in the literature; see e.g. [11], [12], [14], [16], [19] and references therein. The set up of Poincaré-Perron is often generalized to the case of Poincaré difference systems; namely, consider an iteration scheme

$$\mathbf{u}(n+1) = [A + B(n)]\,\mathbf{u}(n), \tag{1.5}$$

where $\mathbf{u}(n)$ is a vector in $\mathbb{C}^k$, A and $B(n)$, $n = 0, 1, \ldots$ are $k \times k$-matrices such that $||B(n)|| \to 0$ as $n \to \infty$. For example, one quite recent result in this direction having a very strong resemblance to Theorem 2 is the following; cf. [19, Theorem 1].

Theorem 3. *Assume that* $||B(n)|| \to 0$ *as* $n \to \infty$ *and let* $\mathbf{u}$ *be a solution of the system* (1.5)*. Then, either* $\mathbf{u}(n) = 0$ *for all sufficiently large* n *or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{||u(n)||}$$

exists and is equal to the modulus of one of the eigenvalues of the matrix A.

In connection with the present project, the second author earlier obtained [5, Theorem 1.2] the following generalization of the Poincare-Perron theorem for the case of Poincaré difference systems, which apparently covers the majority of results in this direction known at present. Let $\mathrm{M_k}(\mathbb{F})$ and $\mathrm{GL_k}(\mathbb{F})$ denote the spaces of all and all invertible $(k \times k)$-matrices over a field $\mathbb{F}$, respectively.

Theorem 4. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of regular matrices in* $\mathrm{GL}_k(\mathbb{C})$ *converging to some (possibly singular) matrix $T \in \mathrm{M}_k(\mathbb{C})$. Assume, furthermore, that T has a positive spectral radius $\rho(T)$ and that the circle $\{z \in \mathbb{C} \mid |z| = \rho(T)\}$ contains exactly one eigenvalue λ_{max} of T that is a simple root of its characteristic equation. Let $\mathbf{u}_{max}$ denote an eigenvector of T corresponding to λ_{max}; i.e., $T\mathbf{u}_{max} = \lambda_{max}\mathbf{u}_{max}$, $\mathbf{0} \neq \mathbf{u}_{max} \in \mathbb{C}^k$. Then the complex line spanned by the product $T_nT_{n-1}\cdots T_1 \in \mathrm{M}_k(\mathbb{C})$ converges to the complex line spanned by $\mathbf{u}_{max}\mathbf{w}^t \in \mathrm{M}_k(\mathbb{C})$, for some fixed vector $\mathbf{0} \neq \mathbf{w} \in \mathbb{C}^k$. Hence, for any vector $\mathbf{x}_0 \in \mathbb{C}^k$ such that $\mathbf{w}^t\mathbf{x}_0 \neq 0$, the complex line in $\mathbb{C}^k$ spanned by $T_nT_{n-1}\cdots T_1\mathbf{x}_0$ converges to the complex line in $\mathbb{C}^k$ spanned by $\mathbf{u}_{max}$ as $n \to \infty$.*

Remark 3. As with Theorem 2, there seems to be no easy way to determine the vector $\mathbf{w}$ in Theorem 4 explicitly.

The goal of this paper is to present an extension of Theorem 4 for sequences of invertible matrices depending continuously or analytically on auxiliary parameters. In other words, we are looking for a *parametric Poincaré-Perron theorem.* Our main result is the following.

Theorem 5. *Let $\{T_n(\mathbf{x})\}_{n\in\mathbb{N}}$ be a sequence of families of regular matrices in* $\mathrm{GL}_k(\mathbb{C})$ *that depend continuously on $\mathbf{x} \in \mathcal{D} \subset \mathbb{R}^d$. Assume that this sequence converges uniformly, on any compact set in $\mathcal{D}$, to a matrix $T(\mathbf{x})$. Suppose furthermore that for each $\mathbf{x} \in \mathcal{D}$, $T(\mathbf{x})$ has exactly one simple eigenvalue $\lambda_{max}(\mathbf{x})$ of maximal modulus and $\hat{\mathbf{u}}_{max}(\mathbf{x}) \in \mathbb{P}^{k-1}$ is the corresponding eigenvector. Then the product $T_n(\mathbf{x})T_{n-1}(\mathbf{x})\ldots T_2(\mathbf{x})T_1(\mathbf{x})$, viewed as an automorphism of $\mathbb{P}^{k-1}$, converges to the transformation $\hat{\mathbf{u}}_{max}(\mathbf{x})\hat{\mathbf{w}}^{\mathrm{t}}(\mathbf{x})$ on $\mathbb{P}^{k-1}\backslash\widehat{H(\mathbf{x})}$, where $\hat{\mathbf{w}}(\mathbf{x}) \in \mathbb{P}^{k-1}$ is continuous in $\mathcal{D}$ and $H(\mathbf{x})$ is the hyperplane given by $\mathbf{w}(\mathbf{x})^{\mathrm{t}}\mathbf{v} = 0$. Here $\mathbb{P}^{k-1}$ is the complex projective space of dimension $k-1$.*

Remark 4. In Theorem 5, we consider continuous dependence of $\{T_n(\mathbf{x})\}$ on $\mathbf{x}$ and uniform convergence of this sequence to the limiting $T(\mathbf{x})$. The same result holds in the analytic category if $\{T_n(\mathbf{x})\}$ depends analytically on $\mathbf{x}$ and converges uniformly to $T(\mathbf{x})$. One can also get the same result in the smooth category under the assumption that $\{T_n(\mathbf{x})\}$ together with its partial derivatives of all orders, depends smoothly on $\mathbf{x}$ and converges uniformly to $T(\mathbf{x})$. Since we consider families of complex matrices depending continuously on parameters, we use $\mathbb{R}^d$ as a parameter space. Choosing the parameters to belong to $\mathbb{C}^d$ would be equally natural.

Remark 5. In a sense, Theorems 4 and 5 can be considered a far reaching and parametric generalization of the well-known power method in linear algebra, which is a simple iterative procedure allowing one to determine the dominating eigenvalue and eigenvector of a given square matrix possessing a unique eigenvalue with maximal modulus. see, e.g., [25]. On the other hand, the fact that we consider sequences of matrices (instead of one and the same matrix) which, in addition, depend on extra parameters creates substantial technical difficulties.

In spite of its simple formulation, Theorem 5 seems to have no prototypes in the existing literature. We can now apply Theorems 2, 4 and 5 to sequences of functions satisfying finite linear recurrence relations. The set up is as follows.

Problem. Given a positive integer $k \geq 2$, let $\{\phi_{i,n}(\mathbf{x})\}$, $1 \leq i \leq k, n \in \mathbb{Z}_+$, be a sequence of k-tuples of complex-valued functions of a (multi-)variable $\mathbf{x} = (x_1, ..., x_d)$, defined in some domain $\Omega \subseteq \mathbb{R}^d$. Describe the *asymptotics* when $n \to \infty$ of the ratio $\Psi_n(\mathbf{x}) = f_{n+1}(\mathbf{x})/f_n(\mathbf{x})$ for a family of complex-valued functions $\{f_n(\mathbf{x}) \mid n \in \mathbb{Z}_+\}$ satisfying

$$f_{n+k}(\mathbf{x}) + \sum_{i=1}^{k} \phi_{i,n}(\mathbf{x}) f_{n+k-i}(\mathbf{x}) = 0, \quad n \geq k-1. \tag{1.6}$$

In other words, given a family $\{f_n(\mathbf{x})\}$ of functions satisfying (1.6), calculate the asymptotic ratio $\Psi(\mathbf{x}) = \lim_{n\to\infty} \Psi_n(\mathbf{x})$ (if it exists).

To formulate our results further, we need some notions. Denote by Pol_k the set $\left\{t^k + a_1 t^{k-1} + \ldots + a_k \mid a_i \in \mathbb{C}, 1 \leq i \leq k\right\}$ of all monic polynomials of degree k with complex coefficients.

Definition 4. The subset $\Xi_k \subset Pol_k$ consisting of all maxmod-nongeneric polynomials is called the **standard maxmod-discriminant**; see Definition 2. For any family

$$\Gamma(t, \alpha_1, \ldots, \alpha_q) = \left\{t^k + a_1(\alpha_1, \ldots, \alpha_q)t^{k-1} + \ldots + a_k(\alpha_1, \ldots, \alpha_q)\right\}$$

of monic polynomials of degree k in t ,we define the **induced maxmod-discriminant** Ξ_Γ to be the set of all parameter values $(\alpha_1, \ldots, \alpha_q) \in \mathbb{C}^q$ for which the corresponding polynomial in t is maxmod-nongeneric, i.e. belongs to Ξ_k.

Some local properties of Ξ_k can also be derived from [3] and [15].

Example. For $k = 2$, the maxmod-discriminant $\Xi_2 \subset Pol_2$ is the real hypersurface consisting of the set of all pairs (a_1, a_2) such that there exists $\epsilon \in [1, \infty)$ that solves the equation $\epsilon a_1^2 - 4a_2 = 0$; (see Lemma 7). More information on Ξ_k is given in Section 6.

Now take a family

$$\left\{\overline{\phi}_n := \big(\phi_{1,n}(\mathbf{x}), \phi_{2,n}(\mathbf{x}), \dots, \phi_{k,n}(\mathbf{x})\big) \mid n \in \mathbb{Z}_+\right\}, \ \mathbf{x} = (x_1, ..., x_d)$$

of k-tuples of complex-valued functions defined on some domain $\Omega \subset \mathbb{R}^d$ such that

(i) $\phi_{k,n}(\mathbf{x})$ is non-vanishing in Ω;

(ii) $\overline{\phi}_n$ converges to a fixed k-tuple of functions $\widetilde{\phi} = \big(\widetilde{\phi}_1(\mathbf{x}), \widetilde{\phi}_2(\mathbf{x}), \dots, \widetilde{\phi}_k(\mathbf{x})\big)$ pointwise in Ω.

Choose some initial k-tuple of functions $I = \big(f_0(\mathbf{x}), \dots, f_{k-1}(\mathbf{x})\big)$, defined on Ω, and determine the family of functions $\{f_n(\mathbf{x}) \mid n \in \mathbb{Z}_+\}$ that satisfies the recurrence relation (1.6) for all $n \geq k$ and coincides with $I = \big(f_0, \dots, f_{k-1}\big)$ for all $0 \leq n \leq k-1$.

Theorems 2, 4 and 5 imply the following result.

Theorem 6. *In the above notation, there exists a unique subset $\Sigma_I \subseteq \Omega \setminus \Xi_{\widetilde{\phi}}$ that is minimal with respect to inclusion and such that the following holds.*

(i) *For each $\mathbf{x} \in \Omega \setminus (\Xi_{\widetilde{\phi}} \cup \Sigma_I)$,*

$$\lim_{n\to\infty} \frac{f_{n+1}(\mathbf{x})}{f_n(\mathbf{x})} = \Psi_{max}(\mathbf{x}),$$

where $\Psi_{max}(\mathbf{x})$ is the leading root of the **asymptotic symbol equation**

$$\Psi^k + \widetilde{\phi}_1(\mathbf{x})\Psi^{k-1} + \widetilde{\phi}_2(\mathbf{x})\Psi^{k-2} + \dots + \widetilde{\phi}_k(\mathbf{x}) = 0 \tag{1.7}$$

and $\Xi_{\widetilde{\phi}}$ denotes the induced maxmod-discriminant of (1.7)*, considered as a family of monic polynomials in the variable Ψ; (cf. Definition 4).*

(ii) *If $\overline{\phi}_n$ consists of continuous functions and $\overline{\phi}_n \rightrightarrows \widetilde{\phi} = \big(\widetilde{\phi}_1(\mathbf{x}), \widetilde{\phi}_2(\mathbf{x}), \dots, \widetilde{\phi}_k(\mathbf{x})\big)$ in Ω, then*

$$\frac{f_{n+1}(\mathbf{x})}{f_n(\mathbf{x})} \rightrightarrows \Psi_{max}(\mathbf{x}) \ \text{in} \ \Omega \setminus (\Xi_{\widetilde{\phi}} \cup \Sigma_I),$$

where $\rightrightarrows$ stands for uniform convergence on compact subsets in Ω.

Remark 6. Notice that since we assume that convergence in (ii) is uniform, each $\widetilde{\phi}_i(\mathbf{x})$ is continuous in Ω. Moreover, since a uniformly convergent sequence of analytic functions necessarily converges to a analytic function, (ii) holds in the analytic category as well. To get the latter result in the smooth category requires the uniform convergence $\overline{\phi}_n \rightrightarrows \widetilde{\phi}$ together with uniform convergence of partial derivatives of all orders.

Definition 5. The set Σ_I introduced in Theorem 6 is called the **set of slowly growing initial conditions**; cf. Definition 3. If the functional coefficients in

(1.6) are fixed (i.e., independent of n), then Σ_I is exactly the set of all points $p \in \Omega$ such that the initial k-tuple $I(p) = \big(f_0(p), \dots, f_{k-1}(p)\big)$ is slowly growing with respect to the recurrence (1.6) evaluated at p.

Remark 7. The exact description of Σ_I in the case of varying coefficients can be found in the proof of Theorem 6; see Section 2. Note that convergence of $(f_{n+1}(\mathbf{x}))/(f_n(\mathbf{x}))$ to the leading root of (1.7) might actually occur at some points $\mathbf{x}$ lying in $\Xi_{\tilde{\phi}}$, where the leading root is still unique but multiple.

When both the coefficients $\phi_{i,n}(z)$ and the functions $f_n(z)$ are complex analytic, one can associate to each meromorphic ratio $(f_{n+1}(z))/(f_n(z))$ the following useful complex-valued distribution ν_n; see [1, p. 249]. (Since we only use this distribution in one-dimensional case, we define it for $\mathbb{C}$. For the multi-dimensional case, cf. e.g. [23].)

Definition 6. Given a meromorphic function g in some open set $\Omega \subseteq \mathbb{C}$, we construct its (complex-valued) **residue distribution** ν_g as follows. Let $\{z_m \mid m \in \mathbb{N}\}$ be the (finite or infinite) set of all the poles of g in Ω. Assume that the Laurent expansion of g at z_m has the form $g(z) = \sum_{-\infty < l \le l_m} T_{m,l}/(z - z_m)^l$. Then the distribution ν_g is given by

$$\nu_g = \sum_{m\ge 1} \left(\sum_{1\le l\le l_m} \frac{(-1)^{l-1}}{(l-1)!} T_{m,l} \frac{\partial^{l-1}}{\partial z^{l-1}} \delta_{z_m} \right), \tag{1.8}$$

where δ_{z_m} is the Dirac mass at z_m. The above sum is meaningful, as a distribution on Ω, since it is locally finite there.

Remark 8. The distribution ν_g is a complex-valued *measure* if and only if g has all simple poles; see [1, p. 250]. If the latter holds, then in the notation of Definition 6, the value of this complex measure at z_m equals $T_{m,1}$; i.e., the residue of g at z_m.

Definition 7. If $\{f_n(z)\}$ consists of functions that are analytic in Ω and ν_n denotes the residue distribution of the meromorphic function $(f_{n+1}(z))/(f_n(z))$ in Ω, then the limit $\nu = \lim_{n\to\infty} \nu_n$ (if it exists in the sense of weak convergence) is called the **asymptotic ratio distribution** of the family $\{f_n(z)\}$.

Remark 9. Observe that the support of ν describes the asymptotics of the zero loci of the family $\{f_n(z)\}$.

Proposition 1. *Under the assumptions of Theorem 6 (ii), the following hold in the complex analytic category.*

(i) *The support of* ν *belongs to* $\Xi_{\tilde{\phi}}$, *where, as before,* $\Xi_{\tilde{\phi}}$ *denotes the induced maxmod-discriminant. The set* Σ_I *of slowly growing initial conditions is irrelevant to the support of* ν.

(ii) *Suppose that there exists a nonisolated point* $p_0 \in \Xi_{\tilde{\phi}}$ *such that equation* (1.7), *considered at* p_0, *has the property that among its roots with maximal absolute value, there exist at least two roots with the same maximal multiplicity. If the sequence* $\{(f_{n+1}(p_0))/(f_n(p_0)) \mid n \in \mathbb{Z}_+\}$ *diverges, then the support of* ν *coincides with* $\Xi_{\tilde{\phi}}$.

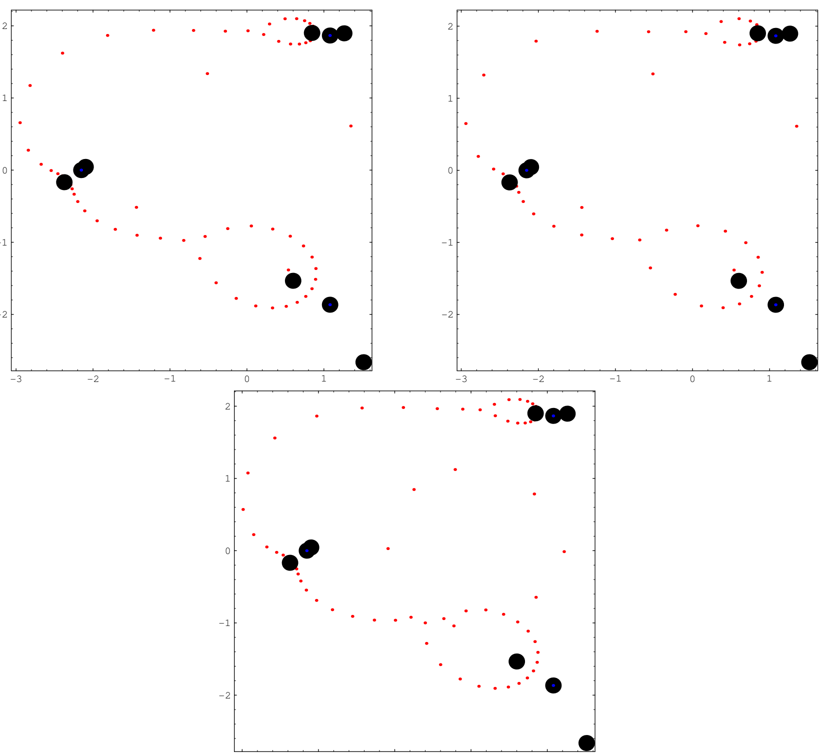

Figure 1. Zeros of polynomials satisfying the 4-term recurrence relation $p_{n+1}(z) = p_n(z) + (z+1)(z-i)p_{n-1}(z) + (z^3+10)p_{n-2}(z)$.

Let us illustrate the latter result in a concrete situation. Consider a sequence of polynomials $\{p_n(z)\}$ satisfying the 4-term recurrence relation

$$p_{n+1}(z) = p_n(z) + (z+1)(z-i)p_{n-1}(z) + (z^3+10)p_{n-2}(z),$$

with fixed coefficients, and starting with some initial triple of polynomials $I = (p_0(z), p_1(z), p_2(z))$. Consider the sequence $\{\mathbb{Z}(p_n)\}$ of the zero loci of $p_n(z)$. Then one can roughly divide the zeros in $\mathbb{Z}(p_n)$ into 2 parts. The zeros in the first part fill (when $n \to \infty$) the maxmod-discriminant $\Xi_{\widetilde{\phi}} \subset \mathbb{C}$, which is a continuous curve. The second part, consisting of finitely many points, depends of the initial triple and represents the set Σ_I of slowly growing conditions.

Explanations to Figure 1. The two upper pictures show the zeros of $p_{64}(z)$ and $p_{45}(z)$, for the same initial triple $p_0(z) = 0,\ p_1(z) = z^4 - 5i,\ p_2(z) = z$. The lower picture shows the zeros of $p_{64}(z)$, for the same recurrence relation, but with another initial triple $p_0(z) = z^8 - z^5 + i,\ p_1(z) = z - 5i,\ p_2(z) = 5iz^2 + z - 10$. Observe that on all three pictures, the zeros split into two parts, where the first part forms a pattern close to a smooth curve $\Xi_{\widetilde{\phi}}$ and a the second part consists of a number of isolated points. In the upper pictures, there are four isolated points, which practically coincide on both pictures, although the polynomials themselves are coprime. On the lower picture, there are seven isolated points, which also form a very stable set, as these points hardly change, if one takes different polynomials $p_n(z)$, with the same initial triple. The nine fat points in these three pictures are the branching points of the symbol equation $\Psi^3 = \Psi^2 + (z+1)(z-i)\Psi + (z^3 + 10)$.

Remark 10. The isolated points on Figure 1 have a strong resemblance to the spurious poles that were considered in a substantial number of papers on Padé approximation; see [21] and references therein. The study of the exact relation between these two objects can be found in [2].

This paper is organized as follows. In Section 2, we prove our parametric Poincaré-Perron theorem. We settle the remaining results in Section 3. In Section 4, we consider concrete examples of 3-term recurrence relations, with polynomial coefficients related to the theory of biorthogonal polynomials [10]. In Section 5, we discuss a number of related topics and open problems. Finally, in the appendix, we study the topological structure of the standard maxmod-discriminant $\Xi_k \subset Pol_k$.

2 Proving a parametric Poincaré-Perron theorem

For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, denote by $\mathbb{F}^k, M_k(\mathbb{F}), \mathrm{GL}_k(\mathbb{F})$ the k-dimensional vector space, the algebra of $(k \times k)$-matrices and the subgroup of $k \times k$ invertible matrices over the field $\mathbb{F}$, respectively. Denote by $||\cdot||$ any vector norm on $\mathbb{F}^k$ or on $M_k(\mathbb{F})$. Let $||\cdot||_2$ be the ℓ_2-norm on $\mathbb{F}^k$ induced by the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{y}^*\mathbf{x}$ on $\mathbb{F}^k$ and denote by $||\cdot||_2$ the induced operator norm on $M_k(\mathbb{F})$. For $T \in M_k(\mathbb{C})$, denote by $\rho(T)$ the spectral radius of T, i.e., the maximal modulus of the eigenvalues of T. As in [6], the spaces $\mathbb{PF}^k$, $PM_k(\mathbb{F})$, and $P\mathrm{GL}_k(\mathbb{F})$ are obtained by identifying the orbits of the action of $\mathbb{F}^* := \mathbb{F}\backslash\{0\}$ on the nonzero elements of the corresponding sets by multiplication. Then $\mathbb{PR}^k$, $\mathbb{P}M_k(\mathbb{R})$ and $\mathbb{PC}^k$, $\mathbb{P}M_k(\mathbb{C})$ are compact real and complex manifolds, respectively. To keep our notation standard, we set $\mathbb{P}^{k-1} = \mathbb{PC}^k$. For $\mathbf{x} \in \mathbb{F}^k\backslash\{\mathbf{0}\}, T \in M_k(\mathbb{F})\backslash\{0\}$, we denote by $\hat{\mathbf{x}}, \hat{T}$ the induced elements in $\mathbb{PF}^k$, $\mathbb{P}M_k(\mathbb{F})$, respectively. Furthermore, $\hat{T}$ can be viewed as a map from $\mathbb{PF}^k\backslash\widehat{\ker T}$ to $\mathbb{PF}^k$. We denote by $d(\cdot,\cdot) : \mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \to [0,\infty)$ the Fubini-Study metric on $\mathbb{P}^{k-1}$; see e.g. [7].

Consider an iteration scheme

$$\mathbf{x}_n := T_n\mathbf{x}_{n-1}, \quad \mathbf{x}_0 \in \mathbb{F}^k, \ T_n \in M_k(\mathbb{F}), n \in \mathbb{N}. \tag{2.1}$$

This system is called **convergent** if $\mathbf{x}_n, n \in \mathbb{N}$ is a convergent sequence, for each $\mathbf{x}_0 \in \mathbb{F}^k$. An iteration scheme is convergent if and only if the infinite product $\dots T_nT_{n-1}\dots T_2T_1$, which is defined as the limit of $T_nT_{n-1}\dots T_2T_1$ as $n \to \infty$, converges. For the stationary case $T_n = T, n \in \mathbb{N}$, the necessary and sufficient conditions for convergency are well known; namely, (i) the spectral radius $\rho(T)$ cannot exceed 1, (ii) if $\rho(T) = 1$, then 1 is an eigenvalue of T and all its Jordan blocks have size 1, and (iii) all other eigenvalues λ of T different from 1 satisfy $|\lambda| < 1$.

In some cases, as (for example) in problems related to the Lyapunov exponents in dynamical systems, of interest is whether or not the line spanned by the vector $\mathbf{x}_i$ converges, for all $\mathbf{x}_0 \neq 0$ in some homogeneous open Zariski set in $\mathbb{F}^k$. If this condition holds, we call (2.1) **projectively convergent**.

For the stationary case $0 \neq T_n = T \in M_k(\mathbb{C})$, one can easily check that (2.1) is projectively convergent if and only if among all the eigenvalues λ of T satisfying $|\lambda| = \rho(T)$, there exists exactly one eigenvalue λ_0 that has Jordan blocks of maximal size.

The main objective in Theorem 5 is to prove that $\hat{\mathbf{w}}(\mathbf{x})$ is continuous in $\mathcal{D}$. To attain this objective, we use some technique developed in [6]; in particular, the arguments of the proof of Theorem 4, [6, Theorem 1.2, page 258]. Namely, $\lambda_{max}(\mathbf{x})$, the unique simple eigenvalue of $T(\mathbf{x})$ with the maximal modulus, is continuous and nonvanishing in $\mathcal{D}$. To prove Theorem 5, we replace $T(\mathbf{x})$ with $(T(\mathbf{x}))/(\lambda_{max}(\mathbf{x}))$ and, for each n, $T_n(\mathbf{x})$ with $(T_n(\mathbf{x}))/(\lambda_{max}(\mathbf{x}))$. Thus, we can assume that for each $\mathbf{x} \in \mathcal{D}$, the number $\lambda_{max} = 1$ is the simple eigenvalue of $T(\mathbf{x})$ of maximal modulus. For each $\mathbf{x} \in \mathcal{D}$, let $\mathbf{u}(\mathbf{x})$ $v(\mathbf{x}), \neq \mathbf{0}$ be the right and left eigenvectors of $T(\mathbf{x})$ corresponding to 1, i.e.,

$$T(\mathbf{x})\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}), \quad T^{\mathrm{t}}(\mathbf{x})\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}), \quad \mathbf{u}^{\mathrm{t}}(\mathbf{x})\mathbf{v}(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathcal{D}. \tag{2.2}$$

Since $\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x})$ can be chosen continuous in $\mathcal{D}$, it follows that $\hat{\mathbf{u}}(\mathbf{x})$ and $\hat{\mathbf{v}}(\mathbf{x}) \in \mathbb{P}^{n-1}$

are continuous in $\mathcal{D}$ as well. In what follows, we fix a norm $\|\cdot\|$ on $M_k(\mathbb{C})$ satisfying, for any $T \in M_k(\mathbb{C})$, the condition $\|T\| = \|T^{\mathrm{t}}\|$.

We need two additional lemmas. The following generalizes the inequalities given in [6, pp. 254–255].

Lemma 1. *Let $\{T_n(\mathbf{x})\}_{n\in\mathbb{N}}$ be a sequence of complex-valued invertible $(k \times k)$-matrices with continuous entries for $\mathbf{x} \in \mathcal{D} \subset \mathbb{R}^d$. Assume that this sequence converges uniformly, on any compact set in $\mathcal{D}$, to a matrix $T(\mathbf{x})$. Suppose furthermore that for each $\mathbf{x} \in \mathcal{D}$, the number $\lambda_{max} = 1$ is a simple eigenvalue of $T(\mathbf{x})$ and all other eigenvalues of $T(\mathbf{x})$ lie in the open unit disk $|\lambda| < 1$. Then for any fixed $\mathbf{x}_0$ and for each $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon), m = m(\varepsilon) \in \mathbb{N}$ such that*

$$\|T_{n+m}(\mathbf{x})T_{n+m-1}(\mathbf{x})\dots T_{n+1}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)\mathbf{v}^{\mathrm{t}}(\mathbf{x}_0)\| < \varepsilon, \qquad \text{for each } n \geq N \text{ and } |\mathbf{x}-\mathbf{x}_0| \leq \delta. \tag{2.3}$$

Proof. Since 1 a simple eigenvalue of $T(\mathbf{x}_0)$ and all other eigenvalues of $T(\mathbf{x}_0)$ lie in the open unit disk, $\lim_{m\to\infty} T^m(\mathbf{x}_0) = \mathbf{u}(\mathbf{x}_0)\mathbf{v}^{\mathrm{t}}(\mathbf{x}_0)$. Hence, there exists $m = m(\varepsilon)$ such that $\|T^m(\mathbf{x}_0) - \mathbf{u}(\mathbf{x}_0)\mathbf{v}^{\mathrm{t}}(\mathbf{x}_0)\| < \varepsilon/2$. Also, since the map $(M_k(\mathbb{C}))^m \to M_k(\mathbb{C})$ sending a m-tuple $X_1, \dots, X_m$ to its product is continuous, there exists $\delta_1 > 0$ such that for $i = 1, \dots, m$, $\|X_1X_2\dots X_m - T(\mathbf{x}_0)^m\| \leq \varepsilon/2$ whenever $\|X_i - T(\mathbf{x}_0)\| \leq \delta_1$. Since $T_n(\mathbf{x})$ converges uniformly to $T(\mathbf{x})$ on any compact subset of $\mathcal{D}$, there exist $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon) \in \mathbb{N}$ such that $\|T_n(\mathbf{x}) - T(\mathbf{x}_0)\| \leq \varepsilon/2$, for $n \geq N$ and $\|\mathbf{x} - \mathbf{x}_0\| \leq \delta$. □

Next we give a modification of [6, Lemma 5.1].

Lemma 2. *Let $E \in M_k(\mathbb{C})$ be a matrix of rank one with $\rho(E) > 0$, i.e. $E = \mathbf{v}\mathbf{u}^{\mathrm{t}}, \mathbf{u}^{\mathrm{t}}\mathbf{v} = 1$. Set $O_r := \{\hat{\mathbf{x}} \in \mathbb{PC}^k : d(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq r\}$ such that $O_r \cap \widehat{\ker E} = \varnothing$, so that $\hat{E} : O_r \to \{\hat{\mathbf{v}}\}$. Then there exists $\varepsilon = \varepsilon(r)$ such that whenever $B \in M_k(\mathbb{C})$ satisfies $\|B - E\| \leq \varepsilon$,*

1. $d(\hat{B}\hat{\mathbf{x}}, \hat{\mathbf{v}}) \leq r/2$, *for each* $\hat{\mathbf{x}} \in \mathbb{O}_r$.
2. $d(\hat{B}\hat{\mathbf{x}}, \hat{B}\hat{\mathbf{y}}) \leq d(\hat{\mathbf{x}}, \hat{\mathbf{y}})/2$, *for each* $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in O_r$.

Proof. Clearly, $\widehat{E\mathbf{x}} = \hat{\mathbf{v}}$ if $\mathbf{u}^{\mathrm{t}}\mathbf{x} \neq 0$; hence $\hat{E} : O_r \to \{\hat{\mathbf{v}}\}$. Recall that for any $B \in M_k(\mathbb{C})\backslash\{0\}$, the transformation $\hat{B} : \mathbb{PC}^k\backslash\widehat{\ker B} \to \mathbb{PC}^k$ is analytic. Hence there exists $\varepsilon_0 > 0$ such that whenever $\|B - E\| \leq \varepsilon_0$, $B \neq 0$ and $\widehat{\ker B} \cap O_r = \varnothing$. Thus, $\hat{B}$ is analytic on O_r. For small $\epsilon \in (0, \epsilon_0)$, any $\hat{B}$ that satisfies $\|B - E\| \leq \varepsilon$ is a small perturbation of $\hat{E}$. Since $\hat{E}O_r = \{\hat{\mathbf{v}}\}$, we deduce condition 1, for any $\varepsilon \in (0, \varepsilon_1)$, for some $0 < \varepsilon_1 < \varepsilon_0$. Since $d(\hat{E}\hat{\mathbf{x}}, \hat{E}\hat{\mathbf{y}}) = 0$ for any $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in O_r$, we deduce condition 2, for some $\varepsilon \in (0, \varepsilon_1)$. □

Proof of Theorem 5. Recall that Theorem 1.2 of [6] implies that the map $\widehat{T_n(\mathbf{x})}\widehat{T_{n-1}(\mathbf{x})}\ldots\widehat{T_2(\mathbf{x})}\widehat{T_1(\mathbf{x})}$, viewed as an automorphism of $\mathbb{P}^{k-1}$, converges pointwise on $\mathbb{P}^{k-1}\backslash H(\mathbf{x})$ to the transformation $\hat{\mathbf{u}}(\mathbf{x})\hat{\mathbf{w}}^{\mathrm{t}}(\mathbf{x})$. Now fix some $\mathbf{x}_0 \in \mathcal{D}$. Then there exists $\delta_1 > 0$ such that the eigenvectors $\mathbf{u}(\mathbf{x}), \mathbf{v}(x) \in \mathbb{C}^k$ satisfying (2.2) can be chosen to be continuous in $|\mathbf{x}-\mathbf{x}_0| \leq \delta_1$. Since $\mathbf{v}^{\mathrm{t}}(\mathbf{x})\mathbf{u}(x) = 1$, we can choose $\delta_2 \in (0, \delta_1)$ such that $\mathbf{v}(\mathbf{x}_0)\mathbf{u}(\mathbf{x}) \neq 0$ for $|\mathbf{x}-\mathbf{x}_0| \leq \delta_1$. To prove the continuity of $\hat{\mathbf{w}}(\mathbf{x})$ in $|\mathbf{x}-\mathbf{x}_0| < \delta$, it is enough to prove the continuity of the limit of $\widehat{\mathbf{v}^{\mathrm{t}}(\mathbf{x}_0)}\widehat{T_n(\mathbf{x})}\widehat{T_{n-1}(\mathbf{x})}\ldots\widehat{T_2(\mathbf{x})}\widehat{T_1(\mathbf{x})}$ in $|\mathbf{x}-\mathbf{x}_0| < \delta_3$, for some $\delta_3 \in (0, \delta_2)$. Since each $T_j(\mathbf{x}) \in \mathrm{GL}(k, \mathbb{C})$ is continuous, it is enough to show the continuity of the limit of $\widehat{\mathbf{v}^{\mathrm{t}}(\mathbf{x}_0)}\widehat{T_{n+N}(\mathbf{x})}\widehat{T_{n+N-1}(\mathbf{x})}\ldots\widehat{T_{N+2}(\mathbf{x})}\widehat{T_{N+1}(\mathbf{x})}$ in $|\mathbf{x}-\mathbf{x}_0| < \delta_3$. What is required is to prove that every subsequence of $\widehat{\mathbf{v}^{\mathrm{t}}(\mathbf{x}_0)}\widehat{T_{n+N}(\mathbf{x})}\widehat{T_{n+N-1}(\mathbf{x})}\ldots\widehat{T_{N+2}(\mathbf{x})}\widehat{T_{N+1}(\mathbf{x})}$ converges uniformly in $|\mathbf{x}-\mathbf{x}_0| \leq \delta_3$, in the Fubini-Study metric $d(\cdot,\cdot)$ on $\mathbb{P}^{k-1}$.

This is done by using Lemmas 1-2. Namely, let $E = \mathbf{v}(\mathbf{x}_0)\mathbf{u}^{\mathrm{t}}(\mathbf{x}_0)$. Choose $r > 0, \varepsilon > 0$ such that conditions 1-2 of Lemma 2 hold. Next choose N, δ such that condition (2.3) holds. Let $\delta_3 = \min(\delta, \delta_2)$. Define

$$B_j(\mathbf{x}) = T^{\mathrm{t}}_{(j-1)m+N+1}(\mathbf{x})\ldots T^{\mathrm{t}}_{jm+N}(\mathbf{x}), \quad j = 1, \ldots.$$

Assume that $|\mathbf{x}-\mathbf{x}_0| \leq \delta_3$. We claim that the sequence $\widehat{B_1(\mathbf{x})}\ldots\widehat{B_l(\mathbf{x})}\widehat{\mathbf{v}(\mathbf{x}_0)}$ converges uniformly to $\mathbf{w}(\mathbf{x}) \in O_r$, in the Fubini-Study metric. First, observe that (2.3) implies that $\widehat{B_j(\mathbf{x})}O_r \subset O_{r/2}$, by condition 1 of Lemma 2. Hence, $\hat{\mathbf{y}}_l(\mathbf{x}) = \widehat{B_1(\mathbf{x})}\ldots\widehat{B_l(\mathbf{x})}\widehat{\mathbf{v}(\mathbf{x}_0)} \subset O_{r/2}$. Condition 1 of Lemma 2 yields

$$\begin{aligned}
&d(\widehat{B_1(\mathbf{x})}\ldots\widehat{B_l(\mathbf{x})}\widehat{\mathbf{v}(\mathbf{x}_0)}, \widehat{B_1(\mathbf{x})}\ldots\widehat{B_{l+p}(\mathbf{x})}\widehat{\mathbf{v}(\mathbf{x}_0)}) \\
&\qquad\qquad \leq \frac{1}{2^l}d(\widehat{\mathbf{v}(\mathbf{x}_0)}, \widehat{B_{l+1}(\mathbf{x})}\ldots\widehat{B_{l+p}(\mathbf{x})}\widehat{\mathbf{v}(\mathbf{x}_0)}) \leq \frac{l}{2^l}.
\end{aligned}$$

Hence, the sequence $\hat{\mathbf{y}}_l$, $l \in \mathbb{N}$, converges uniformly in $|\mathbf{x}-\mathbf{x}_0| \leq \delta_3$ to $\hat{\mathbf{y}}(\mathbf{x})$. Since each $\hat{\mathbf{y}}_l(\mathbf{x})$ is continuous in $|\mathbf{x}-\mathbf{x}_0| \leq \delta_3$, $\mathbf{y}(\mathbf{x})$ is continuous in the open disk $|\mathbf{x}-\mathbf{x}_0| < \delta_3$. □

3 Proving remaining results

Consider a recurrence relation with varying coefficients of the form

$$u_{n+k} + \alpha_{1,n}u_{n+k-1} + \alpha_{2,n}u_{n+k-2} + \ldots + \alpha_{k,n}u_n = 0, \tag{3.1}$$

where $\alpha_{k,n} \neq 0$. Assume that for all $i \in \{1, \ldots, k\}$, $\lim_{n\to\infty}\alpha_{i,n} =: \alpha_i$ and denote the limiting recurrence relation of (3.1) by

$$v_{n+k} + \alpha_1 v_{n+k-1} + \alpha_2 v_{n+k-2} + \ldots + \alpha_k v_n = 0. \tag{3.2}$$

Lemma 3. *If* (3.2) *is maxmod-generic then* $\lim_{n\to\infty}(v_{n+1})/(v_n)$ *exists and equals* λ_{max} *on the complement of a complex hyperplane* $H \subset \mathbb{C}^k$*, where* λ_{max} *denotes the leading spectral number of* (3.2).

Proof. The result is an immediate corollary of Theorem 4 applied to the family of linear operators $\{T_n \mid n \in \mathbb{N}\}$, whose action on $\mathbb{C}^{k+1}$ is given by

$$T_n = \begin{pmatrix} 1 & \alpha_{1,n} & \alpha_{2,n} & \alpha_{3,n} & \dots & \alpha_{k,n} \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad n \in \mathbb{N}.$$

Notice that $\alpha_{k,n} \neq 0$. □

The exceptional complex hyperplane H is called the **hyperplane of slow growth**. From Theorem 4, we deduce the following; cf. [5, Proposition 2.1].

Corollary 1. *Under the assumptions of Theorem 4, there exists a sequence of real numbers* $\{\theta_n\}_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} e^{i\theta_n} \frac{T_n T_{n-1} \cdots T_1}{||T_n T_{n-1} \cdots T_1||} = \mathbf{u}_{max}\mathbf{w}^t,$$

the convergence taking place in $\mathrm{M}_k(\mathbb{C})$.

We now settle Theorem 6.

Theorem 6 (i). Let us first determine Σ_I and, at the same time, prove the pointwise convergence of $f_{n+1}(x_1, \dots, x_d)/f_n(x_1, \dots, x_d)$ to $\Psi_{max}(x_1, \dots, x_d)$ in the complement $\Omega \setminus (\Xi_{\tilde{\phi}} \cup \Sigma_I)$. One can view the set-up of Theorem 6 as the situation in which the recurrence relation (3.1) depends on additional parameters $\mathbf{x} = (x_1, \dots, x_d)$. Thus, if at a given point $p \in \Omega \subseteq \mathbb{R}^d$, the limit $\lim_{n\to\infty} f_{n+1}(p)/f_n(p)$ does not exist, then p lies in the induced maxmod-discriminant $\Xi_{\tilde{\phi}}$; cf. Definition 4. On the other hand, if $\lim_{n\to\infty}(f_{n+1}(p))/(f_n(p))$ exists but is not equal to $\Psi_{max}(p)$ and, additionally, p lies in $\Omega \setminus \Xi_{\tilde{\phi}}$, then the corresponding initial k-tuple $I(p) = \big(f_0(p), \dots, f_{k-1}(p)\big)$ belongs to the hyperplane $H(p)$ of slow growth at the given point p; see Lemma 3. The latter set of points p is, by definition, the set Σ_I of slowly growing initial conditions. For varying coefficients, the hyperplane of slow growth at a given point $p \in \Omega \setminus \Xi_{\tilde{\phi}}$ is determined by $\lim_{n\to\infty} e^{i\theta_n} T_n T_{n-1} \cdots T_1/||T_n T_{n-1} \cdots T_1||$; see Corollary 1. Indeed, Theorem 4 implies that in this case, the hyperplane of slow growth $H(p)$ at p consists of all vectors $\mathbf{x}(p) \in \mathbb{C}^k$ such that $\mathbf{w}^t(p)\mathbf{x}(p) = 0$. □

To settle Theorem 6 (ii), we assume that $\overline{\phi}_n \rightrightarrows \widetilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_k)$ and all functions considered are continuous in Ω. Let $p_0 \in \Omega \setminus (\Xi_{\widetilde{\phi}} \cup \Sigma_I)$. From Corollary 1 and the arguments of [5, Sections 4 and 5], it follows that one can find a sufficiently small neighborhood $\mathcal{O}_{p_0} \subset \Omega \setminus (\Xi_{\widetilde{\phi}} \cup \Sigma_I)$ of p_0 such that for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ satisfying

$$\left|\left| e^{i\theta_n(p)} \frac{K_n(p)}{||K_n(p)||} - \mathbf{u}_{max}(p)\mathbf{w}^t(p) \right|\right| \leq \epsilon, \qquad \text{for } p \in \mathcal{O}_{p_0},\ n \geq N_\epsilon, \tag{3.3}$$

where $K_n(p) := T_n(p)T_{n-1}(p)\cdots T_1(p)$. Clearly, $\mathbf{u}_{max}(p)\mathbf{w}^t(p)$ is a rank one matrix and therefore, for all $p \in \mathcal{O}_{p_0}$, has precisely one simple eigenvalue of maximum modulus. From (3.3), we deduce that there exists $N \in \mathbb{N}$ such that, for all $p \in \mathcal{O}_{p_0}$ and $n \geq N$, the polynomial $\det(\lambda I_k - K_n(p))$ has precisely one simple eigenvalue $\lambda_{max,n}(p)$ of maximum modulus. The Implicit Function Theorem then implies that $\lambda_{max,n}(p)$ is continuous in $\mathcal{O}_{p_0}$ for all $n \geq N$, so that

$$(*) \qquad \left\{ \frac{K_n(p)}{\lambda_{max,n}(p)} \right\}_{n \geq N}$$

is a sequence of matrix-valued functions that are continuous in $\mathcal{O}_{p_0}$. On the other hand, it is not difficult to see that this sequence converges pointwise in $\mathcal{O}_{p_0}$ to $\mathbf{u}_{max}(p)\mathbf{w}^t(p)$; cf. the proof of Theorem 4 given in [5]. Together with (3.3), this implies that

$$(**) \qquad \left\{ \frac{||K_n(p)||}{|\lambda_{max,n}(p)|} \right\}_{n \geq N}$$

is a sequence of (continuous) functions that converges pointwise to the constant function 1 on $\mathcal{O}_{p_0}$. It follows that there exists some open subset $\mathcal{O}'_{p_0} \subset \mathcal{O}_{p_0}$ with $p_0 \in \mathcal{O}'_{p_0}$ such that (**) is a uniformly bounded sequence of functions on $\mathcal{O}'_{p_0}$. Thus (*) is a bounded sequence of continuous matrix-valued functions that converges pointwise to the function $\mathbf{u}_{max}(p)\mathbf{w}^t(p)$ in $\mathcal{O}'_{p_0}$. Invoking the Vitali Convergence Theorem [22, p. 168], we conclude that

$$\frac{K_n(p)}{\lambda_{max,n}(p)} \rightrightarrows \mathbf{u}_{max}(p)\mathbf{w}^t(p) \text{ in } \mathcal{O}'_{p_0}. \tag{3.4}$$

□

Notation. In the notation of Theorem 6, let $V = \Omega \times \mathbb{C}^k$ denote the Cartesian product of Ω and the linear space $\mathbb{C}^k$ of all initial k-tuples at a point in Ω. Any initial k-tuple $I = (f_0(\mathbf{x}), \ldots, f_{k-1}(\mathbf{x}))$ of smooth (respectively, analytic) functions in Ω can be considered as a smooth (respectively, analytic) section of V viewed as a trivial vector bundle over Ω. Denote by $\widetilde{V}$ the restriction of the bundle V to the subset $\Omega \setminus \Xi_{\widetilde{\phi}}$ of the base.

Definition 8. Let Δ_ϕ denote the subset of $\widetilde{V}$ consisting of all pairs $(p, I(p))$ such that the asymptotic ratio of the recurrence (1.6) evaluated at p with initial k-tuple $I(p) \in \mathbb{C}^k$ does not coincide with the leading root of (1.7) at p. The set Δ_ϕ is called the **subvariety of slow growth**.

Corollary 2. *In the above notation, the subvariety Δ_ϕ of slow growth is a smooth (respectively, analytic) subbundle of complex hyperplanes in $\widetilde{V}$, provided that the family of k-tuples*

$$\left\{\overline{\phi}_n = \left(\phi_{1,n}(\mathbf{x}), \phi_{2,n}(\mathbf{x}), \dots, \phi_{k,n}(\mathbf{x})\right) \mid n \in \mathbb{Z}_+\right\},$$

with $\phi_{k,n}(\mathbf{x})$ nonvanishing in Ω and its limit k-tuple $\widetilde{\phi} = \left(\widetilde{\phi}_1(\mathbf{x}), \widetilde{\phi}_2(\mathbf{x}), \dots, \widetilde{\phi}_k(\mathbf{x})\right)$, consist of smooth (respectively, analytic) functions that satisfy $\overline{\phi}_n \rightrightarrows \widetilde{\phi}$, in Ω, in the corresponding category.

Proof. Indeed, by Theorem 6 (ii), the family of hyperplanes of slow growth depends smoothly (respectively, analytically) on $p \in \Omega \setminus \Xi_{\widetilde{\phi}}$, in the corresponding category. □

Definition 9. Let $\xi : E \to B$ be a (complex) vector bundle over a base B. A smooth (respectively, analytic) section $S : B \to E$ is called **transversal** to a given smooth (respectively, analytic) submanifold $\mathcal{H} \subset E$ if, at each point p of the intersection $S \cap \mathcal{H}$, the sum of the tangent spaces at p to S and $\mathcal{H}$ coincides with the tangent space at p to the ambient space E. A subset X of a topological space Y is called **massive** if X can be represented as the intersection of at most countably many open dense subsets in Y.

Remark 11. Thom's Transversality Theorem (see, e.g., [8]) implies that the set of all smooth sections of the bundle $\xi : E \to B$ that are transversal to a given smooth subbundle $\mathcal{H} \subset E$ is a massive subset of the set of all sections. The same holds in the analytic category, provided that the fibration is trivial.

Lemma 4. *If an initial k-tuple $I = \left(f_0(\mathbf{x}), \dots, f_{k-1}(\mathbf{x})\right)$ of smooth (respectively, analytic) functions in Ω is transversal to the subvariety Δ_ϕ of slow growth, then the set Σ_I of slowly growing initial conditions is either empty or a smooth (respectively, analytic) subvariety in Ω of real codimension two.*

Proof. By Thom's Transversality Theorem, the set of all smooth initial k-tuples that are transversal to Δ_Φ is a massive subset of the set of all possible smooth initial k-tuples; cf. Remark 11. Combined with Proposition 2, this implies, in particular, that for a given initial k-tuple I transversal to Δ_ϕ, I (considered as a section of $\widetilde{V}$) and Δ_Φ are smooth submanifolds of $\widetilde{V}$ transversal to each other, where $\widetilde{V}$ is the restriction of the bundle $V = \Omega \times \mathbb{C}^k$ to the subset $\Omega \setminus \Xi_{\widetilde{\phi}}$ of the base; cf. Notation 3. This transversality property implies that the intersection $I \cap \Delta_\phi$ is either empty or a smooth submanifold of $\widetilde{V}$ of real codimension 2. Notice that the image $\xi(I \cap \Delta_\phi)$ of the projection $\xi : \widetilde{V} \to \Omega \setminus \Xi_{\widetilde{\phi}}$ to the base is exactly Σ_I. Actually, ξ induces a diffeomorphism between $I \cap \Delta_\phi$ and Σ_I. To see this, simply note that ξ maps the section I diffeomorphically to $\Omega \setminus \Xi_{\widetilde{\phi}}$ and that the intersection $I \cap \Delta_\phi$ is a smooth submanifold of the section I. The proof of the corresponding result in the analytic category is analogous. □

In particular, in the one-dimensional case one has the following result.

Lemma 5. *If the coefficients* $\big(\phi_{1,n}(z), \dots, \phi_{k,n}(z)\big)$, $n \in \mathbb{Z}_+$, *and the initial* k*-tuple* $I = \big(f_0(z), \dots, f_{k-1}(z)\big)$ *of recurrence* (1.6) *are complex analytic functions on an open set* $\Omega \subseteq \mathbb{C}$, *and* $\phi_{k,n}(z) \neq 0$ *in* Ω, *then either* $\Xi_{\widetilde{\phi}} = \Omega$ *or* $\Xi_{\widetilde{\phi}}$ *is a union of real analytic curves and* Σ_I *is either all of* Ω *or consists of isolated points.*

Proof. By Corollary 2, the set Δ_Φ is analytic in the analytic category. Thus, in this case, the asymptotic symbol equation (1.7) is either maxmod-nongeneric everywhere in $\Omega \subseteq \mathbb{C}$ or maxmod-nongeneric on a one-dimensional real analytic subset, which proves the statement in part (i) concerning $\Xi_{\widetilde{\phi}}$. Note that, in fact, in the analytic category, $\Xi_{\widetilde{\phi}}$ is always a real semianalytic set and Σ_I is analytic. Therefore, Σ_I is either a complex analytic curve, in which case it coincides with Ω, or an analytic zero-dimensional subset of Ω, i.e., the union of isolated points.□

We finally turn to Proposition 1. To settle it, we need an additional lemma, which a simple consequence of Theorem 1.

Notation. Let Rec_k be the k-dimensional complex linear space consisting of all $(k+1)$-term recurrence relations with constant coefficients of the form (1.1). As before, we denote by $\mathbb{C}^k$ the k-dimensional complex linear space of all initial k-tuples $(u_0, ..., u_{k-1})$.

Definition 10. A maxmod-nongeneric recurrence relation in Rec_k with initial k-tuple $in_k \in \mathbb{C}^k$ is said to be of **dominant type** if the following conditions are satisfied. Let $\lambda_1, \dots, \lambda_r$, $r \leq k$, denote all distinct spectral numbers with maximal absolute value and assume that these have multiplicities $m_1, \dots, m_r$, respectively. Then there exists a unique index $i_0 \in \{1, \dots, r\}$ such that $m_i < m_{i_0}$ for $i \in \{1, \dots, r\} \setminus \{i_0\}$ and the initial k-tuple in_k is **fast growing**, in the sense that the degree of the polynomial P_{i_0} in (1.3) corresponding to λ_{i_0} is precisely $m_{i_0} - 1$. The number λ_{i_0} is called the **dominant spectral number** of this recurrence relation.

Lemma 6. *The following hold.*

(i) *The set of all slowly growing initial k-tuples with respect to a given maxmod-generic recurrence relation in Rec_k is a complex hyperplane $\mathcal{SG}_k$ in $\mathbb{C}^k$. The set $\mathcal{SG}_k$ is called the* **hyperplane of slow growth**.

(ii) *For any maxmod-generic recurrence relation in Rec_k and any fast growing initial k-tuple $(u_0, \dots, u_{k-1})$, the limit $\lim_{n\to\infty} u_{n+1}/u_n$ exists and coincides with the leading spectral number λ_{max}, that is, the (unique) root of the characteristic equation* (1.2) *with maximal absolute value.*

(iii) *Given a maxmod-nongeneric recurrence relation of dominant type in Rec_k, the limit $\lim_{n\to\infty} u_{n+1}/u_n$ exists and coincides with the dominant spectral number.*

(iv) *For any maxmod-nongeneric recurrence relation of nondominant type in Rec_k, the set of initial k-tuples for which $\lim_{n\to\infty} u_{n+1}/u_n$ exists is a union of complex subspaces of $\mathbb{C}^k$ of positive codimensions. This union is called the* **exceptional variety**.

Proof. To prove (i), note that the coefficient κ_{max} in Definition 3 is a nontrivial linear combination of the entries of the initial k-tuple $(u_0, \dots, u_{k-1})$, with coefficients depending on $\alpha_1, \dots, \alpha_k$. Therefore, the condition $\kappa_{max} = 0$ determines a complex hyperplane $\mathcal{SG}_k$ in $\mathbb{C}^k$. It is easy to see that the hyperplane of slow growth is the direct sum of all Jordan blocks corresponding to the spectral numbers of a given recurrence (1.1) other than the leading one.

The assumptions of part (ii) together with (1.3) yield $u_n = \kappa_{max}\lambda_{max}^n + \dots$ for $n \in \mathbb{Z}_+$, where $\dots$ stands for the remaining terms in (1.3) corresponding to the spectral numbers whose absolute values are strictly smaller than $|\lambda_{max}|$. Therefore, the quotient u_{n+1}/u_n has a limit as $n \to \infty$ and this limit coincides with λ_{max}, as required. By definition, λ_{max} is a root of (1.2), which completes the proof of (ii). Alternatively, this last step can be carried out by dividing both sides of (1.1) by u_{n-k+1} and then letting $n \to \infty$. In view of Definition 10, the same arguments show that the assertion in (iii) is true as well.

For the proof of part (iv), we proceed as follows. Take any maxmod-nongeneric recurrence relation of the form (1.1) and let $\lambda_1, \dots, \lambda_r$, $r \le k$, be all its distinct spectral numbers with maximal absolute value. Thus $|\lambda_i| = |\lambda_{max}|$ if and only if $1 \le i \le r$. Choose an initial k-tuple $IT = (u_0, \dots, u_{k-1})$ and denote by $P_1, ..., P_r$ the polynomials in (1.3) corresponding to $\lambda_1, \dots, \lambda_r$, respectively, for the sequence $\{u_n \mid n \in \mathbb{Z}_+\}$ constructed using the given recurrence with initial k-tuple IT as above. Assuming, as we may, that our recurrence relation is nontrivial, we get from (1.3) that $\lambda_i \neq 0$ if $1 \le i \le r$. We may further assume that the degrees $d_1, \dots, d_r$ of the polynomials $P_1, ..., P_r$, respectively, satisfy $d_1 \ge \dots \ge d_r$. Under these conditions, we now prove that $\lim_{n\to\infty} u_{n+1}/u_n$ exists if and only if exactly one of the polynomials $P_1, \dots, P_r$ is nonvanishing. A direct check, analogous to the proof of part (ii), shows that if P_1 is the only nonvanishing polynomial, then $\lim_{n\to\infty} u_{n+1}/u_n = \lambda_1$. If $r \geq 2$ and $s \in \{2, \dots, r\}$ is such that $P_1, \dots, P_s$ are all nonvanishing polynomials among $P_1, \dots, P_r$, then using again (1.3), we get

$$\frac{u_{n+1}}{u_n} = \frac{P_1(n+1) + P_2(n+1)\left(\frac{\lambda_2}{\lambda_1}\right)^{n+1} + \dots + P_s(n+1)\left(\frac{\lambda_s}{\lambda_1}\right)^{n+1} + o(1)}{P_1(n) + P_2(n)\left(\frac{\lambda_2}{\lambda_1}\right)^{n} + \dots + P_s(n)\left(\frac{\lambda_s}{\lambda_1}\right)^{n} + o(1)}.$$

Since $|\lambda_i/\lambda_1| = 1$ and $\lambda_i \neq \lambda_1$, $2 \leq i \leq s$, it follows that if $d_1 = d_2$, then the expression on the right-hand side has no limit as $n \to \infty$. Therefore, if such a limit exists, then $d_1 > d_2$, which gives us a complex subspace of $\mathbb{C}^k$ of (positive) codimension equal to $d_1 - d_2$. Thus, the exceptional variety is a union of complex subspaces of $\mathbb{C}^k$ of (in general) different codimensions. □

Now we prove Proposition 1.

Proof. Under the assumptions of the Proposition 1, the leading root of the asymptotic symbol equation (1.7) is a well-defined analytic function on a sufficiently small neighborhood of Σ_I. Therefore, the residue distribution (1.8) associated to this leading root vanishes in a neighborhood of Σ_I. Thus, the set Σ_I of slowly growing initial conditions can be deleted from the support of the asymptotic ratio distribution ν, which proves (i).

To prove that the support of ν coincides with $\Xi_{\widetilde{\Phi}}$ under the nondegeneracy assumptions of (ii), we show that in this case, the sequence $f_{n+1}(p)/f_n(p)$ diverges for almost all $p \in \Xi_{\widetilde{\Phi}}$. Indeed, by analyticity, the first condition of (ii) implies that the recurrence relation is of nondominant type almost everywhere in $\Xi_{\widetilde{\Phi}}$. The second condition of (ii) then implies that the sequence $f_{n+1}(p)/f_n(p)$ diverges almost everywhere in $\Xi_{\widetilde{\Phi}}$, which settles part (ii) of the proposition. □

4 Application to biorthogonal polynomials

Here, we calculate explicitly the induced maxmod-discriminant $\Xi_{\widetilde{\Phi}}$ and the asymptotic ratio distribution ν in a number of cases. We illustrate the relation of $\Xi_{\widetilde{\Phi}}$ and Σ_I with the roots of polynomials satisfying certain recurrence relations. Let us first recall some definitions from [10]. There the authors introduced two types of polynomial families related to what they call R_I- and R_{II}-type continued fractions, respectively. These families were later studied in [26].

A polynomial family of type R_I is a system of monic polynomials generated by

$$p_{n+1}(z) = (z - c_{n+1})p_n(z) - \lambda_{n+1}(z - a_{n+1})p_{n-1}(z),$$

with $p_{-1}(z) = 0$, $p_0(z) = 1$ and $p_n(a_{n+1})\lambda_{n+1} \neq 0$, for $n \in \mathbb{Z}_+$.

A polynomial family of type R_{II} is a system of monic polynomials generated by

$$p_{n+1}(z) = (z - c_{n+1})p_n(z) - \lambda_{n+1}(z - a_{n+1})(z - b_{n+1})p_{n-1}(z)$$

with $p_{-1}(z) = 0$, $p_0(z) = 1$ and $p_n(a_{n+1})p_n(b_{n+1})\lambda_{n+1} \neq 0$ for $n \in \mathbb{Z}_+$.

Assuming that

$$\lim_{n\to\infty} a_n = A, \ \lim_{n\to\infty} b_n = B, \ \lim_{n\to\infty} c_n = C \text{ and } \lim_{n\to\infty} \lambda_n = \Lambda,$$

where A, B, C, Λ are complex numbers, we describe the asymptotic ratio $\Psi_{max}(z) = \lim_{n\to\infty} p_{n+1}(z)/p_n(z)$ and the asymptotic ratio measure ν, for R_I- and R_{II}-type polynomial families. Using our previous results, we restrict ourselves to a recurrence relation of the form

$$p_{n+1}(z) = Q_1(z)p_n(z) + Q_2(z)p_{n-1}(z), \tag{4.1}$$

with the initial triple $p_{-1}(z) = 0$, $p_0(z) = 1$, $\deg Q_1(z) = 1$, and $\deg Q_2(z) \leq 2$. By Theorem 6 and Corollary 5, the asymptotic ratio $\Psi_{max}(z)$ satisfies the symbol equation

$$\Psi_{max}^2(z) = Q_1(z)\Psi_{max}(z) + Q_2(z)$$

everywhere in the complement of the union of the induced maxmod-discriminant $\Xi_{\tilde{\phi}}$ and a finite set of (isolated) points. Notice also that the asymptotic ratio distribution ν satisfies the relation

$$\nu = \frac{\partial \Psi_{max}(z)}{\partial \bar{z}},$$

where the right hand side is interpreted as a distribution. By our previous results, the support of ν coincides with $\Xi_{\tilde{\phi}}$. In order to describe $\Xi_{\tilde{\phi}}$, we need the following simple lemma.

Lemma 7. *A quadratic polynomial $t^2 + a_1 t + a_2$ has two roots with the same absolute value if and only if there exists a real number $\epsilon \in [1, \infty)$ such that $\epsilon a_1^2 - 4a_2 = 0$.*

Proof. The roots of the polynomial are given by $t_{1,2} = (-a_1 \pm \sqrt{a_1^2 - 4a_2})/2$, so that if $|t_1| = |t_2|$ then the complex numbers a_1 and $\sqrt{a_1^2 - 4a_2}$, considered as vectors in $\mathbb{C}$, must be orthogonal. Thus $\left|\operatorname{Arg}(a_1) - \operatorname{Arg}\sqrt{a_1^2 - 4a_2}\right| = \pi/2$, which is equivalent to saying that $\sqrt{a_1^2 - 4a_2}/a_1$ is purely imaginary and therefore, $a_1^2 - 4a_2/a_1^2 \in (-\infty, 0]$. The converse statement is obvious. □

Corollary 3. *The induced maxmod-discriminant* $\Xi_{\tilde{\phi}} \subset \mathbb{C}$ *is the set of all solutions* z *of the equation*

$$\epsilon Q_1^2(z) + 4Q_2(z) = 0, \tag{4.2}$$

where $\epsilon \in [1, \infty)$.

We start with the situation that $Q_1(z)$ and $Q_2(z)$ are real polynomials. Let C denote the unique (real) root of $Q_1(z)$ and D, E denote the branching points of the symbol equation, i.e., the roots of $Q_1^2(z) + 4Q_2(z) = 0$. In the case $Q_1(z)$ and $Q_2(z)$ are real, there are three basic cases:

1. D and E are real, $D < E$, and $C \in [D, E]$;
2. D and E are complex conjugates;
3. D and E are real, $D < E$, and $C \notin [D, E]$.

Case 1. The discriminant $\Xi_{\tilde{\phi}}$ is the interval $[D, E]$. The density ρ_ν of ν in this case is positive and equals

$$\rho_\nu(x) = \frac{i}{2\pi}\sqrt{Q_1^2(x) + 4Q_2(x)}dx, \tag{4.3}$$

where the value of $\sqrt{Q_1^2(x) + 4Q_2(x)}$ is taken with negative imaginary part. (The value of $Q_1^2(x) + 4Q_2(x)$ on the interval $[D, E]$ is negative.)

Remark 12. Note that $\int_D^E d\nu$ is not necessarily equal to 1. In fact, $\int_D^E d\nu = (E - D)^2/16$. For comparison, take the sequence of inverse ratios $p_n(z)/p_{n+1}(z)$, for the above sequence of monic polynomials $\{p_n(z)\}$. Let $\tilde{\nu}_n$ denote the residue distribution of $p_n(z)/p_{n+1}(z)$ and set $\tilde{\nu} = \lim_{n\to\infty} \tilde{\nu}_n$. One sees easily that the distribution $\tilde{\nu}$ is actually a probability measure (i.e., $\int_D^E d\tilde{\nu} = 1$) which has the same support as ν. Its density $\rho_{\tilde{\nu}}$ is given by

$$\rho_{\tilde{\nu}}(x) = -\frac{i}{2\pi}\frac{\sqrt{Q_1^2(x) + 4Q_2(x)}}{Q_2(x)}dx.$$

For the concrete example depicted in Figure 2, the asymptotic ratio measure ν equals $[(2\sqrt{-x^2 - 3x + 4})/(25\pi)]dx$ and is supported on the interval $[-4, 1]$.

Case 2. The discriminant $\Xi_{\tilde{\phi}}$ is the arc $D - C - E$ of the unique circle passing through all three points D, C, E. In order to simplify the notation while calculating ν, assume that $\deg Q_1(z) = 1$. Then, by an affine change of the variable z, we can

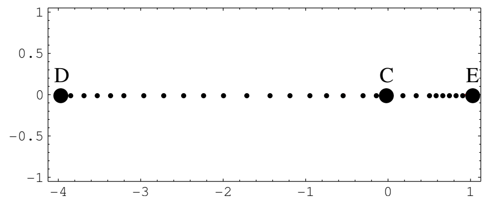

Figure 2. Zeros of $p_{41}(z)$ satisfying the 3-term recurrence relation $p_{n+1}(z) = zp_n(z)+ \frac{(3z-4)}{4}p_{n-1}(z)$ with $p_{-1}(z) = 0$ and $p_0(z) = 1$.

normalize $Q_1(z)$ such that $Q_1(z) = z$. Set $Q_2(z) = az^2+bz+c$, where a, b, c are real numbers. The condition that $Q_1^2(z)+4Q_2(z) = 0$ has complex conjugate roots gives $b^2 - c(4a+1) < 0$. Then, the center of the circle passing through D, C, E is given by $(-c/b, 0)$ and its radius is $|c/b|$. If γ is the angle from the real positive half-axis to the ray through the center of the circle and a point (x, y) on it, then we have the parametrizations $x = |c/b| \cos\gamma - c/b$, $y = |c/b| \sin\gamma$, and $n = \cos\gamma + i \sin\gamma$ of the unit normal to the circle at the point (x, y).

To calculate the asymptotic ratio measure $\nu = \partial\Psi_{max}(z)/\partial\overline{z}$, note that if $\Psi_{max}(z)$ is a piecewise analytic function with smooth curves separating the domains, where $\Psi_{max}(z)$ coincides with a given analytic function, then $\nu = \partial\Psi_{max}(z)/\partial\overline{z}$ is concentrated on these separation curves. Additionally, at each smooth point where two branches Ψ_1 and Ψ_2 meet, ν satisfies the condition $\nu = ((\Psi_1 - \Psi_2)nds)(2\pi)$ (up to sign). Applying this result to the case under consideration yields

$$\nu = \frac{\partial\Psi_{max}(z)}{\partial\overline{z}} = \frac{\sqrt{Q_1^2(z)+4Q_2(z)n}}{2\pi}\left|\frac{c}{b}\right| d\gamma.$$

Explicit computations give

$$\nu = \frac{\sqrt{2}}{b}(\cos\gamma + i\sin\gamma)^{\frac{3}{2}}\sqrt{c(2b^2-(1+4a)c(1-\cos\gamma))}d\gamma. \tag{4.4}$$

For the concrete example considered in Figure 3,

$$\nu = \sqrt{2\cos\gamma}(\cos\gamma + i\sin\gamma)^{\frac{3}{2}}d\gamma,$$

where γ is the angle from the real positive half-axis to the ray emanating from $\big(-c/b, 0\big) = (1, 0)$ (i.e., the center of the circle of radius $|c/b| = 1$ containing the points D, C, E in this case) and passing through a variable point (x, y) lying on the left half-circle.

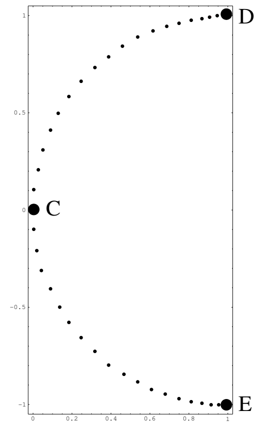

Figure 3. Zeros of $p_{41}(z)$ satisfying the 3-term recurrence relation $p_{n+1}(z) = zp_n(z) + (\frac{1-z}{2})p_{n-1}(z)$, with $p_{-1}(z) = 0$ and $p_0(z) = 1$.

Case 3. The discriminant $\Xi_{\tilde{\phi}}$ is the union of the interval $[D, E]$ and the circle is given (as in Case 2) by the equation $x(x - x_0) + y^2 = 0$, where x (respectively, y) is the real (respectively, imaginary) part of z and $x_0 = -2c/b$. The density ρ_ν of $\nu = \partial\Psi_{max}(z)/\partial\bar{z}$ is given by formula (4.3) on the interval and by formula (4.4) on the circle. A concrete example is depicted in Figure 4.

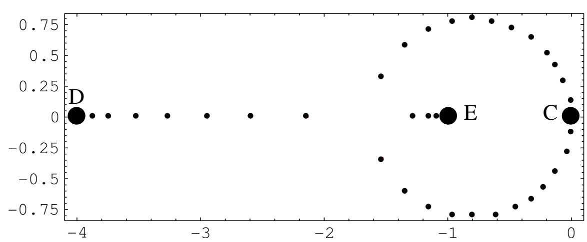

Figure 4. Zeros of $p_{41}(z)$ satisfying the 3-term recurrence relation $p_{n+1}(z) = zp_n(z) + (\frac{5z+4}{4})p_{n-1}(z)$, with $p_{-1}(z) = 0$ and $p_0(z) = 1$.

In the case that either of the polynomials $Q_1(z)$ and $Q_2(z)$ has complex coefficients, it seems difficult to give a description of the support of ν more precise than the one obtained in Corollary 3. Figure 5 illustrates possible forms of this support.

Note that all examples in this section deal with the standard initial polynomials $p_{-1}(z) = 0$ and $p_0(z) = 1$. Computer experiments show no isolated zeros of $p_n(z)$, in these cases.

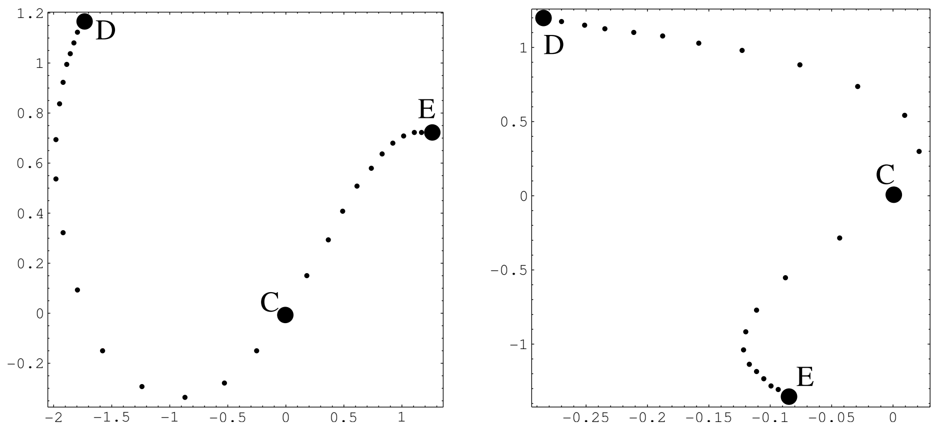

Figure 5. Zeros of $p_{41}(z)$ for the recurrence relations $p_{n+1}(z) = zp_n(z) + (iz^2 + 2z - 1 - 3i)p_{n-1}(z)$ (left) and $p_{n+1}(z) = zp_n(z) + ((2-i)z^2 + z + 4 - i)p_{n-1}(z)$ (right), with $p_{-1}(z) = 0$ and $p_0(z) = 1$

Lemma 8. *For any recurrence relation of the form* (4.1)*, with initial values* $p_{-1}(z) = 0$ *and* $p_0(z) = 1$*, the set* Σ_I *of slowly growing initial conditions is empty; cf.* [2].

Proof. Every recurrence relation of the form (4.1) can be rewritten as

$$\begin{pmatrix} p_{n+1}(z) \\ p_n(z) \end{pmatrix} = T(z) \begin{pmatrix} p_n(z) \\ p_{n-1}(z) \end{pmatrix}, \quad \text{where } T(z) = \begin{pmatrix} Q_1(z) & Q_2(z) \\ 1 & 0 \end{pmatrix}.$$

It is easily checked that for any $z \in \mathbb{C}$, the vector $(1, 0)^t \in \mathbb{C}^2$ is not an eigenvector of $T(z)$. This implies the desired result immediately. □

We point out that any change to the initial pair $(p_{-1}(z), p_0(z))$ introduces additional isolated zeros away from $\Xi_{\widetilde{\phi}}$ to the polynomials $p_n(z)$.

5 Related topics and open problems

Here, we briefly mention some related topics and open problems.

5.1 Does Theorem 5(ii) hold in the algebraic category, i.e., when the varying coefficients are algebraic functions of bounded degree?

5.2 Although one of the assumptions of Theorem 6 is that all $\phi_{k,n}(\mathbf{x})$ are nonvanishing in $\mathbf{\Omega}$, computer experiments show that the theorem holds in many cases that some $\phi_{k,n}(\mathbf{x})$ vanish. Apparently one can construct complicated counterexamples that render Theorem 6 false, in general. However, it would be important for applications to find some sufficient conditions, weaker than $\phi_{k,n}(\mathbf{x}) \neq 0$, under which the conclusions of Theorem 6(ii) are still valid.

5.3 The theory of multiple orthogonal polynomials, which is currently expereincing rapid development, is a natural area of application our results; see e.g. [4] and the references therein. In this paper, we consider only the rather simple instance of biorthogonal polynomials. It would be nice to calculate the maxmod-discriminant $\Xi_{\tilde{\phi}}$ and the asymptotic ratio distribution in more examples.

5.4 It is natural to ask what topological information about the induced maxmod-discriminant can be obtained from the coefficients of a given recurrence relation.

Problem 1. Describe the topological properties of $\Xi_{\tilde{\phi}}$ depending on the coefficients of the recurrence relation (say assumed fixed and algebraic) and establish necessary and sufficient conditions for the set $\Xi_{\tilde{\phi}}$ to be compact. Are there any characteristic numbers associated with the singularities on $\Xi_{\tilde{\phi}}$?

6 Appendix. On topology and geometry of maxmod-discriminants

Definition 11. Denote by $\widetilde{\Xi}_k \subset Pol_k = \{t^k + a_1 t^{k-1} + \ldots + a_k\}$ the set of all monic polynomials of degree k with complex coefficients having at least two roots with the same absolute value.

It is obvious from Definitions 2, 4, and 11 that $\widetilde{\Xi}_k$ contains the standard maxmod-discriminant Ξ_k.

Proposition 2. *$\widetilde{\Xi}_k$ is a real semialgebraic hypersurface of degree at most $(4k-1)(4k-2)$ in Pol_k. Furthermore, the hypersurface $\widetilde{\Xi}_k$ is quasihomogeneous with quasihomogeneous weight equal to i for both the real and the imaginary parts of a_i, $i = 1, 2, \ldots, k$. (Here Pol_k is considered as a $2k$-dimensional real affine space with the real and imaginary parts of all a_i's chosen as coordinates.)*

Proof. In order to calculate the degree of $\widetilde{\Xi}_k$, let us describe an algorithm that gives the equation for the analytic continuation of $\widetilde{\Xi}_k$. This algorithm may be presented as the superposition of two resultants. Let $u = x + iy \in \mathbb{C}$. Consider first the resultant of $P(t) = t^k + a_1 t^{k-1} + \ldots + a_k$ and $P(ut) = (ut)^k + a_1(ut)^{k-1} + \ldots + a_k$, which we denote by $R(u, a_1, \dots, a_k)$. (Recall that all a_i's are complex.) Clearly, $R(u, a_1, \dots, a_k)$ is a polynomial in the variable u. For any fixed value of u, the resultant $R(u, a_1, \dots, a_k)$ vanishes if and only if $P(t)$ and $P(ut)$, considered as polynomials in t, have a common zero. One can easily see that $R(u, a_1, \dots, a_k)$ is divisible by $(u-1)^k$ and that the quotient $\widetilde{R}(u, a_1, \dots, a_k) = R(u, a_1, \dots, a_k)/(u-1)^k$ is coprime with $u-1$. We want to find an equation for the set of all monic complex polynomials $P(t)$ such that $P(ut)$ and $P(t)$ have a common zero, for some $u \neq 1$ with $|u| = 1$. The standard rational parametrization of the unit circle is given by $u = ((1-i\theta)^2)/(1+\theta^2)$, where $\theta \in \mathbb{R}$. The result of the substitution $u = x + iy = \frac{(1-i\theta)^2}{1+\theta^2}$ into $\widetilde{R}(u, a_1, \dots, a_k)$ gives a complex-valued rational function of the real variable θ whose denominator equals to $(1+\theta^2)^{2k}$. By taking the resultant of the real and imaginary parts of the numerator of the latter rational function, one gets the required algebraic equation for the analytic continuation of $\widetilde{\Xi}_k$. This recipe allows us to calculate (an upper bound for) the degree of $\widetilde{\Xi}_k$. It is easy to see that $\widetilde{R}(u, a_1, \dots, a_k)$ is a polynomial of degree $2k$ in u and of degree at most $2k-1$ in the variables $a_1, \dots, a_k$. Making the substitution $u = x + iy = ((1-i\theta)^2)/(1+\theta^2)$ and taking the real and imaginary parts of the above rational function yields polynomials of degrees at most $2k-1$ in the variables $\Re a_1, \dots, \Re a_k, \Im a_1, \dots, \Im a_k$ and of degrees $4k$ and $4k-1$, respectively, in θ. These polynomials have proportional leading terms and, therefore, their resultant is a polynomial in the variables $\Re a_1, \dots, \Re a_k, \Im a_1, \dots, \Im a_k$ of degree at most $(2(4k)-2)(2k-1) = (4k-1)(4k-2)$.

The quasihomogeneity of $\widetilde{\Xi}_k$ with quasihomogeneous weights as specified in the statement of the proposition follows from the fact that $\widetilde{\Xi}_k$ is preserved under multiplication of all roots of a polynomial by a nonnegative real number. □

Remark 13. The proof of Proposition 2 contains an algorithm which gives an explicit equation for the analytic continuation of $\widetilde{\Xi}_k$. However, the resulting expression contains several hundred terms, even for $k = 3$, and does not seem to be of much use. For $k = 3$, the estimate $(4k-1)(4k-2)$ for the degree is sharp. It is very plausible that this estimate is actually sharp for arbitrary k, although in the general case, it seems very difficult to check the necessary nondegeneracy conditions while performing the superposition of the two resultants described in the algorithm.

Proposition 3. *$\widetilde{\Xi}_k$ is a singular real hypersurface with a (singular) boundary which coincides with the usual discriminant $\mathcal{D}_k \subset Pol_k$, where $\mathcal{D}_k$ is the set of all polynomials in Pol_k with multiple roots. Moreover, $\widetilde{\Xi}_k$ has no local singularities (i.e., singularities on a given branch) outside $\mathcal{D}_k$. The list of singularities of $\widetilde{\Xi}_k$ is* *finite, for any given $k \in \mathbb{N}$; i.e., its singularities have no moduli.*

Proof. Indeed, to show that $\mathcal{D}_k \subset Pol_k$ is the boundary of $\widetilde{\Xi}_k$, consider the standard Vieta map $Vi : \mathbb{C}^k \to Pol_k$ sending a k-tuple of (labeled) roots to the coefficients of the monic polynomial with these roots; i.e, to the elementary symmetric functions of these roots with alternating signs. It is known that the Vieta map induces a local diffeomorphism on the complement $\mathbb{C}^k \setminus \mathcal{T}_k$, where $\mathcal{T}_k$ is the standard Coxeter hyperplane arrangement consisting of $\binom{k}{2}$ hyperplanes $L_{i,j}$ given by $L_{i,j} : x_i = x_j$, $1 \leq i < j \leq k$. An easy observation is that $\mathcal{T}_k$ coincides with the preimage $Vi^{-1}(\mathcal{D}_k)$ of the discriminant $\mathcal{D}_k$. Consider now the arrangement of quadratic cones $\mathfrak{C}_k = \cup_{i<j}\mathfrak{C}_{i,j}$ in $\mathbb{C}^k$, where $\mathfrak{C}_{i,j}$ is given by the equation $|x_i| = |x_j|$. Obviously, $\mathfrak{C}_k$ coincides with the preimage $Vi^{-1}(\widetilde{\Xi}_k)$. From the defining equation, it is clear that each $\mathfrak{C}_{i,j}$ is smooth outside the origin. Therefore, $\widetilde{\Xi}_k$ is locally smooth (that is, it consists of smooth local branches) outside $\mathcal{D}_k$. Note also that for $1 \leq i < j \leq k$, the quadratic cone $\mathfrak{C}_{i,j}$ contains the complex hyperplane $L_{i,j}$ and that the restriction $Vi\big|_{\mathfrak{C}_{i,j}}$ has a fold near a generic point of $L_{i,j}$. Therefore, $\mathcal{D}_k$ is the boundary of $\widetilde{\Xi}_k$. The absence of the moduli for the singularities of $\widetilde{\Xi}_k$ can be derived from that for the singularities of $\mathcal{D}_k$. The type of a (multi)singularity of $\widetilde{\Xi}_k$ near a polynomial $P(z) \in \widetilde{\Xi}_k$ is encoded in the following combinatorial information about the roots of $P(z)$. Determine first the multiplicity (possibly vanishing) of the root of $P(z)$ at the origin and the set of all distinct positive absolute values for all the roots of $P(z)$. For each such positive absolute value, determine the number and multiplicity of distinct roots having this absolute value. If all the roots are simple, then (as above) $\widetilde{\Xi}_k$ locally consists of (in general, nontransversal) smooth branches. Otherwise, $P(z)$ lies in $\mathcal{D}_k$, which is the boundary of $\widetilde{\Xi}_k$, and local branches of $\widetilde{\Xi}_k$ near $P(z)$ might have boundary singularities. □

Let Ω be an open subset of $\mathbb{R}^2$ and assume that the coefficients of the recurrence relation (1.6) are sufficiently generic. The arguments in the proof of Proposition 3, applied to the induced maxmod-discriminant $\Xi_{\widetilde{\phi}}$ of the associated symbol equation (1.7), imply that the only possible singularities on the curve $\Xi_{\widetilde{\phi}}$ are the endpoints and "Y-type" singularities, i.e., triple (local) rays emanating from a given point; see Figures 1–5. Note, however, that the transversal intersection of smooth branches (that is, the interval and the circle) in Figure 4 is actually unstable, since it disappears under a small perturbation of the coefficients of the recurrence relation.

Given a topological space X, let $\widehat{X}$ be its one-point compactification.

Proposition 4. *For all $k \in \mathbb{N}$, we have the following.*

(i) *The hypersurfaces $\widetilde{\Xi}_k$ and Ξ_k are contractible in Pol_k; thus, their usual (co)homology groups are trivial.*

(ii) *The one-point compactification* $\widehat{\Xi}_k$ *is Alexander dual in the sphere* S^{2k} *to a circle* S^1. *Therefore,* $\widehat{\Xi}_k$ *is homotopically equivalent to a sphere* S^{2k-2}, *so that* $H_i\big(\widehat{\Xi}_k, \mathbb{Z}\big) \cong \mathbb{Z}$ *if* $i \in \{0, 2k-2\}$ *and* $H_i\big(\widehat{\Xi}_k, \mathbb{Z}\big) = 0$ *otherwise.*

(iii) *The one-point compactification* $\widehat{\widetilde{\Xi}}_k$ *is Alexander dual in the sphere* S^{2k} *to a* $(k-1)$*-dimensional torus* $\mathcal{T}^{k-1}$. *Hence* $\widetilde{H}_i\big(\widehat{\widetilde{\Xi}}_k, \mathbb{Z}\big) \cong \widetilde{H}^{2k-i-1}\big(\mathcal{T}^{k-1}, \mathbb{Z}\big)$ *for every* i. *Here,* $\widetilde{H}_i(X)$ *(respectively,* $\widetilde{H}^i(X)$*) is the reduced mod point homology (respectively, cohomology) of the topological space* X*; see e.g.* [24].

Proof. The contractibility of $\widetilde{\Xi}_k$ and Ξ_k follows directly from their quasihomogeneity. To show (ii) and (iii), we use the standard Alexander duality in Pol_k. Let $\Upsilon_k = Pol_k \setminus \Xi_k$ and $\widetilde{\Upsilon}_k = Pol_k \setminus \widetilde{\Xi}_k$ denote the complements in Pol_k of Ξ_k and $\widetilde{\Xi}_k$, respectively. Then

$$\widetilde{H}^{2k-i}(\Upsilon_k, \mathbb{Z}) \cong \widetilde{H}_i\big(\widehat{\Xi}_k, \mathbb{Z}\big) \text{ and } \widetilde{H}^{2k-i}\big(\widetilde{\Upsilon}_k, \mathbb{Z}\big) \cong \widetilde{H}_i\big(\widehat{\widetilde{\Xi}}_k, \mathbb{Z}\big).$$

Obviously, the space $\widetilde{\Upsilon}_1 \simeq \Upsilon_1 \simeq \mathbb{C}$ is contractible. The next lemma describes the topology of Υ_k and $\widetilde{\Upsilon}_k$ for $k > 1$.

Lemma 9. *For every* $k \in \mathbb{N}$, $k \geq 2$,

(i) $\widetilde{\Upsilon}_k$ *is an open* $2k$*-dimensional manifold which is homotopically equivalent to the* $(k-1)$*-dimensional torus* $\mathcal{T}^{k-1}$*;*

(ii) Υ_k *is an open* $2k$*-dimensional manifold* Υ_k *which is homotopically equivalent to a circle* S^1.

Proof. (i) The space $\widetilde{\Upsilon}_k$ consists of all k-tuples of complex numbers with distinct absolute values. Let $X_k = \{r_1 < r_2 < \ldots < r_k \mid r_1 \geq 0\}$ denote the set of all possible k-tuples of distinct absolute values. Then $\widetilde{\Upsilon}_k$ is "fibered" over X_k with a "fiber" which is isomorphic to $\mathcal{T}^k$ if $r_1 \neq 0$ and isomorphic to $\mathcal{T}^{k-1}$ if $r_1 = 0$. To obtain an actual fibration, consider the set $\widehat{X}_k = \{0 < r_2 < \ldots < r_k\}$ of the absolute values of the roots, starting from the second smallest. Now $\widetilde{\Upsilon}_k$ is actually fibered over $\widehat{X}_k$ with a fiber isomorphic to $\mathcal{T}^{k-1} \times D_{r_2}$, where D_{r_2} is the open disk of radius $r_2 > 0$ centered at the origin. The observation that $\widehat{X}_k$ is contractible now implies that $\widetilde{\Upsilon}_k$ is homotopically equivalent to $\mathcal{T}^{k-1}$.

(ii) The space Υ_k consists of all k-tuples of complex numbers such there exists a unique number with largest absolute value in the considered k-tuple. Let $0 < r_{max}$ denote this largest absolute value. Then Υ_k is fibered over $\mathbb{R}^+ \simeq \{r_{max}\}$ with a fiber given by the product $S^1 \times Pol_{k-1}(r_{max})$, where $Pol_{k-1}(r_{max})$ is the set of all polynomials of degree $k-1$ whose roots lie in the open disk of radius r_{max} centered at the origin. Since both $\mathbb{R}^+$ and $Pol_{k-1}(r_{max})$ are contractible, it follows that the space Υ_k is homotopically equivalent to S^1. $\square$

Lemma 9 and the fact that S^1 is always unknotted in Pol_k for $k \geq 2$ complete the proof of Proposition 4. $\square$

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