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Classifying Real Polynomial Pencils

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1 Introduction and main results

In what follows, by a *pencil* $L = \{\alpha P + \beta Q\}$ we will always mean a *real polynomial pencil* of degree n homogeneous polynomials in two real variables, that is, a real line in $\mathbb{RP}^n$ identified with the space of all homogeneous degree n real polynomials considered up to a constant factor. Here, $(\alpha : \beta) \in \mathbb{RP}^1$ is a projective parameter. In order to use derivatives, it will often be convenient to view homogeneous degree n polynomials in two variables as inhomogeneous polynomials of degree at most n in one variable. Any choice of a basis (P, Q) in L allows us to consider the real rational function P/Q; a different choice of basis leads to a rational function of the form $(AP + BQ)/(CP + DQ)$, which can be viewed as the postcomposition of the rational function P/Q with the real linear fractional transformation $(Az + B)/(Cz + D)$ in the target $\mathbb{CP}^1$. Thus all properties of real rational functions which are invariant under real linear fractional transformations in the target space are naturally inherited by real polynomial pencils. For instance, the graph of a real rational function P/Q restricted to $\mathbb{RP}^1$ defines a finite branched covering $\mathbb{RP}^1 \to \mathbb{RP}^1$. We call two rational functions P_1/Q_1 and P_2/Q_2 *graph-equivalent* if there exist diffeomorphisms of the source $\mathbb{RP}^1$ and the target $\mathbb{RP}^1$ sending the graph of P_1/Q_1 to that of P_2/Q_2. As a property which is invariant under the postcomposition with a linear fractional transformation of the target space, the above graph-equivalence can be defined for the pencils $\alpha P_1 + \beta Q_1$ and $\alpha P_2 + \beta Q_2$.

The real pencils depicted in Figure 1.1 are represented as subsets of $\mathbb{R}^2$ of the form $\{(P(x), Q(x)) \mid x \in \mathbb{R}\}$. The leftmost pencil and the central pencil are graph-equivalent while the rightmost pencil is not graph-equivalent to them.

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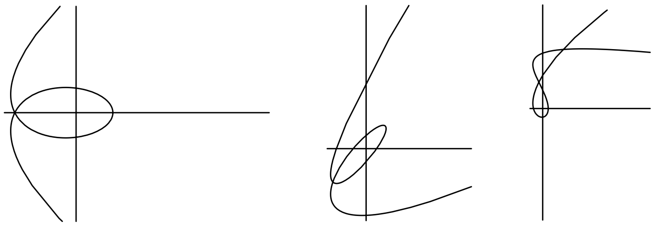

Figure 1.1 Graph-equivalent and nonequivalent real pencils.

The most classical notion of genericity for meromorphic functions requires that the function under consideration should have the maximal possible number of (simple) critical points with all distinct critical values. We will refer to this notion as *Hurwitz-genericity*, see Section 3. The classification of Hurwitz-generic real rational functions was carried out in detail in [9]. The violation of Hurwitz-genericity essentially occurs for two basic reasons. Either several critical points collapse and form a degenerate critical point or some critical values collide but their corresponding critical points are still distinct. In the present paper, we study a weaker notion of genericity than Hurwitz-genericity requiring only that all real critical points of the considered real rational functions stay simple, see Definition 1.2 below. This notion is the natural counterpart of the absence of the collapse of critical points in the realm of real algebraic geometry. It still keeps some important information about the behavior of real rational functions and is closely related to the natural analog of the classical discriminant for the Grassmannian of two-dimensional subspaces. One more important observation is that the violation of such genericity is detected in the source space instead of the target which is always more difficult. The above notions of genericity are invariant under the postcomposition with a linear fractional transformation of the target space and can therefore be defined for pencils as well.

Our notion of genericity allows us in particular to give a complete solution to the following problem.

Problem 1.1. For which pencils $L = \{\alpha P + \beta Q\}$ is the number of real zeros in this pencil constant, that is, when is the number of real solutions (counted with multiplicities) of the equation $\alpha P + \beta Q = 0$ independent of α/β?

An example of such a situation is provided by a well-known result of Obreschkoff, see [10], saying that a pencil $L = \{\alpha P + \beta Q\}$ consists of polynomials with only real (distinct) zeros if and only if both P and Q have real (distinct) and interlacing zeros. However, there exist pencils with a constant number of real zeros which are not covered by

Obreschkoff's result. For instance, one may consider the pencil $L = \{\alpha P + \beta P'\}$, where $P(x) = x^4 + x^2 - 5x - 4$.

An easy observation is that a pencil $L = \{\alpha P + \beta Q\}$ has a constant number of real zeros if and only if the Wronskian $W(P, Q) = PQ' - QP'$ has no real zeros, or, in other words, P and Q form a fundamental system for some second-order linear ordinary differential equation. Indeed, since the zeros of pencil L are the level sets of P/Q, both conditions are equivalent to the absence of real critical points of P/Q.

Let $G_{2,n+1}$ denote, as usual, the Grassmannian of lines in $\mathbb{RP}^n$. The fact that the behavior of the number of real zeros in a pencil $L = \{\alpha P + \beta Q\}$ is closely related to the properties of real zeros of the Wronskian $W(P, Q) = PQ' - QP'$ justifies the following definition.

Definition 1.2. A real polynomial pencil $L = \{\alpha P + \beta Q\}$ is called *generic* if the Wronskian $W(P, Q) = PQ' - QP'$ has no multiple real zeros and it is called *nongeneric* otherwise. The set $\mathcal{D}_{2,n+1}$ of all nongeneric real pencils in $G_{2,n+1}$ is called the *Grassmann discriminant*.

Clearly, the degree of the Wronskian of almost any pencil in $\mathbb{RP}^n$ equals $2n - 2$. If the Wronskian of a pencil in $\mathbb{RP}^n$ is of the degree $2n - 4$ or less, then we consider this pencil as degenerate (since its Wronskian has a double zero at ∞).

Definition 1.3. Two generic pencils are called *equivalent* if they can be connected by a path through generic pencils, that is, if they belong to the same connected component of the set $\widetilde{G_{2,n+1}} = G_{2,n+1} \setminus \mathcal{D}_{2,n+1}$ of all generic pencils in $\mathbb{RP}^n$.

Note that, as defined above, the equivalence of two generic real pencils does not necessarily imply their graph-equivalence since real critical values can collide. The main question that we address below is the following.

Problem 1.4. Enumerate the equivalence classes of all generic pencils in $\mathbb{RP}^n$.

The study of this topic originated from our attempt to solve the following intriguing conjecture of Craven, Csordas, and Smith (cf. [1]; see also [12]).

Conjecture 1.5 (Hawaii conjecture)**.** If a real polynomial P has $2s$ nonreal zeros, then the Wronskian $W(P, P') = PP'' - (P')^2$ has at most $2s$ real zeros. □

Our ultimate goal was to get a complete description of all connected components in $\widetilde{G_{2,n+1}} = G_{2,n+1} \setminus \mathcal{D}_{2,n+1}$ where pencils of the form $\{\alpha P + \beta P'\}$ can lie. Note that such a description would immediately prove or disprove the Hawaii conjecture.

The main result of this paper—Theorem 1.6—completely solves Problem 1.4. The answer is given in terms of boundary-weighted gardens of total weight n, a notion that we define and study in detail in Sections 2 and 3. The notion of garden of a real rational function provides also a natural topological context for studying Conjecture 1.5 and related questions, see Conjectures 5.1 and 5.2 in Section 5.

Theorem 1.6. The connected components in the space $\widetilde{G_{2,n+1}} = G_{2,n+1} \setminus \mathcal{D}_{2,n+1}$ of all generic pencils in $\mathbb{RP}^n$ are in one-to-one correspondence with the set of equivalence classes of all boundary-weighted gardens of total weight n. $\square$

From Theorem 1.6 and the arguments involving the Wronskian that we mentioned earlier, we deduce the following answer to Problem 1.1.

Corollary 1.7. There exist $[(n+1)/2]$ different components in $\widetilde{G_{2,n+1}} = G_{2,n+1} \setminus \mathcal{D}_{2,n+1}$ where the Wronskian $W(P,Q)$ has no real zeros at all. $\square$

The values of the number of connected components for small values of n are $1, 2, 4, 8, 14, 28$ for n equal to $1, 2, 3, 4, 5, 6$, respectively (see Figure 1.2). This sequence of integers was not recognized by the On-line Encyclopedia of Integer Sequences.

The structure of the paper is as follows. In Section 2, we define the notions of garden, boundary-weighted garden, and Morse perestroika and list some of their properties. We further study these notions in Section 3, where we prove the main results of the paper. In Section 4, we build on some of the aforementioned ideas and obtain a simple new proof of a generalization of the famous Hermite-Biehler theorem. Finally, Section 5 contains a number of conjectures and open problems.

2 Preliminaries on gardens and gardening

We first recall the notion of garden of a real polynomial pencil as defined in [3, 9].

Definition 2.1. The *garden* $\mathcal{G}(L)$ of a real polynomial pencil $L = \{\alpha P(z) + \beta Q(z)\}$ is the set of all $z \in \mathbb{CP}^1$ for which the rational function $f_L = P(z)/Q(z)$ attains real values.

Note that the defining property of $\mathcal{G}(L)$ is actually independent of the choice of real basis (P, Q) of the real polynomial pencil L.

Let z be an affine coordinate on $\mathbb{CP}^1$. Observe that $\mathcal{G}(L) \subset \mathbb{CP}^1$ is an algebraic curve in the coordinates $(\Re z, \Im z)$ which necessarily contains $\mathbb{RP}^1 \subset \mathbb{CP}^1$ and is invariant under the complex conjugation map $\tau : \mathbb{CP}^1 \to \mathbb{CP}^1$. Note that the singularities of any garden occur exactly at the critical points of $f_L = P(z)/Q(z)$, where f_L attains a real value. If such a critical point has multiplicity $m \geq 2$, then at that point the garden has a transversal intersection of m nonsingular branches with angle π/m between any two neighboring branches. A critical point with real critical value is called *simple* if its multiplicity

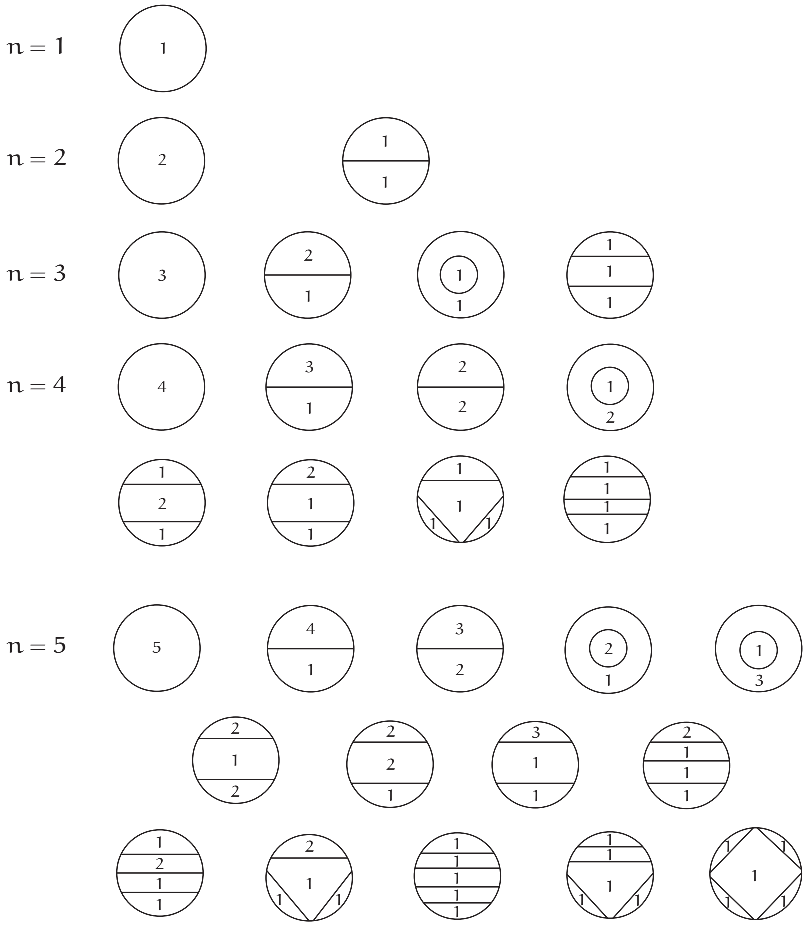

Figure 1.2 Enumeration of equivalence classes for $n \le 5$.

equals 2. A pencil $L = \{\alpha P(z) + \beta Q(z)\}$ is called *nonsingular* if the only critical points of f_L with real critical values are real and simple. This implies that the only singularities of its garden $\mathcal{G}(L)$ are transversal intersections of $\mathbb{RP}^1$ with other branches of $\mathcal{G}(L)$. The garden $\mathcal{G}(L)$ of a nonsingular pencil L will be called *nonsingular* as well. The aforementioned transversal intersections are called the *vertices* of the garden. Note that the vertices of $\mathcal{G}(L)$ correspond exactly to the real zeros of $W(P, Q)$.

We need to describe nonsingular gardens in more detail. A nonsingular garden $\mathcal{G}(L)$ is the disjoint union of two basic parts $\mathcal{C}(L) \cup \mathcal{O}(l)$, the *chord part* $\mathcal{C}(L)$ and the (possibly empty) *oval part* $\mathcal{O}(L)$. The chord part $\mathcal{C}(L)$ is the connected component of $\mathcal{G}(L)$ containing $\mathbb{RP}^1$ while the oval part $\mathcal{O}(L)$ is the complement $\mathcal{G}(L) \setminus \mathcal{C}(L)$. We call the edges connecting the vertices of $\mathcal{C}(L)$ and not belonging to $\mathbb{RP}^1$ the *chords*. The oval part $\mathcal{O}(L)$ consists of a number of τ-invariant smooth closed curves called *ovals*. The connected

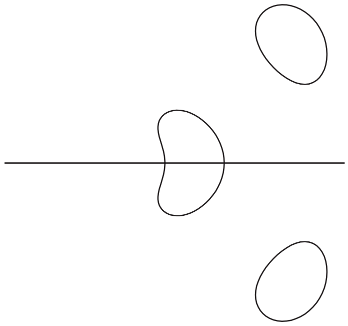

Figure 2.1 Garden of the pencil $L = \{\alpha P + \beta P'\}$, where $P(x) = x^5 - 3x^4 + 10x^3 + 10x^2 + 9x + 13$.

components of $\mathbb{CP}^1 \setminus \mathcal{G}(L)$ are called the *faces* of the garden $\mathcal{G}(L)$ (see the example depicted in Figure 2.1). We fix the standard metric on the image $\mathbb{CP}^1$ such that the length of $\mathbb{RP}^1$ equals 1. If we choose some basis (P, Q) of the nonsingular pencil L under consideration, then by using the rational function $f_L = P(z)/Q(z) : \mathbb{CP}^1 \to \mathbb{CP}^1$, we can assign an extra piece of information to all elements of the garden $\mathcal{G}(L)$.

Definition 2.2. The *edge-weighted garden* $\mathcal{EG}(P/Q)$ of the rational function P/Q is the garden $\mathcal{G}(L)$ of the pencil $L = \{\alpha P + \beta Q\}$ together with all edges, chords, and ovals, each of these objects being endowed with the *weight* given by the length of its respective image in the target $\mathbb{RP}^1$ under the rational function P/Q. The *total weight* of an edge-weighted garden is the sum of the weights of all its edges, chords, and ovals.

Remark 2.3. Note that the image of an edge, chord, or oval can cover some interval of $\mathbb{RP}^1$ several times. The lengths/weights considered in Definition 2.2 are total lengths obtained by counting multiplicities. In particular, this implies that the total weight of the edge-weighted garden $\mathcal{EG}(P/Q)$ equals the degree of P/Q as a map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$.

Definition 2.4. A *boundary-weighted garden* is a nonsingular garden with positive integer weights assigned to each boundary component and satisfying the additional requirement that τ-symmetric faces be assigned equal weights. The *total weight* of a boundary-weighted garden is the sum of the weights of all boundary components contained in the closed upper hemisphere other than ovals plus twice the weight of all ovals in the closed upper hemisphere.

There is an obvious map λ from edge-weighted gardens to boundary-weighted gardens obtained by assigning to each boundary component the sum of the weights of the elements contained in this boundary component. Note that the latter sum is either

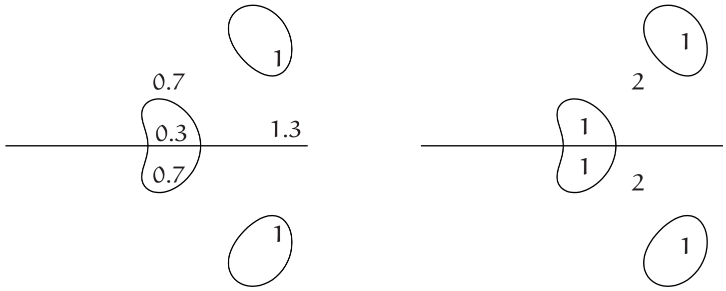

Figure 2.2 An edge-weighted garden and its boundary-weighted image.

the sum of the weights of all edges and chords if the boundary component contains them, or just the weight of an oval if the boundary component is an oval, see Figure 2.2. One can easily see that the image under λ of an edge-weighted garden $\mathcal{EG}(P/Q)$ is invariant under postcompositions of the rational function P/Q with real linear fractional transformations. We may therefore associate to each nonsingular pencil a canonical boundary-weighted garden in the following way.

Definition 2.5. The *boundary-weighted garden* $\mathcal{FG}(L)$ of a given nonsingular pencil L is the image under the map λ of the edge-weighted garden $\mathcal{EG}(P/Q)$, where (P, Q) is some basis of L.

Note that the integer placed in each face on Figure 2.2 is the weight of the outer boundary component of the face (if the face is multiconnected). In order to describe connected components in the space of generic real pencils, we need to introduce the following equivalence relation on the set of all boundary-weighted gardens of given total weight. In what follows, we will work with the half of a garden contained in the upper hemisphere and assume that all operations are performed symmetrically.

Definition 2.6. The following operation is called a *Morse perestroika* of a boundary-weighted garden. Choose any face whose boundary contains either two chords, two ovals, or a chord and an oval. Having chosen two chords, two ovals, or a chord and an oval on the boundary of this face—call them chosen elements—connect them by a simple path lying entirely in the face. Now deform this face by contracting the path connecting the chosen elements to a point. Under this deformation, the two chosen elements will have a common point; more precisely, their local intersection near this point will become a node (which explains the presence of Morse in the terminology). Then resolve this node in the opposite direction by gluing together the two chosen elements. The new element resulting from this operation will have a weight equal to the sum of the weights of the former

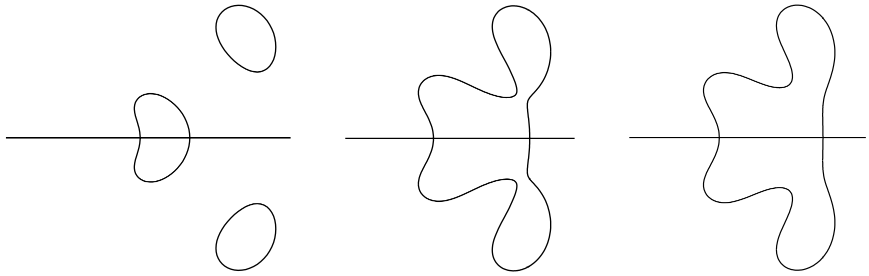

Figure 2.3 Morse perestroika in action.

components. If the original face was simply connected, then it would be cut into two new faces. Its boundary will be cut into two new boundary components whose weights are two arbitrary positive integers which add up to the weight of the initial boundary. Two boundary-weighted gardens that can be obtained from each other by a sequence of Morse perestroikas are called *equivalent*.

Note that Morse perestroikas are performed simultaneously and symmetrically in both hemispheres, see Figure 2.3. It is not difficult to see that the equivalence relation introduced in Definition 2.6 preserves the total weight and the number of chords of a garden.

We have defined all the notions mentioned in Theorem 1.6 and are now ready to prove this theorem.

3 Proofs

We start with some generalities about $\mathcal{D}_{2,n+1}$ which can be easily extended to linear polynomial families of higher dimensions. The following important mapping is called the *Wronski map*, see, for example, [2]. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Introducing an affine coordinate z on $\mathbb{KP}^1$, we can identify $\mathbb{KP}^n$ with the space of inhomogeneous polynomials of degree at most n in the variable z. Consider now the map

$$W : G_{2,n+1} \longrightarrow \mathbb{KP}^{2n-2} \tag{3.1}$$

that sends a 2-dimensional linear polynomial subspace of $\mathbb{KP}^n$ to the linear span of its Wronskian, that is, the determinant of the 2×2 matrix $\left(\begin{smallmatrix} P(z) & Q(z) \\ P'(z) & Q'(z) \end{smallmatrix}\right)$, where $(P(z), Q(z))$ is some basis of the chosen subspace. Note that a change of basis in the given subspace amounts to multiplying the Wronskian by a nonzero constant and that all such Wronskians are polynomials in z of degree at most $2n - 2$.

Several important facts are known about the map W. Over $\mathbb{C}$ the map W is finite and its degree equals the degree of $G_{2,n+1}$ under its Plücker embedding. The latter number equals the nth Catalan number $\mathcal{C}_n = (1/n)\binom{2n-2}{n-1}$, see [5]. Moreover, the Wronski map is perfectly adjusted to the Schubert cell decomposition of $G_{2,n+1}$ constructed by using the natural complete flag in $\mathbb{KP}^n$ whose i-dimensional subspaces consist of all polynomials of degree at most i, where $i = 0, 1, \dots, n$. It turns out that over $\mathbb{C}$ the degree of the restriction of W to any of the above Schubert cells equals the degree of this cell under the Plücker embedding of $G_{2,n+1}$, see [2].

Denote by $\mathcal{D}_{2n-2} \subset \mathbb{KP}^{2n-2}$ the standard discriminant in $\mathbb{KP}^{2n-2}$, that is, the set of all polynomials having a multiple zero over $\mathbb{K}$. The Grassmann discriminant $\mathcal{D}_{2,n+1}$ introduced in Definition 1.2 may alternatively be characterized as follows.

Definition 3.1. The Grassmann discriminant $\mathcal{D}_{2,n+1} \subset G_{2,n+1}$ is the inverse image $W^{-1}(\mathcal{D}_{2n-2})$ of $\mathcal{D}_{2n-2}$ under the Wronski map W.

Lemma 3.2. The discriminant $\mathcal{D}_{2,n+1}$ consists of two irreducible components $\mathcal{U}$ and V. The first component $\mathcal{U}$ is the closure of the set of all lines in $\mathbb{KP}^n$ tangent to $\mathcal{D}_n \subset \mathbb{KP}^n$ at its smooth points. The second component V is the set of all lines passing through the stratum $\Sigma_3 \subset \mathcal{D}_n$, where Σ_3 consists of all polynomials having a root over $\mathbb{K}$ of multiplicity exceeding 2 (compare with [6] and see the example depicted in Figure 3.1 and the accompanying explanations). □

Proof. Take a pencil $L = \{\alpha P + \beta Q\}$ and consider the matrix

$$M_L = \begin{pmatrix} P(z) & P'(z) & P''(z) \\ Q(z) & Q'(z) & Q''(z) \end{pmatrix}. \tag{3.2}$$

If the Wronskian $W(P, Q) = \begin{vmatrix} P(z) & P'(z) \\ Q(z) & Q'(z) \end{vmatrix}$ has a multiple zero at some z_0, then

$$\begin{vmatrix} P(z_0) & P'(z_0) \\ Q(z_0) & Q'(z_0) \end{vmatrix} = \begin{vmatrix} P(z_0) & P''(z_0) \\ Q(z_0) & Q''(z_0) \end{vmatrix} = 0. \tag{3.3}$$

The latter conditions can be satisfied in two different ways. Either there exists z_0 such that $P(z_0) = Q(z_0) = 0$, that is, the first column in M_L vanishes at z_0, or the first column never vanishes but there exists z_0 such that the first and second rows are linearly dependent. The first situation corresponds to the case when the rational curve $(P(z), Q(z))$ passes through the origin and the corresponding pencil in $\mathbb{KP}^n$ is tangent to

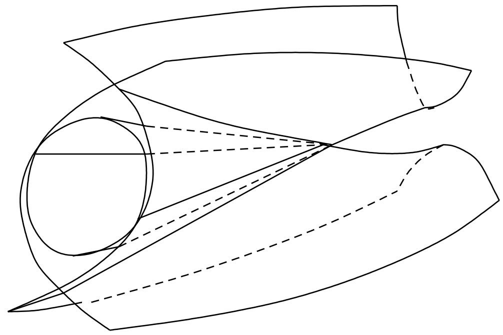

Figure 3.1 Section of the $\mathcal{D}_{2,4}$-discriminant transversal to the lift of ρ to $G_{2,4}$ using tangent lines.

$\mathcal{D}_n$. The second situation means that there exists a linear combination of P and Q which vanishes up to a cubic term, that is, the pencil intersects Σ_3, which means geometrically that the curve $(P(z), Q(z))$ has a tangent line at some inflection point passing through the origin. ∎

For the sake of completeness, we present without proof yet another characterization of $\mathcal{D}_{2,n+1}$. The *standard rational normal curve* $\rho : \mathbb{KP}^1 \to \mathbb{KP}^n$ is the curve consisting of all degree n polynomials with an n-tuple root. Given a complete projective flag f in $\mathbb{KP}^n$, we associate to f the standard Schubert cell decomposition $\mathfrak{S}_f$ of $G_{2,n+1}$ whose cells consist of all 2-dimensional projective subspaces with a given set of dimensions of intersections with the subspaces of f. The cells are labeled by Young diagrams with at most two rows of length not exceeding $n-1$. Given a rational curve $\gamma : \mathbb{KP}^1 \to \mathbb{KP}^n$, one defines its *flag lift* $\gamma_F : \mathbb{KP}^1 \to F_{n+1}$ to be the curve consisting of all osculating flags to γ. As is well known, the same definition applies in fact to any projective algebraic curve.

Proposition 3.3. The component U (resp., V) of $\mathcal{D}_{2,n+1}$ is the union of the Schubert cells $\bigcup_{f\in\rho_F} C_{1,1}(f)$ (resp., $\bigcup_{f\in\rho_F} C_{2,0}(f)$), where f runs over the flag lift ρ_F of the standard rational curve ρ. Here, $C_{1,1}(f)$ is the cell of codimension two in $G_{2,n+1}$ whose Young diagram with respect to f is $(1,1)$, while $C_{2,0}$ is the cell whose Young diagram with respect to f is $(2,0)$. □

Figure 3.1 shows a 3-dimensional local section of $G_{2,4}$ transversal to the lift $\widetilde{\rho}$ of the standard rational curve ρ obtained by taking tangent lines. Locally, the $\mathcal{D}_{2,4}$-discriminant is a cylinder in the direction of $\widetilde{\rho}$. Thus we are considering the section of a cylinder transversal to its cylindrical direction. This section consists of two Whitney umbrellas representing components U and V and intersecting each other along a pair of lines. In the case of $G_{2,4}$, components U and V are isomorphic as complex algebraic varieties, which is no longer true for $n \geq 4$.

In order to complete the proof of Theorem 1.6, we need several additional definitions and constructions. We first recall the following classical definition.

Definition 3.4. A pencil $L = \{\alpha P(z) + \beta Q(z)\}$ of degree n polynomials is called *Hurwitz-generic* if the rational function $f_L = P(z)/Q(z)$ has $2n - 2$ distinct critical points with distinct critical values, and it is called *Hurwitz-nongeneric* otherwise.

Remark 3.5. As we already noted in the introduction, a real rational function of the form $(AP + BQ)/(CP + DQ)$ may be viewed as the postcomposition of the rational function P/Q with the linear fractional transformation $(Az + B)/(Cz + D)$ in the target $\mathbb{CP}^1$. This shows that the property introduced in Definition 3.4 is independent of the choice of basis of the pencil L.

Definition 3.6. The *Hurwitz discriminant* is the subset $\mathcal{H}_{2,n+1} \subset G_{2,n+1}$ consisting of all Hurwitz-nongeneric pencils in the Grassmannian of lines in $\mathbb{RP}^n$.

Clearly, any real Hurwitz-generic pencil L is generic in the sense of Definition 1.2. Moreover, such a pencil is also nonsingular, that is, it has a nonsingular garden $\mathcal{G}(L)$. Indeed, any complex critical point together with its complex conjugate forms a pair that cannot have a real critical value. This proves the following lemma.

Lemma 3.7. The Grassmann discriminant $\mathcal{D}_{2,n+1}$ is always contained in the Hurwitz discriminant $\mathcal{H}_{2,n+1}$. $\square$

We say that a nonsingular garden is *directed* if its edges, chords, and ovals are directed in such a way that the boundary of each face becomes a directed cycle. This means that any given face will lie either to the right of any of its boundary components or to the left of any such component whenever we follow the direction that has been assigned to a boundary component. The faces that lie to the left of all of their boundary components are called *positive* while faces lying to the right of their boundary components are called *negative*. All neighbors of positive faces are negative and vice versa. Obviously, in order to direct a garden, it suffices to direct any one of its edges. Therefore, there exist exactly two possible ways of directing a garden and these are opposite to each other in the sense that the one is obtained from the other by reversing the direction of every edge.

By a *proper (cyclic) labeling* of a directed boundary-weighted garden with $2k$ vertices (equivalently, with k chords), we understand the labeling of its vertices by symbols $1, \dots, 2k$ satisfying the following condition: for each boundary component, the number of decreases (downs) between consecutive labels when we traverse the labels of the vertices in the order prescribed by the component's direction does not exceed the weight of this component. An *involution* of a properly labeled directed boundary-weighted garden is an operation that reverses both its orientation and the (cyclic) order of the labels by sending label i to label $2k - i$.

Given a Hurwitz-generic rational function $f = P/Q$ and fixing the orientation of the target $\mathbb{RP}^1$, one gets the properly directed and labeled boundary-weighted garden $\widetilde{\mathcal{G}}(f)$ of f by taking its garden with the induced orientation of all elements and their induced weights together with the cyclic labeling of its vertices coming from the real critical values of f. Note that for any other choice of basis of the pencil $L = \{\alpha P + \beta Q\}$, the resulting garden either coincides with $\widetilde{\mathcal{G}}(f)$ or may be obtained from $\widetilde{\mathcal{G}}(f)$ by an involution.

An equivalent version of the following result was stated and proved by means of rational functions in [9].

Theorem 3.8. Let $\mathcal{H}_{2,n+1}$ denote the divisor of all Hurwitz-nongeneric pencils. The connected components in the space $\widetilde{H}_{2,n+1} = G_{2,n+1} \setminus \mathcal{H}_{2,n+1}$ of all Hurwitz-generic pencils are in one-to-one correspondence with the set of all properly directed and cyclicly labeled gardens of weight n modulo the action of the involution. □

Recall from Lemma 3.7 that the Grassmann discriminant $\mathcal{D}_{2,n+1}$ and the Hurwitz discriminant $\mathcal{H}_{2,n+1}$ satisfy $\mathcal{H}_{2,n+1} \supset \mathcal{D}_{2,n+1}$. For our further purposes, we need the following description of $\mathcal{H}_{2,n+1}$.

Theorem 3.9. The Hurwitz discriminant $\mathcal{H}_{2,n+1}$ is the union of four real discriminants U, V, W, and Z, where U and V are defined in Lemma 3.2 and W and Z are two real algebraic hypersurfaces with the same complexification, namely, the hypersurface of all coinciding critical values. More precisely, W is the set of all real pencils $L = \{\alpha P + \beta Q\}$ for which the rational function $f_L = P(z)/Q(z)$ has two real critical points with coinciding real critical value, while Z is the set of all real pencils $L = \{\alpha P + \beta Q\}$ for which the rational function $f_L = P(z)/Q(z)$ has two complex conjugate critical points with coinciding (and therefore real) critical value. □

Our plan is as follows. We will show that by crossing W one can realize any admissible relabeling of a given cyclicly labeled boundary-weighted garden and that by crossing Z we can realize any of its admissible Morse perestroikas. These two facts will be easy corollaries of the following statements.

Theorem 3.10. Any edge-weighted garden $\mathcal{G}$ of total weight n is realized by a real rational function of degree n. Moreover, the set of all real rational functions with a given edge-weighted oriented garden is path-connected. □

Here, by an edge-weighted garden of total weight n, we understand an abstract embedded τ-invariant "graph" containing $\mathbb{RP}^1$ with vertices only of even multiplicity and possibly containing a number of τ-invariant ovals considered up to a diffeomorphism of the plane. All edges, chords, and ovals of this "graph" are equipped with positive weights. Moreover, ovals have integer weights. Finally, for any boundary component, the sum of all weights in this component is a positive integer and the sum of the weights of all elements in this "graph" equals n. It is important to note that in Theorem 3.10 we *do not assume* that $\mathcal{G}$ is a nonsingular garden and that we actually allow arbitrary complex critical points with real critical values.

Proof of Theorem 3.10. The proof is based on ideas similar to those used in the proof of [9, Theorem 1], and so we will only sketch it here. (The only major difference compared to [9] is that we allow singular gardens.) We want to construct a topological branched covering $\mathbb{CP}^1 \to \mathbb{CP}^1$ which is invariant under complex conjugation and whose garden is isomorphic to $\mathcal{G}$. This will prove the realization theorem, since, by Riemann's uniqueness theorem, there exists a unique complex structure on $\mathbb{CP}^1$ for which this topological covering is holomorphic. The orientation of the garden uniquely specifies which of its faces should be mapped to the upper hemisphere and also which faces should be mapped to the lower hemisphere. (Neighboring faces are always mapped to opposite hemispheres.) Each open face of the garden is a topological surface of genus 0. The normalization of its closure is a closed topological surface with boundary. Now, for any face of $\mathcal{G}$, consider the total weight of its boundary components, that is, the number of times each boundary component should traverse $\mathbb{RP}^1$. The Riemann-Hurwitz formula determines the number of simple complex critical points the face under consideration should contain. We also know in which hemisphere the corresponding critical values should lie. We now recall some definitions from [9, Section 3.1]. Denote by Λ^+ the upper hemisphere $\{z \in \mathbb{CP}^1 \mid \operatorname{Im} z \geq 0\}$ and by P a genus g topological surface with a boundary consisting of k connected components. Consider the set $\mathcal{H}^k_{g,m}$ of all generic degree m branched coverings of the form $\phi : P \to \Lambda^+$ and let $a_1, \dots, a_k$ be all the distinct connected components of ∂P. Given a partition $(m_1, \dots, m_k) \vdash m$, denote by $\mathcal{H}^k_{g,m}(m_1, \dots, m_k) \subset \mathcal{H}^k_{g,m}$ the subset of maps $\phi : P \to \Lambda^+$ such that $\deg \phi|_{a_i} = m_i$ for $i = 1, \dots, k$. Obviously,

$$\mathcal{H}^k_{g,m} = \bigcup_{(m_1,\dots,m_k)\vdash m} \mathcal{H}^k_{g,m}(m_1, \dots, m_k). \tag{3.4}$$

Let f be a face in the upper hemisphere of $\mathbb{CP}^1 \setminus \mathcal{G}$ and consider the space $\mathcal{H}_f$ of all branched coverings from the normalization of the closure of f to $\Lambda^\pm$, where $\Lambda^\pm$ is the upper or lower hemisphere depending on where f should be mapped according to the chosen orientation. Lemma 2 in [9] shows that for any partition $(m_1, \dots, m_k) \vdash m$, the space $\mathcal{H}^k_{g,m}(m_1, \dots, m_k)$ is path-connected. In particular, this implies that each space $\mathcal{H}_f$ is path-connected. We need the following result.

Lemma 3.11. Let $\mathcal{G}$ be an edge-weighted oriented garden and fix an arbitrary set of (critical) values for the vertices belonging to its chords. Denote by $\mathrm{Rat}_{\mathcal{G}}$ the set of all real rational functions with edge-weighted oriented garden $\mathcal{G}$ and having these prescribed critical values. Then $\mathrm{Rat}_{\mathcal{G}}$ is homeomorphic to $\Pi_{f \in \mathrm{Ind}_{\mathcal{G}}} \mathcal{H}_f \times (\mathbb{RP}^1)^q$, where q is the number of different connected components of $\mathcal{G}$ containing vertices—that is, critical points with real critical values—and $\mathrm{Ind}_{\mathcal{G}}$ is the index set of all faces f in the upper hemisphere of $\mathbb{CP}^1 \setminus \mathcal{G}$. □

Remark 3.12. Note that for a nonsingular garden with a positive number of vertices, one has $q = 1$ since all its vertices belong to $\mathbb{RP}^1$. However, the singular gardens considered in Theorem 3.10 might contain singular ovals with vertices which are not connected to $\mathbb{RP}^1$.

Proof of Lemma 3.11. We show first that by assigning all real critical values and picking an arbitrary map ϕ_f from each space $\mathcal{H}_f$ for $f \in \mathrm{Ind}_{\mathcal{G}}$, we can glue together all the ϕ_f's into precisely one half of a unique real rational function from $\mathrm{Rat}_{\mathcal{G}}$. This follows simply from the fact that the real critical values determine exactly which parts of the boundary components of ϕ_{f_i} and ϕ_{f_j} for any two neighboring faces f_i and f_j should be identified (glued together). Indeed, by gluing together all the ϕ_f's for all $f \in \mathrm{Ind}_{\mathcal{G}}$ according to this recipe, we get a unique map from Λ^+ to $\mathbb{CP}^1$. We may then take the conjugate copy of the latter map and glue the two halves together along $\mathbb{RP}^1$ into a sphere $\mathbb{CP}^1$, thus obtaining a unique final map $\mathbb{CP}^1 \to \mathbb{CP}^1$. One can easily see that the final map is the topological branched covering that satisfies all the properties required. It just remains to notice that in order to assign all real critical values for an edge-weighted garden, it is necessary and sufficient to assign arbitrarily just one real critical value for each connected component of $\mathcal{G}$ containing vertices. The critical values of the remaining vertices in each such component will then be automatically restored from the set of weights of the chords and ovals in the respective component. ■

To finish the proof of Theorem 3.10, just notice that the Cartesian product of path-connected topological spaces is path-connected. ■

Corollary 3.13. Any admissible Morse perestroika of a given nonsingular boundary-weighted garden is realizable. □

Proof. Any singular garden that occurs while performing an arbitrary generic perestroika contains just two simple complex conjugate critical points with a common real critical value. It follows from Theorem 3.10 that such a garden can be realized by a rational function. Any small generic 1-parameter deformation of this rational function will necessarily produce the required perestroika. Indeed, in any such deformation, the imaginary part of the interesting critical value will necessarily change signs while the rest of the garden will topologically stay the same. ■

Theorem 3.14. The set of all real rational functions with a given boundary-weighted oriented garden is path-connected. □

Proof. We use an argument similar to that of [3]. Let $\mathcal{G}$ be an oriented boundary-weighted garden and denote by $\mathcal{ELG}$ the set of all possible edge-weighted gardens whose boundary-weighted gardens coincide with $\mathcal{G}$, see Figure 2.2. Enumerating arbitrarily all chords and edges in $\mathcal{G}$ and denoting the weight of the ith chord by $w_{c,i}$, the weight of the jth edge by $w_{e,j}$, and the weight of the mth boundary component by $w(B_m)$, we get the following system of linear inequalities (one for each edge and chord) and linear equations (one for each boundary component other than an oval) satisfied by edge weights for all gardens in $\mathcal{ELG}$:

$$\begin{aligned} & w_{c,i} > 0, \\ & w_{e,j} > 0, \\ & \sum_{i_l \in B_m} w_{c,i_l} + \sum_{j_l \in B_m} w_{e,j_l} = w(B_m). \end{aligned} \tag{3.5}$$

Let $\mathrm{Sol}_{\mathcal{G}}$ denote the set of all solutions to system (3.5). Obviously, $\mathrm{Sol}_{\mathcal{G}}$ is a nonempty convex polytope. For any solution of (3.5), we get an edge-weighted oriented garden. By Theorem 3.10, the set of all real rational functions realizing such a garden is path-connected. Therefore, the set of real rational functions with a given oriented boundary-weighted garden is actually fibered over a contractible base with isomorphic path-connected fibers. (Note that, by Lemma 3.11, the topology of the fiber does not depend on the particular weights of the chords.) Thus the total space of fibration is path-connected. ■

Corollary 3.15. Any admissible relabeling of a given boundary-weighted and cyclicly labeled garden is realizable. □

Proof. Take any admissible labeling of a given boundary-weighted garden. Place its labels arbitrarily on $\mathbb{RP}^1$ in an order-preserving way, that is, assign real critical values to all real critical points. Then one can restore the weights of all the chords and edges of the garden. These weights will necessarily satisfy system (3.5). Having done so for two different labelings and using the fact that the set of rational functions in Theorem 3.14 is path-connected, we conclude that we can find a path from the first rational function to the second through rational functions with the same boundary-weighted garden. ■

Corollary 3.15 completes the proof of Theorem 1.6. We now use this theorem to deduce Corollary 1.7.

Proof of Corollary 1.7. For a real polynomial pencil $L = \{\alpha P + \beta Q\}$, the Wronskian $W(P, Q)$ has no real zeros if and only if the garden $\mathcal{G}(L)$ (as well as its equivalence class) has no chords. Among all equivalence classes of boundary-weighted gardens of total weight n, there are exactly $[(n+1)/2]$ classes with no chords. This is because the boundary-weighted garden of every such class consists of $\mathbb{RP}^1$ and at most one additional oval whose respective weights are integers k and l that satisfy $1 \leq k \leq n$, $0 \leq l$, $k \equiv n \bmod 2$, and $k + 2l = n$ (cf. Definition 2.4 and Remark 2.3). The cases when $n \leq 5$ are illustrated in Figure 1.2. ■

4 Real pencils and the Hermite-Biehler theorem

The properties of a real pencil $\{\alpha P + \beta Q\}$ or, equivalently, of the plane rational curve $\gamma = (P, Q)$ are also involved in the following well-known result. The classical Hermite-Biehler theorem asserts that given two polynomials P and Q with real coefficients and of degrees n and $n-1$, respectively, the zeros of the complex polynomial $S = P + iQ$ have (nonzero) imaginary parts of the same sign if and only if P and Q have real distinct and interlacing zeros. In fact, if μ is an arbitrary complex number and $\sharp_+$ denotes the number of zeros of the polynomial $S_\mu := P + \mu Q$ lying in the upper half-plane, then the following more general result is known to be true, see [4].

Proposition 4.1. In the above notation, consider the plane real rational curve γ_μ given by $(P + \Re\mu \cdot Q, \Im\mu \cdot Q)$. Then $\sharp_+$ equals the winding number of γ_μ around the origin. □

Below we give a new proof of the generalized Hermite-Biehler theorem for all pairs (P, Q) of real polynomials. In particular, our method yields a simple proof of the main result in [7].

Proposition 4.2. For given polynomials P and Q with real coefficients, the complex polynomial $S_\mu = P + \mu Q$ with $\mu \notin \mathbb{R}$ has a real zero if and only if P and Q have a common real zero. □

Proof. Indeed, if $P(x)$ and $Q(x)$ have a common real zero x_0, then $S_\mu(x_0) = 0$. On the other hand, if, for some $x_0 \in \mathbb{R}$, one has $S_\mu(x_0) = 0$, then $P(x_0) + \Re\mu \cdot Q(x_0) = 0$ and $\Im\mu \cdot Q(x_0) = 0$, which immediately imply $P(x_0) = Q(x_0) = 0$ since $\Im\mu \neq 0$. ■

A convenient geometric reformulation of this statement is as follows. Denote by Pol_n the space of all monic degree n polynomials with complex coefficients of the form $S(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ and let $\mathcal{RD} \subset \mathrm{Pol}_n$ be the hypersurface of all polynomials S that have at least one real zero. Finally, let $\mathrm{Res} \subseteq \mathrm{Pol}_n$ be the hypersurface of all $S = P + iQ$ such that P and Q have a real common zero. In the literature on singularities, Res is often called the (generalized) Whitney umbrella.

Corollary 4.3. The discriminant $\mathcal{RD}$ coincides with the resultant hypersurface Res. □

Remark 4.4. In the definition of Res, we disregard the subvariety of real codimension two where P and Q have common complex zeros.

Given an arrangement of black and white distinct points on $\mathbb{R}$, we define its *canonical reduction* to be the interlacing (possibly empty) arrangement obtained in the following way: if our arrangement contains a pair of neighboring points of the same color, then we remove these points and we continue this procedure until no such removals can be performed. Note that the resulting canonical reduction depends only on the initial (relative) order of the points in the given arrangement and not on their exact locations on $\mathbb{R}$.

Corollary 4.5 (cf. [11]). The number of connected components in $\mathrm{Pol}_n \setminus \mathcal{RD}$ equals $n+1$ and these components can be labeled by the canonical reductions as follows. Let P and Q be polynomials with real coefficients of degrees n and $n-1$, respectively. Assume that they have no common real zeros and that the leading coefficient of Q is positive. Then, for the polynomial $S_\mu = P + \mu Q$ with $\Im\mu \neq 0$, one has $\sharp_+ - \sharp_- = \kappa T$, where $\sharp_+$ (resp., $\sharp_-$) is the number of zeros of S_μ in the upper (resp., lower) half-plane, κ is the sign of $\Im\mu$, and T is the number of zeros of P appearing in the canonical reduction of the real zeros of P and Q. □

5 Final remarks

As we already mentioned in the introduction, the notion of garden of a real rational function provides a natural topological framework for investigating the Hawaii conjecture (Conjecture 1.5). Indeed, given a polynomial P of degree n with real coefficients, we consider the garden $\mathcal{G}_P$ of the rational function P'/P (cf. Definitions 2.1 and 2.2). Obviously, all zeros of P lie on $\mathcal{G}_P$. We make the following conjecture.

Conjecture 5.1. Each chord of $\mathcal{G}_P$ contains at least one nonreal zero of P. □

Note that Conjecture 5.1 would immediately imply the Hawaii conjecture since the real critical points of P'/P are the same as the real zeros of the Wronskian $W(P, P')$ and the latter are precisely the endpoints of the chords in $\mathcal{G}_P$ (cf. Section 2).

It is natural to ask whether the Hawaii conjecture extends to classes of rational functions other than logarithmic derivatives. Let n be a positive integer and denote by $\mathcal{QP}_n$ the set of all nonidentically vanishing rational functions of the form

$$f(x) = \sum_{i=1}^{n} c_i P_i^{\alpha}(x), \tag{5.1}$$

where $c_i \in \mathbb{R}$ for $1 \leq i \leq n$, α is a real number satisfying $\alpha \leq -1$, and $P_1, \ldots, P_n$ are second-degree monic polynomials with real coefficients without real zeros. Based on extensive numerical experiments, we propose the following analog of Conjecture 1.5 for the class $\mathcal{QP}_n$.

Conjecture 5.2. If $f \in \mathcal{QP}_n$, then f has at most $2n - 1$ real critical points. Moreover, if α is a negative integer, then the following analog of Conjecture 5.1 holds: each chord of the garden $\mathcal{G}_f$ of the real rational function f contains at least one nonreal zero of the polynomial $\prod_{i=1}^{n} P_i(x)$. □

A possible way to attack Conjecture 5.2 might be as follows. We first recall the definition of a Tchebycheff system as given in, for example, [8].

Definition 5.3. A linear n-dimensional space V of smooth real-valued functions defined on some interval (a, b) (a might be equal to $-\infty$ and b to $+\infty$) is called a *Tchebycheff system* if any nonidentically vanishing function $f \in V$ has at most $n - 1$ real zeros on (a, b) counted with multiplicities.

Problem 5.4. Let $P_1, \ldots, P_n$ be as in (5.1) and $\alpha \in \mathbb{R}$ with $\alpha \leq -1$. Is it possible to extend the n-tuple of functions $(P_1^{\alpha}(x))', \ldots, (P_n^{\alpha}(x))'$ to a Tchebycheff system of dimension $2n$ on $(-\infty, +\infty)$?

Note that an affirmative answer to Problem 5.4 would automatically confirm the validity of Conjecture 5.2.

The main question about the classification of generic pencils (Problem 1.4) extends straightforwardly to polynomial families with more than one parameter. However, a solution to the problem of enumerating connected components in other Grassmannians similar to Theorem 1.6 would first require an appropriate definition of the notion of garden in these cases.

To conclude, we formulate some related questions.

Problem 5.5. What can one say about the topology of the space of generic pencils $\widetilde{G_{2,n+1}} = G_{2,n+1} \setminus \mathcal{D}_{2,n+1}$? For instance, are connected components in $\widetilde{G_{2,n+1}}$ contractible? Note that this is true for polynomials without multiple real roots.

A real rational function of degree n is called an M-*function* if all its $2n-2$ critical points and critical values are real and distinct. Any M-function of degree n induces a degree n map $\mathbb{RP}^1 \to \mathbb{RP}^1$ with exactly $2n-2$ branching points.

Problem 5.6. What type of maps $\mathbb{RP}^1 \to \mathbb{RP}^1$ of degree n with $2n-2$ branching points can occur from M-functions?

More precisely, given a map $\mathbb{RP}^1 \to \mathbb{RP}^1$ of degree n with $2n-2$ simple branching points, we label its n real critical points and the corresponding n critical values cyclicly. Then we can associate to this map the unique cyclic permutation of length $2n-2$ sending each critical point to its critical value. Problem 5.6 may therefore be reformulated as follows.

Problem 5.7. What cyclic permutation can an M-function have?

Note that Problem 5.7 is actually asking for a description of all possible shapes of the graphs of real rational M-functions—a topic which is standardly considered in elementary calculus courses if one omits “M-” in the above formulation. However, in the general case, the answer to Problem 5.7 seems to be unknown and quite nontrivial.

Problem 5.8. Enumerate the connected components in the space of real rational functions having only simple real critical points with distinct critical values.

The arguments given in the introduction show that all real pencils are necessarily graph-equivalent in each such component. This project is the intermediate situation between the one covered in [9] and the one described in the present paper.

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