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A Note on Planarity Stratification of Hurwitz Spaces

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Abstract. One can easily show that any meromorphic function on a complex closed Riemann surface can be represented as a composition of a birational map of this surface to $\mathbb{CP}^2$ and a projection of the image curve from an appropriate point $p \in \mathbb{CP}^2$ to the pencil of lines through p. We introduce a natural stratification of Hurwitz spaces according to the minimal degree of a plane curve such that a given meromorphic function can be represented in the above way and calculate the dimensions of these strata. We observe that they are closely related to a family of Severi varieties studied earlier by J. Harris, Z. Ran, and I. Tyomkin.

1 Basic Definitions and Facts

In what follows we will always work over the field $\mathbb{C}$ of complex numbers, and by a genus $p_g(\mathcal{C})$ of a (singular) curve $\mathcal{C}$ we mean its geometric genus, *i.e.,* the genus of its normalization. We start with the following statement.

Proposition 1.1 *Any meromorphic function $f\colon \mathcal{C} \to \mathbb{CP}^1$ on a complex closed Riemann surface $\mathcal{C}$ can be represented as $f = \pi_p \circ \nu$ where $\nu\colon \mathcal{C} \to \mathbb{CP}^2$ is a birational mapping of $\mathcal{C}$ to its image and $\pi_p\colon \nu(\mathcal{C}) \to \mathbb{CP}^1$ is the projection of the image curve $\nu(\mathcal{C})$ from a point $p \in \mathbb{CP}^2$ to the pencil of lines through p.*

Proof Let $\mathbb{C}(\mathcal{C})$ be the field of meromorphic functions on $\mathcal{C}$. Consider its subfield $\mathbb{C}(f) \subset \mathbb{C}(\mathcal{C})$ generated by f (*i.e.,* the set of all rational functions of f). Since $\mathcal{C}$ is one-dimensional, the field extension $\mathbb{C}(\mathcal{C}) : \mathbb{C}(f)$ is finite. Assume that $\mathbb{C}(\mathcal{C}) \neq \mathbb{C}(f)$ and choose any meromorphic function $g\colon \mathcal{C} \to \mathbb{CP}^1$ generating this extension. Removing a point from $\mathbb{CP}^1$ and its inverse images under f and g, we get a birational mapping $\mathcal{C} \setminus \{\text{finite set}\} \to \mathbb{C}^2$ given by the pair (f, g). Its compactification gives a birational mapping $\nu\colon \mathcal{C} \to \mathbb{CP}^2$. Projection "along the second coordinate" gives a presentation of the original meromorphic function $f\colon \mathcal{C} \to \mathbb{CP}^1$ as $f = \pi_p \circ \nu$. Finally observe that $\mathbb{C}(\mathcal{C}) = \mathbb{C}(f)$ only if $\mathcal{C}$ has genus 0 and $\deg f = 1$, the latter case being trivial. ■

Obviously, if ν maps $\mathcal{C}$ birationally on its image and $f = \pi_p \circ \nu$ for some point $p \in \mathbb{CP}^2$, then $\deg(\nu(\mathcal{C})) = \deg f$ if and only if $p \notin \nu(\mathcal{C})$ and $\deg(\nu(\mathcal{C})) > \deg f$ if $p \in \nu(\mathcal{C})$.

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Definition 1.2 The *planarity defect* $\operatorname{pdef}(f)$ of a meromorphic function $f\colon \mathcal{C} \to \mathbb{CP}^1$ equals

$$\operatorname{pdef}(f) := \min_{\nu}(\deg(\nu(\mathcal{C})) - \deg(f)$$

such that $f = \pi_p \circ \nu$, as above.

We start with the following simple observation.

Lemma 1.3 *Given $f\colon \mathcal{C} \to \mathbb{CP}^1$, $\operatorname{pdef}(f) = 0$ if and only if $h^0(f^\star(\mathcal{O}_1)) \geq 3$, and for almost any point $p \in \mathcal{C}$ and any other point $q \neq p$,*

$$h^0\big(f^\star(\mathcal{O}(1)) - p - q\big) = h^0\big(f^\star(\mathcal{O}(1))\big) - 2.$$

Proof Indeed, observe that f determines a 1-dimensional linear subsystem in the complete linear system $f^\star(\mathcal{O}(1))$. (We count dimensions of linear systems projectively.) Moreover, if $r_f = h^0(f^\star(\mathcal{O}(1))) \geq 3$, then the system $f^\star(\mathcal{O}(1))$ defines a map $\phi_f\colon \mathcal{C} \to \mathbb{CP}^{r_f-1}$ with $r_f - 1 \geq 2$. If additionally, sections of $f^\star(\mathcal{O}(1))$ separate each generic point on $\mathcal{C}$ from all other points, then ϕ_f is birational on the image. The latter condition is made explicit above. Choosing an appropriate 3-dimensional subsystem of $f^\star(\mathcal{O}(1))$ including f, we get the required statement. ∎

Unfortunately, the second condition is not easy to check in concrete situations, see Remark 1.5. We say that a linear system $\mathcal{L}$ on a curve $\mathcal{C}$ is birationally very ample if the image of $\mathcal{C}$ in the projectivized space of its sections is birationally equivalent to $\mathcal{C}$; compare [17].

The following sufficient condition of the birational very ampleness of $f^\star(\mathcal{O}(1))$ is valid. Recall that a point $z \in \mathbb{CP}^1$ is called a *simple branching point* (resp. *complicated branching point*) of a meromorphic function $f\colon \mathcal{C} \to \mathbb{CP}^1$ of degree d if the number of its geometrically distinct preimages in $f^{-1}(p)$ equals $d-1$ (resp. is smaller than $d-1$).

Lemma 1.4 *If $f\colon \mathcal{C} \to \mathbb{CP}^1$ has at most one complicated branching point, then $\operatorname{pdef}(f) = 0$ if and only if $h^0(f^\star(\mathcal{O}(1))) \geq 3$. In particular, under the above assumptions, if $\deg(f) = d \geq g + 2$, where g is the genus of $\mathcal{C}$, then $\operatorname{pdef}(f) = 0$.*

Proof As in Lemma 1.3, the necessary condition for $\operatorname{pdef}(f) = 0$ is

$$r_f := h^0\big(f^\star(\mathcal{O}(1))\big) \geq 3.$$

By the Riemann–Roch formula

$$r_f := h^0\big(f^\star(\mathcal{O}(1))\big) = d - g + 1 + h^0(K \setminus (f)_\infty), \tag{1.1}$$

where $(f)_\infty$ is the pole divisor of f. The linear system $f^\star(\mathcal{O}(1))$ determines the mapping $\phi_f\colon \mathcal{C} \to \mathbb{CP}^{r_f-1}$. Moreover, if $r_f \geq 3$ and f has at most one complicated branching point, then ϕ_f defines a birational mapping of $\mathcal{C}$ on its image $\phi_f(\mathcal{C})$. Indeed, since $r_f \geq 3$, the only thing that we have to exclude is that $\phi_f\colon \mathcal{C} \to \phi_f(\mathcal{C})$ is a non-trivial covering. Assume that $\phi_f\colon \mathcal{C} \to \phi_f(\mathcal{C})$ is a non-trivial covering. Notice that independently of the fact whether ϕ_f is birational on the image or not, $f = \pi_p \circ \phi_f$, where π_p is a projection of $\mathbb{CP}^2 \setminus p \to \mathbb{CP}^1$ from some point $p \in \mathbb{CP}^2$. Also the map f can be lifted in the standard way to $f = \widetilde{\pi}_p \circ \widetilde{\phi}_f$ where $\widetilde{\phi}_f\colon \mathcal{C} \to \widetilde{\phi}_f(\mathcal{C})$ is the standard lift

of ϕ_f to the normalization $\widetilde{\phi}_f(\mathcal{C})$ of the image $\phi_f(\mathcal{C})$, and $\widetilde{\pi}_p$ is the composition of the standard map from the normalization $\widetilde{\phi}_f(\mathcal{C})$ to the image curve $\phi_f(\mathcal{C})$ with the projection π_p. Branching points of f are either the images under $\widetilde{\pi}_p$ of the branching points of $\widetilde{\phi}_f$ or the branching points of $\widetilde{\pi}_p$ itself. But each branching point of $\widetilde{\pi}_p$ is a non-simple branching point of f. Therefore, in the case of a non-trivial covering we get at least two non-simple branching points of f, which contradicts our assumptions. The case when $\phi_f(\mathcal{C})$ is a line in $\mathbb{CP}^2$ is obviously impossible due to the dimension of the linear system $f^\star(\mathcal{O}(1))$. Finally observe that if $d \geq g+2$, then r_f is at least 3 by the Riemann–Roch formula (1.1). ∎

Remark 1.5 Observe that for $d \geq g+1$, any curve $\mathcal{C}$ of genus g admits a meromorphic function of degree d with all simple branching points, *i.e.,* the natural map $\mathcal{H}_{g,d} \to \mathcal{M}_g$ where $\mathcal{M}_g$ is the moduli space of curves of genus g is surjective; see [20]. Also for $d \geq 2g+1$, no genericity assumptions whatsoever on f are required for birational ampleness, since $f^\star(\mathcal{O}(1))$ becomes very ample and defines an embedding $\mathcal{C} \to \mathbb{CP}^{r_f-1}$. However, in the interval $g+2 \leq d \leq 2g$ this linear system might define a non-trivial covering on the image as shown by the next classical example; see [11, Proposition 5.3]. This circumstance shows that one needs some additional assumption on the branching points to avoid such coverings.

Example Let $\mathcal{C}$ be a hyperelliptic curve of genus $g > 2$ and let $|L|\colon \mathcal{C} \to \mathbb{CP}^1$ be the hyperelliptic map. Let s_0 and s_1 be a basis for $H^0(L)$. The Riemann–Roch formula gives that $h^0(gL) = g+1 < 2g$. Note that there are precisely $\binom{d+n-1}{n-1}$ monomials of degree d in n variables. Therefore, there are precisely $d+1$ monomials of degree d in s_0 and s_1. The map $|L|\colon \mathcal{C} \to \mathbb{CP}^1$ is given by

$$\mathcal{C} \ni p \longmapsto \big[s_0(p) : s_1(p)\big] \in \mathbb{CP}^1,$$

while the map $|mL|\colon \mathcal{C} \to \mathbb{CP}^g$ is given by

$$P \longmapsto \big[s_0(p)^g : s_0(p)^{g-1}s_1(p) : \cdots : s_1(p)^g\big].$$

But it is now clear that $|mL|\colon \mathcal{C} \to \mathbb{CP}^g$ can be factored as $|L|\colon \mathcal{C} \to \mathbb{CP}^1$ followed by the Veronese embedding $V\colon \mathbb{CP}^1 \to \mathbb{CP}^g$. Hence, the image of $\mathcal{C}$ under the map $|mL|$ is a rational normal curve. Now suppose that $m > g$. Then the Riemann–Roch formula gives $h^0(mL) = 2mg+1 > m+1$. Thus, s_0 and s_1 only generate a subspace of $H^0(mL)$ and the above argument no longer works (which is good, since $|mL|$ determines a closed embedding).

We now characterize the vanishing of the planarity defect in different terms. Consider the push-forward sheaf $f_\star(\mathcal{O}_\mathcal{C})$ on $\mathbb{CP}^1$. Since f is a finite map of compact curves, $f_\star(\mathcal{O}_\mathcal{C})$ is a vector bundle on $\mathbb{CP}^1$ whose dimension equals $\deg(f)$. By the well-known result of Grothendieck, $f_\star(\mathcal{O}_\mathcal{C}) = \mathcal{O} \oplus \sum_i \mathcal{O}(a_i)$, where a_i are integers; see *e.g.,* [12]. Observe that all a_i must be negative, since $h^0(\mathcal{O}_\mathcal{C}) = h^0(f_\star(\mathcal{O}_\mathcal{C})) = 1$.

Proposition 1.6 *For any meromorphic function $f\colon \mathcal{C} \to \mathbb{CP}^1$ with at most one complicated branching point, its planarity defect* $\operatorname{pdef}(f)$ *vanishes if and only if $a_{\max} = -1$, where $a_{\max}$ is the maximal of all a_i's in the above notation.*

Proof Let us show that under our assumptions $\operatorname{pdef}(f) = 0 \leftrightarrow a_{\max} = -1$. We need to check that $h^0(f^\star(\mathcal{O}(1))) \geq 3$ if and only if $a_{\max} = -1$. Consider $f_\star(f^\star(\mathcal{O}(1)))$. Observe that, $h^0(f_\star(f^\star(\mathcal{O}(1)))) = h^0(f^\star(\mathcal{O}(1)))$, since f is a finite map of compact algebraic curves. Now by projection formula, see [11, Ex. 8.3]

$$f_\star\big(f^\star(\mathcal{O}(1))\big) = \mathcal{O}(1) \otimes f_\star(\mathcal{O}_\mathcal{C}) = \mathcal{O}(1) \oplus \sum_i \mathcal{O}(a_i + 1).$$

Since $a_{\max} = -1$, at least one of the terms $\mathcal{O}(a_i + 1)$ equals $\mathcal{O}$. Therefore

$$h^0(f_\star(f^\star(\mathcal{O}(1)))) = h^0(\mathcal{O}(1)) + \sum_i h^0(\mathcal{O}(a_i + 1)) \geq 2 + 1.$$

In fact, $h^0(f_\star(f^\star(\mathcal{O}(1)))) = 2 +$ the number of indices i such that $a_i = -1$. ∎

Proposition 1.6 shows that there is a connection of the planarity defect with the slope invariants of meromorphic functions and with the Maroni strata (*cf.* [6,18]). In fact, the following statement is true.

Proposition 1.7 *Given a meromorphic function $f\colon \mathcal{C} \to \mathbb{CP}^1$ of degree d, its planarity defect* $\operatorname{pdef}(f)$ *equals $d' - d$ where d' is the minimal degree of a linear system $\mathcal{L}$ such that* (a) *$\mathcal{L}$ is birationally very ample and* (b) *the (effective) divisor of $f^\star(\mathcal{O}(1))$ is contained in the (effective) divisor of $\mathcal{L}$.*

Proof If $f^\star(\mathcal{O}(1))$ can serve as $\mathcal{L}$, then there is nothing to prove. Otherwise the divisor of $\mathcal{L}$ must be strictly larger than that of $f^\star(\mathcal{O}(1))$. In the latter case one can choose a 1-dimensional linear subsystem of $\mathcal{L}$ defining a meromorphic function $g\colon \mathcal{C} \to \mathbb{CP}^1$ that is not proportional to f. Considering the map $\psi\colon \mathcal{C} \to \mathbb{C}^2$ given by (f, g) and extending it to the map $\widetilde{\psi}\colon \mathcal{C} \to \mathbb{CP}^2$, we get the required planarity defect. ∎

1.1 Planarity Stratification of Small Hurwitz Spaces

The small Hurwitz space of degree d functions of genus g curves is defined as:

$$\mathcal{H}_{g,d} = \big\{ f\colon \mathcal{C} \to \mathbb{CP}^1 \mid f \text{ has only simple branched points,} \quad \deg f = d \geq 2, \operatorname{gen}(\mathcal{C}) = g \geq 0 \big\}.$$

Recall that $\dim \mathcal{H}_{g,d}$ equals the number of (simple) branching points of a function from $\mathcal{H}_{g,d}$ and is given by the formula

$$\dim \mathcal{H}_{g,d} = 2d + 2g - 2.$$

Small Hurwitz spaces were introduced and studied in substantial details by Clebsch [5] and Hurwitz [13] at the end of the 19-th century as a tool for investigation of the moduli space $\mathcal{M}_g$ of genus g curves.

Proposition 1.1 allows us to introduce the *planarity stratification* of $\mathcal{H}_{g,d}$:

$$\mathcal{H}_{g,d}^{m(g,d)} \subset \mathcal{H}_{g,d}^{m(g,d)+1} \subset \cdots \subset \mathcal{H}_{g,d}^{M(g,d)} = \mathcal{H}_{g,d}, \tag{1.2}$$

where $\mathcal{H}_{g,d}^{l}$ consists of all meromorphic functions in $\mathcal{H}_{g,d}$ whose planarity defect does not exceed l. Here $m(g,d)$ (resp. $M(g,d)$) is the minimal (resp. maximal) value

of the planarity defect for degree d meromorphic functions with all simple branching points on curves of genus g.

We present some information about this stratification.

Proposition 1.8 *For any pair (g,d) where $g \geq 0$ and $d \geq 2$,*

$$m(g,d) = \min_{l \geq 0} \binom{d+l-1}{2} - \binom{l}{2} \geq g, \tag{1.3}$$

which gives

$$m(g,d) = \max\left(0, \left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil\right). \tag{1.4}$$

Moreover, the following result holds.

Theorem 1.9 *In the above notation, given g, d, and $l \geq m(g,d)$, the stratum $\mathcal{H}^l_{g,d}$ is irreducible and its dimension is given by*

$$\dim \mathcal{H}^l_{g,d} = \min(3d+g+2l-4, 2d+2g-2). \tag{1.5}$$

The substantial part of the proof of Theorem 1.9 consists of the following generalization of the famous result by J. Harris [11] showing that the space of plane curves of genus g and degree d where $g \leq \binom{d-1}{2}$ is an irreducible variety whose dense subset consists of nodal curves of genus g (irreducibility of the Severi varieties). Fixing as above a point $p \in \mathbb{CP}^2$, denote by $S_{g,d,l}$ the variety of reduced irreducible plane curves of degree d having genus g and order l at p, where $g \leq \binom{d+l-1}{2} - \binom{l}{2}$. (The order of a plane curve at a given point is the multiplicity of its local intersection at p with a generic line passing through p.) Denote by $W_{g,d,l} \subset S_{g,d,l}$ its subset consisting of curves having an ordinary singularity of order l at p (*i.e.,* transversal intersection of l smooth local branches) and only usual nodes outside p.

Theorem 1.10 (i) *$W_{g,d,l}$ is a smooth manifold of dimension $3d+g+2l-1$.*
(ii) *$W_{g,d,l}$ is dense in $S_{g,d,l}$.*
(iii) *$S_{g,d,l}$ is irreducible.*

The main result of [11] is the proof of the same statement in the basic case $l = 0$. Theorem 1.10 follows from already known results of Z. Ran [19] and I. Tyomkin [21]. We first prove Proposition 1.8 and Theorem 1.10 followed by Theorem 1.9.

Lemma 1.11 *The genus of a plane curve decreases by at least $\binom{l}{2}$ by a singularity of order l. Moreover the ordinary singularity of order l decreases the genus by exactly $\binom{l}{2}$.*

Proof The following algorithm describes by which number the genus of a plane curve of degree d is decreased due to a singularity of order l.

Step 1 Subtract $\binom{l}{2}$ from $\binom{d-1}{2}$.
Step 2 Blow up the singularity in the plane. The strict transform of the curve will intersect the exceptional divisor at l points (counting multiplicities). If each

of these (geometrically distinct) points is smooth on the strict transform, then the genus drops by exactly $\binom{l}{2}$.

Step 3 If among these points there exist singular points, we have to repeat the previous step, *i.e.,* if the order of singularity is s, then we decrease the genus by $\binom{s}{2}$, then we blow up this point etc.

After finitely many such steps the curve becomes smooth. (Further blow-ups will not change the genus.) Thus the minimal decrease of genus equals $\binom{l}{2}$. ∎

Proof of Proposition 1.8 The necessity of (1.3) is obvious. Indeed, we need to construct a plane curve of degree $d+l$ such that it has a singularity of order l at p (so that its projection from p will be a covering of degree d) and has a genus of normalization equal to g. Having a singularity of order l at p decreases the genus by at least $\binom{l}{2}$ compared to $\binom{d+l-1}{2}$, which is the genus of a smooth curve of degree $d+l$; see Lemma 1.11. Thus, inequality (1.3) must be satisfied. To show that the least value of l satisfying (1.3) is enough, consider first a configuration of l generic lines through p and additionally d lines in $\mathbb{CP}^2$ in general position. This curve has genus 0. A slight deformation of this curve by a polynomial vanishing up to order $l+1$ at p will resolve all nodes outside p and given $g = \min_{l\geq 0}\binom{d+l-1}{2} - \binom{l}{2}$. A more careful deformation will resolve any number of nodes between 0 and $\binom{d}{2}$; see the proof of Theorem 1.10. The classical case $g \leq \binom{d-1}{2}$ is well presented in [10, Appendix E] and the general case in [19]. ∎

We will need some information about the Hirzebruch surfaces and the Severi varieties on them. For a given non-negative integer n, let $\Sigma_n = \mathrm{Proj}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(n))$ be the Hirzebruch surface and let $\kappa\colon \Sigma_n \to \mathbb{CP}^1$ be the natural projection. Consider two non-zero sections $(1,0), (0,\sigma) \in H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(n))$. They define the maps

$$\mathbb{CP}^1 \setminus \mathcal{Z}(\sigma) \to \Sigma_n,$$

where $\mathcal{Z}(\sigma)$ is the zero locus of σ. We denote the closures of the images of these maps by L_0 and L_∞, respectively. (It is clear that the homological class of L_∞ is independent of the choice of σ.) The following facts are standard.

Proposition 1.12 (i) *The Picard group* $\mathrm{Pic}(\Sigma_n)$ *is a free abelian group generated by the classes F and L_∞, where F denotes the fiber of projection κ. (Observe that $L_0 = nF + L_\infty$.)*

(ii) *The intersection form on* $\mathrm{Pic}(\Sigma_n)$ *is given by $F^2 = 0, L_\infty^2 = -n$, and $F \cdot L_\infty = 1$.*

(iii) *Any effective divisor $M \in \mathrm{Div}(\Sigma_n)$ is linearly equivalent to a linear combination of F and L_∞ with non-negative coefficients. Moreover, if M does not contain L_∞, then it is linearly equivalent to a combination of F and L_0 with non-negative coefficients.*

(iv) *The canonical class is given by*

$$K_{\Sigma_n} = -(2L_\infty + (2+n)F) = -(L_0 + L_\infty + 2F).$$

(v) *Any smooth curve $\mathcal{C}$ with the class $dL_0 + kF$ has genus* $g(\mathcal{C}) = \frac{(d-1)(dn+2k-2)}{2}$.

Let g, d, k be non-negative integers. We define the Severi variety

$$V_{g,d,k} \subseteq |\mathcal{O}_{\Sigma_n}(sL_0 + kF)|$$

to be the closure of the locus of reduced nodal curves of genus g that do not contain L_∞, and we define $V^{\mathrm{irr}}_{g,d,k} \subset V_{g,d,k}$ to be the union of the irreducible components whose generic points correspond to irreducible curves.

The main result of [21, Theorem 3.1] is as follows.

Theorem 1.13 *For any triple g, k, d of non-negative integers, the variety $V^{\mathrm{irr}}_{g,d,k} \subset V_{g,d,k}$ (if non-empty) is irreducible and of expected dimension.*

Proof of Theorem 1.10 Let us first naively count the expected dimension of $S_{g,d,l}$. Indeed, the dimension of the space $S_{g,d,l}$ of plane curves of degree $d+l$ with a singularity at p of order l equals $\frac{(d+l)(d+l+3)}{2} - \binom{l+1}{2}$. The number of nodes on such a curve under the assumptions that it has genus g equals

$$\sharp_{nodes} = \binom{d+l-1}{2} - \binom{l}{2} - g. \tag{1.6}$$

Assuming that each node decreases the dimension by 1, we get

$$\exp\dim S_{g,d,l} = 3d + g + 2l - 1.$$

We finish our proof with a reference to Theorem 1.13. Indeed, if one blows up the point $p \in \mathbb{CP}^2$, then one gets the first Hirzebruch surface Σ_1. Observe that plane curves of degree $d+l$ having a singularity of order l at p will after the blow-up lie in the class $(d+l)L_0 - lL_\infty = dL_0 + lF$. Therefore, the above set $W_{g,d,l}$ of irreducible plane curves having the singularity of order l at the point p after this blow-up will transform into the space $V^{\mathrm{irr}}_{g,d,l}$ in the above notation. (We consider only the strict transform of each curve disregarding the exceptional divisor.) Thus, by the latter theorem, the variety $S_{g,d,l}$ is irreducible and of expected dimension. Another proof of essentially the same result directly in the plane $\mathbb{CP}^2$ can be found in [19, Irreducibility Theorem, p. 122]. ∎

Observe that, fixing a point $p \in \mathbb{CP}^2$, there exists a three-dimensional group $G_p \subset PGL_3$ of projective transformations preserving p as well as the pencil of lines through p. In other words, each line through p will be mapped to itself. Obviously G_p acts on plane curves and curves from the same orbit define equivalent branched coverings of $\mathbb{CP}^1$ obtained by projection from p.

Proof of Theorem 1.9 To settle this theorem we need to prove that curves from $S_{g,d,l}$ that give equivalent branched coverings of $\mathbb{CP}^1$ when projecting from p do not appear in families of G_p-orbits. If this is true, then $\dim \mathcal{H}^l_{g,d} = \dim S_{g,d,l} - 3$.

Let (as above) $S_{g,d,l}$ be the Severi variety of all plane curves of degree $d+l$, genus g and ordinary singularity of order l at point p. Let $H(g,d)$ be the Hurwitz space of all branched coverings of degree d and genus g. (Observe that, in general, $H(g,d)$ is different from $\mathcal{H}_{g,d}$, which only includes branching coverings with only simple branching points.) Let $B\colon S_{g,d,l} \to H(g,d)$ be the map sending each plane curve $\Gamma \in S_{g,d,l}$ to the branched covering of $\mathbb{CP}^1$ whose source is the normalization N of Γ and which is obtained by projecting Γ from the point p.

Proposition 1.14 *The dimension of the fiber of the above map over a branched covering $f: N \to \mathbb{CP}^1$, where N is the normalization of a generic curve $\Gamma \in S_{g,d,l}$ equals $h^0(N, \mathcal{O}_N(E))$. Here E is the divisor of degree $d + 2l$ on N obtained as the pull-back of p together with the pull-back of the intersection of Γ with a general line in $\mathbb{CP}^2$. (For an arbitrary curve $\Gamma \in S_{g,d,l}$, the dimension of the fiber is at most $h^0(N, \mathcal{O}_N(E))$.)*

(Observe that since $B: S_{g,d,l} \to H(g,d)$ is a smooth map at a generic curve $\Gamma \in S_{g,d,l}$, the dimension of a generic fiber is well defined.)

Proof Let $\pi: \Sigma_1 \to \mathbb{CP}^2$ be the standard projection of the first Hirzebruch surface Σ_1 obtained by the blow-up of the point p to $\mathbb{CP}^2$. We have natural maps

$$\begin{array}{ccc} N & \xrightarrow{\phi} & \Sigma_1 \\ & \searrow^{f} & \downarrow \\ & & \mathbb{CP}^1 \end{array}$$

and exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_N & \longrightarrow & \phi^* T_{\Sigma_1} & \longrightarrow & N_\phi & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_N & \longrightarrow & f^* T_{\mathbb{CP}^1} & \longrightarrow & N_f & \longrightarrow & 0, \end{array} \tag{1.7}$$

where N_f (resp. N_ϕ) is the normal sheaf of f (resp. ϕ), *i.e.*, $N_f = \mathrm{Coker}(T_N \to f^* T_{\mathbb{CP}^1})$ (resp. $N_\phi = \mathrm{Coker}(T_N \to \phi^* T_{\Sigma_1})$. It is known that

$$\mathrm{Def}^1(N, \phi) = H^0(N, N_\phi) \quad \text{and} \quad \mathrm{Def}^1(N, f) = H^0(N, N_f)$$

are the tangent spaces to the space of deformations of the pairs (N, ϕ) and (N, f) respectively. The first one is the tangent space to the Severi variety if ϕ is an immersion; the second one is the tangent space to the Hurwitz space. Sequence (1.7) implies that the kernels $\alpha: \phi^* T_{\Sigma_1} \to f^* T_{\mathbb{CP}^1}$ and $N_\phi \to N_f$ coincide, since $\phi^* T_{\Sigma_1} \twoheadrightarrow f^* T_{\mathbb{CP}^1}$. Since the $\mathbb{CP}^1$-bundle $\Sigma_1 \to \mathbb{CP}^1$ admits two non-intersecting sections (the line L and the inverse image of p in Σ_1), $\mathrm{Ker}\, \alpha = \phi^* \mathcal{O}_{\Sigma_1}(L + \pi^{-1}(p))$. ∎

For a small number of nodes compared to the degree of the irreducible plane curve, Theorem 1.9 is immediate from the following fact; see [2, Exercise 20(iii), § 1, Appendix A, Ch. 1]. (Moreover a stronger statement is valid.) It claims that if the number δ of nodes of an irreducible plane nodal curve $\Gamma \subset \mathbb{CP}^2$ of degree d satisfies the inequality $\delta < d - 3$, then the linear system g^2_d cut out on Γ by lines is complete and unique on the normalization N of Γ. This fact immediately implies that under the above assumptions two plane curves whose normalizations are isomorphic will be projectively equivalent. Then for degree at least 4 it will be straightforward that if the isomorphism of their normalizations is induced by the equivalence of the meromorphic functions obtained by projection from the same point p, then the projective transformation realizing this equivalence belongs to G_p, see the proof of Proposition 2.1. In general, one

should show that for a generic curve $\Gamma \in S_{g,d,l}$, one has $h^0(N, \mathcal{O}_N(E)) = 3$. This fact is also valid and will appear in a forthcoming publication by the second author. ■

Corollary 1.15 *Given g, d as above,*

$$M(g,d) = \max\left(0, \left\lceil \frac{g-d+2}{2} \right\rceil\right). \tag{1.8}$$

In particular, $m(g,d) = M(g,d) = 0$ if and only if $d \geq g+2$.

Proof Theorem 1.9 implies that $M(g,d)$ equals the minimal non-negative integer l for which

$$3d + g + 2l - 4 \geq 2d + 2g - 2 \quad \Longleftrightarrow \quad 2l \geq g - d + 2.$$

The latter inequality implies that $M(g,d) = \max(0, \lceil \frac{g-d+2}{2} \rceil)$. This formula shows that $M(g,d) = 0$ if and only if $d \geq g+2$. ■

Corollary 1.16 *The planarity stratification of $\mathcal{H}_{g,d}$ consists of one term in the following two cases. Either $d \geq g+2$ in which case the planarity defect vanishes, or $d = 3$, in which case the planarity defect equals $\lceil \frac{g-1}{2} \rceil$.*

Proof We have that $\mathcal{H}_{g,d}$ consists of one term if and only if $m(g,d) = M(g,d)$. By Proposition 1.8 and Theorem 1.9 (unless $M(g,d)$ vanishes, which happens if and only if $d \geq g+2$) this corresponds to the case when

$$\left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil = \left\lceil \frac{g-d+2}{2} \right\rceil.$$

If $d > 3$, then the denominator of the left-hand side is smaller than that of the right-hand side, and the numerator of the left-hand side is bigger than that of the right-hand side, which means that the equality never holds. For $d = 3$ the left-hand side and the right-hand side coincide giving the planarity defect equal to $\lceil \frac{g-1}{2} \rceil$. ■

1.2 Stratification of Hurwitz Spaces with One Complicated Branching Point

Analogously to the above, given a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n) \vdash d$ of a positive integer d, let

$$\mathcal{H}_{g,\mu} = \{f : \mathcal{C} \to \mathbb{CP}^1 \mid f \text{ has all simple branched points except at } \infty$$
$$\text{whose profile is given by } \mu,\ \deg f = d \geq 2, \operatorname{gen}(\mathcal{C}) = g \geq 0\}$$

the Hurwitz space of all degree d functions on genus g curves with one complicated branching point at ∞ having a given profile μ. Recall that $\dim \mathcal{H}_{g,\mu}$ equals the number of simple branching points of a function from $\mathcal{H}_{g,\mu}$ and is given by the formula

$$\dim \mathcal{H}_{g,\mu} = 2d + 2g - 2 - \sum_{i=1}^{n} (\mu_i - 1).$$

Proposition 1.1 allows us to introduce the *planarity stratification* of $\mathcal{H}_{g,\mu}$:

$$\mathcal{H}_{g,\mu}^{m(g,\mu)} \subset \mathcal{H}_{g,\mu}^{m(g,\mu)+1} \subset \cdots \subset \mathcal{H}_{g,\mu}^{M(g,\mu)} = \mathcal{H}_{g,\mu}. \tag{1.9}$$

Here $\mathcal{H}^l_{g,\mu}$ consists of all meromorphic functions in $\mathcal{H}_{g,\mu}$ whose defect does not exceed l. Observe that by Lemma 1.4, $M(g,\mu) \le d+2$.

Proposition 1.17 *For any pair $(g, \mu \vdash d)$, where $g \ge 0$ and $d \ge 2$,*

$$m(g,\mu) = \min_{l\ge 0} \binom{d+l-1}{2} - \binom{l}{2} \ge g, \tag{1.10}$$

which gives

$$m(g,\mu) = \left\lceil \frac{g - \binom{d-1}{2}}{d-1} \right\rceil.$$

(Observe that $m(g,\mu) = m(g,d)$, given by (1.3).)

Proof By definition the stratum $\mathcal{H}^{m(g,\mu)}_{g,u}$ should lie in at least $\mathcal{H}^{m(g,d)}_{g,d}$ or, possibly in the higher strata of the planarity stratification of $\mathcal{H}_{g,d}$. Therefore, $m(g,\mu)$ is at least equal to the minimal l given by the right-hand side of (1.10). The fact that $m(g,\mu)$ is exactly equal to the minimal l satisfying the latter condition is explained in the proof of Theorem 1.18. ∎

We have the following result above the dimensions of the strata of (1.9).

Theorem 1.18 *In the above notation, given g, d, and $l \ge m(g,\mu)$, the stratum $\mathcal{H}^l_{g,\mu}$ is equidimensional and its dimension is given by:*

$$\dim \mathcal{H}^l_{g,\mu} = \min\left(3d+g+2l-4-\sum_{i=1}^n(\mu_i-1), 2d+2g-2-\sum_{i=1}^n(\mu_i-1)\right).$$

Proof Theorem 1.18 follows directly from Lemmas 1.19 and 1.20. ∎

Fix a flag $p \in L_0 \subset \mathbb{CP}^2$, positive integers g, d, l, and a partition $\mu \vdash d$. Consider the locus $V \subset |\mathcal{O}^2_{\mathbb{CP}}(d+l)|$ of plane curves $\mathcal{C}$ such that: (1) $\deg \mathcal{C} = d+l$; (2) $\mathcal{C}$ is reduced and irreducible; (3) $\mathrm{mult}_p \mathcal{C} = l$; (4) $p_g(\mathcal{C}) = g$; (5) $\kappa^{-1}L_0 = \sum_i \mu_i q_i$ where $\kappa\colon N \to \mathcal{C}$ is the normalization map.

Again, let $\Sigma_1 = Bl_p\mathbb{CP}^2$ be the first Hirzebruch surface obtained by the blow-up of $\mathbb{CP}^2$ at p. Let $F_0 \subset \Sigma_1$ be the strict transform of L_0, and let F be the class of F_0. Denote by $L \subset \Sigma_1$ the class of the preimage of a general line in $\mathbb{CP}^2$, and denote by $E \subset \Sigma_1$ the exceptional divisor. Then V can be identified with the locus of curves $\mathcal{C} \in |\mathcal{O}_{\Sigma_1}((d+l)L - lE)| = |\mathcal{O}_{\Sigma_1}(dL + lF)|$ such that (i) $\mathcal{C}$ is reduced and irreducible; (ii) $p_g(\mathcal{C}) = g$; (iii) $\kappa^{-1}F_0 = \sum_i \mu_i q_i$. (Here $p_g(\mathcal{C})$ is the geometric genus.)

Let $V_1 \subset V$ be an irreducible component of V.

Lemma 1.19 $\dim V_1 \ge \exp\dim := -K_{\Sigma_1}\cdot\mathcal{C} + g - 1 - \sum_{i=1}^n(\mu_i - 1)$.

Proof Let $o \in V_1$ be a general point and let $\mathcal{C}_o$ be the corresponding curve. By [14, Lemma A.3] there exists a neighborhood W of $o \in V_1$ over which the family $\mathcal{C}_W \to W$ is equinormalizable, *i.e.,* if $N_W \to \mathcal{C}_W$ is the normalization, then for all $a \in W$, $(N_W)_a \to (\mathcal{C}_W)_a = \mathcal{C}_a$ is the normalization. Thus $\dim V_1$ is equal to the dimension of (a component of) the deformation space of $f\colon N_0 \to \Sigma_1$ satisfying condition (iii). Notice that condition (iii) has codimension less than or equal to

$\sum_{i=1}^{n}(\mu_i - 1)$ in the space of all deformations of the pair (N_0, f_0). Thus, it suffices to show that (any component of) $\mathrm{Def}(N_0, f_0)$ has dimension at least $-K_{\Sigma_1} \cdot \mathcal{C} + g - 1$. By the standard deformation theory, any component of the latter space has dimension $\geq \dim \mathrm{Def}'(N_0, f_0) - \dim \mathrm{Ob}(N_0, f_0)$. In our case $\mathrm{Def}'(N_0, f_0) = H^0(N_0, N_{f_0})$ and $\mathrm{Ob}(N_0, f_0) = H^1(N_0, N_{f_0})$ where N_{f_0} is the normal sheaf of f_0, *i.e.*, $N_{f_0} = \mathrm{Coker}(T_{N_0} \to f_0^* T_{\Sigma_1})$. This implies the statement, since $h^0(N_0, N_{f_0}) - h^1(N_0, N_{f_0}) = \chi(N_0, N_{f_0}) = -K_{\Sigma_1} \cdot \mathcal{C} + g - 1$ by the Riemann–Roch theorem. ∎

Lemma 1.20 $\dim V_1 \leq \exp\dim V_1$.

Proof If $\dim V_1 > \exp\dim V_1$, then there exists a configuration of n points on F_0 such that $\{\mathcal{C} \in V_1 | \mathcal{C} \cap F_0 = \text{given configuration}\}$ has dimension greater than $-K_{\Sigma_1} \cdot \mathcal{C} + g - 1 - \sum_{i=1}^{n}(\mu_i - 1) - n = -K_{\Sigma_1} \cdot \mathcal{C} + g - 1 - F_0 \cdot \mathcal{C}$, which is a contradiction with [21, Lemma 2.9]. ∎

Corollary 1.21 *Given g, μ as above,*

$$M_{g,\mu} = \max\left(0, \left\lceil \frac{g - d + 2}{2} \right\rceil\right).$$

In particular, $m_{g,\mu} = M_{g,\mu} = 0$ if and only if $d = \sum_{i=1} \mu_i \geq g + 2$.

Proof See the proof of Corollary 1.15. ∎

Stratification (1.2) is (almost) the special case of (1.9), the difference being that one simple branching point is placed at ∞.

Remark 1.22 According to the information the second author obtained from I. Tyomkin, he can prove that each stratum $\mathcal{H}^l_{g,\mu}$ is irreducible for $g = 0$ and $g = 1$, and he has an idea of the proof for other genera if $\mu \vdash d$ is not very complicated. Whether $\mathcal{H}^l_{g,\mu}$ could be irreducible for an arbitrary partition μ is unclear at present and might be a difficult problem.

2 Hurwitz Numbers of the Planarity Stratification and Zeuten-type Problems

Due to irreducibility of strata of (1.2) and equidimensionality of strata of (1.9) we can introduce the correspoding notion of Hurwitz numbers related to these strata. Recall that the *branching morphism*

$$\delta_{g,d} : \mathcal{H}_{g,d} \longrightarrow \mathbb{CP}^{2d+2g-2} \setminus \Delta$$

is, by definition, the map sending a meromorphic function f to the unordered set of its branching points (which are distinct by definiton). Here, $\Delta \subset \mathbb{CP}^{2d+2g-2}$ is the hypersurface of unordered $(2d + 2g - 2)$-tuples of points in $\mathbb{CP}^1$, where not all of them are pairwise distinct. It is well known that $\delta_{g,d}$ is a finite covering and its degree $h_{g,d}$ is called the (small) Hurwitz number. In particular, for $g = 0$ the corresponding

Hurwitz number $h_{0,d}$ equals $(2d-2)!d^{d-3}$. In general, however, closed formulas for $h_{g,d}$ (as well as for many other Hurwitz numbers) are unknown.

Analogously, the *branching morphism*

$$\delta_{g,\mu}\colon \mathcal{H}_{g,\mu} \to \mathbb{C}^{w_\mu} \setminus \Delta$$

is, by definition, the map sending a meromorphic function $f \in \mathcal{H}_{g,\mu}$ to the unordered set of its simple branching points (which are distinct by definiton). Here, $\Delta \subset \mathbb{C}^{w_\mu}$ is the hypersurface of unordered w_μ-tuples of points in $\mathbb{C}$ where not all of them are pairwise distinct and $w_\mu = 2d + 2g - 2 - \sum_{i=1}^{n}(\mu_i - 1)$. It is well known that $\delta_{g,\mu}$ is a finite covering and its degree $h_{g,\mu}$ is called the single Hurwitz number. In particular, for $g = 0$ the corresponding Hurwitz number $h_{0,\mu}$ equals

$$(d+n-2)! \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} d^{n-3}.$$

Stratifications (1.2)–(1.9) allow us to introduce Hurwitz numbers that take into account these filtrations. Before we introduce this notion in general, let us start with a motivating example.

Example Recall that, fixing a point $p \in \mathbb{CP}^2$, we have a three-dimensional group $G_p \subset PGL_3$ of projective transformations preserving p as well as the pencil of lines through p. Obviously, for $d > 1$, G_p acts (locally) freely on the space $S_{d,p}$ of all smooth plane curves of degree d not passing through p. The following simple statement holds.

As usual, two meromorphic functions $f_1\colon \mathcal{C}_1 \to \mathbb{CP}^1$ and $f_2\colon \mathcal{C}_2 \to \mathbb{CP}^1$ are called *equivalent* if there exists an isomorphism is: $\mathcal{C}_1 \to \mathcal{C}_2$ such that $f_2 \circ \mathrm{is} = f_1$.

Proposition 2.1 *Suppose that $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{CP}^2$ are smooth projective curves of degree at least 4 not passing through p. Then the meromorphic functions $\pi_p\mathcal{C}_1$ and $\pi_p\mathcal{C}_2$ are equivalent if and only if there exists an isomorphism* $\mathrm{is} \in G_p$ *such that* $\mathrm{is}(\mathcal{C}_1) = \mathcal{C}_2$.

Proof The "if" part being obvious, suppose that $\pi_p\mathcal{C}_1$ and $\pi_p\mathcal{C}_2$ are equivalent and that this equivalence is performed by the isomorphism is: $\mathcal{C}_1 \to \mathcal{C}_2$. For each line $\ell \ni p$, the isomorphism is maps $\mathcal{C}_1 \cap \ell$ to $\mathcal{C}_2 \cap \ell$; thus, is maps hyperplane sections of $\mathcal{C}_1$ to hyperplane sections of $\mathcal{C}_2$. Since both $\mathcal{C}_1$ and $\mathcal{C}_2$ are embedded in $\mathbb{CP}^2$ by the complete linear system of plane sections, this implies that is is a projective automorphism, *i.e.,* $\mathrm{is} \in PGL_3$. It remains to check that $\mathrm{is} \in G_p$; to that end, consider a generic $\ell \ni p$; this line intersects $\mathcal{C}$ at $d = \deg \mathcal{C} > 1$ points $p_1, \dots, p_d$, and these points are mapped by is to d distinct points on ℓ. So, $\mathrm{is}(\ell) = \ell$ for the generic (whence for any) $\ell \ni p$. If $\ell_1, \ell_2 \ni p$, then

$$is(p) = is(\ell_1 \cap \ell_2) = is(\ell_1) \cap is(\ell_2) = \ell_1 \cap \ell_2 = p,$$

which completes the proof. ∎

Denote by $\mathfrak{h}_d$ the number of different 3-dimensional orbits of the above action on the space $S_{d,p}$ with the same set of $d(d-1)$ tangent lines (*e.g.,* branching points of the projection).

Example One can easily observe that $\mathfrak{h}_2 = 2$, $\mathfrak{h}_3 = 40$ (which are the usual Hurwitz numbers for degree d and genus $\binom{d-1}{2}$). But beginning with $d = 4$, the situation changes. So far the only calculated non-trivial example is $d = 4$ (see [22, 23]) for which $\mathfrak{h}_4 = 120 \times (3^{10} - 1)$. Numbers $\mathfrak{h}_d$ for $d > 4$ are unknown at present.

Observe a straightforward analogy of the calculation of $\mathfrak{h}_d$ with (a special case) of the classical Zeuten's problem; see [1, 24]. Namely, given $d \geq 2$ and $0 \leq k \leq \frac{d(d+3)}{2}$ define the number $N_k(d)$ as the number of smooth plane curves of degree d passing through k points in general position and tangent to $\frac{d(d+3)}{2} - k$ lines in general position. In [24] H. G. Zeuten predicted these numbers for d up to 4. His predictions were rigorously proven only about twenty years ago; see [1] and references therein. The above problem of calculation of $\mathfrak{h}_d$ is similar to Zeuten's problem for $k = \frac{d(d+3)}{2}$. But instead of taking $\frac{d(d+3)}{2}$ generic lines we should take $\frac{d(d+3)}{2} - 3$ generic lines through a given point p and count the number of 3-dimensional orbits under the action of G_p.

Definition 2.2 Introduce the Hurwitz number $\mathfrak{h}^l_{g,\mu}$ as the degree of the restriction of the morphism $\delta_{g,\mu}$ to the (irreducible component of the) stratum $\mathcal{H}^l_{g,\mu}$ where $m(g,\mu) \leq l \leq M(g,\mu)$.

By definition, we have $\mathfrak{h}^{M(g,\mu)}_{g,\mu} = h_{g,\mu}$. Also, the number $\mathfrak{h}_d$ introduced above equals $\mathfrak{h}^0_{(d-1)(d-2)/2,1^d}$.

3 Final Remarks

(1) It would be very interesting to prove/disprove the irreducibility of the strata $\mathcal{H}^l_{g,\mu}$.
(2) It is important to develop tools that help to calculate the Hurwitz numbers of $\mathcal{H}^l_{g,d}$ and/or $\mathcal{H}^l_{g,\mu}$ due to the fact that they are naturally related to Zeuten-type problems. In the case of the usual single Hurwitz numbers there exists a standard combinatorial approach to their calculation that is not always very useful for practical computations but is theoretically important. Other standard tools for the usual Hurwitz numbers are the cut-and-join equation (see *e.g.,* [8]) and the ELSV-formula (see *e.g.,* [7]). It might be possible to find analogs of the latter tools by using an appropriate compactification of the above strata similar to those already existing in the literature.
(3) Another approach to the calculation of the Hurwitz strata of the planarity filtration might come from the correspondence theorem in tropical algebraic geometry. Recently in [3] the authors developed some tropical tools for finding the answers to a similar class of Zeuten-type problems.
(4) Finally, we want to mention a recent preprint [4], which gives a criterion when meromorphic functions of degree d on a certain class of plane curves of degree d with only nodes and some additional non-degeneracy assumptions might be realized by a projection from a point outside the curve.

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