CURVATURE-FORM ALGEBRAS OF PARTIAL FLAG VARIETIES: COARSENING DUALITY AND THE FAMILY $(k,1,1)$

GLEB NENASHEV, ALEXANDER POSTNIKOV, BORIS SHAPIRO, AND MICHAEL SHAPIRO

ABSTRACT. Let $\mathcal{A}_{\mathbf{a}}$ be the algebra generated by the invariant Chern–Weil forms of the tautological quotient bundles on the partial flag variety $\mathrm{Fl}(\mathbf{a})$, $\mathbf{a}=(a_1,\dots,a_k)$. We conjecture that $\dim \mathcal{A}_{\mathbf{a}}$ equals the number of terms in the expansion of $\prod_{i<j}\prod_{p\in A_i,\,q\in A_j}(x_p+x_q)$ in products of Schur polynomials of the block alphabets. We prove a coarsening duality for invariant Chern forms, $c(E_J)\,c(E_{J^c})=1$ at the level of forms, and derive from it universal relations and nonvanishing statements in $\mathcal{A}_{\mathbf{a}}$. We observe that the conjecture holds for complete flags, and in a natural graphical generalisation for all graphs on singleton blocks, by combining results of Postnikov–Shapiro–Shapiro and Stanley. Our main result proves the conjecture for the first infinite family of genuinely multipartite flags, $\mathbf{a}=(k,1,1)$:

$$\dim \mathcal{A}_{k,1,1}=\binom{k+3}{3}+\binom{k+2}{2}=\frac{(k+1)(k+2)(k+6)}{6},$$

and we give the Hilbert series. The proof combines skew Howe duality for (GL_k,GL_2), a reduction to torus invariants of 2×2 matrices, and the Legendre polynomials, which appear as eigenfunctions of an $\mathfrak{sl}_2$-Casimir operator. The same method proves the graphical conjecture for the path with blocks $(1,k,1)$.

1. INTRODUCTION

The cohomology ring of a flag variety is generated by the Chern classes of its tautological bundles. A finer object is the algebra generated by distinguished differential-form representatives of these classes. For the complete flag variety SL_n/B this curvature-form algebra was introduced in [11]. It was then computed in [9, 10]: it is strictly larger than cohomology, and its dimension equals the number of forests on n labelled vertices. This led to the theory of Postnikov–Shapiro algebras and their relatives [8, 5, 6]. The present note studies the analogous algebras for partial flag varieties.

Let $\mathbf{a}=(a_1,\dots,a_k)$ with $a_i>0$ and $n=a_1+\cdots+a_k$. Let

$$\mathrm{Fl}(\mathbf{a})=SL_n(\mathbb{C})/P_{\mathbf{a}}\simeq SU(n)/S\big(U(a_1)\times\cdots\times U(a_k)\big)$$

be the partial flag variety with successive quotient ranks a_i. The standard Hermitian form on $\mathbb{C}^n$ determines canonical homogeneous connections on the tautological quotient bundles $L_1,\dots,L_k$. Their Chern–Weil forms $\phi_j^{(i)}$ are invariant, and we call them the *invariant Chern forms*. Let $\mathcal{A}_{\mathbf{a}}$ be the graded algebra they generate, with $\deg\phi_j^{(i)}=j$.

Fix the consecutive block decomposition $[n]=A_1\sqcup\cdots\sqcup A_k$, $|A_i|=a_i$, and put $X_i=\{x_p : p\in A_i\}$. The polynomial

$$F_{\mathbf{a}}=\prod_{1\le i<j\le k}\ \prod_{p\in A_i,\,q\in A_j}(x_p+x_q) \tag{1.1}$$

is separately symmetric in the alphabets X_i, so it expands uniquely in the products $\prod_i s_{\lambda^{(i)}}(X_i)$. Let $\mathrm{SchurSupp}(F_{\mathbf{a}})$ be the set of tuples $(\lambda^{(1)},\dots,\lambda^{(k)})$ that occur with nonzero coefficient.

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Conjecture 1.1 (Schur-support conjecture)**.** *For every composition* $\mathbf{a}$*,* $\dim \mathcal{A}_{\mathbf{a}} = \#\,\mathrm{SchurSupp}(F_{\mathbf{a}})$.

The conjecture is known for $k = 2$, where $\mathcal{A}_{a,b} \cong H^*(\mathrm{Gr}(a, a+b); \mathbb{C})$ (Theorem 4.2). Exact computations confirm it in all the cases we could reach. The present paper adds the following.

Theorem A (Coarsening duality; Theorem 3.2)**.** *For every set* J *of blocks, let* Q_J *be the curvature matrix of the coarsened tautological bundle* E_J *with block* $A_J = \bigsqcup_{i\in J} A_i$*. Then*

$$\det(I + tQ_J) \cdot \det(I + tQ_{J^c}) = 1$$

identically in the exterior algebra. Moreover $\operatorname{tr} Q_J = \sum_{i\in J} \phi_1^{(i)}$*, and*

$$\Big(\sum_{i\in J} \phi_1^{(i)}\Big)^{a_J(n-a_J)} \neq 0, \qquad \Big(\sum_{i\in J} \phi_1^{(i)}\Big)^{a_J(n-a_J)+1} = 0, \qquad a_J = |A_J|.$$

The conjecture extends naturally to a graph Γ on the set of blocks (Section 4). When all blocks are singletons, the graphical conjecture follows from known results: the curvature algebra is the Postnikov–Shapiro algebra of Γ, whose dimension is the number of forests of Γ [10, 8], and the Schur support is the set of lattice points of the graphical zonotope, which has the same cardinality by Stanley [12]. This is Proposition 4.3. Our main result treats the simplest family in which blocks of size greater than one interact with two other blocks.

Theorem B (Theorem 5.10 and Proposition 5.12)**.** *For every* $k \geq 1$*, Conjecture 1.1 holds for* $\mathbf{a} = (k, 1, 1)$*, and for every permutation of it. More precisely,*

$$\mathrm{Hilb}(\mathcal{A}_{k,1,1}; t) = \sum_{0\leq\nu_2\leq\nu_1\leq k} (\nu_1 - \nu_2 + 1)\, t^{\nu_1+\nu_2} \; + \; \sum_{j=0}^{k} (k + 1 - j)\, t^{k+1+j},$$

and

$$\dim \mathcal{A}_{k,1,1} = \#\,\mathrm{SchurSupp}(F_{k,1,1}) = \binom{k+3}{3} + \binom{k+2}{2} = \frac{(k+1)(k+2)(k+6)}{6}.$$

Moreover $\mathcal{A}_{k,1,1}$ *is generated by the two line-bundle forms* x, y *together with one form* z *of degree* 2*. The Chern forms* p_j *of the rank-*k *block satisfy*

$$\sum_{j\geq 0} p_j t^j = \frac{1}{1 + st + (s^2 - z)t^2}, \qquad s = x + y.$$

Theorem C (Theorem 5.11)**.** *For the path graph on blocks of sizes* $(1, k, 1)$*, in which the two singleton blocks are joined only to the middle block, the graphical curvature algebra has dimension* $\binom{k+3}{3}$*. This equals the Schur-support cardinality of* $\prod_{x\in X}(x + u)(x + v)$*.*

The proof of Theorem B goes as follows. Coarsening duality expresses everything through the 2×2 matrix $M = BA$ of odd bilinear contractions, plus one nilpotent correction w coming from the edge between the two singleton blocks. Skew Howe duality identifies the algebra generated by the entries of M with a quotient of $\mathbb{C}[\mathrm{End}\,\mathbb{C}^2]$ by the isotypic components $\mathbb{S}_\nu \otimes \mathbb{S}_\nu^*$ with $\nu_1 > k$. The curvature algebra becomes the torus-invariant part of this quotient, together with the image of a derivation ∂. The derivation ∂ is infinitesimal conjugation by E_{12} divided by the matrix entry M_{21}. The Casimir operator of $\mathfrak{sl}_2$ acting by conjugation turns into the Legendre operator $\partial\,(X - m_1)(X - m_2)\,\partial$, where m_1, m_2 are the eigenvalues of M. The classical three-term recurrence and derivative identities for Legendre polynomials then compute the image of ∂ exactly.

Organisation. Section 2 recalls the exterior-algebra model and its Eulerian expansion. Section 3 proves coarsening duality and its consequences. Section 4 formulates the graphical conjecture and collects the known cases. Section 5 proves Theorems B and C. Section 6 records computations and open problems.

2. The exterior-algebra model

2.1. **The curvature matrix.** We work at the standard flag. For p, q in different blocks let ω^{pq} be the corresponding complex Maurer–Cartan 1-form, and let

$$E_{\mathbf{a}} = \Lambda_{\mathbb{C}}\big(\omega^{pq} : \ p, q \text{ in different blocks}\big).$$

An invariant form is determined by its value at the base point, and the isotropy-invariant elements of $E_{\mathbf{a}}$ are exactly these values. Up to a common nonzero normalisation of each degree, which is irrelevant for all algebraic statements, the curvature of L_i is (compare [2, §4])

$$Q_i = (q_{\alpha\beta})_{\alpha,\beta\in A_i}, \qquad q_{\alpha\beta} = \sum_{t\notin A_i} \omega^{\alpha t}\wedge\omega^{t\beta}, \qquad \det(I + tQ_i) = \sum_{j\geq 0} \phi_j^{(i)} t^j. \tag{2.1}$$

Thus $\mathcal{A}_{\mathbf{a}}$ is the subalgebra of $E_{\mathbf{a}}$ generated by all $\phi_j^{(i)}$. The model is symmetric under permutations of the blocks, so $\mathcal{A}_{\mathbf{a}}$ depends on $\mathbf{a}$ only up to reordering. The same holds for $\#\,\mathrm{SchurSupp}(F_{\mathbf{a}})$.

2.2. **Eulerian and compatible-cycle expansions.** Fix a block A_i. Let $\mathcal{E}_i^{(j)}$ be the set of Eulerian subgraphs with $2j$ edges of the complete bipartite digraph on $A_i \sqcup ([n] \setminus A_i)$, containing both orientations of every edge, such that every vertex of A_i has indegree at most one. For $G \in \mathcal{E}_i^{(j)}$ and $\alpha \in A_i$ with $\mathrm{indeg}_G\,\alpha = 1$, let $t_{\mp}(\alpha)$ be the neighbours of α along G. Put

$$m_i(G) = \prod_{\alpha} \omega^{t_-(\alpha)\alpha}\wedge\omega^{\alpha t_+(\alpha)}, \qquad cp(G) = \prod_{v\notin A_i} \mathrm{indeg}_G(v)!.$$

The factors of $m_i(G)$ are even, so their order is immaterial. Here $cp(G)$ counts the decompositions of G into closed walks, with traversals recorded.

Theorem 2.1 (Eulerian expansion)**.** $\phi_j^{(i)} = (-1)^j \sum_{G\in\mathcal{E}_i^{(j)}} cp(G)\, m_i(G)$.

Proof. Expand the principal $j \times j$ minors of Q_i. A term is indexed by distinct $\alpha_1, \dots, \alpha_j \in A_i$, a permutation π, and vertices $t_s \notin A_i$; it is the graph with edges $\alpha_s \to t_s \to \alpha_{\pi(s)}$. Recovering the term from the graph amounts to choosing, at each $v \notin A_i$, a bijection between incoming and outgoing edges, which gives the factor $cp(G)$. The determinant sign $(-1)^{j-c(\pi)}$ combines with one sign per cycle of π, from rewriting each cycle in the normal form of m_i, to give $(-1)^j$. □

Let $\mathcal{C}_i$ be the set of directed closed walks, up to rotation, that visit each vertex of A_i at most once. Two such walks are *in conflict* if they share a vertex of A_i.

Corollary 2.2 (Compatible-cycle expansion)**.** $\det(I + tQ_i) = \sum_{\mathcal{S}} \prod_{C\in\mathcal{S}} m_i(C)\,(-t)^{|C|/2}$, *where* $\mathcal{S}$ *runs over the conflict-free families in* $\mathcal{C}_i$.

Proof. Conflict-free families are edge-disjoint, and their unions are exactly the graphs in $\mathcal{E}_i$. Each $G \in \mathcal{E}_i^{(j)}$ arises from precisely $cp(G)$ families. □

Remark 2.3. The sign is -1 per vertex of A_i visited, not per walk. The conflict condition cannot be dropped: an unrestricted product over all closed walks gives wrong coefficients already for $a_i = 1$.

3. Coarsening duality

Lemma 3.1 (Opposite determinants)**.** *Let A be an $m \times r$ and B an $r \times m$ matrix with odd entries in a supercommutative algebra. Then* $\det(I_m + tAB)\det(I_r + tBA) = 1$.

Proof. The entries of AB and BA are even and nilpotent. Moving the first odd factor cyclically past the other $2q-1$ factors gives $\mathrm{tr}((BA)^q) = -\mathrm{tr}((AB)^q)$. Hence $\log\det(I+tAB)+\log\det(I+tBA) = 0$. □

For a set J of blocks put $A_J = \bigsqcup_{i\in J} A_i$, $a_J = |A_J|$, and

$$Q_J = (q^J_{\alpha\beta})_{\alpha,\beta\in A_J}, \qquad q^J_{\alpha\beta} = \sum_{t\notin A_J} \omega^{\alpha t}\wedge\omega^{t\beta}.$$

The projection $\mathrm{Fl}(\mathbf{a}) \to \mathrm{Fl}(\mathbf{a}_J)$ that merges the blocks of J pulls back the canonical connection of the merged tautological bundle E_J. Its curvature at the base point is Q_J. In particular $Q_{\{i\}} = Q_i$.

Theorem 3.2 (Coarsening duality)**.** *For every set J of blocks:*

(i) $\det(I + tQ_J)\,\det(I + tQ_{J^c}) = 1$*;*

(ii) $\operatorname{tr} Q_J = \sum_{i\in J} \phi^{(i)}_1$*;*

(iii) with $N_J = a_J(n - a_J)$*, one has* $(\operatorname{tr} Q_J)^{N_J} \neq 0$ *and* $(\operatorname{tr} Q_J)^{N_J+1} = 0$*.*

Proof. (i) Put $A = (\omega^{\alpha t})_{\alpha\in A_J, t\notin A_J}$ and $B = (\omega^{t\beta})_{t\notin A_J,\beta\in A_J}$. Then $Q_J = AB$, and $(BA)_{tt'} = \sum_{\alpha\in A_J}\omega^{t\alpha}\omega^{\alpha t'} = (Q_{J^c})_{tt'}$. Apply Lemma 3.1. (ii) In $\sum_{i\in J}\operatorname{tr} Q_i$, the terms with both indices in A_J but in different blocks occur in pairs $\omega^{\alpha t}\omega^{t\alpha} + \omega^{t\alpha}\omega^{\alpha t} = 0$. (iii) We have $\operatorname{tr} Q_J = \sum z_{\alpha t}$ with $z_{\alpha t} = \omega^{\alpha t}\omega^{t\alpha}$, $\alpha \in A_J$, $t \notin A_J$. These are N_J commuting square-zero elements in pairwise disjoint sets of generators, so $(\sum z_{\alpha t})^{N_J} = N_J!\prod z_{\alpha t} \neq 0$, and the next power vanishes. □

Corollary 3.3. *If $|J| = 1$ or $|J^c| = 1$, all coefficients of $\det(I + tQ_J)$ lie in $\mathcal{A}_\mathbf{a}$. In particular, for three blocks the Chern forms of every merged bundle lie in $\mathcal{A}_\mathbf{a}$, and the Chern forms of any one block are polynomials in those of the merged complementary bundle.*

Proposition 3.4 (Cohomological lower bound)**.** *The map sending a form to its de Rham class is a surjective graded algebra homomorphism $\mathcal{A}_\mathbf{a} \to H^{2*}(\mathrm{Fl}(\mathbf{a});\mathbb{C})$. In particular* $\dim \mathcal{A}_\mathbf{a} \geq n!/(a_1!\cdots a_k!)$.

Proof. Chern–Weil forms are closed, and the classes of the $\phi^{(i)}_j$ are the Chern classes of the L_i, which generate $H^*(\mathrm{Fl}(\mathbf{a});\mathbb{C})$ by Borel's theorem [1]. □

Remark 3.5. For complete flags the relations of Theorem 3.2(iii), $(\sum_{i\in J} x_i)^{|J|(n-|J|)+1} = 0$, are exactly the defining relations of the Postnikov–Shapiro presentation [10, 8]. For general $\mathbf{a}$ they are far from generating all relations: when $k = 2$ the relation ideal is generated in degrees $b+1,\dots,a+b$, and not only in the top degree $ab+1$.

4. The Schur-support conjecture and known cases

4.1. **Graphical version.** Let Γ be a simple graph on the set of blocks $[k]$. Define $\mathcal{A}_{\mathbf{a},\Gamma} \subset E_\mathbf{a}$ as the algebra generated by the coefficients of $\det(I + tQ^\Gamma_i)$, where Q^Γ_i is given by (2.1) with t restricted to the blocks adjacent to i in Γ. Put

$$F_{\mathbf{a},\Gamma} = \prod_{ij\in E(\Gamma)}\ \prod_{p\in A_i,\, q\in A_j} (x_p + x_q).$$

For the complete graph these reduce to $\mathcal{A}_\mathbf{a}$ and $F_\mathbf{a}$.

Conjecture 4.1 (Graphical Schur-support conjecture)**.** $\dim \mathcal{A}_{\mathbf{a},\Gamma} = \#\,\mathrm{SchurSupp}(F_{\mathbf{a},\Gamma})$.

Some caution is needed: the analogous statement with product *monomial* support is false already for $\mathbf{a} = (2,2)$, where $\prod_{i,j\leq 2}(x_i + y_j)$ has seven distinct monomials while $\dim \mathcal{A}_{2,2} = 6$. Likewise a product over ordered pairs $i \neq j$ in (1.1) does not give the right count.

4.2. **Two blocks.**

Theorem 4.2. *For* $\mathbf{a} = (a, b)$ *one has* $\mathcal{A}_{a,b} \cong H^*(\mathrm{Gr}(a, a+b); \mathbb{C})$*, and Conjecture 1.1 holds.*

Proof. By Theorem 3.2(i) the two Chern families generate the same algebra. On the compact Hermitian symmetric space $\mathrm{Gr}(a, a+b)$ every invariant form is harmonic, so invariant forms inject into cohomology. By Proposition 3.4 the map is also surjective. On the other side, the dual Cauchy identity $\prod_{x\in X, y\in Y}(x+y) = \sum_{\mu\subseteq b^a} s_\mu(X) s_{\tilde{\mu}}(Y)$, with $\tilde{\mu}$ the complement of μ' in a^b rotated by 180°, has $\binom{a+b}{a}$ terms, all with coefficient one. □

4.3. **Singleton blocks.**

Proposition 4.3. *Conjecture 4.1 holds for* $\mathbf{a} = (1^n)$ *and every graph* Γ *on* $[n]$*; both sides equal the number of forests in* Γ*. In particular Conjecture 1.1 holds for complete flags.*

Proof. Put $z_{ij} = \omega^{ij}\omega^{ji}$. These elements commute, $z_{ij} = -z_{ji}$, $z_{ij}^2 = 0$, and their products over distinct unordered pairs are linearly independent. The generators of $\mathcal{A}_{1^n,\Gamma}$ are $x_i = \sum_{j\sim i} z_{ij}$. This is precisely the algebra of [10] (for Γ complete) and [8] (for general Γ), whose dimension is the number of forests of Γ. On the other side, Schur polynomials in one variable are monomials, and all coefficients of $F_{1^n,\Gamma} = \prod_{ij\in\Gamma}(x_i + x_j)$ are positive. Its support is therefore the set of indegree vectors of orientations of Γ, that is, the set of lattice points of the graphical zonotope $Z_\Gamma = \sum_{ij\in\Gamma}[e_i, e_j]$. Their number is the number of forests of Γ [12] (see also [7]). □

4.4. **A Littlewood–Richardson criterion.** For $i < j$ put $R_{ij} = (a_j^{a_i})$. By the dual Cauchy identity,

$$\prod_{p\in A_i,\, q\in A_j} (x_p + x_q) = \sum_{\mu\subseteq R_{ij}} s_\mu(X_i)\, s_{R'_{ij}/\mu'}(X_j).$$

Since Littlewood–Richardson coefficients are nonnegative, there is no cancellation in the product of these expansions.

Proposition 4.4. *A tuple* $(\lambda^{(i)})$ *lies in* $\mathrm{SchurSupp}(F_{\mathbf{a},\Gamma})$ *if and only if there are* $\mu_{ij} \subseteq R_{ij}$*,* $ij \in E(\Gamma)$*,* $i < j$*, such that for every* i *the function* $s_{\lambda^{(i)}}(X_i)$ *occurs in* $\prod_{j>i} s_{\mu_{ij}}(X_i) \prod_{j<i} s_{R'_{ji}/\mu'_{ji}}(X_i)$*.*

5. The family $(k, 1, 1)$

5.1. **Reduction to a** 2×2 **matrix.** Let the blocks be $A_0 = \{1, \dots, k\}$, $\{u\}$ and $\{v\}$. Introduce the odd generators

$$a_\alpha = \omega^{\alpha u}, \quad b_\alpha = \omega^{u\alpha}, \quad c_\alpha = \omega^{\alpha v}, \quad d_\alpha = \omega^{v\alpha} \quad (\alpha \in A_0), \qquad e = \omega^{uv}, \quad f = \omega^{vu}.$$

Write $E' = \Lambda(a, b, c, d)$, so that $E_{k,1,1} = E' \otimes \Lambda(e, f)$. Let $\mathsf{A} = (a\ c)$ be the $k \times 2$ matrix and $\mathsf{B} = \binom{b}{d}$ the $2 \times k$ matrix with these entries, and set

$$M = \mathsf{BA} = \begin{pmatrix} P & R \\ S & T \end{pmatrix}, \qquad P = \sum_\alpha b_\alpha a_\alpha,\ R = \sum_\alpha b_\alpha c_\alpha,\ S = \sum_\alpha d_\alpha a_\alpha,\ T = \sum_\alpha d_\alpha c_\alpha.$$

Put $w = ef$, and $s = P + T$, $D = PT - RS$. By (2.1),

$$x := \phi_1^{(u)} = P + w, \qquad y := \phi_1^{(v)} = T - w, \qquad Q_0 = \mathsf{AB},$$

so $x + y = s$. Lemma 3.1 gives

$$\sum_{j\ge 0} p_j t^j := \det(I + tQ_0) = \frac{1}{\det(I + tM)} = \frac{1}{1 + st + Dt^2}. \tag{5.1}$$

Proposition 5.1. $\mathcal{A}_{k,1,1} = \mathbb{C}[x, s, D]$*. With* $z = p_2 = s^2 - D$*, the Chern forms of the rank-k block satisfy* $p_1 = -s$ *and* $p_j = -s\, p_{j-1} - (s^2 - z)\, p_{j-2}$ *for* $j \ge 2$*. The vanishing of* p_{k+1} *and* p_{k+2} *gives relations in* $\mathcal{A}_{k,1,1}$*.*

Proof. This is immediate from (5.1). Since $\det(I + tQ_0)$ is a polynomial of degree at most k, its coefficients beyond degree k vanish. □

This proves the recurrence that was conjectured in an earlier version of this work. Write $R := \mathbb{C}[s, D, X]$ for a polynomial ring with $\deg s = \deg X = 1$ and $\deg D = 2$. Let $\varphi : R \to E'$ be the homomorphism $s \mapsto s$, $D \mapsto D$, $X \mapsto P$, and let $\Phi : R \to E_{k,1,1}$ be the homomorphism with $X \mapsto x = P + w$. Since $w^2 = 0$,

$$\Phi(F) = \varphi(F) + w\,\varphi(\partial F), \qquad \partial = \partial/\partial X. \tag{5.2}$$

Because $w \notin E'$, $\Phi(F) = 0$ exactly when $\varphi(F) = \varphi(\partial F) = 0$. Writing $I = \ker \varphi$, we obtain:

Lemma 5.2. *$\mathcal{A}_{k,1,1} \cong R/K$, where $K = \{F \in I : \partial F \in I\}$. Hence*

$$\dim \mathcal{A}_{k,1,1} = \dim R/I + \dim N, \qquad N := (I + \partial I)/I,$$

and the analogous identity holds for Hilbert series, with N shifted by $\deg w = 1$.

Proof. The map $I \to R/I$, $F \mapsto \partial F \bmod I$, has kernel K and image N. □

Note that $\varphi(R) = \mathbb{C}[P, T, RS]$, since $T = s - X$ and $RS = X(s - X) - D$. In particular R/I is the graphical curvature algebra of the path $u - A_0 - v$. Indeed, removing the edge uv replaces x, y by P, T and leaves (5.1) unchanged.

5.2. **Skew Howe duality.** Let $U = \mathbb{C}^2$, and let $\mathbb{C}[\operatorname{End} U] = \mathbb{C}[P, R, S, T]$ be the polynomial ring in the entries of a generic matrix M, graded by degree. The group $GL(U) \times GL(U)$ acts by $M \mapsto gMh^{-1}$, and

$$\mathbb{C}[\operatorname{End} U] = \bigoplus_{\nu} \mathsf{M}_\nu, \qquad \mathsf{M}_\nu \cong \mathbb{S}_\nu U \otimes \mathbb{S}_\nu U^*,$$

where ν runs over partitions with at most two parts and M_ν is the space of matrix coefficients of $\mathbb{S}_\nu U$, of degree $|\nu|$ [3]. Let $\pi : \mathbb{C}[\operatorname{End} U] \to E'$ send $M \mapsto \mathsf{BA}$, and let $\mathcal{J} = \pi(\mathbb{C}[\operatorname{End} U])$.

Proposition 5.3. *$\ker \pi = I_k := \bigoplus_{\nu_1 > k} \mathsf{M}_\nu$, so $\mathcal{J} \cong \bigoplus_{\nu \subseteq (k,k)} \mathsf{M}_\nu$.*

Proof. Let $V = \mathbb{C}^k$. Here $\mathsf{A} \in V \otimes U^*$ and $\mathsf{B} \in V^* \otimes U$ have independent odd entries, and π is $GL(U) \times GL(U)$-equivariant for $\mathsf{B} \mapsto g\mathsf{B}$, $\mathsf{A} \mapsto \mathsf{A}h^{-1}$. Its image lies in $E'^{GL(V)}$. By the skew Cauchy decomposition $\Lambda(V \otimes U^*) = \bigoplus_\lambda \mathbb{S}_\lambda V \otimes \mathbb{S}_{\lambda'} U^*$, and similarly for $\Lambda(V^* \otimes U)$. Hence

$$E'^{GL(V)} \cong \bigoplus_{\lambda} \mathbb{S}_{\lambda'} U \otimes \mathbb{S}_{\lambda'} U^*,$$

where λ runs over partitions with at most k rows and at most 2 columns; equivalently $\nu = \lambda' \subseteq (k, k)$. By the first fundamental theorem for skew $(GL(V), GL(U))$-duality [4], $E'^{GL(V)}$ is generated by the contractions $(\mathsf{BA})_{ij}$, so π maps onto it. Because both decompositions are multiplicity free, Schur's lemma shows that π is injective on M_ν for $\nu \subseteq (k, k)$ and kills M_ν otherwise. □

Remark 5.4. The vanishing of $\pi(I_k)$ can also be seen directly. I_k is generated by $\mathsf{M}_{(k+1)}$, which is spanned by the coefficients of $(Pu_1 + Ru_2)^i (Su_1 + Tu_2)^{k+1-i}$. Under π this becomes a product of $k + 1$ factors of the form $\sum_\alpha (\text{odd})_\alpha\, \xi_\alpha$ with $\xi_\alpha = a_\alpha u_1 + c_\alpha u_2$ and $\xi_\alpha^2 = 0$, so it vanishes by the pigeonhole principle.

The diagonal torus $\mathbb{T} \subset GL(U)$ acts by conjugation, and $\mathbb{C}[\operatorname{End} U]^{\mathbb{T}} = \mathbb{C}[P, T, RS]$. Identifying this ring with R via $X = P$, $s = P + T$, $D = PT - RS$, we have $I = I_k^{\mathbb{T}} = \bigoplus_{\nu_1 > k} \mathsf{M}_\nu^{\mathbb{T}}$. Since conjugation acts on M_ν as on $\mathbb{S}_\nu U \otimes \mathbb{S}_\nu U^*$, the zero-weight space $\mathsf{M}_\nu^{\mathbb{T}}$ has dimension $\nu_1 - \nu_2 + 1$.

Corollary 5.5. $\operatorname{Hilb}(R/I; t) = \displaystyle\sum_{0 \le \nu_2 \le \nu_1 \le k} (\nu_1 - \nu_2 + 1)\, t^{\nu_1 + \nu_2}$ *and* $\dim R/I = \binom{k+3}{3}$. *Moreover $I \cap \mathbb{C}[s, D] = \operatorname{span}\{s_\nu(m_1, m_2) : \nu_1 > k\}$, so $\mathbb{C}[s, D]/(I \cap \mathbb{C}[s, D]) \cong H^*(\operatorname{Gr}(2, k + 2); \mathbb{C})$. Here m_1, m_2 are the eigenvalues of M, with $s = m_1 + m_2$ and $D = m_1 m_2$.*

Proof. For the dimension, $\sum_{\nu_1=0}^{k}\sum_{\nu_2=0}^{\nu_1}(\nu_1-\nu_2+1)=\sum_{\nu_1}\binom{\nu_1+2}{2}=\binom{k+3}{3}$. The conjugation-invariant line in M_ν is spanned by the character $s_\nu(m_1,m_2)$. □

5.3. **The derivation ∂ and the Legendre operator.** Infinitesimal conjugation by E_{12} and E_{21} gives the derivations of $\mathbb{C}[\operatorname{End} U]$

$$\varepsilon = S\partial_P + (T-P)\partial_R - S\partial_T, \qquad \phi = -R\partial_P + (P-T)\partial_S + R\partial_T.$$

They preserve every $GL(U)$-stable subspace, in particular I_k. Put

$$\Pi(X) = X^2 - sX + D = (X-m_1)(X-m_2) = -RS.$$

A direct computation in the coordinates (P,T,RS) shows that for $F\in R$

$$\varepsilon F = S\,\partial F, \qquad \phi F = -R\,\partial F, \qquad \mathcal{L}F := \phi\varepsilon F = \partial\big(\Pi\,\partial F\big). \tag{5.3}$$

In particular $\partial F = \varepsilon F/S$: the derivation in (5.2) is infinitesimal conjugation divided by a matrix entry. The operator $\mathcal{L}$ maps R to itself and preserves $I = I_k\cap R$.

Define the Legendre polynomials adapted to Π by the Rodrigues formula

$$L_l = \frac{1}{l!}\,\partial^l\,\Pi^l \in \mathbb{C}[s,D][X], \qquad \Delta = s^2-4D = (m_1-m_2)^2.$$

Substituting $X = \frac12\big(s+(m_1-m_2)\xi\big)$ gives $L_l = (m_1-m_2)^l P_l(\xi)$, where P_l is the classical Legendre polynomial. The classical identities [13] therefore translate into the following polynomial identities over $\mathbb{C}[s,D]$.

Lemma 5.6. *(i) L_l is homogeneous of degree l and of degree l in X, with leading coefficient $\binom{2l}{l}$; hence $R=\bigoplus_{l\ge0}\mathbb{C}[s,D]\,L_l$.*
(ii) $\mathcal{L}L_l = l(l+1)L_l$.
(iii) $(2l+1)L_1L_l = (l+1)L_{l+1} + l\,\Delta\,L_{l-1}$, where $L_1 = 2X-s$.
(iv) $\partial L_{l+1} - \Delta\,\partial L_{l-1} = 2(2l+1)L_l$; also $\partial L_1 = 2L_0$ and $\partial L_0 = 0$.

Proposition 5.7. *For $l\ge0$ let $J_l = \{a\in\mathbb{C}[s,D] : aL_l\in I\}$. Then:*
(a) $I=\bigoplus_l J_lL_l$, and each J_l is an ideal of $\mathbb{C}[s,D]$;
(b) $J_0\subseteq J_1\subseteq J_2\subseteq\cdots$ and $\Delta J_l\subseteq J_{l-1}$ for $l\ge1$;
(c) $\operatorname{Hilb}(\mathbb{C}[s,D]/J_l;t) = \sum t^{|\nu|-l}$, summed over $\nu\subseteq(k,k)$ with $\nu_1-\nu_2\ge l$. In particular $\operatorname{codim} J_l = \binom{k-l+2}{2}$ for $l\le k$, and $J_l=\mathbb{C}[s,D]$ for $l>k$.

Proof. (a) Each graded piece R_d is finite dimensional. By Lemma 5.6(i),(ii), $\mathcal{L}$ acts on R_d diagonalisably, with distinct eigenvalues $l(l+1)$ on the summands $(\mathbb{C}[s,D]L_l)_d$. Since I_d is $\mathcal{L}$-stable, it is the sum of its intersections with these eigenspaces. The J_l are ideals because I is an ideal of R.

(b) If $a\in J_l$, then $L_1\cdot aL_l\in I$. By Lemma 5.6(iii) its components in the $\mathcal{L}$-eigenspaces are $\frac{l+1}{2l+1}aL_{l+1}$ and $\frac{l}{2l+1}\Delta aL_{l-1}$, and by (a) both lie in I.

(c) Each $\mathsf{M}_\nu^{\mathbb{T}}$ is $\mathcal{L}$-stable. By Clebsch–Gordan, conjugation makes $\mathsf{M}_\nu\cong\mathbb{S}_\nu U\otimes\mathbb{S}_\nu U^*$ isomorphic, as an $\mathfrak{sl}_2$-module, to $\bigoplus_{l=0}^{m}V(2l)$ with $m=\nu_1-\nu_2$. The Casimir element acts on the zero-weight vector of $V(2l)$ as a multiple of fe, and these values are distinct for different l. Since $\mathcal{L}=\phi\varepsilon$ is the operator fe (up to an overall sign convention for ε and ϕ, which cancels), $\mathcal{L}$ has $m+1$ distinct eigenvalues on $\mathsf{M}_\nu^{\mathbb{T}}$. Elements of $\mathsf{M}_\nu^{\mathbb{T}}$ are of the form $D^{\nu_2}G$ with G a polynomial in P,T,RS of degree m, so their X-degree is at most m; hence these eigenvalues are $l(l+1)$ for $l=0,\dots,m$. Thus $\mathsf{M}_\nu^{\mathbb{T}}\cap\mathbb{C}[s,D]L_l$ is a line $\mathbb{C}\,a_{\nu,l}L_l$ for $l\le m$ and is zero otherwise, with $\deg a_{\nu,l} = |\nu|-l$. Since $R=\bigoplus_\nu\mathsf{M}_\nu^{\mathbb{T}}$ and $I=\bigoplus_{\nu_1>k}\mathsf{M}_\nu^{\mathbb{T}}$, the classes of $a_{\nu,l}$ with $\nu\subseteq(k,k)$ and $\nu_1-\nu_2\ge l$ form a basis of $\mathbb{C}[s,D]/J_l$. Finally, $\#\{\nu\subseteq(k,k):\nu_1-\nu_2\ge l\} = \sum_{\nu_1=l}^{k}(\nu_1-l+1) = \binom{k-l+2}{2}$. □

Remark 5.8. One can show that $a_{(m),l}$ is proportional to $h_{m-l}(m_1^{\times(l+1)},m_2^{\times(l+1)})$, the complete homogeneous polynomial in the alphabet (m_1,m_2) repeated $l+1$ times; that is, $\sum_m a_{(m),l}u^{m-l}\propto(1-su+Du^2)^{-l-1}$. Moreover $a_{\nu,l}=D^{\nu_2}a_{(\nu_1-\nu_2),l}$. This explicit form is not needed below.

5.4. **The image of ∂.**

Proposition 5.9. $\dim N = \operatorname{codim} J_0 = \binom{k+2}{2}$, *and* $\operatorname{Hilb}(N;t) = \sum_{j=0}^{k}(k+1-j)\,t^{k+j}$.

Proof. For $a \in \mathbb{C}[s,D]$ we have $\partial(aL_l) = a\,\partial L_l$, because $\partial s = \partial D = 0$. By Lemma 5.6(iv), ∂L_l is a $\mathbb{C}[s,D]$-combination of $L_{l-1}, L_{l-3}, \dots$, and the coefficient of L_{l-1} is $2(2l-1)$. Consider the map

$$\Psi : \bigoplus_{l\ge 1} J_l \longrightarrow R/I = \bigoplus_i (\mathbb{C}[s,D]/J_i)\,L_i, \qquad (a_l) \mapsto \sum_l \partial(a_l L_l) \bmod I,$$

whose image is N (note $\partial(J_0 L_0) = 0$). For each $l \ge 1$ choose a graded complement S_l of J_{l-1} in J_l; this is possible by Proposition 5.7(b).

Spanning. For $a \in J_{l-1}$ and $l \ge 2$, Lemma 5.6(iv) gives

$$\partial(aL_l) = \partial(\Delta a\, L_{l-2}) + 2(2l-1)\,aL_{l-1}.$$

The last term lies in I, and $\Delta a \in J_{l-2}$ by Proposition 5.7(b). For $l = 1$ we get $\partial(aL_1) = 2aL_0 \in I$. By induction on l, $\Psi(J_l) \subseteq \sum_{l'\le l}\Psi(S_{l'})$, so $N = \Psi(\bigoplus_l S_l)$.

Injectivity on $\bigoplus S_l$. Suppose $\Psi((a_l)) = 0$ with $a_l \in S_l$ not all zero, and let l_0 be maximal with $a_{l_0} \ne 0$. The L_{l_0-1}-component of $\Psi((a_l))$ is $2(2l_0-1)a_{l_0} \bmod J_{l_0-1}$, since all other terms only involve L_i with $i < l_0 - 1$. This component is nonzero because $a_{l_0} \in S_{l_0} \setminus \{0\}$, a contradiction.

Hence $N \cong \bigoplus_{l\ge 1} S_l$, with $a \in S_l$ contributing in degree $\deg a + l - 1$. Telescoping gives $\dim N = \sum_{l\ge 1}(\operatorname{codim} J_{l-1} - \operatorname{codim} J_l) = \operatorname{codim} J_0 = \binom{k+2}{2}$. For the Hilbert series, Proposition 5.7(c) and the bijection $\nu \mapsto \nu - (1,0)$ give

$$\operatorname{Hilb}(J_l/J_{l-1};t) = \sum_{j=0}^{k-l+1} t^{k+j-l+1},$$

corresponding to the partitions (k,j). Multiplying by t^{l-1} and summing over $1 \le l \le k+1$ gives the claim. □

5.5. **Main theorems.**

Theorem 5.10. *For every $k \ge 1$,*

$$\operatorname{Hilb}(\mathcal{A}_{k,1,1};t) = \sum_{0\le\nu_2\le\nu_1\le k} (\nu_1 - \nu_2 + 1)\,t^{\nu_1+\nu_2} + \sum_{j=0}^{k}(k+1-j)\,t^{k+1+j},$$

and $\dim \mathcal{A}_{k,1,1} = \binom{k+3}{3} + \binom{k+2}{2} = \frac{(k+1)(k+2)(k+6)}{6}$. *The top degree is* $2k+1 = \dim_{\mathbb{C}} \operatorname{Fl}(k,1,1)$, *and the top-degree component is one-dimensional.*

Proof. Combine Lemma 5.2, Corollary 5.5 and Proposition 5.9. □

Theorem 5.11. *For the path graph $u - A_0 - v$ on blocks of sizes $(1,k,1)$, the graphical curvature algebra is $R/I \cong \mathcal{J}^{\mathbb{T}}$. It has dimension $\binom{k+3}{3}$ and palindromic Hilbert series $\sum_{\nu\subseteq(k,k)}(\nu_1-\nu_2+1)t^{|\nu|}$. This dimension equals $\#\operatorname{SchurSupp}\big(\prod_{x\in X}(x+u)(x+v)\big)$.*

Proof. The first statements are Corollary 5.5. The Hilbert series is palindromic under $\nu \mapsto (k-\nu_2, k-\nu_1)$. The support count is Proposition 5.12(i). □

Proposition 5.12. *Let X be an alphabet of size k, and let u, v be single variables.*

(i) $\#\operatorname{SchurSupp}\big(\prod_{x\in X}(x+u)(x+v)\big) = \sum_{q=0}^{k}(k+1-q)(q+1) = \binom{k+3}{3}$.

(ii) $\#\operatorname{SchurSupp}(F_{k,1,1}) = \sum_{q=0}^{k}(k+1-q)(q+2) = \binom{k+3}{3} + \binom{k+2}{2}$.

Proof. We have $\prod_x (x+u)(x+v) = \sum_{0\le i,j\le k} e_i(X)e_j(X)\,u^{k-i}v^{k-j}$, with nonnegative Schur coefficients. By the Pieri rule, $e_i e_j = \sum_r s_{(2^r 1^{i+j-2r})}(X)$, summed over $\max(0, i+j-k) \le r \le \min(i,j)$. Fix $\lambda = (2^r 1^q)$ with $r+q \le k$. It occurs exactly for $i+j = 2r+q$ and $r \le i \le r+q$, which gives $q+1$ distinct monomials in u, v; this proves (i). For (ii), $F_{k,1,1} = (u+v)\prod_x (x+u)(x+v)$. For fixed λ the u-exponents that occur are $k-i+1$ and $k-i$ for $r \le i \le r+q$, that is, the $q+2$ integers in $[k-r-q,\, k-r+1]$. The closed forms follow from $\sum_{q=0}^{k}(k+1-q)(q+1) = \binom{k+3}{3}$ and $\sum_{q=0}^{k}(k+1-q) = \binom{k+2}{2}$. □

Corollary 5.13. *Conjecture 1.1 holds for* $\mathbf{a} = (k,1,1)$ *and all its permutations, and Conjecture 4.1 holds for the path on* $(1,k,1)$.

Remark 5.14 (Two filtrations). Exact computations for $k \le 6$ show that the filtration of $\mathcal{A}_{k,1,1}$ by the $\mathbb{C}[s,D]$-submodules $\sum_{i\le j} \mathbb{C}[s,D]\,x^i$ has associated graded

$$H^*(\mathrm{Gr}(2,k+2)) \;\oplus\; \bigoplus_{j=1}^{k+1} t^j\, H^*(\mathrm{Gr}(2,k+3-j)).$$

Similarly, the filtration of R/I by powers of X has associated graded $\bigoplus_{j=0}^{k} t^j H^*(\mathrm{Gr}(2,k+2-j))$. The second statement follows from Proposition 5.7. It would be interesting to obtain the first directly and to match it with the Littlewood–Richardson description of $\mathrm{SchurSupp}(F_{k,1,1})$.

Remark 5.15. The draft formula $\frac{1}{6}(k^3+12k^2+5k+30)$ for $\#\,\mathrm{SchurSupp}(F_{k,1,1})$ in an earlier version was incorrect; the correct value is $\frac{1}{6}(k+1)(k+2)(k+6)$. The two agree only for $k \in \{2,3\}$. The corrected formula also covers $k=1$, giving the 7 forests on three vertices.

6. Computations and open problems

All computations use exact (or large-prime modular, double-checked) linear algebra in a sparse exterior-monomial basis. The curvature generators are built from (2.1), and Schur supports are computed by multiplying by the block Vandermonde determinants. In every case below the two sides of Conjecture 4.1 agree.

$\mathbf{a}$	Γ	$\mathrm{Hilb}(\mathcal{A}_{\mathbf{a},\Gamma};t)$	dim
$(2,2,1)$	K_3	$1+2t+5t^2+8t^3+12t^4+11t^5+8t^6+3t^7+t^8$	51
$(2,2,2)$	K_3	(degrees 0–12)	230
$(k,1,1),\ k\le 6$	K_3	Theorem 5.10	$16, 30, 50, 77, 112$
$(2,1,2)$	path, center 1	$1+2t+3t^2+2t^3+t^4$	9
$(2,2,1)$	path, center 2	$1+2t+5t^2+5t^3+5t^4+2t^5+t^6$	21
$(2,1,1,1)$	star, center 2	$1+3t+7t^2+10t^3+9t^4+3t^5+t^6$	34
$(1,2,1,1)$	path $1{-}2{-}1{-}1$	$1+3t+6t^2+6t^3+3t^4+t^5$	20

Problem 6.1. *Prove Conjecture 1.1 for* $(k,l,1)$ *and for* $(k,1,1,1)$. *The method of Section 5 suggests replacing* GL_2 *by* GL_{n-a_1} *and the torus by the Levi subgroup of the remaining blocks. The obstacle is that, for three or more blocks, the forms* $\phi_j^{(i)}$ *with* $i \neq 0$ *are no longer obtained from a single coarse matrix by a rank-one nilpotent shift.*

Problem 6.2. *Do the Chern forms of all coarsened bundles* E_J *lie in* $\mathcal{A}_{\mathbf{a}}$*? By Corollary 3.3 this holds when* $|J|$ *or* $|J^c|$ *equals* 1, *in particular for every partial flag with at most three blocks.*

Problem 6.3. *Find a basis, or a degeneration, of* $\mathcal{A}_{\mathbf{a}}$ *indexed by the Littlewood–Richardson-feasible tuples of Proposition 4.4. In the singleton case this specialises to the forest bases of* [8].

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Department of Mathematics and Computer Science, St. Petersburg State University, 14th Line V.O. 29B, 199178 St. Petersburg, Russia

Email address: g.nenashev@spbu.ru

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA

Email address: apost@math.mit.edu

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden

Email address: shapiro@math.su.se

Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

Email address: mshapiro@math.msu.edu