THE $(n-2,2)$-SPECTRUM OF A GRAPH

B. SHAPIRO

Abstract. We study a representation-theoretic refinement of the ordinary Laplacian spectrum of a graph. Given a graph G on n vertices, one may associate to it the element

$$X_G = \sum_{ij \in E(G)} (ij) \in \mathbb{C}[S_n].$$

The action of X_G in irreducible representations of S_n produces spectral invariants of graphs. The standard representation $(n-1,1)$ recovers the ordinary graph Laplacian spectrum, up to the elementary affine change $X_G = mI - L_G$, where $m = |E(G)|$. The next component, $(n-2,2)$, gives the first representation-theoretic correction. We give an explicit edge-space model for this component, derive a concrete coordinate formula for the induced operator, give a conceptual formula for all trace moments, specialize it to trees as universal linear combinations of support-forest counts, and then compute the first three moments explicitly. The third moment is expressed in terms of three-edge subgraph counts. We also introduce a weighted trace polynomial and prove that this weighted refinement already reconstructs every tree from the second moment, except for a single exceptional value of n where the fourth moment suffices. Finally we discuss the relation with the invariant-theoretic approach of Thiéry [10] and formulate a more explicit support-forest-profile conjecture for the unweighted graph isomorphism problem for trees.

1. Introduction

Spectral graph theory traditionally studies eigenvalues of matrices naturally associated with a graph, such as adjacency matrices, graph Laplacians, signless Laplacians and normalized Laplacians; see, for example, [4, 11]. These operators act on functions on the vertex set. Representation theory of S_n suggests a natural hierarchy of spectral refinements; see, for instance, [3] for the general representation-theoretic viewpoint.

Let G be a graph on the vertex set $[n] = \{1, \dots, n\}$. Unless otherwise stated, we assume $n \geq 4$ when referring to the partition $(n-2,2)$. Put

$$X_G = \sum_{ij \in E(G)} (ij) \in \mathbb{C}[S_n].$$

For every irreducible representation $\rho_\lambda : S_n \to GL(V_\lambda)$ one obtains an operator

$$X_G^\lambda := \rho_\lambda(X_G) = \sum_{ij \in E(G)} \rho_\lambda((ij)).$$

The spectrum of X_G^λ is an isomorphism invariant of G. The same group-algebra element, or equivalently the shifted Laplacian $mI - X_G$, appears in the spectral theory of Cayley graphs of S_n generated by transpositions and in the interchange process; see, for example, [2, 1]. Here we use this construction in a different direction, namely as a hierarchy of graph invariants obtained by looking at individual irreducible components.

The case $\lambda = (n-1,1)$ is the ordinary Laplacian spectrum in disguise. Thus the first genuinely new irreducible component is

$$\lambda = (n-2,2).$$

This note studies the corresponding graph invariant. The main new point beyond the basic construction is a general character formula for all trace moments. For trees this gives a universal expansion of every moment as a linear combination of embedded support-forest counts. The explicit first three moments are then obtained as examples; the first two are comparatively coarse, while the third moment already sees three-edge configurations such as triangles, claws, three-edge paths and disjoint edge triples.

2020 *Mathematics Subject Classification.* 05C50, 05C05, 20C30, 05E10, 13A50.

Key words and phrases. spectral graph theory, symmetric group, Cayley graph, interchange process, graph isomorphism, tree reconstruction, cospectral graphs, invariant theory.

Main contributions. The results of the paper may be summarized as follows.

(i) We identify the $(n-2,2)$-space with the zero-degree edge space.
(ii) We write the operator X_G on this space explicitly in edge coordinates.
(iii) We give a closed character-sum formula for every trace moment.
(iv) For trees, we show that every trace moment is a universal linear combination of embedded support-forest counts.
(v) We introduce the weighted trace polynomials. Their support coefficients encode the edge-line graph of a tree; consequently the weighted refinement reconstructs every tree from the quadratic trace polynomial, except when $n = 7$, where the quartic trace polynomial separates adjacent from disjoint edge pairs.
(vi) We keep the first three unweighted trace moments as explicit examples: the first and second moments are determined by elementary Laplacian data, while the third moment gives the first concrete source of new information beyond the ordinary Laplacian spectrum.

2. The graph representation

Let

$$V_n = \mathbb{C}^{\binom{n}{2}}$$

be the edge space of the complete graph K_n, with basis e_{ij} indexed by unordered pairs $ij \subset [n]$. The symmetric group S_n acts by permuting vertices:

$$\sigma e_{ij} = e_{\sigma(i)\sigma(j)}.$$

The classical decomposition of this permutation representation is [8]

$$V_n \cong (n) \oplus (n-1,1) \oplus (n-2,2). \tag{2.1}$$

The trivial representation is generated by the complete graph vector

$$\mathbf{1} = \sum_{i<j} e_{ij}.$$

The sum of the first two components is generated by the star vectors

$$E_i = \sum_{j \neq i} e_{ij}, \qquad i = 1, \dots, n.$$

These n vectors are linearly independent for $n \geq 3$, and their span is isomorphic to the permutation representation $(n) \oplus (n-1,1)$. Hence the last summand has dimension

$$\binom{n}{2} - n = \frac{n(n-3)}{2}.$$

Remark 1. This decomposition is used explicitly by Thiéry [10] in his study of the invariant ring of weighted graphs. In that setting the component $(n-2,2)$ is described as the space of weighted graphs whose vertex degrees are all zero. The spectral theory of the operator X_G on this component seems to be a natural refinement of that invariant-theoretic viewpoint.

3. The standard representation and the Laplacian

Let A_G be the adjacency matrix of G, let D_G be the diagonal degree matrix, and let

$$L_G = D_G - A_G$$

be the graph Laplacian. Denote $m = |E(G)|$.

Proposition 1. *On the standard representation $(n-1,1)$ one has*

$$X_G^{(n-1,1)} = mI - L_G.$$

Consequently, for $n \geq 4$, the spectrum of $X_G^{(n-1,1)}$ is equivalent to the ordinary Laplacian spectrum of G. More generally, this equivalence holds once the number m of edges is known.

Proof. In the permutation representation on $\mathbb{C}^n$, the transposition (ij) exchanges the basis vectors e_i, e_j. Therefore $I - (ij)$ is the rank-one elementary Laplacian attached to the edge ij. Summing over all edges gives

$$\sum_{ij \in E(G)} (I - (ij)) = L_G.$$

Thus $X_G = mI - L_G$ on $\mathbb{C}^n$. The trivial line is invariant and carries X_G by multiplication by m; after quotienting by the trivial line, or restricting to its orthogonal complement, one obtains the standard representation. Since the trace of $X_G^{(n-1,1)}$ equals $m(n-3)$, the integer m is recovered from this spectrum for $n \geq 4$. □

4. A concrete model for the $(n-2,2)$-component

The component $(n-2,2)$ has a particularly simple realization inside the edge space.

Definition 1. Let

$$W_n = \left\{ z = (z_{ij})_{i<j} \in V_n : \sum_{j \neq i} z_{ij} = 0 \text{ for every } i = 1, \ldots, n \right\},$$

where $z_{ij} = z_{ji}$. We call W_n the zero-degree edge space.

Proposition 2. *For $n \geq 4$, W_n is an S_n-invariant subspace of V_n, and*

$$W_n \cong (n-2,2).$$

For $n = 3$ this space is zero and the component is absent. Equivalently,

$$V_n = \operatorname{span}\{E_1, \ldots, E_n\} \oplus W_n.$$

Proof. The defining equations of W_n say precisely that z is orthogonal to each star vector E_i. Therefore

$$W_n = \operatorname{span}\{E_1, \ldots, E_n\}^{\perp}.$$

Since the star span is S_n-invariant, so is its orthogonal complement. The star span is the permutation representation $(n) \oplus (n-1,1)$. Using the decomposition (2.1), its orthogonal complement is the remaining irreducible component $(n-2,2)$. □

This proposition gives the first main point of the paper: the $(n-2,2)$-spectrum is the spectrum of X_G on zero-degree weighted graphs.

Definition 2. The $(n-2,2)$-spectrum of G is

$$\operatorname{Spec}_{(n-2,2)}(G) := \operatorname{Spec}\left(X_G|_{W_n}\right).$$

5. Coordinate formula for the operator

The action of X_G on the full edge space V_n can be written explicitly. For an unordered pair ab, let d_a be the degree of a in G, and let $\mathbf{1}_{ab \in E(G)}$ be 1 if ab is an edge of G and 0 otherwise.

Proposition 3. *For every basis vector e_{ab} of V_n one has*

$$X_G e_{ab} = \big(m - d_a - d_b + 2 \cdot \mathbf{1}_{ab \in E(G)}\big) e_{ab} + \sum_{\substack{c \neq b \\ ac \in E(G)}} e_{bc} + \sum_{\substack{c \neq a \\ bc \in E(G)}} e_{ac}. \tag{5.1}$$

The restriction of this operator to W_n is $X_G^{(n-2,2)}$.

Proof. Fix ab. An edge-transposition $ij \in E(G)$ acts on e_{ab} in three ways. If ij is disjoint from ab, then it fixes e_{ab}. If $ij = ab$, it also fixes e_{ab}. If ij shares exactly one endpoint with ab, it replaces that endpoint by the other endpoint of ij. Counting these three possibilities gives (5.1). Since W_n is invariant under each permutation of vertices, it is invariant under the sum X_G. □

Remark 2. Formula (5.1) shows that the relation with line graphs is subtler than a literal equality with the line graph spectrum of G. The operator acts on the edge space of the complete graph K_n, not only on the edge set of G. It is better viewed as an edge-replacement operator: an edge ab may move to bc or ac by using an edge of G incident to one of its endpoints. Thus line-graph type adjacency appears, but on the full pair space and with a degree-dependent diagonal term.

6. A universal trace formula

The low-order trace computations below are best viewed as examples of a more general formula. We record it first because it gives a conceptual answer to the "higher trace" problem.

Let $\tau_e = (ij)$ denote the transposition corresponding to an edge $e = ij$. For a word

$$w = (e_1, \dots, e_r) \in E(G)^r$$

put

$$\pi(w) = \tau_{e_1} \cdots \tau_{e_r} \in S_n.$$

If $\sigma \in S_n$, let $c_1(\sigma)$ be the number of fixed points of σ and let $c_2(\sigma)$ be the number of two-cycles in its cycle decomposition.

Lemma 1 (Character of $(n-2,2)$)**.** *For every $\sigma \in S_n$,*

$$\chi^{(n-2,2)}(\sigma) = \binom{c_1(\sigma)}{2} + c_2(\sigma) - c_1(\sigma). \tag{6.1}$$

Proof. The permutation representation of S_n on two-element subsets of $[n]$ has character

$$\binom{c_1(\sigma)}{2} + c_2(\sigma),$$

because a two-subset is fixed either when both of its elements are fixed points of σ, or when it is a two-cycle of σ. This permutation representation decomposes as

$$(n) \oplus (n-1,1) \oplus (n-2,2).$$

The character of $(n) \oplus (n-1,1)$ is the character of the natural permutation representation on vertices, namely $c_1(\sigma)$. Subtracting it gives (6.1). ☐

Theorem 1 (Universal trace formula)**.** *For every graph G on n vertices and every $r \geq 1$,*

$$M_r^{(2)}(G) := \operatorname{tr}\left((X_G|_{W_n})^r\right) = \sum_{(e_1,\dots,e_r)\in E(G)^r} \left[\binom{c_1(\pi(e_1,\dots,e_r))}{2} + c_2(\pi(e_1,\dots,e_r)) - c_1(\pi(e_1,\dots,e_r))\right]. \tag{6.2}$$

Equivalently,

$$M_r^{(2)}(G) = \sum_{(e_1,\dots,e_r)\in E(G)^r} \chi^{(n-2,2)}(\tau_{e_1} \cdots \tau_{e_r}). \tag{6.3}$$

Proof. Expanding X_G^r in the group algebra gives

$$X_G^r = \sum_{(e_1,\dots,e_r)\in E(G)^r} \tau_{e_1} \cdots \tau_{e_r}.$$

Taking the trace in the irreducible representation $(n-2,2)$ gives (6.3). Formula (6.2) follows from the preceding character formula. ☐

6.1. **Specialization to trees: support forests.** For trees the universal formula has a useful interpretation in terms of forest counts. This is slightly more accurate than saying only "subtree counts": if a word uses several mutually separated edges, its support is a forest, whose connected components are subtrees of T.

If T is a tree and $w = (e_1, \dots, e_r)$ is a word in edges of T, let $F(w)$ be the edge-induced forest whose edge set is the set of distinct edges occurring in w. Thus $F(w)$ is a disjoint union of embedded subtrees of T.

Proposition 4 (Tree trace expansion). *For every tree T on n vertices and every $r \geq 1$,*

$$M_r^{(2)}(T) = \sum_{\mathcal{F}} N_{\mathcal{F}}(T)\, C_{r,\mathcal{F}}(n), \tag{6.4}$$

where $\mathcal{F}$ runs over isomorphism types of finite forests with at most r edges, $N_{\mathcal{F}}(T)$ is the number of embedded edge-induced subforests of T isomorphic to $\mathcal{F}$, and $C_{r,\mathcal{F}}(n)$ is a universal coefficient depending only on r, n, and the forest type $\mathcal{F}$.

Proof. Group the words $w \in E(T)^r$ according to the isomorphism type of the support forest $F(w)$. Once this forest is identified with a fixed model $\mathcal{F}$, the contribution of all words whose support is that copy depends only on multiplication of transpositions inside $\mathcal{F}$ and on the number $n - |V(\mathcal{F})|$ of outside vertices, all of which are fixed by the resulting permutation. Thus each embedded copy of $\mathcal{F}$ contributes the same universal quantity $C_{r,\mathcal{F}}(n)$. □

Remark 3. This proposition is the conceptual form of the trace theory for trees: every $(n-2,2)$-trace is a universal linear combination of embedded forest counts. The connected forest components are precisely subtrees, so this is still a subtree-counting invariant, but with separated components retained. The first few explicit trace formulas below should be understood as the first terms of this general support-forest package, rather than as isolated computations.

6.2. **Weighted trace polynomials and a sharper reconstruction package.** The unweighted spectrum gives one number $M_r^{(2)}(T)$ in each degree. For conceptual purposes it is useful to consider a slightly richer object first. Assign an independent variable y_e to every edge e of T and put

$$X_T(y) = \sum_{e \in E(T)} y_e \tau_e \in \mathbb{C}[y_e : e \in E(T)][S_n].$$

Define the weighted trace polynomial

$$\mathcal{M}_r^{(2)}(T; y) = \operatorname{tr}_{(n-2,2)}\left(X_T(y)^r\right). \tag{6.5}$$

The ordinary trace is the diagonal specialization

$$M_r^{(2)}(T) = \mathcal{M}_r^{(2)}(T; 1, \dots, 1).$$

For a finite forest F and a function $\alpha : E(F) \to \mathbb{Z}_{>0}$, write $|\alpha| = \sum_{e \in E(F)} \alpha_e$. Let $C_{F,\alpha}(n)$ be the following universal character sum. Choose any labelling of the vertices of F inside $[n]$ and sum

$$\chi^{(n-2,2)}(\tau_{e_1} \cdots \tau_{e_r}) \tag{6.6}$$

over all words $(e_1, \dots, e_r)$ in the edges of F in which each edge e occurs exactly α_e times. The result is independent of the chosen labelling of F.

Theorem 2 (Weighted support-forest expansion). *For every tree T on n vertices and every $r \geq 1$,*

$$\mathcal{M}_r^{(2)}(T; y) = \sum_{\substack{(F,\alpha) \\ |\alpha| = r}} C_{F,\alpha}(n) \sum_{\substack{(F',\alpha') \subseteq T \\ (F',\alpha') \cong (F,\alpha)}} \prod_{e \in E(F')} y_e^{\alpha'_e}, \tag{6.7}$$

where the outer sum runs over isomorphism classes of finite forests equipped with positive edge multiplicities. The inner sum runs over embedded edge-induced subforests $F' \subseteq T$, together with the transported multiplicity function α', counted once as embedded weighted subforests. This convention avoids any overcounting coming from automorphisms of F.

Proof. Expand $X_T(y)^r$. Each ordered word in edges contributes the monomial $y_{e_1} \cdots y_{e_r}$ multiplied by the character value of the product of the corresponding transpositions. Collect together all words with the same support forest and the same multiplicity vector. Since T is acyclic, the isomorphism type of the resulting weighted support forest determines the universal character sum for every embedded weighted copy. This gives (6.7). ☐

6.3. **A weighted reconstruction theorem for trees.** The weighted trace polynomial is not merely a convenient bookkeeping device. For trees it already contains enough information to recover the tree itself, up to the natural permutation of the edge variables. This gives a rigorous version of the reconstruction mechanism behind the support-forest expansion.

Let

$$\mathcal{M}_2^{(2)}(T;y) = \mathrm{tr}_{(n-2,2)}(X_T(y)^2).$$

For two distinct edges e, f of a tree T, the coefficient of $y_e y_f$ in $\mathcal{M}_2^{(2)}(T;y)$ is

$$2\chi^{(n-2,2)}(\tau_e\tau_f).$$

If e and f are adjacent, then $\tau_e\tau_f$ is a 3-cycle; if they are disjoint, then $\tau_e\tau_f$ has cycle type $(2, 2, 1^{n-4})$. Therefore

$$[y_e y_f]\mathcal{M}_2^{(2)}(T;y) = \begin{cases} 2\alpha_n, & e \cap f \neq \varnothing, \\ 2\beta_n, & e \cap f = \varnothing, \end{cases} \tag{6.8}$$

where

$$\alpha_n = \frac{n^2 - 9n + 18}{2}, \qquad \beta_n = \frac{n^2 - 11n + 32}{2}.$$

Thus $\alpha_n - \beta_n = n - 7$.

Theorem 3 (Weighted reconstruction of trees)**.** *Let T be a tree on n vertices. If $n \neq 7$, then the weighted quadratic trace polynomial $\mathcal{M}_2^{(2)}(T;y)$, considered up to permutation of the edge variables, determines T up to isomorphism. For $n = 7$, the pair*

$$\left(\mathcal{M}_2^{(2)}(T;y), \mathcal{M}_4^{(2)}(T;y)\right)$$

determines T up to isomorphism.

Proof. Assume first that $n \neq 7$. By (6.8), the coefficient of $y_e y_f$ in $\mathcal{M}_2^{(2)}(T;y)$ distinguishes whether two distinct edges e and f of T are adjacent or disjoint. Hence the polynomial determines the line graph $L(T)$: its vertices are the edge variables of T, and two such vertices are adjacent exactly when the corresponding coefficient is $2\alpha_n$ rather than $2\beta_n$.

By Whitney's line-graph theorem [12], a connected graph is determined by its line graph, apart from the classical ambiguity between K_3 and $K_{1,3}$. Since we are working in the class of trees, this ambiguity causes no difficulty: K_3 is not a tree, and $K_{1,3}$ is recovered as the unique tree with line graph K_3. Therefore $L(T)$ determines T.

It remains to discuss the exceptional value $n = 7$, where $\alpha_7 = \beta_7$ and the quadratic trace no longer distinguishes adjacent from disjoint edge pairs. In this case consider the coefficient of $y_e^2 y_f^2$ in $\mathcal{M}_4^{(2)}(T;y)$ for two distinct edges e, f. If e and f are disjoint, the corresponding transpositions commute, so all six words with two occurrences of each edge multiply to the identity. The coefficient is therefore

$$6d, \qquad d = \dim(n-2,2) = \frac{n(n-3)}{2}.$$

If e and f are adjacent, four of the six words multiply to the identity and the two alternating words multiply to 3-cycles. Hence the coefficient is

$$4d + 2\alpha_n.$$

For $n = 7$ one has $d = 14$ and $\alpha_7 = 2$, so the two values are

$$84 \qquad \text{and} \qquad 60,$$

respectively. Thus the quartic weighted trace again reconstructs the line graph of T, and the same line-graph argument recovers T. ☐

Remark 4. This theorem should be viewed as the weighted counterpart of the tree conjecture for the unweighted spectrum. It proves that before diagonal specialization $y_e = 1$, the $(n-2,2)$ trace theory has full reconstructive power for trees. The remaining and subtler problem is to understand how much of this information survives after all edge weights are specialized to one.

Corollary 1. *For a tree T, the weighted trace polynomials $\mathcal{M}_r^{(2)}(T; y)$ determine, degree by degree, universal weighted forest-counting polynomials. Their diagonal specialization gives the unweighted moments $M_r^{(2)}(T)$.*

Remark 5. The preceding theorem is the cleanest form of the reconstruction mechanism. It separates two questions. First, the weighted theory gives a rigorous support-forest expansion with explicit universal coefficients. Second, the ordinary $(n-2,2)$-spectrum is obtained by setting all weights equal to one. The tree-isomorphism conjecture below asserts that this diagonal specialization is still rich enough, especially when combined with the ordinary Laplacian spectrum, see § 9.

7. First moments

7.1. **The first two moments.** The trace moments of $X_G|_{W_n}$ are graph invariants:

$$M_k^{(2)}(G) := \operatorname{tr}\left((X_G|_{W_n})^k\right).$$

These are the power sums of the $(n-2,2)$-spectrum.

Let

$$p(G) = \sum_{v=1}^{n} \binom{d_v}{2}$$

be the number of unordered adjacent pairs of edges of G.

Theorem 4. *Let $d = \dim(n-2,2) = n(n-3)/2$. Then*

$$M_1^{(2)}(G) = m\,\frac{(n-3)(n-4)}{2}. \tag{7.1}$$

Moreover

$$M_2^{(2)}(G) = md + 2p(G)\,\alpha_n + \big(m(m-1) - 2p(G)\big)\beta_n, \tag{7.2}$$

where

$$\alpha_n = \frac{n^2 - 9n + 18}{2}, \qquad \beta_n = \frac{n^2 - 11n + 32}{2}.$$

Proof. For an irreducible character $\chi^{(n-2,2)}$ we have

$$M_k^{(2)}(G) = \sum_{e_1,\dots,e_k \in E(G)} \chi^{(n-2,2)}(e_1 \cdots e_k),$$

where an edge ij is identified with the transposition (ij).

For $k = 1$, all summands are transpositions. The character value of $(n-2,2)$ on a transposition is

$$\chi^{(n-2,2)}(2, 1^{n-2}) = \frac{(n-3)(n-4)}{2}.$$

This gives (7.1).

For $k = 2$, ordered pairs of edges split into three types. If the two edges are equal, their product is the identity; this contributes $m \dim(n-2,2) = md$. If the two edges are distinct and adjacent, their product is a 3-cycle. There are $2p(G)$ such ordered pairs. If they are disjoint, their product has cycle type $(2,2,1^{n-4})$. There are $m(m-1) - 2p(G)$ such ordered pairs.

It remains only to record the two character values

$$\chi^{(n-2,2)}(3, 1^{n-3}) = \frac{n^2 - 9n + 18}{2},$$

and

$$\chi^{(n-2,2)}(2, 2, 1^{n-4}) = \frac{n^2 - 11n + 32}{2}.$$

They follow immediately by subtracting from the edge permutation character the characters of (n) and $(n-1,1)$. ∎

Remark 6. The quadratic moment is a useful consistency check but not yet a source of very new information. Indeed, the ordinary Laplacian spectrum already determines m and $p(G)$, because

$$\mathrm{tr}(L_G) = 2m, \qquad \mathrm{tr}(L_G^2) = \sum_v d_v^2 + 2m,$$

and

$$p(G) = \frac{1}{2}\left(\sum_v d_v^2 - 2m\right).$$

Thus genuinely new information in the $(n-2,2)$-spectrum should begin at third and higher moments.

7.2. **The cubic moment.** We now compute the third moment explicitly. This is the first calculation in which genuine three-edge configurations appear.

Let

$$t(G)$$

be the number of triangles of G. Let $s(G)$ be the number of three-edge stars $K_{1,3}$ contained in G, namely

$$s(G) = \sum_{v\in V(G)} \binom{d_v}{3}.$$

Let $r(G)$ be the number of three-edge paths P_4 contained in G, and let $q(G)$ be the number of three-edge subsets of $E(G)$ whose selected-edge union is a disjoint union of a two-edge path and one isolated edge. Finally, let $d_3(G)$ be the number of three-edge subsets consisting of three pairwise disjoint edges. Throughout this paragraph, "contained" refers to the subgraph determined by the selected edges; no inducedness condition is imposed on the ambient graph G. Thus

$$\binom{m}{3} = t(G) + s(G) + r(G) + q(G) + d_3(G).$$

For convenience put

$$\begin{aligned}
c_2 &= \chi^{(n-2,2)}(2,1^{n-2}) = \frac{(n-3)(n-4)}{2},\\
c_4 &= \chi^{(n-2,2)}(4,1^{n-4}) = \frac{n^2-11n+28}{2},\\
c_{32} &= \chi^{(n-2,2)}(3,2,1^{n-5}) = \frac{n^2-13n+42}{2},\\
c_{222} &= \chi^{(n-2,2)}(2,2,2,1^{n-6}) = \frac{n^2-15n+60}{2}.
\end{aligned}$$

Theorem 5 (Cubic trace formula)**.** *For every graph G on n vertices one has*

$$\begin{aligned}
M_3^{(2)}(G) &= c_2\big(m + 3m(m-1) + 6t(G)\big)\\
&\quad + 6c_4\big(s(G)+r(G)\big) + 6c_{32}q(G) + 6c_{222}d_3(G).
\end{aligned}\tag{8.1}$$

Equivalently, after eliminating $d_3(G)$,

$$\begin{aligned}
M_3^{(2)}(G) &= c_2\big(m + 3m(m-1) + 6t(G)\big) + 6c_{222}\binom{m}{3}\\
&\quad + 6(c_4 - c_{222})(s(G)+r(G)) + 6(c_{32}-c_{222})q(G) - 6c_{222}t(G).
\end{aligned}\tag{8.2}$$

Proof. We expand

$$M_3^{(2)}(G) = \sum_{e,f,h\in E(G)} \chi^{(n-2,2)}(efh),$$

where an edge is identified with the corresponding transposition. The ordered triples of edges are divided according to the isomorphism type of the underlying three-edge multiset.

If all three edges are equal, or if exactly two are equal, then the product of the three transpositions is again a transposition. These cases give $m + 3m(m-1)$ ordered triples. If the three distinct edges form a triangle, the product of the three transpositions is also a transposition; this gives $6t(G)$ further ordered triples.

If the three distinct edges form either a path P_4 or a star $K_{1,3}$, their product is a 4-cycle, independently of the order of multiplication. This contributes $6(r(G)+s(G))$ ordered triples. If their union is a disjoint union of a two-edge path and one isolated edge, the product has cycle type $(3, 2, 1^{n-5})$, and this contributes $6q(G)$ ordered triples. Finally, if the three edges are pairwise disjoint, the product has cycle type $(2, 2, 2, 1^{n-6})$, and this contributes $6d_3(G)$ ordered triples. This proves (8.1). Formula (8.2) follows from $\binom{m}{3} = t + s + r + q + d_3$. □

Remark 7. The character values used above follow from the elementary identity

$$\chi^{(n-2,2)}(\sigma) = \binom{f_1(\sigma)}{2} + f_2(\sigma) - f_1(\sigma),$$

where $f_1(\sigma)$ is the number of fixed points of σ and $f_2(\sigma)$ is the number of two-cycles of σ. This identity is obtained by subtracting the trivial and standard characters from the permutation character of S_n on two-subsets.

Corollary 2. *The $(n-2,2)$-spectrum determines the linear combination of three-edge subgraph counts appearing in* (8.1). *In particular, for trees it determines*

$$6c_4(s(T) + r(T)) + 6c_{32}q(T) + 6c_{222}d_3(T)$$

up to the elementary terms depending only on $m = n - 1$.

8. What the moments say for trees

For a tree T on n vertices the formulas above become more transparent. Since $m = n - 1$ and $t(T) = 0$, the first moment is completely fixed by n. The second moment is equivalent to the number

$$p(T) = \sum_{v \in V(T)} \binom{d_v}{2},$$

of two-edge paths in T. This is already determined by the Laplacian spectrum, since the Laplacian spectrum determines $\sum_v d_v^2$. Thus the first genuinely new tree information begins at the cubic moment.

The two connected tree shapes with three edges are the claw $K_{1,3}$ and the path P_4. In a tree their numbers are

$$s(T) = \sum_{v \in V(T)} \binom{d_v}{3}, \qquad r(T) = \sum_{uv \in E(T)} (d_u - 1)(d_v - 1). \tag{8.3}$$

Here $s(T)$ counts three-edge stars and $r(T)$ counts length-three paths. The remaining disconnected three-edge shapes are a two-edge path together with an isolated edge, counted by $q(T)$, and a three-matching, counted by $d_3(T)$. More explicitly,

$$q(T) = \sum_{u-v-w} \big((n-1) - d_u - d_v - d_w + 2\big), \tag{8.4}$$

where the sum is over unordered two-edge paths $u - v - w$ in T, and

$$d_3(T) = \binom{n-1}{3} - s(T) - r(T) - q(T). \tag{8.5}$$

Consequently the third $(n-2,2)$-moment determines the explicit combination

$$(c_4 - c_{222})(s(T) + r(T)) + (c_{32} - c_{222})q(T), \tag{8.6}$$

up to a known function of n. Since

$$c_4 - c_{222} = 2n - 16, \qquad c_{32} - c_{222} = n - 9,$$

this may be written as

$$(2n - 16)(s(T) + r(T)) + (n - 9)q(T). \tag{8.7}$$

Thus the cubic moment sees a concrete mixture of local branching ($K_{1,3}$), length-three paths, and separated edge configurations.

The general trace formula gives a systematic continuation of this pattern. For each r, the moment $M_r^{(2)}(T)$ is a universal linear combination of embedded support-forest counts. Therefore the full $(n-2,2)$-spectrum gives a finite list of universal support-forest-counting constraints on T, namely the power sums

$$M_r^{(2)}(T), \qquad 1 \le r \le \dim W_n = \frac{n(n-3)}{2}.$$

Higher moments involve larger support forests and increasingly refined information about the arrangement of branches in T.

Definition 3. The $(n-2,2)$ *support-forest profile* of a tree T is the collection of universal support-forest-counting quantities obtained from the moments $M_r^{(2)}(T)$ for $1 \le r \le \dim W_n$.

The preceding discussion can be summarized as follows.

Proposition 5. *For trees, the $(n-2,2)$-spectrum determines the $(n-2,2)$ support-forest profile. The first nontrivial entry of this profile beyond Laplacian data is the cubic quantity (8.7), a concrete linear combination of the number of claws, the number of three-edge paths, and the number of two-edge paths separated from an edge. Equivalently, using the total number of three-edge subsets, it also constrains the number of three-matchings.*

9. Trees and graph isomorphism

Classically, cospectrality is abundant among trees; in particular Schwenk proved that almost all trees are adjacency-cospectral [9]. The motivating question here is whether the pair

$$\Big(\mathrm{Spec}_{(n-1,1)}(G), \mathrm{Spec}_{(n-2,2)}(G)\Big)$$

distinguishes trees. The trace formula makes this question much more concrete than a bare spectral conjecture. For a tree, the moments of $X_T|_{W_n}$ are not mysterious spectral quantities: they are universal linear combinations of embedded support-forest counts.

Thus the conjecture can be reformulated as a support-forest-counting assertion. The Laplacian spectrum gives the usual vertex-level spectral data, while the $(n-2,2)$-spectrum gives a hierarchy of edge-subtree and support-forest statistics. The first terms are as follows.

- $M_1^{(2)}(T)$ gives no information beyond n.
- $M_2^{(2)}(T)$ gives the number of two-edge paths $\sum_v \binom{d_v}{2}$, already determined by the Laplacian spectrum.
- $M_3^{(2)}(T)$ gives the explicit combination

  $$(2n-16)(s(T)+r(T)) + (n-9)q(T),$$

  where $s(T)$ counts claws, $r(T)$ counts length-three paths, and $q(T)$ counts a two-edge path separated from an isolated edge.
- In general, $M_r^{(2)}(T)$ gives a universal linear combination of embedded support forests with at most r edges.

This leads to the following more explicit version of the tree conjecture.

Conjecture 1 (Spectral support-forest-profile conjecture)**.** *Let T_1 and T_2 be trees on n vertices. If T_1 and T_2 have the same Laplacian spectrum and the same $(n-2,2)$ support-forest profile, then T_1 and T_2 are isomorphic. Equivalently, no two non-isomorphic trees have identical $(n-1,1)$- and $(n-2,2)$-spectra.*

A still stronger, but perhaps more approachable, form is the following.

Conjecture 2 (Subtree separation form)**.** *The universal support-forest-counting combinations arising from $M_r^{(2)}(T)$, together with the Laplacian spectrum, determine all embedded forest counts of T. In particular they distinguish trees.*

The second formulation is useful because it gives a concrete program: one can try to invert, degree by degree, the linear transformation from support-forest counts to trace moments. The cubic formula is the first example of this transformation. It is not yet invertible by itself, but it shows which statistics enter first. The fourth and fifth moments should involve all four-edge and five-edge tree shapes and may already give substantially stronger separation inside Laplacian-cospectral families.

10. Preliminary computations and Godsil–McKay switching

We briefly record two computational observations which support the usefulness of the $(n-2,2)$-spectrum as a refinement of the ordinary Laplacian spectrum. These computations are not meant to replace the structural conjectures above, but they indicate that the invariant is sensitive to phenomena invisible to classical spectra.

10.1. **Laplacian-cospectral trees.** We enumerated non-isomorphic trees on n vertices and grouped them by their ordinary Laplacian spectrum. Inside each Laplacian-cospectral class we then computed the spectrum of $X_T|_{W_n}$. In the tested range, every Laplacian-cospectral tree class was split by the $(n-2,2)$-spectrum.

n	number of trees	Laplacian-cospectral classes	trees in these classes	unresolved by $(n-2,2)$
$2 \le n \le 10$	–	0	0	0
11	235	3	6	0
12	551	3	6	0
13	1301	9	18	0
14	3159	15	30	0
15	7741	24	48	0

Thus, at least in this range, the pair

$$(\mathrm{Spec}(L_T), \mathrm{Spec}(X_T|_{W_n}))$$

distinguishes all trees. In terms of moments, this says that for every pair of Laplacian-cospectral trees found in this range, some universal support-forest-count combination coming from the traces $M_r^{(2)}(T)$ separates the pair. This gives a concrete computational form of the support-forest-profile conjecture.

10.2. **Godsil–McKay switching.** Godsil–McKay switching is a standard source of adjacency-cospectral graphs [5]; see also the enumeration work of Haemers and Spence [6]. The $(n-2,2)$-spectrum is not preserved by this operation in general. This is already true in a regular example, hence not merely a consequence of changed degrees.

Consider the following two 4-regular graphs on 10 vertices, written in the standard `graph6` format used by nauty/Traces [7]:

`Ir_GYkuy?`, `I]HTOYRRO`.

They are related by a Godsil–McKay switch. They are non-isomorphic and have the same adjacency spectrum. Since they are regular, they also have the same Laplacian spectrum. However their $(n-2,2)$-spectra are different. Equivalently, their traces agree for the first few low moments but eventually separate:

$$M_r^{(2)}(G) = M_r^{(2)}(G'), \qquad r = 1, 2, 3, 4, 5,$$

while

$$M_6^{(2)}(G) = 175466984, \qquad M_6^{(2)}(G') = 175467176.$$

This example is important conceptually. It shows that the new spectrum is not just another form of the adjacency or Laplacian spectrum. It can detect higher-order edge-interaction information which survives classical cospectral constructions.

Problem 1. *Classify Godsil–McKay switches which preserve the $(n-2,2)$-spectrum. More generally, determine which switching constructions preserve all moments $M_r^{(2)}$ and which are detected by the first non-vanishing difference.*

11. Relation with Thiéry's system of parameters

Thiéry [10] conjectured the existence of a low-degree homogeneous system of parameters for the invariant ring

$$I_n = \mathbb{C}[x_{ij} : 1 \le i < j \le n]^{S_n}.$$

His proposed parameters are built from two families: symmetric power sums in all edge variables x_{ij} and symmetric power sums in the star variables

$$X_i = \sum_{j \ne i} x_{ij}.$$

This exactly reflects the decomposition

$$V_n = (n-2, 2) \oplus \operatorname{span}\{E_1, \dots, E_n\}.$$

The present spectral construction does not by itself prove Thiéry's conjecture. However it suggests a complementary family of invariants: the coefficients of

$$\det\left(tI - X_G|_{W_n}\right).$$

These are polynomial S_n-invariants of degrees $1, \dots, \dim W_n$. They are not expected to generate the full invariant ring, but they give a natural finite collection of invariants attached to the missing component $(n-2, 2)$.

Problem 2. *Determine whether the union of*

- *the edge-count invariant,*
- *the ordinary Laplacian spectral invariants,*
- *the characteristic coefficients of $X_G|_{W_n}$,*

forms a system of parameters for I_n, or at least separates a large and natural class of graphs such as trees.

This problem is weaker than full graph isomorphism but strong enough to connect the present approach with invariant theory.

12. Outlook

Several questions remain open.

1. Compute explicit closed formulas for the fourth and higher moments, especially their support-forest-counting form for trees.
2. Determine exactly which classical graph statistics are determined by $\operatorname{Spec}_{(n-2,2)}(G)$ together with the Laplacian spectrum.
3. Compare $X_G|_{W_n}$ with edge Laplacians, line graph operators and Hodge-theoretic decompositions on graphs.
4. Study higher partitions such as
$$(n-3, 3), \qquad (n-3, 2, 1), \qquad (n-4, 4).$$
5. Investigate the parallel problem for spectra of Cayley graphs of S_n generated by transpositions, in the direction of [2, 1]. This is closely related but should be developed separately.

Acknowledgements. The author is grateful to Dmitrii Pasechnik for discussions of this topic about a decade ago.

References

[1] P. Caputo, T. M. Liggett and T. Richthammer, *Proof of Aldous' spectral gap conjecture*, J. Amer. Math. Soc. **23** (2010), no. 3, 831–851.

[2] F. Cesi, *On the eigenvalues of Cayley graphs on the symmetric group generated by a complete multipartite set of transpositions*, J. Algebraic Combin. **32** (2010), 155–185.

[3] P. Diaconis, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics Lecture Notes–Monograph Series, vol. 11, Institute of Mathematical Statistics, Hayward, CA, 1988.

[4] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, vol. 207, Springer, New York, 2001.

[5] C. D. Godsil and B. D. McKay, *Constructing cospectral graphs*, Aequationes Math. **25** (1982), 257–268.

[6] W. H. Haemers and E. Spence, *Enumeration of cospectral graphs*, European J. Combin. **25** (2004), no. 2, 199–211.

[7] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94–112.
[8] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Graduate Texts in Mathematics, vol. 203, Springer, New York, 2001.
[9] A. J. Schwenk, *Almost all trees are cospectral*, in: F. Harary (ed.), *New Directions in the Theory of Graphs*, Academic Press, New York, 1973, pp. 275–307.
[10] N. M. Thiéry, *Algebraic invariants of graphs; a study based on computer exploration*, ACM SIGSAM Bull. **34** (2000), no. 3, 9–20.
[11] E. R. van Dam and W. H. Haemers, *Which graphs are determined by their spectrum?*, Linear Algebra Appl. **373** (2003), 241–272.
[12] H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), no. 1, 150–168.

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
Email address: shapiro@math.su.se