

# ON GLOBAL NON-OSCILLATION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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ABSTRACT. Based on a new explicit upper bound for the number of zeros of exponential polynomials in a horizontal strip, we obtain a uniform upper bound for the number of zeros of solutions to an ordinary differential equation near its Fuchsian singular point, provided that any two distinct characteristic exponents at this point have distinct real parts. The latter result implies that a Fuchsian differential equation with polynomial coefficients is globally non-oscillating in  $\mathbb{C}P^1$  if and only if every its singular point satisfies the above condition.

## 1. INTRODUCTION

Let us recall the classical notion of non-oscillation of a linear ordinary differential equation, see e.g. [4].

**Definition 1.** A linear ordinary differential equation of order  $k$

$$a_k(z)y^{(k)} + a_{k-1}(z)y^{(k-1)} + \dots + a_0(z)y = 0, \quad (1)$$

with continuous coefficients  $a_j(z)$ ,  $j = 0, \dots, k$  defined in some finite or infinite interval  $I \subseteq \mathbb{R}$  is called *non-oscillating* in  $I$ , if every nontrivial solution of (1) has finitely many zeros in  $I$  counted with multiplicities. Assuming that coefficients  $a_j(z)$ ,  $j = 0, \dots, k$  are analytic, one can define the same notion in a simply-connected domain  $\Omega \subseteq \mathbb{C}$ .

Observe that in the real case, every equation (1) is non-oscillating in any compact interval  $I \subset \mathbb{R}$  free from the roots of  $a_k(z)$ . Analogously, in the complex-analytic case, every equation (1) is non-oscillating in any compact simply-connected domain  $\Omega \subset \mathbb{C}$  free from the roots of  $a_k(z)$ .

In this paper, for a linear differential equation with polynomial coefficients, we introduce the notion of its global non-oscillation in  $\mathbb{C}P^1$ . By this, we mean its classical non-oscillation in an arbitrary *open contractible* domain obtained after the removal from  $\mathbb{C}P^1$  of an appropriate cut connecting all the singular points. Although oscillation/non-oscillation in the complex domain have been studied since the 1920's, (see e.g. [6]), the notion of global non-oscillation seems to be new.

Consider a linear homogeneous differential equation

$$P_k(z)y^{(k)} + P_{k-1}(z)y^{(k-1)} + \dots + P_0(z)y = 0, \quad (2)$$

with polynomial coefficients  $P_k(z), P_{k-1}(z), \dots, P_0(z)$ , and  $GCD(P_k, P_{k-1}, \dots, P_0) = 1$ . Let  $S$  be the set of all singular points of (2) in  $\mathbb{C}P^1$ , i.e., the set of all roots of  $P_k(z)$  (together with  $\infty$  if some of the limits  $\lim_{z \rightarrow \infty} z^{2j}P_{k-j}(z)/P_k(z)$ ,  $j = 1, \dots, k$  are infinite). For a given equation (2), let  $d$  denote the cardinality of  $S$ .

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**Definition 2.** A system  $\bar{\mathcal{C}} := \{\mathcal{C}_j\}_{j=1}^{d-1}$  of smooth Jordan curves in  $\mathbb{C}P^1$ , each of them connecting a pair of distinct singular points, is called *an admissible cut* for equation (2) if and only if: a) for any  $i \neq j$ , the intersection  $\mathcal{C}_i \cap \mathcal{C}_j$  is either empty or consists of their common endpoint; b) a union  $\cup_{j=1}^{d-1} \mathcal{C}_j$  is topologically a tree in  $\mathbb{C}P^1$ , i.e., the complement  $\mathbb{C}P^1 \setminus \cup_j \mathcal{C}_j$  is contractible; c) each  $\mathcal{C}_j$  has a well-defined tangent vector at each of its two endpoints.

In particular, there exist admissible cuts consisting of straight segments connecting the singular points of (2).

**Definition 3.** Equation (2) is called *globally non-oscillating* if, for any its admissible cut  $\bar{\mathcal{C}}$ , each nontrivial solution has finitely many zeros in  $\mathbb{C}P^1 \setminus \bar{\mathcal{C}}$ .

The main result of this paper is the following criterion of global non-oscillation.

**Theorem 4.** *Equation (2) with only Fuchsian singularities is globally non-oscillating if and only if at each singular point all distinct characteristic exponents have pairwise distinct real parts.*

The major part of the proof of Theorem 4 is a local non-oscillation result dealing with a sufficiently small neighbourhood of a Fuchsian singular point. Namely, assume that the coefficients  $a_j(z)$  of (1) are holomorphic in some neighborhood of  $p \in \mathbb{C}P^1$  which is a Fuchsian singular point of (1).

**Proposition 5.** *Assume that the real parts of any two distinct characteristic exponents of (1) at  $p$  are pairwise different. Then for any  $\alpha > 0$ , there exists  $\epsilon > 0$  and an integer  $N > 0$  such that the number of zeros of any solution of (2) in the sector  $\{|z - p| < \epsilon, |\arg(z - p)| < \alpha\}$  is at most  $N$ .*

**Example 6.** Consider Euler equation:

$$t^2 y'' + aty' + by = 0.$$

Assuming that its characteristic exponents  $\lambda_1, \lambda_2$  are distinct, its solutions  $C_1 t^{\lambda_1} + C_2 t^{\lambda_2}$  have infinitely many zeros on spirals  $B \ln t + \phi \in \mathbb{R}$ ,  $B = (\lambda_1 - \lambda_2)/2\sqrt{-1}$ . If  $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2$ , then  $B \in \mathbb{R}$ ; each spiral degenerates into a ray through the origin implying that solutions have infinitely many zeros in sectors. On the other hand, if  $\operatorname{Re} \lambda_1 \neq \operatorname{Re} \lambda_2$ , each spiral has a non-zero constant angle with each ray through the origin; its intersection with any sector with apex at the origin is a finite arc. In this case, one can take  $N = \frac{\alpha}{\pi} |B| |\sin(\arg B)|^{-1} + 1$ .

**Remark 7.** If (2) has a non-Fuchsian singularity at  $p \in \mathbb{C}P^1$ , then, for any sufficiently small  $\epsilon > 0$  and for almost all  $a \in \mathbb{C}$ , almost any solution of (2) has infinitely many  $a$ -values in the  $\epsilon$ -neighbourhood of  $p$  with a straight segment connecting  $p$  with some point on the bounding circle removed. However, there could be some exceptional cases, like  $y' - y = 0$ , when the number of zeros is still finite. The question of characterization of such exceptions is difficult and lies beyond the scope of our paper.

**Remark 8.** An important class of Fuchsian equations consists of those having only real characteristic exponents. For example, Abelian integrals appearing in the infinitesimal Hilbert 16-th problem satisfy Fuchsian equations of this type. For real characteristic exponents the assumption of Theorem 4 is obviously satisfied, and an analog of Theorem 4 was essentially obtained by M. Roitman and S. Yakovenko, [9]. Their approach is based on the so-called Petrov operators and seems to be non-applicable in the more general situation considered in the present paper.

On the other hand, pseudo-Abelian integrals, which generalize Abelian integrals to Darboux integrable systems, see [8, 3], can have non-real characteristic exponents, but their properties are not well-understood.

**Remark 9.** Let us also mention that Proposition 14 below, which is an important technical tool used to prove Theorem 4, is a new result in the classical area of the upper bounds for the number of zeros of exponential polynomials. Therefore, it is of independent interest. Such upper bounds are required in a wide range of mathematical disciplines, from applied mathematics to number theory. Essential progress in this area has been made in the 70's in the papers [12], [15], [16]. But, to the best of our knowledge, in all the previous literature one only considered compact subdomains in  $\mathbb{C}$ , mainly disks and rectangles, while Proposition 14 considers the case of an infinite strip.

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## 2. PROOFS

**2.1. Necessity of assumptions in Theorem 4.** Assume that (2) has a Fuchsian singularity at 0 with two distinct characteristic exponents of the form  $\lambda_j = a + b_j \sqrt{-1}$ ,  $j = 1, 2$ . Then there exist two solutions of (2) of the form  $w_j(z) = z^{a+b_j \sqrt{-1}} g_j(z)$ ,  $j = 1, 2$ ,  $g_j(z)$  being holomorphic near the origin and  $g_j(0) = 1$ . Therefore their sum has the leading term  $z^{a+\frac{(b_1+b_2)\sqrt{-1}}{2}} \cos\left(\frac{b_1-b_2}{2} \ln z\right)$  near the origin:

$$w(z) = w_1 + w_2 = z^{a+\frac{(b_1+b_2)\sqrt{-1}}{2}} [\cos(\rho \ln z) + \epsilon(z)], \quad \rho = \frac{b_1 - b_2}{2} \in \mathbb{R},$$

where  $\epsilon(z)$  tends to zero as  $z \rightarrow 0$  uniformly in any sector  $\{|\arg z| < \alpha\}$ . Such a function has infinitely many roots accumulating to 0 and located close to the real line. (In fact, the arguments of these roots tend either to 0 or to  $\pi$  as they approach the origin). This contradicts the global non-oscillation.

## 2.2. Reduction to Proposition 5.

**2.2.1. Logarithmic charts and the de la Vallée-Poussin theorem.** Set

$$\Pi_{\alpha,\beta} = \{\operatorname{Re} t \leq \beta, |\operatorname{Im} t| \leq \alpha\}, \quad \Pi_{\infty,\beta} = \{\operatorname{Re} t \leq \beta\}, \quad \Pi_{\alpha,\infty} = \{|\operatorname{Im} t| \leq \alpha\}.$$

Most of further considerations will be carried out in logarithmic charts of neighbourhoods of singular points. Let  $z = p$  be a Fuchsian singular point of (2). Consider the logarithmic chart  $t = \log(z - p)$  (or  $t = -\log z$  for  $p = \infty$ ) in some neighbourhood of  $p$ . In this chart (2) takes the form

$$y^{(k)} + c_1(t)y^{(k-1)} + \dots + c_0(t)y = 0, \quad (3)$$

where  $c_j(t)$  are  $2\pi\sqrt{-1}$ -periodic and holomorphic in some  $\Pi_{\infty,\beta}$ . It is well-known that the Fuchsian condition translates into the requirement that each  $c_j(t)$  tends to some finite limit  $C_j$  as  $\operatorname{Re} t \rightarrow -\infty$ ,  $t \in \Pi_{\infty,\beta}$ . In particular,  $c_j(t)$  are uniformly bounded in  $\Pi_{\infty,\beta}$ .

The following complex analog of the classical de la Vallée Poussin theorem [13] is proved in [17, Corollary 2.7].

**Lemma 10.** *If the coefficients  $c_i(z)$  of the equation*

$$y^{(k)} + c_1(t)y^{(k-1)} + \dots + c_k(t)y = 0, \quad z \in \gamma \subset \mathbb{C}$$

are analytic on a finite line segment  $\gamma \subset \mathcal{C}$  of length  $\ell$  and  $|c_j(t)| \leq C$  along  $\gamma$  for some constant  $C \geq 1$ , then the variation of the argument of any nontrivial solution  $w(t)$  of this equation along  $\gamma$  is bounded by

$$\text{VarArg } w(t)|_\gamma \leq \pi(n+1) \left(1 + \frac{\ell C}{\ln 3 - \ln 2}\right). \quad \square$$

For any sufficiently small  $\epsilon > 0$ , construct a simply-connected domain  $U_\epsilon \subset \mathbb{C}P^1$  by: a) taking the large disk  $\{|z| < \epsilon^{-1}\}$  with the  $\epsilon$ -neighbourhoods of all zeros of  $P_k$  removed, b) making cuts by straight segments between the bounding circles so that the obtained domain becomes contractible.

**Corollary 11.** *There is an explicit upper bound  $B(\epsilon)$  for the number of zeros of any solution  $w(t)$  of (2) in  $U_\epsilon$ .*

*Proof.* Clearly, one can explicitly estimate from above the supremum norm of the coefficients  $a_j(z) = \frac{P_{k-j}}{P_k}$ ,  $j = 1, \dots, k$  on the straight segments which form one part of boundary of  $U_\epsilon$ . By Lemma 10, this gives an explicit upper bound on the increment of the argument of  $w(z)$  along these segments.

To estimate the increment of the argument of  $w(z)$  along the arcs of circles forming the remaining part of the boundary of  $U_\epsilon$ , use the logarithmic chart  $t = \log(z - p)$  in a neighbourhood of each Fuchsian singular point  $p$  of (2). The arcs  $\{|z - p| = \epsilon, \alpha_1 \leq \arg(z - p) \leq \alpha_2\}$  become straight segments  $\gamma = \{\text{Re } t = \log \epsilon, \alpha_1 \leq \text{Im } t \leq \alpha_2\} \subset \Pi_{\infty, \beta}$ . Using Lemma 10, we see that the increment of the argument of  $w(t)$  along  $\gamma$  is uniformly bounded (i.e. independently of the choice of  $w(t)$ ).

Taken together, it means that the total increment of the argument of  $w(z)$  along the boundary of  $U_\epsilon$  (and therefore its number of zeros in  $U_\epsilon$ ) is uniformly bounded by an explicitly given expression.  $\square$

**Remark 12.** The above proof implies that the latter bound is a polynomial in  $\epsilon^{-1}$ .

Observe that, for any admissible system of cuts  $\bar{\mathcal{C}}$  and any sufficiently small  $\epsilon$ , the domain  $\mathbb{C}P^1 \setminus \bar{\mathcal{C}}$  can be covered by finitely many  $U_\epsilon$  (choosing different straight lines connecting the bounding circles) and finitely many sectors of finite radii centered at the singular points of (2). This observation reduces the proof of Theorem 4 to providing finite upper bounds for the number of zeros of solutions of (2) in these sectors, i.e. to Proposition 5.

**2.3. Proof of Proposition 5.** The rest of the paper is devoted to the proof of Proposition 5. Our approach is inspired by the Wiman-Valiron theory. The main construction below introduced in [14] is called the *Newton-Hadamard polygon*. It has a strong resemblance with the notion of a tropical polynomial in modern tropical geometry, see e.g., [5]. It was occasionally used earlier in the literature on differential equations, see e.g., [11] and references therein.

**2.3.1. Newton-Hadamard polygon.** For a given function  $w(t) = \sum_{j=1}^m r_j e^{\lambda_j t}$ ,  $r_j, \lambda_j \in \mathbb{C}$ , consider the family

$$\widehat{\mathcal{NH}}_w = \{\kappa_j\}_{j=1}^m, \text{ where } \kappa_j = (\mu_j, -\ln |r_j|)$$

of  $m$  points in the plane  $\mathbb{R}^2$  with coordinates  $(\mu, \phi)$ . Let  $\widehat{\phi}_w$  be the piecewise-linear continuous function defined on  $[\mu_1, \mu_m]$ , which is linear on each  $[\mu_j, \mu_{j+1}]$  and satisfies the requirement  $\widehat{\phi}_w(\mu_j) = \ln |r_j|$ ,  $j = 1, \dots, m$ . In other words, all points  $\kappa_j$  lie on the graph of  $\widehat{\phi}_w$ .

Define  $\phi_0(\mu)$  as the *greatest convex minorant* of  $\widehat{\phi}_w$  on the interval  $[\mu_1, \mu_m]$ . We now define the *Newton-Hadamard polygon* of  $w(t)$  as

$$\mathcal{NH}_w = \{(\phi \leq \phi_0(\mu), \mu \in [\mu_1, \mu_m])\}.$$

For a given real number  $t$ , define

$$\phi_t(\mu) = \phi_0(\mu) - t\mu, \quad \mu \in [\mu_1, \mu_k], \quad (4)$$

so  $|r_j e^{\lambda_j t}| = \exp(-\phi_t(\mu_j))$ . Note that  $\phi_t(\mu_j) \leq -\ln |r_j e^{\lambda_j t}|$ , and the equality holds if and only if  $\mu_j$  lies on the boundary of  $\mathcal{NH}_w$ .

Define the *central index* of  $\phi_t(\mu)$  by the formula:

$$i(t) := \max\{i \mid \mu_i \text{ is the point of the global minimum for } \phi_t(\mu)\},$$

comp. [14, Ch.9]. The central index corresponds to the maximal at the point  $t$  term  $\mathcal{M}(t)$  in the sum  $w(t) = \sum r_j e^{\lambda_j t}$ :

$$\mathcal{M}(t) := \max_j \{|r_j e^{\lambda_j t}|\} = |r_{i(t)} e^{\lambda_{i(t)} t}|.$$

Indeed, as the graph of  $\phi_t(\mu)$  is convex,  $\phi_t(\mu)$  is strictly monotone decreasing on  $\{\mu \leq \mu_{i(t)}\}$  and is strictly monotone increasing on  $\{\mu \geq \mu_{i(t)}\}$ . Thus  $\mu_{i(t)}$  is clearly the boundary point of  $\mathcal{NH}_w$ , as all other points  $\mu_j, j \neq i(t)$ , lie above the line  $\{\phi = \phi_t(\mu_{i(t)}) + t(\mu - \mu_{i(t)})\}$ . Therefore  $\phi_t(\mu_{i(t)}) = -\ln |r_{i(t)} e^{\lambda_{i(t)} t}|$  and  $\mathcal{M}(t) = e^{-\phi_t(\mu_{i(t)})}$ .

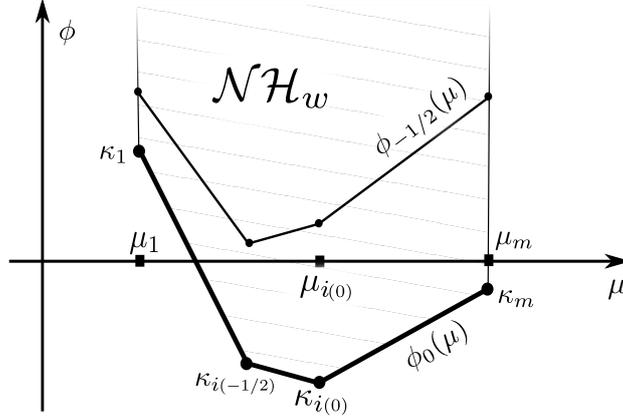


FIGURE 1. Example of a Newton-Hadamard polygon  $\mathcal{NH}_w$ .

For a piecewise-linear function  $f$ , denote by  $\Psi^f$  the set of its slopes. Note that

$$\Psi^{\phi_t} = \Psi^{\phi_0} - t.$$

Note that  $\log \mathcal{M}(t) = \max_{\mu} (t\mu - \phi_t(\mu))$  is the Legendre transform of  $\phi_0(\mu)$ , so it is a convex piece-wise linear function of  $t$ . The values of  $t$  corresponding to its vertices are exactly the slopes  $\Psi^{\phi_0}$ .

**Lemma 13.** *If, for  $j = 1, \dots, k-1$ ,*

$$|\phi_t(\mu_{j+1}) - \phi_t(\mu_j)| \geq \ln 3, \quad (5)$$

*then*

$$\sum_{j \neq i(t)} |r_j e^{\lambda_j t}| < |r_{i(t)} e^{\lambda_{i(t)} t}|.$$

*In particular,  $w(t) \neq 0$ .*

*Proof.* As  $\phi_t(\mu)$  is strictly decreasing on  $\mu < \mu_{i(t)}$ , for  $j < i(t)$  we have

$$-\ln |r_j e^{\lambda_j t}| \geq \phi_t(\mu_j) \geq \phi_t(\mu_{i(t)}) + (i(t) - j) \ln 3,$$

or, equivalently,

$$|r_j e^{\lambda_j t}| \leq 3^{j-i(t)} |r_{i(t)} e^{\lambda_{i(t)} t}|.$$

Summing up over  $j < i(t)$ , we get

$$\sum_{0 \leq j < i(t)} |r_j e^{\lambda_j t}| \leq |r_{i(t)} e^{\lambda_{i(t)} t}| \sum_{j < i(t)} 3^{j-i(t)} < \frac{1}{2} |r_{i(t)} e^{\lambda_{i(t)} t}|,$$

Similarly, for  $j > i(t)$  we get

$$\sum_{0 \leq j < i(t)} |r_j e^{\lambda_j t}| < \frac{1}{2} |r_{i(t)} e^{\lambda_{i(t)} t}|.$$

Taken together, this proves Lemma 13.  $\square$

Now we return to our proof of Proposition 5 and first consider the case of Euler equation. In the logarithmic chart  $t = \log z$  near  $z = 0$ , Euler equation becomes an equation with constant coefficients, and any solution  $w(z)$  becomes an exponential polynomial  $w(t)$ . Our goal is to bound the number of its zeros in a strip  $\Pi_{\alpha, \infty}$ . Evidently, if for some  $t$  one of the terms of  $w(t)$  dominates, i.e. is bigger than the sum of all other terms, then  $w(t)$  cannot vanish. This leads naturally to the Newton-Hadamard polygon introduced above with slopes corresponding to the domains without dominating term (in the resonant case considered in § 2.3.4 the situation is slightly more complicated). In this way we prove that all zeros of  $w(t)$  except at most  $k$  lie in an effectively bounded number of boxes  $\{|\operatorname{Im} t| \leq \alpha, \beta'_j \leq \operatorname{Re} t \leq \beta''_j\}$ , with an explicit bound on their total width  $\sum(\beta''_j - \beta'_j)$ , see Lemma 15 below. The key point is that, although the boxes themselves depend on the choice of a particular solution  $w(t)$ , the upper bounds on their number and on the total width of these boxes are *independent of  $w(t)$* .

Applying results of [7] to each box, we get an explicit upper bound on  $N$ . This bound seems to be of correct magnitude if the roots of the characteristic equation are simple.

Secondly, we study a general Fuchsian singularity as a perturbation of Euler equation. In § 2.3.5 we modify the arguments used for Euler equation to tackle the general case.

**2.3.2. Euler equation.** In the logarithmic chart near the origin, Euler equation transforms into

$$EQ: \quad y^{(k)} + C_1 y^{(k-1)} + \dots + C_k y = 0, \quad C_j \in \mathbb{C}, \quad (6)$$

with constant coefficients  $C_j$ . Let  $\lambda_j = \mu_j + \nu_j \sqrt{-1}$ ,  $j = 1, \dots, m \leq k$ , be the roots of its characteristic equation ordered in the ascending order of their real parts

$$\mu_1 = \operatorname{Re} \lambda_1 \leq \dots \leq \mu_m = \operatorname{Re} \lambda_m, \quad \nu_j = \operatorname{Im} \lambda_j.$$

We assume that all  $\lambda_j$  are pairwise different and have multiplicities  $n_j + 1$ .

**Proposition 14.** *For any  $\alpha \geq 0$  and for any equation (6) such that all its distinct characteristic roots have distinct real parts*

$$\mu_j < \mu_{j+1}, \quad j = 1, \dots, m-1,$$

*there exists an upper bound  $\sharp(EQ, \alpha)$  for the (counted with multiplicities) number of zeros of any nontrivial solution of (6) in the horizontal strip  $\Pi_{\alpha, \infty}$ .*

*Moreover, if all roots  $\lambda_j = \mu_j + \nu_j \sqrt{-1}$ ,  $j = 1, \dots, k$  of the characteristic equation of (6) are simple, then*

$$\sharp(EQ, \alpha) \leq (k-1)^2 + \frac{2}{\pi} (k-1) \mathcal{L}(EQ) [\alpha(\Xi + 1) + \Theta \ln 3], \quad (7)$$

*where  $\mathcal{L}(EQ)$  is the length of the shortest polygonal path passing through all  $\lambda_j$ . Here*

$$\Theta := \max_{1 \leq j \leq k-1} (\mu_{j+1} - \mu_j)^{-1}, \quad \Xi := \max_{1 \leq j \leq k-1} \left| \frac{\nu_{j+1} - \nu_j}{\mu_{j+1} - \mu_j} \right|.$$

Proof of Proposition 14 occupies the rest of § 2.3.2 and § 2.3.4.

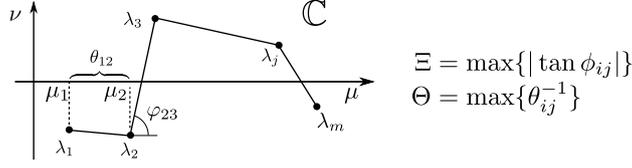


FIGURE 2. The shortest polygonal path passing through all  $\lambda_j$  has length  $\mathcal{L}(\text{EQ})$ .

General solution of (6) is given by:

$$w(t) = \sum_{j=1}^m R_j(t) e^{\lambda_j t}, \text{ where } R_j(t) = \sum_{l=0}^{n_j} a_{j,l} t^{n_j-l}, \sum (n_j + 1) = k. \quad (8)$$

For a solution  $w(t)$ , define the *domain of a single term  $w$ -dominance* in  $\Pi_\alpha$  as

$$G(w, \alpha) := \{t \in \Pi_{\alpha, \infty} \mid \exists j = j(t), \exists \epsilon > 0 : |R_j(t) e^{\lambda_j t}| \geq (1 + \epsilon) \sum_{i \neq j} |R_i(t) e^{\lambda_i t}|\}. \quad (9)$$

Note that  $G(w, \alpha)$  may contain at most  $\min n_j \leq k$  zeros of  $w$ , namely the common zeros of all  $R_j(t)$ . Indeed, if  $|R_j(t) e^{\lambda_j t}| \neq 0$ , then evidently  $w(t) \neq 0$ . If  $|R_j(t) e^{\lambda_j t}| = 0$ , then necessarily  $R_i(t) = 0$  for all  $i$ , so  $t$  is a common zero of all  $R_i(t)$ .

In particular, in case of simple characteristic exponents,  $G(w, \alpha)$  contains no zeros of  $w$  at all.

The following Lemma is the key part of the proof of Proposition 5. Its proof is given in § 2.3.4.

**Lemma 15.** *The complement  $\Pi_\alpha \setminus G(w, \alpha)$  can be covered by at most  $k + k(k+1)^2/2$  horizontal boxes (of height  $2\alpha$ ) whose total width does not exceed*

$$k(k+1)^2(2\Theta \ln k + 2\alpha\Xi + 2\alpha) + 4k^2\Theta.$$

**2.3.3. Non-resonant Euler equation.** We first consider the basic case of simple characteristic exponents  $\lambda_j$ , i.e.  $m = k$ . Then the polynomials  $R_j(t)$  are constants and we will denote them by  $r_j$ . In this case we can give a much better estimate.

**Lemma 16.** *In case of simple characteristic exponents  $\lambda_j$ , the complement  $\Pi_{\alpha, \infty} \setminus G(w, \alpha)$  can be covered by at most  $k-1$  horizontal boxes (of height  $2\alpha$ ) of total width not exceeding*

$$2\alpha(k-1)\Xi + 2(k-1)\Theta \ln 3. \quad (10)$$

The principal case in Lemma 16 is that of  $\alpha = 0$ , i.e.  $\Pi_{0, \infty} = \mathbb{R}$ .

**Lemma 17.** *If  $t$  lies outside the  $(\ln 3 \cdot \Theta)$ -neighbourhood of  $\Psi^{\phi_0}$ , then  $t \in G(w, 0)$ .*

*Proof.* As  $\Psi^{\phi_t} = \Psi^{\phi_0} - t$ , our assumption implies that the absolute values of all slopes of  $\phi_t(\mu)$  exceed  $\ln 3 \cdot \Theta$ . Therefore

$$|\phi_t(\mu_{j+1}) - \phi_t(\mu_j)| \geq \ln 3 \cdot \Theta \cdot |\mu_{j+1} - \mu_j| \geq \ln 3,$$

and the statement follows from Lemma 13.  $\square$

**Lemma 18.** *Under the above assumptions,  $\mathbb{R} \setminus G(w, 0)$  is contained in a union of at most  $k-1$  closed intervals of total length less than or equal to  $2(k-1) \ln 3 \cdot \Theta$ .*

*Proof of Lemma 18.* Lemma 17 implies that the complement of  $G(w, 0)$  lies inside the  $(\ln 3 \cdot \Theta)$ -neighbourhood of  $\Psi^{\phi_0}$ . But it consists of a union of at most  $k-1$  intervals of total length not exceeding  $2(k-1) \ln 3 \cdot \Theta$ .  $\square$

*Proof of Lemma 16.* Set  $t = u + v\sqrt{-1}$  and repeat the above construction of Lemma 18 for every horizontal line  $\text{Im } t = v$  with  $|v| \leq \alpha$ .

Note that

$$w(u + v\sqrt{-1}) = \sum r_j e^{(\mu_j + \nu_j \sqrt{-1})(u + v\sqrt{-1})} = \sum r_j(v) e^{\mu_j u},$$

where

$$r_j(v) = r_j e^{-\nu_j v + (\mu_j u + \nu_j v)\sqrt{-1}}, \quad \ln |r_j(v)| = \ln |r_j| - \nu_j v.$$

As before, for every fixed  $v$ , in  $\mathbb{R}^2$  with coordinates  $(\mu, \phi)$ , consider the point set

$$\widehat{\mathcal{NH}}_w^v = \{\kappa_j^v\}_{j=1}^k \text{ where } \kappa_j^v = (\mu_j, -\ln |r_j(v)|).$$

Define  $\phi_0^v(\mu)$  similarly to the definition of  $\phi_0(\mu)$ , but with  $\widehat{\mathcal{NH}}_w^v$  instead of  $\widehat{\mathcal{NH}}_w$ . Lemma 17 now implies that  $t = u + v\sqrt{-1} \in G(w, \alpha)$  if  $u$  lies outside the  $(\ln 3 \cdot \Theta)$ -neighbourhood of  $\Psi^{\phi_0^v}$ . Taking a union over all  $v \in [-\alpha, \alpha]$ , we get the following.

**Lemma 19.** *The interval  $[u - \alpha\sqrt{-1}, u + \alpha\sqrt{-1}]$  lies entirely in  $G(w, \alpha)$  if  $u$  lies outside the  $(\ln 3 \cdot \Theta)$ -neighbourhood of  $\Psi_\alpha := \cup_{-\alpha \leq v \leq \alpha} \Psi^{\phi_0^v}$ .*

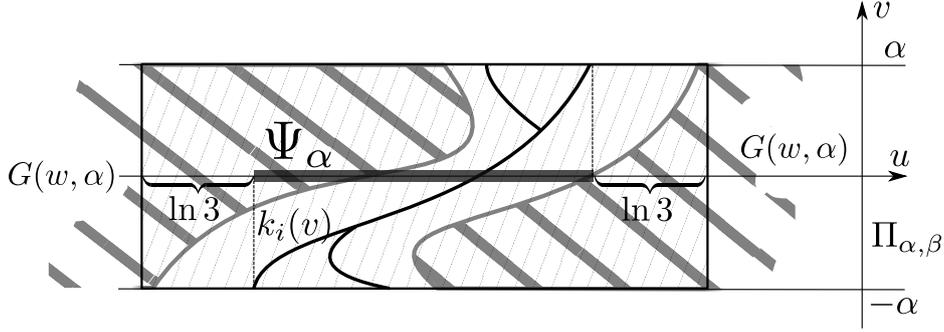


FIGURE 3. A segment of the set  $\Psi_\alpha$ .

We claim that  $\Psi_\alpha$  is a union of at most  $k-1$  closed intervals, with an explicit bound on their total length. Indeed, the set of slopes  $\Psi^{\phi_0^v}$  changes continuously with  $v$ , and consists of no more than  $k-1$  points for each fixed  $v$ .

Moreover, let

$$k(v) = -\frac{(\ln |r_i| - \nu_i v) - (\ln |r_j| - \nu_j v)}{\mu_i - \mu_j}$$

be a slope of  $\Psi^{\phi_0^v}$ . Evidently,

$$\left| \frac{\partial k(v)}{\partial v} \right| = |\xi_{ij}| = \left| \frac{\nu_j - \nu_i}{\mu_j - \mu_i} \right| \leq \Xi.$$

Therefore the total length of  $\Psi_\alpha$  is at most  $2\alpha(k-1)\Xi$ .

Thus the  $(\ln 3 \cdot \Theta)$ -neighbourhood of  $\Psi_\alpha := \cup_{-\alpha \leq v \leq \alpha} \Psi^{\phi_0^v}$  is a union of at most  $k-1$  intervals of total length at most  $2\alpha(k-1)\Xi + 2(k-1)\Theta \ln 3$ . Taken together with Lemma 19, this proves Lemma 16.  $\square$

2.3.4. *Resonant Euler equation.* In this situation the dependence of analogs of points  $\kappa_j^v$  on  $v$  is more complicated. We are forced to consider the slopes of *all* chords connecting these points, and not only those forming the Newton-Hadamard polyline, i.e. those lying on the boundary of  $\mathcal{NH}_w$ . This circumstance apparently results in an excessive upper bound of the total width of boxes covering  $\Pi_{\alpha,\infty} \setminus G(w, \alpha)$ .

*Proof of Lemma 15.* Consider the absolute value

$$r_{ij}(t) = \left| \frac{R_i(t)e^{\lambda_i t}}{R_j(t)e^{\lambda_j t}} \right|$$

of the ratio of any two terms in (8). The complement  $\Pi_{\alpha,\infty} \setminus G(w, \alpha)$  lies in a union  $\Sigma$  of the sets  $\Sigma_{ij}^o = \{|\ln r_{ij}(z)| \leq \ln k\}$  and our goal is to cover  $\Sigma_{ij}^o$  by a union of boxes. We will always assume that  $i > j$ .

We can write

$$\ln r_{ij}(t) = \ln |R_i/R_j| - v\xi_{ij}\theta_{ij} + \theta_{ij}u, \quad (11)$$

where

$$t = u + v\sqrt{-1}, \quad \theta_{ij} = \mu_i - \mu_j, \quad \xi_{ij} = \theta_{ij}^{-1}(\nu_i - \nu_j).$$

Recall that  $\theta_{ij} > 0$ , as  $i > j$ , and  $\Theta = \max_{i>j} \theta_{ij}^{-1}$ ,  $\Xi = \max_{ij} |\xi_{ij}|$ .

Let  $Z = \cup_j \{R_j = 0\}$  be the set of all zeros of all  $R_j$  and define

$$W = \{t \in \Pi_{\alpha,\infty} \mid \forall t_i \in Z \quad |\operatorname{Re}(t - t_i)| \geq 2k\Theta\}. \quad (12)$$

Evidently,  $\Pi_{\alpha,\infty} \setminus W$  is a union of at most  $k$  boxes of total width at most  $4k^2\Theta$ , so it remains to study  $W \cup \Sigma_{ij}^o$ .

For  $t = u + v\sqrt{-1} \in W$ , we have

$$\left| \frac{\partial}{\partial t} \ln \frac{R_i}{R_j}(t) \right| \leq \sum_{t_l \in Z} \left| \frac{1}{t - t_l} \right| \leq \frac{k}{2k\Theta} \leq \frac{\theta_{ij}}{2},$$

implying

$$\left| \frac{\partial}{\partial u} \ln |R_i/R_j| \right| \leq \frac{\theta_{ij}}{2}, \quad (13)$$

$$\left| \frac{\partial}{\partial v} \ln |R_i/R_j| \right| \leq \frac{\theta_{ij}}{2}. \quad (14)$$

We claim that

$$\Sigma_{ij}^o \cap W \subset \Sigma_{ij} = \{u + v\sqrt{-1} \in W : |\ln r_{ij}(u)| \leq \ln k + \alpha|\xi_{ij}|\theta_{ij} + \alpha\theta_{ij}\}.$$

Indeed, from (11), (14) it follows that

$$\begin{aligned} & \left| \ln r_{ij}(u + v\sqrt{-1}) - \ln r_{ij}(u) \right| \leq \\ & \alpha|\xi_{ij}\theta_{ij}| + \left| \ln \left| \frac{R_i}{R_j}(u + v\sqrt{-1}) \right| - \ln \left| \frac{R_i}{R_j}(u) \right| \right| \leq \\ & \alpha|\xi_{ij}\theta_{ij}| + \alpha \frac{\theta_{ij}}{2}. \end{aligned}$$

Note that  $\Sigma_{ij}$  is a union of boxes, since its definition is independent of  $v$ , and it is enough to bound their number and the total width.

Now, (11), (13) imply that

$$\frac{\partial \ln r_{ij}}{\partial u}(u) \geq \theta_{ij} - \left| \frac{\partial}{\partial u} \ln |R_i/R_j| \right| \geq \frac{\theta_{ij}}{2}, \quad u \in W, \quad (15)$$

i.e.  $\ln r_{ij}$  is monotone increasing on each connected component of  $\mathbb{R} \cap W$  faster than  $\frac{\theta_{ij}}{2}u$ .

This implies that  $\Sigma_{ij}$  intersects each connected component of  $\mathbb{R} \cap W$  in an interval of length at most  $4\theta_{ij}^{-1} \ln k + 4\alpha|\xi_{ij}| + 4\alpha$ . In other words,  $\Sigma_{ij} \cap W$  is a union of at most  $k + 1$  boxes of total width not exceeding  $(k + 1)(4|\theta_{ij}|^{-1} \ln k + 4\alpha|\xi_{ij}| + 4\alpha)$ .

Taking a union over all possible pairs  $i > j$ , we conclude that  $\Sigma \setminus W$  lies in a union of at most  $k(k + 1)^2/2$  boxes of total width at most  $k(k + 1)^2(2\Theta \ln k + 2\alpha\Xi + 2\alpha)$ . Adding the boxes  $\Pi_{\alpha, \infty} \setminus W$ , we find that  $\Sigma$  lies in a union of at most  $k + k(k + 1)^2/2$  boxes of total width at most  $k(k + 1)^2(2\Theta \ln k + 2\alpha\Xi + 2\alpha) + 4k^2\Theta$ . This finishes the proof of Lemma 15.  $\square$

Finally let us explain how Lemmas 15 and 16 imply Proposition 14. For a given finite set  $\Lambda = \{\lambda_j\} \subset \mathbb{C}$ , consider the space consisting of exponential polynomials

$$QP_\Lambda = \left\{ \sum_j R_j(t) e^{\lambda_j t}, R_j \in \mathbb{C}[t] \right\}.$$

(Dimension of  $QP_\Lambda$  equals  $k = \sum(1 + \deg R_j)$ .) The following result was proven in [7].

**Theorem 20** ([7]). *The number of zeros of any function  $w \in QP_\Lambda$  in a bounded convex domain  $U$  does not exceed*

$$k - 1 + \frac{1}{\pi} \mathcal{L}(\Lambda) \text{diam}(U), \quad (16)$$

where  $\mathcal{L}(\Lambda)$  is the length of a shortest polygonal path passing through all points of  $\Lambda$ .

Theorem 20 immediately implies an estimate on the number of zeros of  $w(t)$  in the boxes  $B_j$  of Lemma 15 and Lemma 16. We do not write it explicitly in the resonant case of Lemma 15 as we believe it to be too excessive. However, in the case of simple characteristic exponents of Lemma 16 and the second part of Proposition 14, we get

$$\sum \text{diam } B_j \leq 2(k - 1)[\alpha\Xi + \Theta \ln 3] + 2(k - 1)\alpha,$$

and (7) follows.

2.3.5. *Equation with non-constant coefficients in a semistrip.* In general, solutions of (1) considered in the logarithmic chart  $t = \log(z - p)$  near its Fuchsian singular point  $p$  have the form

$$\tilde{w}(t) = \sum_{j=1}^m \tilde{R}_j(t) e^{\lambda_j t}, \quad (17)$$

where

$$\tilde{R}_j(t) = \sum_{l=0}^{n_j} a_{j,l} t^{n_j-l} (1 + \epsilon_{j,l}(t)),$$

$\epsilon_{j,l}(t)$  being  $2\pi\sqrt{-1}$ -periodic,  $\epsilon_{j,l}(t) = O(e^u)$ ,  $t = u + v\sqrt{-1}$ , in some left half-plane  $\Pi_{\infty, \beta}$ .

We will consider  $\tilde{w}$  as a perturbation of (8) with the same  $a_{j,l}$ , and will continue to use the objects defined for  $w$  in §2.3.4.

**Lemma 21.** *There exists  $\beta < 0$  such that*

$$\left| \log \left| \frac{\tilde{R}_j}{R_j} \right| \right| \leq 1/2 \text{ in } W \cap \Pi_{\pi, \beta}. \quad (18)$$

*Proof.* Let  $E = E(\beta)$  be a common upper bound for all  $|\epsilon_{j,l}(t)| e^{-u}$ ,  $t \in \Pi_{\infty, \beta}$ .

Let  $\mathring{R}(t) = \sum |a_{j,l}| |t|^{n_j-l}$ . Vieta's formulas imply that  $\mathring{R}_j(t) \leq \prod_m (|t| + |t_{j,m}|)$ , where  $t_{j,m}$  are the roots of  $R_j(t)$ . Also,

$$\left| \frac{\tilde{R}_j}{R_j} - 1 \right| \leq \sum_{l=0}^{n_j} |a_{j,l} t^{n_j-l} \epsilon_{j,l}| \leq E e^u \mathring{R}_j(t).$$

Therefore, for  $t \in W$ , we get

$$\begin{aligned} \left| \frac{\tilde{R}_j - R_j}{R_j} \right| &\leq Ee^u \frac{\mathring{R}_j(t)}{|R_j(t)|} \leq Ee^u \prod \frac{|t| + |t_m|}{||t| - |t_m||} \leq \\ &\leq Ee^u |t|^\ell \prod_{|t_m| < 2|t|} \frac{1 + |t_m|/|t|}{||t| - |t_m||} \prod_{|t_m| > 2|t|} \frac{|t/t_m + 1|}{1 - |t/t_m|} \leq \\ &Ee^u |t|^\ell \left( \frac{3}{2k\Theta} \right)^\ell 3^{k-\ell} \leq Ee^{\beta/2} M_k, \end{aligned}$$

where  $\ell = \#\{t_m \mid |t_m| < 2|t|\}$  and

$$M_k = \max_{0 \leq \ell \leq k} \max_{t \in \Pi_{\pi,0}} e^{u/2} |t|^\ell \left( \frac{3}{2k\Theta} \right)^\ell 3^{k-\ell} < \infty$$

(recall that  $|t - t_m| \geq 2k\Theta$ , as  $t \in W$ ).

Choose  $\beta < 0$  small enough to obtain  $EM_k e^{\beta/2} < 1 - e^{-1/2}$ . Then

$$\left| \log |\tilde{R}_j/R_j| \right| = \left| \log \left| 1 + \frac{\tilde{R}_j - R_j}{R_j} \right| \right| \leq \frac{1}{2}.$$

□

**Lemma 22.** *For  $\beta$  as in Lemma 21 and  $\alpha \leq \pi$ , the zeros of  $\tilde{w}$  in  $\Pi_{\alpha,\beta}$  lie in at most  $k + k(k+1)^2/2$  boxes of total width at most*

$$\frac{k(k+1)^2}{2} (4\Theta \ln k + 4\alpha\Xi + 4\alpha + 4\Theta) + 4k^2\Theta.$$

*Proof.* We repeat the proof of Lemma 15. Namely, consider the absolute value  $\tilde{r}_{ij}$  of the ratio of two terms in (17). The complement  $\Pi_{\alpha,\beta} \setminus G(\tilde{w}, \alpha)$  lies in a union  $\Sigma$  of the sets

$$\widetilde{\Sigma}_{ij}^o = \{|\ln \tilde{r}_{ij}(t)| \leq \ln k\}.$$

Again, we consider only the set  $W$ . By Lemma 21,  $|\log \tilde{r}_{ij} - \log r_{ij}| \leq 1$ . So it is enough to require  $|\ln r_{ij}(z)| \leq \ln k + 1$ , i.e.

$$\widetilde{\Sigma}_{ij}^o \subset \widetilde{\Sigma}_{ij} = \{u + v\sqrt{-1} \in \Pi_\alpha : |\ln r_{ij}(u)| \leq \ln k + \alpha|\xi_{ij}\theta_{ij}| + \alpha|\theta_{ij}| + 1\}.$$

Repeating the same arguments as in Lemma 15 with  $\widetilde{\Sigma}_{ij}$  instead of  $\Sigma_{ij}$ , we arrive at the required estimates. □

### 2.3.6. Final punch.

*Proof of Theorem 4.* Let

$$y^{(k)} + a_1(t)y^{(k-1)} + \cdots + a_k(t)y = 0$$

be the reduced form of (2) (i.e.  $a_j(t) = P_{k-j}(t)/P_k(t)$ ,  $j = 1, \dots, k$ ) in the logarithmic chart near its Fuchsian singularity. Choose  $\beta$  as in Lemma 21 and denote by  $C$  the common upper bound on  $|a_j(t)|$  in  $\Pi_{\alpha,\beta}$ .

The example illustrating Corollary 2.7 of [17] (Lemma 10 above) claims that if  $D$  is a rectangle of perimeter  $\ell$  and the coefficients of (3) are bounded by  $C \geq 1$  in this rectangle, then any nontrivial solution of (3) has at most

$$2(k+1) + \frac{(k+1)\ell C}{2(\ln 3 - \ln 2)}$$

isolated zeros in  $D$ .

Applying this estimate to each of the boxes of Lemma 22, we conclude that  $w(t)$  has at most  $2(k+1)(k + k(k+1)^2/2) + \frac{(k+1)\ell C}{2(\ln 3 - \ln 2)}$  zeros in  $\Pi_{\alpha,\beta}$ , where

$$\ell \leq k(k+1)^2(4\Theta \ln k + 4\alpha\Xi + 4\alpha + 4\Theta) + 8k^2\Theta + 4(k + k(k+1)^2/2)\alpha,$$

is the total perimeter of all boxes appearing in Lemma 22.

In the original coordinates this translates to an upper bound for the number of zeros of any solution of (2) in the sector  $p + \exp \Pi_{\alpha, \beta} = \{|z - p| \leq e^\beta, |\arg z| \leq \alpha\}$  at the Fuchsian singular point  $p$ . This bound proves Proposition 5 and, therefore, settles Theorem 4.  $\square$

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