INVERSE MOMENT PROBLEM FOR ALGEBRAIC BODIES AND PICARD-LEFSCHETZ THEORY

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To Viktor Vassiliev on the occasion of his 70th birthday

Abstract. in this paper we consider ramified integrals obtained by the analytic continuation of the moments of algebraic bodies, i.e., connected components of the complements of real algebraic hypersurfaces in $\mathbb{R}^n$. We study their properties and relate them to the inverse moment problem.

1. Introduction

Inverse moment problem in potential theory has a long history and is still a very active area of research. Below we address one specific fundamental instance of the latter problem, namely, the question of recovery of an algebraic body from its moments which, to the best of our knowledge, has not been previously considered in complete generality. The main novelty of our paper is that we apply to this question the approach of Picard-Lefschetz theory as suggested in similar situations by V. Arnold and V. Vassiliev, see [2, 15, 16].[1]

Another motivation for our study comes from the publications [6, 7] in which the authors considered the question of recovering a real homogeneous and positive outside the origin polynomial $g(x_1, x_2, \dots, x_n)$ of a given degree k from the moments of its sublevel set which, by definition, is given by the inequality

$$G := \{(x_1, \dots, x_n) \in \mathbb{R}^n |\ g(x_1, \dots, x_n) \le 1\}.$$

The authors of [6, 7] provided an algorithm recovering g from the information about certain moments of G of degree less than or equal to $3k$. Although the suggested recovery algorithm can be reduced to rather simple linear algebra its major drawback is that for a homogeneous polynomial g of degree k the moments of G are far too many compared to the number of degrees of freedom of g and therefore they are highly dependent. This issue is not discussed in the latter publications. Without understanding of these dependences (which are undoubtedly quite complicated) it is impossible to estimate the relevance of the suggested procedure.

Finally, in a recent paper [14], the authors made further progress in the question suggested in [6] which asks about the minimal number of moments of G which uniquely determine g. Namely, they have shown that in the above set-up, the polynomial g is uniquely defined by the collection of all $\binom{n+k-1}{k}$ moments of degree

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[1]A beautiful paper [2] deals with the modern interpretation and generalizations of Lemma 28 from Newton's Principia. In this Lemma Newton "proves" that the area of the portion of a plane oval cut out by a real line can not be an algebraic function of (the coefficients of) the line. In [15, 16] and further publications, V. Vassiliev developed a whole theory of algebraic (non-)integrability of algebraic bodies. A short historical account of discussions around Lemma 28 including among others such celebrities as Leibniz, Bernouili, Huygens, and Routh can be found in [12]. A highly recommended reading!

k of its sublevel set G, see Appendix below. [2](Notice that g has exactly $\binom{n+k-1}{k}$ coefficients.)

{sec:set-up}

2. General set-up

2.1. **Problem formulation and basic constructions.** Throughout the paper we assume that the algebraic hypersurface under consideration is smooth and that the corresponding real domain is compact. Given a real-valued polynomial $p(x_1, x_2, \dots, x_n)$ of degree k, denote by $Y_p \subset \mathbb{R}^n$ its real zero locus. A compact connected component of Y_p is called an *algebraic ovaloid* and connected domain $\Omega \subset \mathbb{R}^n - Y_p$ bounded by an algebraic ovaloid is called an *algebraic body.* We say that algebraic body Ω *has degree* k if k is the minimal degree of a real polynomial whose zeros locus contains its boundary $\partial\Omega$. Such a polynomial exists and is unique up to a scalar. We will refer to it as *defining polynomial of* Ω and denote it by p_Ω. We say that an algebraic body Ω is *smooth* if its boundary $\partial\Omega$ is smooth, i.e., the defining polynomial of Ω has no singular points on $\partial\Omega$.

Assuming that $\mathbb{R}^n$ is endowed with a fixed coordinate system $(x_1, x_2, \dots, x_n)$, let us consider different moments of Ω, i.e., integrals of the form

{eq:moments}
$$m_I(\Omega) = \int_\Omega X^I dX, \qquad (2.1)$$
eq:moments

where $I = (i_1, i_2, \dots, i_n)$ is a multiindex with non-negative integer entries, $X^I = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$, and $dX = dx_1 dx_2 \dots dx_n$. Rather sloppily, the major question in the inverse moment problem can be formulated as follows.

Problem 1. Find an appropriate collection of moments $m_I(\Omega)$ where the multiindex I runs over some finite set which will allow to uniquely recover Ω. (In particular, we want to recover the defining polynomial p_Ω of Ω.) If possible, find an appropriate numerical procedure/algorithm performing this recovery from the collection of moments.

Remark 1. Observe that for fixed k, algebraic bodies of degree k depend on $\binom{n+k}{k} - 1$ parameters. Since the complete infinite collection of moments uniquely define an arbitrary compactly supported measure there exists a finite collection of moments distinguishing all algebraic bodies of a given fixed degree k. On the other hand, if a chosen collection of moments contains more elements than the number of parameters in the family of algebraic bodies then one runs in a difficult problem of describing dependences among the respective moments, i.e., describing the respective moment varieties. Examples of such activities can be found in e.g., [1, 5, 3].

In our situation, for $n \geq 2$, it seems natural to consider the collection of all moments of Ω of degree at most k since the number of moments in this collection equals $\binom{n+k}{k}$ which is the dimension of the linear space $Pol^k(\mathbb{R}^n)$ of all polynomials of degree at most k in n variables. The family of algebraic bodies of degree k depends on one less parameter. One needs to describe the dependencies among the latter moments. For $n = 1$, which is a rather trivial case we can take moments of degree at most $k - 1$.

To an algebraic body $\Omega \subset \mathbb{R}^n$ of degree k we associate a collection of its moments

{eq:Mom}
$$m^\Omega_{i_1, i_2, \dots, i_n} = \int_\Omega x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} dx_1 dx_2 \dots dx_n,\ 0 \leq i_1 + i_2 + \dots + i_n \leq k \qquad (2.2)$$
eq:Mom

[2]The manuscript [14] seems to be unavailable online. For that reason we include the relevant short argument from [14] in Appendix below.

of order less than or equal to k. We arrange the latter moments in a real polynomial of degree (at most) k in n variables by taking its (normalized) *moment generating function*

{eq:gener}
$$\Phi^{\Omega}(t_1,t_2,\dots,t_n) := \sum_{i_1+i_2+i_n\le k} \binom{i_1+i_2+\dots+i_n}{i_1,i_2,\dots,i_n} m^{\Omega}_{i_1,i_2,\dots,i_n} t_1^{i_1}t_2^{i_2}\dots t_n^{i_n}. \quad (2.3)$$
eq:gener

{lm:inl}

lm:inl **Lemma 1.** *Given an algebraic body $\Omega\subset\mathbb{R}^n$ of degree k, its moment generating function is given by the integral transform*

$$\Phi^{\Omega}(t_1,t_2,\dots,t_n) = \int_{\Omega} \frac{(t_1x_1+t_2x_2+\dots+t_nx_n)^{k+1}-1}{t_1x_1+t_2x_2+\dots+t_nx_n-1} dx_1dx_2\dots dx_n. \quad (2.4)$$
{eq:MMM}

Our next goal is to consider how $\Phi^{\Omega}(t_1,t_2,\dots,t_n)$ depends on Ω and complexify it by using the approach of Picard-Lefschetz theory, see [15, 16].

Observe that for any multiindex $I=(i_1,i_2,\dots,i_n)$, the differential n-form

$$X^I dX = x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}dx_1dx_2\dots dx_n$$

is a meromorphic top-dimensional form in $\mathbb{C}P^n$; its polar set coinciding with the hyperplane L_∞ at infinity. Our integration domain Ω can be considered as a cycle in the reduced relative homology $\bar H_n(\mathbb{C}P^n - L_\infty, Y_p^{\mathbb{C}})$ with integer coefficients. Here $Y_p^{\mathbb{C}}$ is the (affine) complexification of the real algebraic hypersurface Y_p bounding the ovaloid Ω. By Stokes' theorem, the value of the moment $m_I(\Omega)=\int_\Omega X^I dX$ depends only on the homology class $\theta(\Omega)\in \bar H_n(\mathbb{C}P^n - L_\infty, Y_p^{\mathbb{C}})$ represented by Ω. To simplify the notation, denote by $\mathcal{H}(p)$ the latter homology group $\bar H_n(\mathbb{C}P^n - L_\infty, Y_p^{\mathbb{C}})$.

{lm:stab}

lm:stab **Lemma 2.** (i) For $n\ge 2$ and any positive integer k, the abelian group $\mathcal{H}(p)$ is independent of the choice of polynomial p if (and only if?) $\bar Y_p^{\mathbb{C}}$ is a smooth projective hypersurface in $\mathbb{C}P^n$ which intersects the hyperplane $L_\infty\subset\mathbb{C}P^n$ transversally. (Here $\bar Y_p^{\mathbb{C}}\subset\mathbb{C}P^n$ is the standard projectivization of $Y_p^{\mathbb{C}}\subset\mathbb{C}^n$).

(ii) Under the above assumptions $\mathcal{H}(p)\simeq\mathbb{Z}^{(k-1)^n}$.

(iii) Each moment $m^{\Omega}_{i_1,i_2,\dots,i_n}$ can be interpreted as a "period" of the affine hypersurface Y_p where p is the defining polynomial of Ω. Namely, by Stokes' theorem, one has

$$m^{\Omega}_{i_1,i_2,\dots,i_n} = \frac{\int_{\partial\Omega} x_1^{i_1+1}x_2^{i_2}\dots x_n^{i_n}dx_2\dots dx_n}{i_1+1} = \dots = \frac{\int_{\partial\Omega} x_1^{i_1}x_2^{i_2}\dots x_n^{i_n+1}dx_1dx_2\dots dx_{n-1}}{i_n+1}.$$

Proof. Part (i) follows from ...

To settle Part (ii) notice that $\mathbb{C}P^n\setminus L_\infty = \mathbb{C}^n$ and consider the long exact sequence of a pair $(\mathbb{C}^n, Y_p^{\mathbb{C}})$

$$\dots\to H_n(Y_p^{\mathbb{C}}) \xrightarrow{i_*} H_n(\mathbb{C}^n) \xrightarrow{j_*} H_n(\mathbb{C}^n, Y_p^{\mathbb{C}}) \xrightarrow{\partial} H_{n-1}(Y_p^{\mathbb{C}}) \to H_{n-1}(\mathbb{C}^n)\to\cdots.$$

Since $\mathbb{C}^n$ is contratible $H_n(\mathbb{C}^n, Y_p^{\mathbb{C}})\simeq H_{n-1}(Y_p^{\mathbb{C}})$. Notice that here $Y_p^{\mathbb{C}}$ is the affine hypersurface. A smooth affine hypersurface in $\mathbb{C}^n$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres. Therefore to calculate $H_{n-1}(Y_p^{\mathbb{C}})$ we just need to know its Euler characteristics. The Euler characteristics of a smooth hypersurface of degree k in $\mathbb{C}P^n$ is given by

$$\chi(\bar Y) = \frac{1}{k}[(1-k)^{n+1}-1]+n+1,$$

see e.g. [4], Ch.5, Exer. 3.7. By additivity of Euler characteristsics, we get to a smooth hypersurface Y of degree k whose compactification intersects the hyperplane at ∞ transversally

$$\chi(Y)=\frac{1}{k}[(1-k)^{n+1}-1]+n+1-\left(\frac{1}{k}[(1-k)^n-1]+n\right)=1-(1-k)^n.$$

Thus $H_{n-1}(Y_p^{\mathbb{C}})\simeq\mathbb{Z}^{(k-1)^n}$.

Part (iii) follows from the fact that

$$\frac{1}{i_1+1}d(x_1^{i_1+1}x_2^{i_2}\dots x_n^{i_n}dx_2\dots dx_n)=\frac{1}{i_2+1}d(x_1^{i_1}x_2^{i_2+1}\dots x_n^{i_n}dx_1dx_3\dots dx_n)=$$

$$=\frac{1}{i_n+1}d(x_1^{i_1}x_2^{i_2}\dots x_n^{i_n+1}dx_1dx_2\dots dx_{n-1})=x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}dx_1dx_2\dots dx_n.$$

$\partial\Omega$ defines a cycle in $H_{k-1}(Y_p)$.

□

Example 1. For $n=1$ and a given k, $Y_p^{\mathbb{C}}$ satisfying the conditions of Lemma 2 consists of k distinct points in $\mathbb{C}$ and therefore $H_0(Y_p^{\mathbb{C}})\simeq\mathbb{Z}^k$. (In this case we need the reduced homology which gives $\mathbb{Z}^{k-1}$). For $n=2$, $Y_p^{\mathbb{C}}$ is a smooth curve of genus $\binom{k-1}{2}$ with k distinct punctures. In this case, $H_1(Y_p^{\mathbb{C}})\simeq\mathbb{Z}^{(k-1)^2}$.

Denote by $\mathcal{P}_{k,n}\subset Pol^k(\mathbb{C}^n)$ the subset of all complex polynomials of degree k in n variables which satisfy the conditions of Lemma 2, i.e., for any $p\in\mathcal{P}_{k,n}$ the projectivized zero locus of p is smooth and intersects L_∞ transversally. Polynomials belonging to $\mathcal{P}_{k,n}$ will be called *nice*. Denote by $\mathcal{A}_{k,n}$ the space of pairs (p,θ) consisting of a nice polynomial $p\in\mathcal{P}_{k,n}$ and an integer homology class $\theta\in\bar{H}_n(\mathbb{C}P^n-L_\infty,Y_p^{\mathbb{C}})$. One has a fibration $\kappa:\mathcal{A}_{k,n}\to\mathcal{P}_{k,n}$ with the fiber being the abelian group $\mathcal{H}(p)\simeq\mathbb{Z}^{(k-1)^n}$. The fundamental group $\pi_1(\mathcal{P}_{k,n})$ acts by the monodromy representation of the fiber. In what follows, we might also need $\mathcal{A}_{k,n}^{\mathbb{C}}$ obtained by complexification of $\mathcal{H}_p$.

2.2. **The fundamental group of $\mathcal{P}_{k,n}$ and the monodromy group of $\mathcal{H}(p)$.** The following information is borrowed from the paper [8] of M. Lönne.

In Lemma 3.6. of [8] a presentation is given for the affine cone over the your space. The divisor D corresponds to singular projective hypersurfaces, the divisor $\hat{A}$ corresponds to projective hypersurfaces with singular intersection with the hyperplane $x_0=0$.

The generators t_i and the relations arise from the Brieskorn-Pham monodromy explained in subsection 4.3 of [8]. This part of the presentation is made explicit in Theorem 7.1.

The generators $\hat{r}_a$ and the relations R_a arise from the affine cone over the space of smooth hypersurfaces in the hyperplane $x_0=0$.

The corresponding presentation is that given in the Main Theorem, but with the dimension smaller by one.

The additional relations were not explicitly studied in the article, but I am quite convinced that the map ϕ is conjugation by the element δ_0.

Going to your space, one has still to divide out the action by $\mathbb{C}^\star$. That simply add one more relation, which is a generalization of relation (5) in the Main Theorem, but which has to involve a similar element featuring the r_a's.

For the generators t_i a fairly explicit description is possible, Namely, if you take the base point of your space to be in family on page 368 similarly for the a's and z.

Then the t_i correspond to geometric generators in the z -line associated to all degenerations, each of which is generic, i.e. the hypersurface is smooth except for a single A_1-singularity.

In case of the generators r_a, the generators can be described by paths from the assertion of Proposition 5.9, but they are more involved.

{sec:depend}

3. Moments of algebraic bodies and their dependencies

Formula (2.2) can be extended from the case of an algebraic body Ω to any homology class $\theta \in H_{n-1}(Y_p^{\mathbb{C}})$ with integer coefficients. Using Gauss-Manin connection, for any multidegree $(i_1, i_2, \ldots, i_n)$, we define

$$m_{i_1,i_2,\ldots,i_n} : \tilde{\mathcal{A}}_{k,n} \to \mathbb{C},$$

where $\tilde{\mathcal{A}}_{k,n} \simeq \mathcal{A}_{k,n}$ consists of pairs (p, θ) with p being a nice polynomial and θ being the integer homology class in $H_{n-1}(Y_p^{\mathbb{C}})$. The value of the moment $m_{i_1,i_2,\ldots,i_n}$ on θ is defined as in Part (iii) of Lemma 2. Obviously, $m_{i_1,i_2,\ldots,i_n}$ is an Abelian integral which is a linear function on the fibers $H_{n-1}(Y_p^{\mathbb{C}})$ of the projection $\kappa : \tilde{\mathcal{A}}_{k,n} \to \mathcal{P}_{k,n}$. Further, as in (2.3), we can define the moment generating function

$$\Phi_{t_1,t_2,\ldots,t_n} : \mathcal{A}_{k,n} \to Pol^k(\mathbb{C}^n).$$

lm:Nilsson {lm:Nilsson}

Lemma 3. *For any $k > 3$, $n \geq 2$ and $\theta \in H_{n-1}(Y_p^{\mathbb{C}})$, every moment $m^{\theta}_{i_1,i_2,\ldots,i_n}$ is a Nilsson-class function in $Pol^k(\mathbb{C}^n)$ unramified over $\mathcal{P}_{k,n} \subset Pol^k(\mathbb{C}^n)$. It has determinacy $(k-1)^n$, see* [13].

Remark 2. $\tilde{\mathcal{A}}_{k,n}$ is a complex analytic variety.

Denote by $S_{k,n}$ the polynomial ring $Sym[Pol^k(\mathbb{C}^n)]$ localized at the divisor $Pol^k(C^n) - \mathcal{P}_{k,n}$. By a slight abuse of notations we will also denote the pullback of $\kappa^{-1}(S_{k,n})$ by $S_{k,n}$ as well. Now denote by $\Xi_{k,n}$ the ring on $\mathcal{A}_{k,n}$ generated by all $m_{i_1,i_2,\ldots,i_n}$ with coefficients in $\Xi_{k,n}$.

conj:module {conj:module}

Conjecture 1. *For given k and n, $\Xi_{k,n}$ is a finitely generated ring over $S_{k,n}$. A system of its generators is given by $m_{i_1,i_2,\ldots,i_n}$ with $0 \leq i_1 + i_2 + \ldots i_n \leq k+1$.*

Lemma 4. *Among the moments $m_{i_1,i_2,\ldots,i_n}$ with $0 \leq i_1+i_2+\ldots i_n \leq k$ there exist following linear dependencies. If $p = \sum_{I=(i_1,i_2,\ldots,i_n),0\leq|I|\leq k} \alpha_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, then*

$$\sum_{I,0\leq|I|\leq k} (i_1+1)\alpha_{i_1,i_2,\ldots,i_n} m_{i_1,i_2,\ldots,i_n} = \sum_{I,0\leq|I|\leq k} (i_2+1)\alpha_{i_1,i_2,\ldots,i_n} m_{i_1,i_2,\ldots,i_n} =$$

$$\cdots = \sum_{I,0\leq|I|\leq k} (i_n+1)\alpha_{i_1,i_2,\ldots,i_n} m_{i_1,i_2,\ldots,i_n} = 0.$$

The above relations are linear in the moments with coefficients being linear functions of coordinates on the base space.

Proof. By Stokes theorem, the first combination is

$$\int_{\Omega} \frac{\partial (x_1 p)}{\partial x_1} dx_1 \wedge \ldots \wedge dx_n = \int_{\partial\Omega} (x_1 p)\, dx_2 \wedge dx_n = 0$$

as $p = 0$ on $\partial\Omega$ (and similarly for the other combinations). □

In general, any form $\eta = p\omega_1 + d\omega_2$ will give the zero value for $\int_{\partial\Omega} \eta$, so one should consider $\Omega^{n-1} / \left(p\Omega^{n-1} + d\Omega^{n-2}\right)$.

In the total space $Pol^k(\mathbb{R}^n) \times \mathbb{C}^n$ consider a small neighborhood $\mathcal{U} = V \times U$ of a cycle $\{p_0\} \times \partial\Omega(p_0)$ and a family of cycles $\{p\} \times \Omega_p$ lying in this neighborhood. Then $\partial\Omega$ generated homologies of $H_{n-1}(\mathcal{U})$, and assume that the period of $\omega_0 = x_1 dx_1 \wedge \cdots \wedge dx_n$ on $\partial\Omega$ does not vanish. Let ω be a holomorphic $(n-1)$-form in this neighborhoods in $\Omega^{n-1}(\mathbb{C}^n) \otimes \mathcal{O}(V)$. Then the period of $\omega - f_0(p)\omega_0$ vanishes on each $\partial\Omega_p$, so we can assume that $i^*\omega$ is exact for each p (where $i : \Omega_p \to \mathbb{C}^n$ is the embedding).

Thus we can find another form η on $\mathcal{U}$ such that $i^*\omega = i^* d\eta$. Thus $\omega = d\eta + dp \wedge \eta$ on Ω_p, so $\omega = d\eta + dp \wedge \eta_2 + p\eta_3$.

3.1. **Generalities about Abelian integrals.**

3.1.1. *Set-up.* In the product $Pol^k(\mathbb{C}^n) \times \mathbb{C}^n$ we consider the subset $\mathcal{Y} = \cup_p \{p\} \times Y_p^{\mathbb{C}}$. The projection $\pi : \mathcal{Y} \to Pol^k(\mathbb{C}^n)$ to the first factor is locally trivial over $\mathcal{P}_{k,n} \subset Pol^k(\mathbb{C}^n)$, so $\pi : \pi^{-1}(\mathcal{P}_{k,n}) \to \mathcal{P}_{k,n}$ is a fiber bundle. Let $\mathcal{B} = \cup\{p\} \times H_{n-1}(Y_p^{\mathbb{C}}, \mathbb{C}) \mapsto \mathcal{P}_{k,n}$ and $\mathcal{B}^* = \cup\{p\} \times H^{n-1}(Y_p^{\mathbb{C}}, \mathbb{C}) \mapsto \mathcal{P}_{k,n}$ be the associated Milnor homological and cohomological vector bundles.

The inclusion $\tilde{\mathcal{A}}_{k,n} \subset \mathcal{B}$ defines uniquely a locally trivial connection ∇ on $\mathcal{B}$, the Gauss-Manin connection, such that $\tilde{\mathcal{A}}_{k,n}$ is a union of its horizontal sections.

For a generic tuple of polynomial $(n-1)$-forms ω_i, $i = 1, \dots, (k-1)^n$, the restrictions of ω_i to $Y_p^{\mathbb{C}}$ generate $H^{n-1}(Y_p^{\mathbb{C}}, \mathbb{C})$ for all p in some Zarisski open subset $\mathcal{U} = \mathcal{U}(\{\omega_i\}) \subset Pol^k(\mathbb{C}^n)$, so provides a trivialization of $\mathcal{B}^*$ (and therefore of $\mathcal{B}$ as well) over $\mathcal{U}$. For a given $\theta \in H_{n-1}(Y_p^{\mathbb{C}})$ the numbers $\int_\theta \omega_i$ are coordinates of θ in this trivialization.

3.1.2. *Picard-Lefschetz formulas.* The Picard-Lefschetz formulas describe the monodromy M_γ of ∇ corresponding to loops $\gamma \in \pi_1(\mathcal{P}_{k,n})$ going once around smooth points of the set $\Sigma = Pol^k(\mathbb{C}^n) \setminus \mathcal{P}_{k,n}$ of not nice polynomials. Namely, assume that $p_0 \in \Sigma$ is a smooth point such that $Y_{p_0}^{\mathbb{C}}$ has a unique singular x point of Morse type. Then

- for any small neighborhood V of x and for all $p \in \mathcal{P}_{k,n}$ close enough to p_0 the intersection $Y_p^{\mathbb{C}} \cap V$ is diffeomorphic to $T\mathbb{S}^{n-1}$,
- the map $H_{n-1}\left(Y_p^{\mathbb{C}} \cap V\right) \to H_{n-1}\left(Y_p^{\mathbb{C}}\right)$ is injective.

The sphere $\mathbb{S}^{n-1}$, i.e. the generator of $H_{n-1}\left(Y_p^{\mathbb{C}} \cap V\right)$, as well as its image $\Delta_x \in H_{n-1}\left(Y_p^{\mathbb{C}}\right)$ are called the "cycle vanishing at p". Let γ be a boundary of a small disc $D \subset Pol^k(\mathbb{C}^n)$ centered at p_0 and transversal to Σ. The Picard-Lefschetz formula claims that

$$M_\gamma(\theta) = \theta + (-1)^{n(n+1)/2} \left(\theta \circ \Delta_x\right) \Delta_x, \tag{3.1}$$

(in particular $M_\gamma(\Delta_x) = (-1)^n \Delta_x$). As the small loops generate $\pi_1(\mathcal{P}_{k,n})$, we get the representation $M : \pi_1(\mathcal{P}_{k,n}) \to GL(H_{n-1}(Y_p^{\mathbb{C}}, \mathbb{Z})) \sim GL(n, \mathbb{Z})$.

{sssec:change of forms}

3.1.3. *Change of forms.* The change of the generic tuple of forms $\{\omega_i\}$ to another tuple $\{\eta_i\}$ leads to a gauge transform $\mathcal{B}^* \to \mathcal{B}^*$, $(p, \omega) \to (p, T(p)\omega)$, with $T(p)$ being rational $GL(n, \mathbb{C})$-valued function in $\mathcal{P}_{k,n}$. Indeed, consider the pairing matrix $P_{\{\omega_i\},\{\theta_i\}} = \{\int_{\theta_j} \omega_i\}$, where $\{\theta_i\}$ is some basis of $H_{n-1}(Y_p^{\mathbb{C}}, \mathbb{C})$. The matrix P is a holomorphic multivalued function in $\mathcal{U}(\{\omega_i\})$, and the monodromy of P is described by Picard-Lefschetz formulas, $M_\gamma(P) = PM(\gamma)$.

So, if $\tilde{P} = \{\int_{\theta_j} \eta_i\}$ is another pairing matrix then $T(p) = \tilde{P} \cdot P^{-1}$ is univalued, and being of moderate growth is therefore rational. Thus $\tilde{P} = T(p)P$, so

{eq:various forms}

$$\eta_i = \sum_j T_{ij}(p)\omega_j \tag{3.2}$$

eq:various forms

as elements of $H^{n-1}(Y_p^{\mathbb{C}}, \mathbb{C})$.

3.1.4. *Gelfand–Leray derivative.* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $df \neq 0$ on the hypersurface $f = 0$. Let

$$\omega = g(x)\, dx_1 \wedge \cdots \wedge dx_n.$$

The Gelfand–Leray form $\frac{\omega}{df}$ is defined by

$$df \wedge \left(\frac{\omega}{df}\right) = \omega.$$

In coordinates, if $\partial_{x_j} f \neq 0$, one has

$$\frac{\omega}{df} = (-1)^{j-1}\frac{g(x)}{\partial_{x_j} f}\, dx_1 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_n.$$

This form is independent of the choice of index j.

In particular, integrals over level sets can be written as

$$\int_{f=0} \frac{\omega}{df} = \int_{f=0} (-1)^{j-1}\frac{g(x)}{\partial_{x_j} f}\, dx_1 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_n.$$

3.1.5. *Gauss-Manin connection is rational.* Choose a generic tuple of forms $\{\omega_i\}$ generating $H^{n-1}(Y_p^{\mathbb{C}})$ for a generic p. Combination of (**??**) and (3.2) implies that the Gauss-Manin connection on $\mathcal{B}$ in the chart given by these forms has rational (in α_I's) coefficients.

Indeed, we have to find the coefficients Γ^i_{Ij} in

$$\frac{\partial}{\partial \alpha_I}\int_\theta \omega_j = \sum_i \Gamma^i_{Ij}\int_\theta \omega_i, \tag{3.3}$$

or, equivalently (by (**??**)), to find the coefficients Γ^i_{Ij} in the representation

$$x^I\frac{d\omega_j}{dp} = \sum_i \Gamma^i_{Ij}\omega_i \tag{3.4}$$

as elements of $H^{n-1}(Y_p^{\mathbb{C}})$. But (a slight generalization of) (3.2) implies that Γ^i_{Ij} are rational functions of α_I.

4. Ellipsoids

sec:quadr {sec:quadr}

Here we investigate the simplest nontrivial case when p has degree 2. Note that the compactness of the ovaloid Ω it defines requires the homogeneous degree 2 part of it to be positive definite, i.e. $p(x) = \langle Ax, Ax\rangle + 2\langle b, x\rangle + c$, where $\det A \neq 0$. Here we write $\langle,\rangle$ for the standard scalar product on $\mathbb{R}^n$.

On the other hand, Ω is defined by the inequality $p(x) \leq d$, thus without loss in generality, choosing c and d appropriately, we may assume that the inequality is $p(x) = \langle Ax + b, Ax + b\rangle \leq d$, that is to say that $p(x) \geq 0$ for any x, and its only zero $A^{-1}b \in \Omega$, which is nonempty iff $d \geq 0$,and consists of a single point if $d = 0$. Excluding the latter case, we can divide both parts by d, and set $d = 1$. As well, A is not unique, as the matrix of the associated positive definite form is A^TA. A positive definite matrix admits unique Cholesky decomposition L^TL, with L lower-triangular. Thus we may assume that A is lower-triangular. Thus Ω is specified, uniquely, by $\binom{n}{2} + 2n = \binom{n+2}{2} - 1$ parameters. To summarise, we state

lem:quaddef {lem:quaddef}

Lemma 5. *Let Ω be a compact ovaloid with non-empty interior defined by a quadratic p. Then it is defined by the inequality $p(x) = \langle Ax+b, Ax+b\rangle \leq 1$, where A is lower-triangular nn matrix and b an n-vector. In particular Ω is uniquely specified by $\binom{n+2}{2} - 1$ parameters.* □

The moments $m_I(\Omega)$ as in (2.1) and the moments of Ω shifted by b are naturally related.

$$m_I(\Omega) = \int\limits_{<Ax+b,Ax+b>\leq 1} x^I dx = \int\limits_{<Ay,Ay>\leq 1} (y - A^{-1}b)^I dy.$$

It is natural to keep track of m_I by using an m-th order Fantappie transformation, i.e. the generating function

$$F_p(u) := \int\limits_{<Ax+b,Ax+b>\leq 1} \frac{dx}{(1-<u,x>)^m} = \int\limits_{<Ax,Ax>\leq 1} \frac{dx}{(1-<u,x-A^{-1}b>)^m}, \tag{4.1}$$

{eq:fangen}

as $(\frac{\partial^{|I|}}{\partial u^I}F(u))(0) = Cm_I$, with C only dependent on I and m.

In general, an affine change of variables of p does not affect F_p much, and the action of such a transformation may be computed as follows.

{lem:affFanta} lem:affFanta

Lemma 6. *Let $x \mapsto Ax + b$ be an invertible affine transformation. Denoting $A^* = (A^T)^{-1}$ and $c := 1 + <b, A^*u>$, one has*

$$F_{g(Ax+b)\leq 1}(u) = \frac{c^m}{|A|}F_{g(x)\leq 1}(c^{-m}A^*u).$$

Proof. Direct computation

$$\int\limits_{g(Ax+b)\leq 1} \frac{dx}{(1-<u,x>)^m} = |A| \int\limits_{g(y)\leq 1} \frac{dy}{(c-<A^*u,x>)^m} = \frac{c^m}{|A|}F_{g(x)\leq 1}(c^{-m}A^*u).$$

using change of variables $y = Ax + b$. □

In view of Lemma 6 it suffices to compute F_p for $p(x) = <x,x>$. In some cases it is easier to work with Laplace transform.

{eq:expgen}

$$L_p(u) := \int\limits_{<Ax+b,Ax+b>\leq 1} e^{<u,x>}dx. \tag{4.2}$$

eq:expgen

{eq:expgen}

$$m_I(\Omega) = \frac{\partial^{|I|}}{\partial u^I}L_p(u)(0). \tag{4.3}$$

Analogously to Lemma 6 one has

{lem:affexp} lem:affexp

Lemma 7. *Let $x \mapsto Ax + b$ be an invertible affine transformation. Denoting $A^* = (A^T)^{-1}$, one has*

$$L_{g(Ax+b)\leq 1}(u) = \frac{e^{-<A^*u,b>}}{|A|}L_{g(x)\leq 1}(A^*u).$$

Proof. Direct computation

$$\int\limits_{g(Ax+b)\leq 1} e^{<u,x>}dx = \frac{1}{|A|}\int\limits_{g(y)\leq 1} e^{<A^*u,y-b>}dy = \frac{e^{-<A^*u,b>}}{|A|}\int\limits_{g(y)\leq 1} e^{<A^*u,y>}dy,$$

using change of variables $y = Ax + b$. □

An explicit formula, attributed to N. Ja. Sonin, for $L_{<x,x>\leq 1}(u)$ may be found in §5.10, Chap.18 of [17].

{eq:sonin}

$$L_{<x,x>\leq 1}(u) = \pi^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{<u,u>^k}{4^k k!\Gamma(\frac{n}{2}+k+1)}. \tag{4.4}$$

eq:sonin

Lemma 7 and (4.4) imply

{eq:soninaff}

$$L_{<Ax+b,Ax+b>\leq 1}(u) = \frac{\pi^{\frac{n}{2}}e^{-<A^*u,b>}}{|A|} \sum_{k=0}^{\infty} \frac{<A^*u,A^*u>^k}{4^k k!\Gamma(\frac{n}{2}+k+1)}. \tag{4.5}$$

eq:soninaff

Going back to a general case now, observe that Lemma 7 for $u = 0$ implies that the moment m_1 of the constant function 1, i.e. the volume of the set defined by

$g(Ax+b) \le 1$, equals $V_1 := m_1(g(x) \le 1)$, the volume of the set defined by $g(x) \le 1$ divided by $|A|$.

{eq:volumeAb}

$$m_1 := m_1(g(Ax+b) \le 1) = \frac{V_1}{|A|}. \tag{4.6}$$

eq:volumeAb

As well, one can reconstruct b and A from moments of $g(Ax+b) \le 1$ using $L_{g(x)\le 1}$ and vanishing of the odd (in particular, multilinear) moments of $g(x) \le 1$. By Lemma 7 one gets

$$\begin{aligned} D_j(u) := \frac{\partial}{\partial u_j} L_{g(Ax+b)\le 1}(u) &= L_{g(x)\le 1}(A^*u)\frac{\partial}{\partial u_j}\frac{e^{-<A^*u,b>}}{|A|} + \frac{e^{-<A^*u,b>}}{|A|}\frac{\partial}{\partial u_j} L_{g(x)\le 1}(A^*u) \\ &= \frac{e^{-<A^*u,b>}}{|A|}\left(\frac{\partial}{\partial u_j} L_{g(x)\le 1}(A^*u) - (A^{-1}b)_j L_{g(x)\le 1}(A^*u)\right) \end{aligned}$$

Thus $m_{x_j} = D_j(0) = -|A|^{-1}V_1(A^{-1}b)_j$ - and so the vector m_x of 1st moments is the image of b under a linear transformation proportional to A^{-1}.

$$m_x := m_x(g(Ax+b) \le 1) = -\frac{V_1}{|A|}A^{-1}b = -m_1 A^{-1}b. \tag{4.7}$$

{eq:linearmomentsAb}

Further,

$$D_{ij}(u) := \frac{\partial}{\partial u_i} D_j(u) = \frac{e^{-<A^*u,b>}}{|A|}$$
$$\left(\frac{\partial^2 L_{g(x)}(A^*u)}{\partial u_i \partial u_j} - (A^{-1}b)_j \frac{\partial L_{g(x)}(A^*u)}{\partial u_i} - (A^{-1}b)_i \frac{\partial L_{g(x)}(A^*u)}{\partial u_j} + (A^{-1}b)_i (A^{-1}b)_j L_{g(x)}(A^*u)\right).$$

Note that, denoting $\beta_{ij} = (A^{-1})_{in}$,

$$\begin{aligned} I(u) := \frac{\partial^2}{\partial u_i \partial u_j} L_{g(x)}(A^*u) &= \frac{\partial}{\partial u_j}\int_{g(x)\le 1} (A^{-1}x)_i e^{<A^*u,x>}dx \\ &= \int_{g(x)\le 1} (A^{-1}x)_i (A^{-1}x)_j e^{<A^*u,x>}dx \\ &= \sum_k \sum_\ell \beta_{ik}\beta_{j\ell}\int_{g(x)\le 1} x_k x_\ell e^{<A^*u,x>}dx. \end{aligned}$$

As for $k \ne \ell$ the moment of $x_k x_\ell$ vanishes,

$$I(0) = \sum_k \beta_{ik}\beta_{jk}\mu_k, \quad \text{where } \mu_k := m_{x_k^2}(g(x) \le 1).$$

Thus, by (4.7)

$$m_{x_i x_j} = D_{ij}(0) = |A|^{-1}\left(I(0) + V_1 (A^{-1}b)_i (A^{-1}b)_j\right) = |A|^{-1}I(0) + \frac{m_{x_i} m_{x_j}}{m_1}.$$

One can put the entries of $I(0) = I_{ij}(0)$ into an nn matrix $A^{-1}\mathrm{diag}(\mu_1, \dots, \mu_n)A^*$. This gives

$$M - \frac{1}{m_1} m_x m_x^T = |A|^{-1}A^{-1}\mathrm{diag}(\mu_1, \dots, \mu_n)A^*, \qquad \text{where} \quad M := (m_{x_i x_j})_{ij}.$$

Expressing $|A|^{-1} = m_1/V_1$, taking determinants on both sides, and using Schur complement formula for the determinant, we obtain an explicit algebraic relation between m_1, m_x, and M.

$$\left| M - \frac{1}{m_1} m_x m_x^T \right| = \frac{1}{m_1}\begin{vmatrix} m_1 & m_x^T \\ m_x & M \end{vmatrix} = \mu_1 \dots \mu_n m_1^3 V_1^{-3} \tag{4.8}$$

{eq:momquadrel}

I.e. one can solve the resulting quartic equation

$$m_1|M| + \sum_{k=1}^{n}(-1)^k m_{x_k}|m_x M|_{-k-1} = \mu_1 \dots \mu_n V_1^{-3} m_1^4.$$

in m_1 to express it in terms of the 1st and 2nd moments.

5. Sublevel moment variety and moment map

In this section we consider a special case studied in [6, 7, 14]. Denote by $Pol^k(\mathbb{R}^n)$ the linear space of all real homogeneous polynomials of degree k in n (real) variables and by $\mathcal{P}^k(\mathbb{R}^n) \subset Pol^k(\mathbb{R}^n)$ an open cone in the former space consisting of all polynomials positive outside the origin. (We will always assume that $\mathbb{R}^n$ is endowed with a fixed coordinate system $(x_1, x_2, \dots, x_n)$).

To each polynomial $g(x_1, \dots, x_n) \in \mathcal{P}^k(\mathbb{R}^n)$ we associate a collection of the moments

{eq:Mom_g}

$$m^{(g)}_{i_1,i_2,\dots,i_n} = \int_G x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} dx_1 dx_2 \dots dx_n,\ i_1 + i_2 + \dots + i_n = k \qquad (5.1)$$

eq:Mom_g

of order k, cf. (2.2). Here $G \subset \mathbb{R}^n$ is the sublevel set of g given by $\{g(x_1, \dots, x_n) \leq 1\}$. Let us arrange the latter moments in a real homogeneous polynomial of degree k in n variables by taking its (normalized) *moment generating function*

$$\Phi^{(g)}(t_1, t_2, \dots, t_n) = \sum_{i_1+i_2+i_n=k} \binom{k}{i_1, i_2, \dots, i_n} m^{(g)}_{i_1,i_2,\dots,i_n} t_1^{i_1} t_2^{i_2} \dots t_n^{i_n} \in Pol^k(\mathbb{R}^n).$$

Denote by $\mathfrak{M}_k : \mathcal{P}^k(\mathbb{R}^n) \to Pol^k(\mathbb{R}^n)$ the *sublevel moment map*[3] sending $g(x_1, \dots, x_n)$ to $\Phi^{(g)}(t_1, t_2, \dots, t_n)$. We call the image

$$\mathcal{S}_{k,n} := \mathfrak{M}_k(\mathcal{P}^k(\mathbb{R}^n)) \subset Pol^k(\mathbb{R}^n)$$

the *sublevel moment variety*. Our nearest goal is to describe some properties of $\mathcal{S}_{k,n}$. Moment varieties of a similar flavor have been previously considered in [1, 5]. The next claim is straight-forward.

{lm:integral}
lm:integral

Lemma 8. *The subselevel moment map $\mathfrak{M}_k : \mathcal{P}^k(\mathbb{R}^n) \to Pol^k(\mathbb{R}^n)$ is given by the integral transform*

{eq:MM}

$$\Phi^{(g)}(t_1, t_2, \dots, t_n) = \int_G (t_1 x_1 + t_2 x_2 + \dots + t_n x_n)^k dx_1 dx_2 \dots dx_n. \qquad (5.2)$$

eq:MM

By a formula due to A. Morozov and Sh. Shakirov [11] we get the following.

{lm:NewInt}
lm:NewInt

Corollary 1. *The above integral transform can be rewritten as*

$$\Phi^{(g)}(t_1, t_2, \dots, t_n) = \int_{\mathbb{R}^n} (t_1 x_1 + t_2 x_2 + \dots + t_n x_n)^k e^{-g(x_1,\dots,x_n)} dx_1 dx_2 \dots dx_n.$$

The next statement follows from the main result of [14].

{pror:bijection}
pror:bijection

Proposition 1. *The moment map $\mathfrak{M}_k : \mathcal{P}^k(\mathbb{R}^n) \to Pol^k(\mathbb{R}^n)$ defined by (5.2) is injective, i.e., defines a diffeomorphism of $\mathcal{P}^k(\mathbb{R}^n)$ on its image $\mathfrak{M}_k(\mathcal{P}^k)$.*

Remark 3. Notice that if $g(x_1, \dots, x_n)$ attains values of both signs then the integrals (5.1) and (5.2) are divergent. However since (5.2) is well-defined on an open subset $\mathcal{P}^k(\mathbb{R}^n)) \subset Pol^k(\mathbb{R}^n)$ it has an interesting analytic continuation related to ramified integrals, see below.

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[3]The term *moment map* has been earlier erroneously used in symplectic geometry where the corresponding notion should be called a *momentum map*, see discussions in [10].

6. Examples

Example 2 (Moments of an ellipse, revisited). Consider the ellipse

$$E = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} \le 1 \right\}, \qquad a,b>0.$$

Its moments are

$$m_{ij}(E) = \iint_E x^i y^j \, dx\, dy.$$

After the affine change of variables

$$x = x_0 + au, \qquad y = y_0 + bv,$$

the ellipse is transformed into the unit disk

$$u^2 + v^2 \le 1,$$

and the Jacobian is equal to ab. Hence

$$m_{ij}(E) = ab \iint_{u^2+v^2\le 1} (x_0+au)^i (y_0+bv)^j \, du\, dv.$$

Expanding by the binomial formula, one gets

$$m_{ij}(E) = ab \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i}{p}\binom{j}{q} x_0^{i-p} y_0^{j-q} a^p b^q \iint_{u^2+v^2\le 1} u^p v^q \, du\, dv.$$

By symmetry,

$$\iint_{u^2+v^2\le 1} u^p v^q \, du\, dv = 0 \qquad \text{if } p \text{ or } q \text{ is odd.}$$

Therefore only even indices contribute. Writing $p = 2r$, $q = 2s$, we obtain

$$m_{ij}(E) = ab \sum_{r=0}^{\lfloor i/2 \rfloor} \sum_{s=0}^{\lfloor j/2 \rfloor} \binom{i}{2r}\binom{j}{2s} x_0^{i-2r} y_0^{j-2s} a^{2r} b^{2s} \, I_{2r,2s},$$

where

$$I_{2r,2s} := \iint_{u^2+v^2\le 1} u^{2r} v^{2s} \, du\, dv.$$

Using polar coordinates,

$$I_{2r,2s} = \frac{1}{r+s+1} \, B\!\left(r+\frac{1}{2}, s+\frac{1}{2}\right) = \frac{\Gamma(r+\frac{1}{2})\Gamma(s+\frac{1}{2})}{(r+s+1)\Gamma(r+s+1)}.$$

Equivalently,

$$I_{2r,2s} = \pi \, \frac{(2r)!(2s)!}{4^{\,r+s}\, r!\, s!\, (r+s+1)!}.$$

Thus the general moment formula for the ellipse is

$$m_{ij}(E) = \pi ab \sum_{r=0}^{\lfloor i/2 \rfloor} \sum_{s=0}^{\lfloor j/2 \rfloor} \binom{i}{2r}\binom{j}{2s} x_0^{i-2r} y_0^{j-2s} a^{2r} b^{2s} \frac{(2r)!(2s)!}{4^{\,r+s}\, r!\, s!\, (r+s+1)!}.$$

In particular,

$$m_{00}(E) = \operatorname{Area}(E) = \pi ab,$$

$$m_{10}(E) = \pi ab\, x_0, \qquad m_{01}(E) = \pi ab\, y_0,$$

$$m_{20}(E) = \pi ab \left(x_0^2 + \frac{a^2}{4}\right), \qquad m_{02}(E) = \pi ab \left(y_0^2 + \frac{b^2}{4}\right),$$

$$m_{11}(E) = \pi ab\, x_0 y_0,$$

$$m_{30}(E) = \pi ab \left(x_0^3 + \frac{3a^2 x_0}{4}\right), \qquad m_{03}(E) = \pi ab \left(y_0^3 + \frac{3b^2 y_0}{4}\right),$$

$$m_{21}(E) = \pi ab\, y_0 \left(x_0^2 + \frac{a^2}{4} \right), \qquad m_{12}(E) = \pi ab\, x_0 \left(y_0^2 + \frac{b^2}{4} \right),$$

and

$$m_{40}(E) = \pi ab \left(x_0^4 + \frac{3}{2} a^2 x_0^2 + \frac{a^4}{8} \right),$$

$$m_{04}(E) = \pi ab \left(y_0^4 + \frac{3}{2} b^2 y_0^2 + \frac{b^4}{8} \right),$$

$$m_{22}(E) = \pi ab \left(x_0^2 + \frac{a^2}{4} \right) \left(y_0^2 + \frac{b^2}{4} \right).$$

In the centered case $x_0 = y_0 = 0$, all odd moments vanish and

$$m_{2r,2s}(E) = \pi ab\, a^{2r} b^{2s} \frac{(2r)!(2s)!}{4^{r+s}\, r!\, s!\, (r+s+1)!}.$$

Reduction to the general case. The above formulas are obtained from the centered axis-parallel ellipse by affine changes of variables. Indeed, any ellipse in the plane can be written in the form

$$E = T(D),$$

where $D = \{u^2 + v^2 \leq 1\}$ is the unit disk and

$$T(u, v) = (x_0, y_0) + A \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A \in \mathrm{GL}_2(\mathbb{R}).$$

In particular, for an axis-parallel ellipse one has $A = \mathrm{diag}(a, b)$, while a general ellipse is obtained by composing this scaling with a rotation.

By the change of variables formula,

$$m_{ij}(E) = |\det A| \iint_D \big(x_0 + (A_{11}u + A_{12}v)\big)^i \big(y_0 + (A_{21}u + A_{22}v)\big)^j \, du\, dv.$$

Expanding the integrand, all moments reduce to linear combinations of the basic integrals

$$\iint_D u^p v^q \, du\, dv,$$

which vanish unless p, q are even and are given explicitly above. Therefore the moments of an arbitrary ellipse are obtained from those of the unit disk by affine invariance and polynomial expansion.

Equivalently, the moment generating function transforms according to

$$\Phi^{T(D)}(t) = |\det A|\, \Phi^D(A^T t)\, e^{\langle t, (x_0, y_0) \rangle},$$

which provides a compact way to recover all moments of a general ellipse from the centered case.

Example 3 (Moments of the compact oval of a Weierstrass cubic). Consider the real nonsingular cubic in Weierstrass form

$$C: \quad y^2 = x(x - \mu)(x - \lambda), \qquad \lambda > \mu > 0,$$

and let $O \subset \mathbb{R}^2$ be its compact oval.

We define the moments

$$M_{\ell,m} := \iint_O x^\ell y^m \, dx\, dy.$$

By symmetry with respect to $y \mapsto -y$, one has

$$M_{\ell,m} = 0 \quad \text{for odd } m.$$

For $m = 2r$ even, using Fubini and the equation of the curve, we obtain

$$M_{\ell,2r} = \frac{2}{2r+1} \int_0^{\mu} x^{\ell} \big(x(\mu - x)(\lambda - x)\big)^{r+\frac{1}{2}} \, dx.$$

With the substitution $x = \mu t$, this becomes

$$M_{\ell,2r} = \frac{2\,\lambda^{r+\frac{1}{2}} \mu^{\ell+2r+2}}{2r+1} \int_0^1 t^{\ell+r+\frac{1}{2}} (1-t)^{r+\frac{1}{2}} \Big(1 - \frac{\mu}{\lambda} t\Big)^{r+\frac{1}{2}} \, dt.$$

This integral is a classical Euler–Gauss hypergeometric integral, hence

$$M_{\ell,2r} = \frac{2\,\lambda^{r+\frac{1}{2}} \mu^{\ell+2r+2}}{2r+1} \, B\Big(\ell + r + \frac{3}{2},\ r + \frac{3}{2}\Big) \, {}_2F_1\Big(-r - \frac{1}{2},\ \ell + r + \frac{3}{2};\ \ell + 2r + 3;\ \frac{\mu}{\lambda}\Big).$$

In particular, all moments of the oval are periods of the elliptic curve C. Since the parameters of the hypergeometric function are half-integers, these expressions reduce to linear combinations of the complete elliptic integrals K and E. Therefore, for all ℓ, r,

$$M_{\ell,2r} = A_{\ell,r}(\lambda,\mu)\, K\Big(\frac{\mu}{\lambda}\Big) + B_{\ell,r}(\lambda,\mu)\, E\Big(\frac{\mu}{\lambda}\Big),$$

where $A_{\ell,r}$ and $B_{\ell,r}$ are algebraic (in fact polynomial) in λ and μ.

As a basic example, the area of the oval is

$$M_{0,0} = 2 \int_0^{\mu} \sqrt{x(\mu - x)(\lambda - x)} \, dx,$$

which is a classical complete elliptic integral.

The relevant monodromy is best understood in Picard–Lefschetz terms rather than as a finite linear monodromy group. Namely, the projection

$$\pi \colon C \to \mathbb{C}P^1, \qquad (x,y) \mapsto x,$$

is a double cover branched at the four points

$$0,\ \mu,\ \lambda,\ \infty.$$

The compact oval corresponds to the lift of the real segment $[0,\mu]$ joining two neighbouring branch points. If one considers the four elementary arcs between neighbouring branch points in cyclic order on $\mathbb{R}P^1$, then their lifts give the basic vanishing cycles for this family. When two adjacent branch points collide, the local monodromy is the corresponding Picard–Lefschetz transformation. These local transformations generate the homological monodromy of the family. On the set of the four elementary vanishing cycles they act by adjacent transpositions, and therefore the induced permutation group is the dihedral group D_4. Thus D_4 should be understood here as the finite combinatorial monodromy of the four basic branches/cycles, while the full monodromy acting on homology is the Picard–Lefschetz group generated by the corresponding transvections.

Example 4 (Moments of the Fermat quartic oval). Consider the algebraic body

$$\Omega_4 = \{(x,y) \in \mathbb{R}^2 \mid x^4 + y^4 \le 1\},$$

bounded by the smooth plane quartic $x^4 + y^4 = 1$. By symmetry, all moments with at least one odd exponent vanish:

$$m_{ij}(\Omega_4) = \int_{\Omega_4} x^i y^j \, dx\, dy = 0 \qquad \text{if } i \text{ or } j \text{ is odd.}$$

For even moments, writing $i = 2a$, $j = 2b$ with $a, b \geq 0$, we get

$$m_{2a,2b}(\Omega_4) = 4\int_0^1 x^{2a}\left(\int_0^{(1-x^4)^{1/4}} y^{2b}\,dy\right)dx$$
$$= \frac{4}{2b+1}\int_0^1 x^{2a}(1-x^4)^{\frac{2b+1}{4}}\,dx.$$

Substituting $u = x^4$, one obtains

$$m_{2a,2b}(\Omega_4) = \frac{1}{2b+1}\int_0^1 u^{\frac{2a-3}{4}}(1-u)^{\frac{2b+1}{4}}\,du$$
$$= \frac{1}{2(a+b+1)}B\left(\frac{2a+1}{4}, \frac{2b+1}{4}\right),$$

i.e.

$$m_{2a,2b}(\Omega_4) = \frac{\Gamma\left(\frac{2a+1}{4}\right)\Gamma\left(\frac{2b+1}{4}\right)}{4\,\Gamma\left(\frac{a+b+3}{2}\right)}.$$

In particular,

$$m_{00}(\Omega_4) = \mathrm{Area}(\Omega_4) = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{\pi}},$$
$$m_{20}(\Omega_4) = m_{02}(\Omega_4) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{4} = \frac{\pi\sqrt{2}}{4},$$
$$m_{40}(\Omega_4) = m_{04}(\Omega_4) = \frac{\Gamma(\frac{1}{4})^2}{12\sqrt{\pi}}, \qquad m_{22}(\Omega_4) = \frac{\Gamma(\frac{3}{4})^2}{3\sqrt{\pi}}.$$

Therefore the moment generating function (2.3) up to degree 4 is

$$\Phi^{\Omega_4}(t_1,t_2) = m_{00} + m_{20}(t_1^2+t_2^2) + m_{40}(t_1^4+t_2^4) + 6m_{22}t_1^2t_2^2$$
$$= \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{\pi}} + \frac{\pi\sqrt{2}}{4}(t_1^2+t_2^2) + \frac{\Gamma(\frac{1}{4})^2}{12\sqrt{\pi}}(t_1^4+t_2^4) + \frac{2\,\Gamma(\frac{3}{4})^2}{\sqrt{\pi}}\,t_1^2t_2^2.$$

Example 5 (Ovaloid of a cubic surface). A convenient model of a smooth cubic ovaloid is given by

$$S_a = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2+az^3 = 1\}, \qquad |a| < \frac{2}{3\sqrt{3}}.$$

For this range of a, the real locus $S_a(\mathbb{R})$ has a unique compact connected component, which bounds the domain

$$\Omega_a = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2+az^3 \leq 1\}.$$

Put

$$R_a(z) = 1 - z^2 - az^3,$$

and denote by $z_-(a) < z_+(a)$ the endpoints of the interval on which $R_a(z) \geq 0$. The section of Ω_a by the plane $z = \mathrm{const}$ is the disk

$$x^2 + y^2 \leq R_a(z).$$

Therefore the moments

$$m_{ijk}(a) = \iiint_{\Omega_a} x^i y^j z^k\,dx\,dy\,dz$$

satisfy

$$m_{ijk}(a) = 0 \qquad \text{if } i \text{ or } j \text{ is odd}.$$

For $i = 2r$, $j = 2s$, one obtains

$$m_{2r,2s,k}(a) = \frac{\Gamma(r+\frac{1}{2})\Gamma(s+\frac{1}{2})}{\Gamma(r+s+2)}\int_{z_-(a)}^{z_+(a)} z^k(1-z^2-az^3)^{r+s+1}\,dz.$$

Thus all moments of Ω_a are expressed as Abelian integrals on the elliptic curve

$$w^2 = 1 - z^2 - az^3.$$

In particular,

$$m_{000}(a) = \pi \int_{z_-(a)}^{z_+(a)} (1 - z^2 - az^3)\, dz, \qquad m_{001}(a) = \pi \int_{z_-(a)}^{z_+(a)} z(1 - z^2 - az^3)\, dz,$$

$$m_{200}(a) = m_{020}(a) = \frac{\pi}{4} \int_{z_-(a)}^{z_+(a)} (1-z^2-az^3)^2\, dz, \qquad m_{002}(a) = \pi \int_{z_-(a)}^{z_+(a)} z^2(1-z^2-az^3)\, dz.$$

For $|a| \ll 1$, one has the expansions

$$m_{000}(a) = \frac{4\pi}{3} + O(a^2), \qquad m_{001}(a) = -\frac{2\pi}{5} a + O(a^3),$$

$$m_{200}(a) = m_{020}(a) = \frac{4\pi}{15} + O(a^2), \qquad m_{002}(a) = \frac{4\pi}{15} + O(a^2).$$

The integrals above depend on the parameter a through the roots of the cubic polynomial $R_a(z) = 1 - z^2 - az^3$. As a varies in the complex domain, these roots undergo analytic continuation, and the corresponding Abelian integrals acquire nontrivial monodromy.

From the Picard–Lefschetz viewpoint, this monodromy is generated by vanishing cycles associated with the degeneration of the affine cubic surface

$$x^2 + y^2 + z^2 + az^3 = 1.$$

The middle homology of this affine surface has rank

$$\mu = (3-1)^3 = 8,$$

coinciding with the Milnor number of the cubic singularity $x^3+y^3+z^3$. Accordingly, one obtains eight vanishing cycles and the monodromy group is generated by the corresponding Picard–Lefschetz reflections.

Passing to the projective cubic surface amounts to factoring out the two isotropic directions coming from the boundary at infinity. The resulting intersection lattice is the root lattice of type E_6, and the induced monodromy group is the Weyl group $W(E_6)$.

Thus the affine situation considered here is naturally an extension of the E_6-geometry by two null directions; in particular, the monodromy group of the Abelian integrals above is the affine extension of $W(E_6)$.

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7. Appendix. Main theorem of [14].

{lm:PZ}

lm:PZ *Lemma* A. If $G = \{x \in \mathbb{R}^n : g(x) \le 1\}$, then for all $q \in \mathcal{P}^k(\mathbb{R}^n)$,

$$\int_G q(x)dx = \frac{1}{n+k} \int_S q(z) g^{-\frac{n+k}{k}}(z) d\sigma(z),$$

where S is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ is the standard area form on it.

Proof. Since g is homogeneous and positive, for each $z \in S$, there exists a unique positive number $\lambda(z)$ such that

$$1 = g(\lambda(z)z) = \lambda^k(z)g(z),$$

implying that $\lambda(z) = \frac{1}{\sqrt[k]{g(z)}}$. Setting $x = tz$ with $0 \le t \le \frac{1}{\sqrt[k]{g(z)}}$, we obtain

$$\int_G q(x)dx = \int_S \int_0^{\frac{1}{\sqrt[k]{g(z)}}} q(tz)t^{n-1}dtd\sigma(z) = \int_S \int_0^{\frac{1}{\sqrt[k]{g(z)}}} q(z)t^{n-1+k}dtd\sigma(z) = \frac{1}{n+k}\int_S q(z)g(z)^{-\frac{n+k}{k}}d\sigma(z).$$

□

{th:PZ}
th:PZ

Theorem A. Given positive homogeneous polynomials g_0 and g_1 of degree k in n variables, set

$$G_0 := \{x \in \mathbb{R}^n : g_0(x) \le 1\} \quad \text{and} \quad G_1 := \{x \in \mathbb{R}^n : g_1(x) \le 1\}.$$

Then if all moments of order k of G_0 coincide with the respective moments of order k of G_1, then $g_0 = g_1$ and $G_0 = G_1$.

Proof. For every $t \in [0, 1]$, define

$$g_t = tg_1 + (1-t)g_0 \quad \text{and} \quad G_t = \{x \in \mathbb{R}^N : G_t(x) \le 1.\}.$$

Now for each $q \in \mathcal{P}^k(\mathbb{R}^n)$, define

$$F(t, q) = \int_{G_t} q(x)dx.$$

The $F(0, q) = F(1, q)$. Indeed, we have

$$F(0,q) = \int_{G_0} \sum_{|\alpha|=k} q_\alpha x^\alpha dx = \sum_{|\alpha|=k} q_\alpha \int_{G_0} x^\alpha dx = \sum_{|\alpha|=k} q_\alpha \int_{G_1} x^\alpha dx = \int_{G_1} \sum_{|\alpha|=k} q_\alpha x^\alpha dx = F(1,q).$$

By Lemma A, we have that for each t and q,

$$F(t,q) = \frac{1}{n+k}\int_S q(z)g_t^{-\frac{n+k}{k}}d\sigma(z) = \frac{1}{n+k}\int_S q(z)(tg_1(z)+(1-t)g_0(z))^{-\frac{n+k}{k}}d\sigma(z),$$

thus

$$\frac{d}{dt}F(t,q) = -\frac{1}{k}\int_S q(z)(tg_1(z) + (1-t)g_0(z))^{-\frac{n+2k}{k}}(g_1(z) - g_0(z))d\sigma(z).$$

To obtain a contradiction, suppose that $g_0 \ne g_1$, and take $q = g_1 - g_0$. Then for all t, we get

$$\frac{d}{dt}F(t,q) = -\frac{1}{k}\int_S (g_1(z) - g_0(z))^2(tg_1(z) + (1-t)g_0(z))^{-\frac{n+2k}{k}}d\sigma(z) < 0.$$

But since $F(0, q) = F(1, q)$, by Rolle's Theorem there exists a $c \in (0, 1)$ such that $\frac{d}{dt}F(c, q) = 0$. Thus $g_0 = g_1$ and $G_0 = G_1$. □

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