Journal of Knot Theory and Its Ramifications

TOPOLOGICAL CLASSIFICATION OF GENERIC REAL RATIONAL FUNCTIONS

SERGEI NATANZON[*,1], BORIS SHAPIRO[†] AND ALEK VAINSHTEIN[‡]

[*] *Moscow State University and Independent University of Moscow, Russia*
[†] *Department of Mathematics, University of Stockholm, Stockholm, Sweden*
[‡] *Department of Mathematics and Department of Computer Science*
University of Haifa, Haifa, Israel 31905.

ABSTRACT
To any real rational function with generic ramification points we assign a combinatorial object, called a garden, which consists of a weighted labeled directed planar chord diagram and of a set of weighted rooted trees each corresponding to a face of the diagram. We prove that any garden corresponds to a generic real rational function, and that equivalent functions have equivalent gardens.

Keywords: Real rational functions; Generic ramification; Topological invariants; Hurwitz numbers.

1. Introduction

Let $f\colon P \to \bar{\mathbb{C}} = \mathbb{C} \cup \infty$ be a meromorphic function of degree n on a compact Riemann surface P of genus g. We say that f is a *generic* (complex) meromorphic function if the preimage $f^{-1}(z)$ of any point $z \in \bar{\mathbb{C}}$ consists of either n or $n-1$ points; equivalently, the singularities of f are of degree two, and at any two distinct singular points f takes distinct values. The points z for which $|f^{-1}(z)| = n-1$ are called *simple ramification points*. The set of all simple ramification points of f is denoted $\Sigma(f)$; by the Riemann–Hurwitz formula, it consists of $2n+2g-2$ points.

Two meromorphic functions $f_i\colon P_i \to \bar{\mathbb{C}}$ $(i = 1, 2)$ are called *equivalent* if there exists a biholomorphic map $\varphi\colon P_1 \to P_2$ such that $f_1 = f_2 \circ \varphi$. Let $\mathbb{C}\mathcal{H}_{g,n}$ be the set of equivalence classes of complex generic meromorphic functions of degree n on surfaces of genus g. The correspondence $f \mapsto \Sigma(f)$ generates a covering $\mathbb{C}\Phi_{g,n}\colon \mathbb{C}\mathcal{H}_{g,n} \to \mathbb{C}Q_{g,n}$, where $\mathbb{C}Q_{g,n}$ is the configuration space of all $(2n+2g-2)$-tuples of unordered distinct points on $\bar{\mathbb{C}}$, or, equivalently, the projectivized space of complex homogeneous degree $2n+2g-2$ polynomials in two variables without

[1]Partly supported by the INTAS grant 00-0259 and by the RFFI grant 01-01-00739.

multiple roots. We assume that $\mathbb{C}\mathcal{H}_{g,n}$ is provided with the weakest topology for which the map $\mathbb{C}\Phi_{g,n}$ is continuous. According to the Hurwitz theorem [Hu], $\mathbb{C}\mathcal{H}_{g,n}$ is a connected space. The degree $\mathbb{C}h_{g,n}$ of the covering $\mathbb{C}\Phi_{g,n}$, and its analogs for arbitrary meromorphic functions, are called the *Hurwitz numbers*. These numbers arise in many situations in mathematical physics; for example, they generate correlators of topological field theory, see [CMR]. In recent years they attracted much attention. For the case $g = 0$, the Hurwitz numbers are, in particular, calculated in [CT]:

$$\mathbb{C}h_{0,n} = \frac{n^{n-3}(2n-2)!}{n!};$$

in fact, this result was apparently known already to Hurwitz himself. For the case $g = 1$, the Hurwitz numbers are calculated in [GJV]:

$$\mathbb{C}h_{1,n} = \frac{1}{24}\left(n^n - n^{n-1} - \sum_{i=2}^{n}\binom{n}{i}(i-2)!n^{n-i}\right).$$

For certain classes of nongeneric rational functions, Hurwitz numbers where studied in [SSV, GL, ELSV]. The former paper exploits the classic approach due to Hurwitz, which links the numbers in question to the characters of the symmetric group. The approach developed in the other two papers is due to Arnold and is based on the singularity theory.

In the present paper we study a similar problem for real meromorphic functions. A real meromorphic function is defined on a *real algebraic curve*, which is a pair (P, τ), where P is a complex algebraic curve (a compact Riemann surface), and $\tau\colon P \to P$ is the antiholomorphic involution (the involution of complex conjugation). A *real meromorphic function* is a complex meromorphic function $f\colon P \to \bar{\mathbb{C}}$ such that $\overline{f(\tau p)} = f(p)$ for any $p \in P$, see e.g. [N2]. A real meromorphic function (P, τ, f) is said to be *generic* if (P, f) is a generic complex meromorphic function. Evidently, for any real meromorphic function f one has $\overline{\Sigma(f)} = \Sigma(f)$. Two real meromorphic functions (P_i, τ_i, f_i) $(i = 1, 2)$ are called *equivalent* if there exists a biholomorphic map $\varphi\colon P_1 \to P_2$ such that $f_1 = f_2 \circ \varphi$ and $\varphi \circ \tau_1 = \tau_2 \circ \varphi$. Let $\mathbb{R}\mathcal{H}_{g,n}$ denote the space of equivalence classes of generic real meromorphic functions of degree n on surfaces of genus g. The topology of $\mathbb{C}\mathcal{H}_{g,n}$ generates a topology on $\mathbb{R}\mathcal{H}_{g,n}$; in this topology $\mathbb{R}\mathcal{H}_{g,n}$ is not connected (see [N2]). The covering $\mathbb{C}\Phi_{g,n}$ generates a covering $\mathbb{R}\Phi_{g,n}\colon \mathbb{R}\mathcal{H}_{g,n} \to \mathbb{R}Q_{g,n}$, where $\mathbb{R}Q_{g,n}$ is the projectivized space of real homogeneous degree $2n + 2g - 2$ polynomials in two variables without multiple roots, see [N2].

In this paper we study connected components of $\mathbb{R}\mathcal{H}_{0,n}$; the points of this space are called equivalence classes of *generic real rational functions*. Allowing a slight abuse of language, we refer to the elements of $\mathbb{R}\mathcal{H}_{0,n}$ as generic real rational functions, and write $f \in \mathbb{R}\mathcal{H}_{0,n}$ meaning that the equivalence class of f belongs to $\mathbb{R}\mathcal{H}_{0,n}$. We define topological invariants that distinguish each connected component $H \subset \mathbb{R}\mathcal{H}_{0,n}$ and find the corresponding Hurwitz number $\mathbb{R}h_H$, that is, the degree of the restriction of $\mathbb{R}\Phi_{0,n}$ to H. For certain classes of nongeneric real rational functions Hurwitz numbers were studied in [Ar, Ba, Sh, SV1]. Note that interesting cell decompositions of the space of complex generic rational (and, more generally, meromorphic) functions were studied in [CP, BC].

The principal result of this note is as follows; see precise definitions in §2.

Main Theorem. *The set of all connected components of the space $\mathbb{R}\mathcal{H}_{0,n}$ is in a 1-1-correspondence with the set of the equivalence classes of all gardens of weight n.*

The structure of the paper is as follows. In §2 we introduce the notion of a garden and construct it for any given generic real rational function. In §3 we prove the main theorem as well as all necessary preliminary results for the calculation of Hurwitz numbers, which is carried out in §4. Finally, §5 contains some open questions and comments.

The authors are sincerely grateful to the Max-Planck Mathematical Institute in Bonn for the financial support and excellent research atmosphere during the fall of 2000 when this project was started. The second author wants to acknowledge the hospitality of IHES, Paris in January 2001 during the final stage of preparation of the manuscript.

2. Topological Invariants of Generic Real Rational Functions

The purpose of this section is to introduce a combinatorial object, which we assign to any generic real rational function. This object, called a *garden*, consists of a weighted labeled directed planar chord diagram and of a set of weighted rooted trees each corresponding to a face of the diagram.

2.1. Defining gardens abstractly. By a *planar chord diagram* (of *order* $2l$) we mean a circle drawn on the plane together with $2l$ points on this circle partitioned into l pairs in such a way that for any two pairs, the chords joining the points from the same pair do not intersect. The above $2l$ points are called the *vertices* of the chord diagram; the chords joining the vertices from the same pair, as well as the arcs of the circle joining adjacent vertices, are called the *edges*. Clearly, a planar chord diagram is a plane graph, so the notion of its *faces* is defined in a usual way (except for the outer face of the graph, which is not a face of the diagram). We say that a planar chord diagram is *directed* if its edges are directed in such a way that the boundary of each face becomes a directed cycle. Obviously, in order to direct a planar chord diagram it suffices to direct any one of its edges. Therefore there exist exactly two possible ways of directing a diagram, which are opposite to each other, i.e. the second one is obtained from the first one by reversing the direction of every edge. Once and for all fixing the standard orientation of the plane, we call a face of a directed planar chord diagram *positive* if the face lies to the left when we traverse its boundary according to the chosen direction, and *negative* otherwise. All neighbors of positive faces are negative, and vice verse.

A planar chord diagram is said to be *weighted* if each edge is equipped with a nonnegative integer weight, and *labeled* if there exists a bijection β (*labeling*) that takes the vertex set of the diagram to the set $\{1, 2, \dots, 2l\}$. Two labelings β_1 and β_2 are said to be *cyclically equivalent* if $\beta_1(v) - \beta_2(v) \mod 2l$ is a constant not depending on the choice of a vertex v.

Consider a labeled directed planar chord diagram. For any face j we denote by d_j the number of descents in the sequence of vertex labels ordered cyclically along the boundary of the face; clearly $d_j \geqslant 1$. If the diagram is also weighted, we denote by t_j the sum of d_j and the weights of all the edges along the boundary of the face j.

Recall that a *rooted tree* is a tree with one distinguished node called the *root*; all the other nodes of the tree are said to be *inner*. Given a rooted tree with the root r and two nodes u and v, we say that u is a *child* of v if the tree contains the edge (u, v), and if v lies on the unique path between u and r. We say that a rooted tree is *weighted* if each of its nodes is equipped with a positive integer weight.

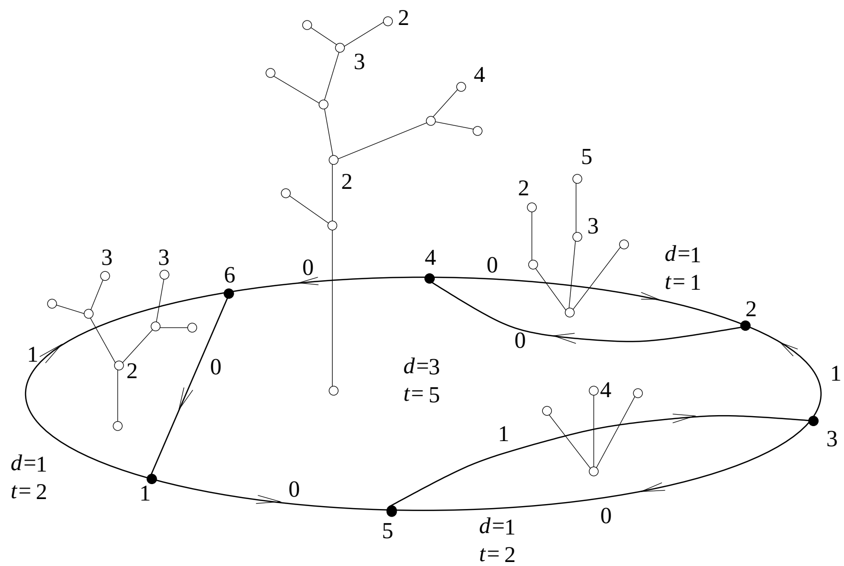

FIG. 1. A GARDEN OF ORDER 6 AND TOTAL WEIGHT 106

IMPORTANT DEFINITION. A *garden* is a weighted labeled directed planar chord diagram with a weighted rooted tree (possibly consisting just of its root) corresponding to each face of the diagram. The weights of the *inner* nodes of the trees are arbitrary positive integers, and the weight of the root of the tree corresponding to the face j equals t_j. The *total weight* of the garden equals twice the sum of the weights of all the inner nodes of all trees plus the sum of the weights of all roots.

An example of a garden is given on Fig. 1. The order of the diagram equals 6. The numbers written near the vertices and the edges are the labels and the weights, respectively. The weights of the nodes are equal to one, unless specified otherwise. The total weight of the garden equals 106.

Two gardens are said to be *equivalent* if there exists a bijection of the vertex sets of the corresponding chord diagrams that preserves chords, their orientation, labels (up to the cyclic equivalence), rooted trees, and weights.

2.2. Getting gardens from rational functions. To each function $f \in \mathbb{R}\mathcal{H}_{0,n}$ we associate the garden $G(f)$ as follows. First of all, represent $\Sigma = \Sigma(f)$ as $\Sigma = \Sigma_R \cup \Sigma_I$, where Σ_R is the set of real critical values of f (not necessary finite), and Σ_I is the set of its non-real critical values. Consider the preimage $S(f)$ of the real line $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ under f. Evidently, $S(f)$ contains $\bar{\mathbb{R}}$ and is invariant under the standard involution. All the critical points of f that correspond to critical values in Σ_R are real as well. Indeed, if x is a critical point with a real critical value, then $\bar{x}$ is a critical point with the same critical value; therefore, $x = \bar{x}$, since f is generic. A similar argument shows that the number of such critical points is even; we denote it $2l(\Sigma)$.

For each critical point as above, $S(f)$ contains exactly four arcs incident to it. Two of these arcs are the arcs of $\bar{\mathbb{R}} \subset S(f)$, while the other two interchange under the standard involution; in particular, the other endpoints of these two arcs coincide. Moreover, these arcs do not intersect outside $\bar{\mathbb{R}} \subset S(f)$, since such an intersection point would be a critical point with a real critical value. Therefore, these arcs together with $\bar{\mathbb{R}} \subset S(f)$ define a 2-dimensional cell complex on $\bar{\mathbb{C}}$. The 2-cells of this complex are called the *faces* of $S(f)$. Besides, $S(f)$ contains a number of closed curves called *ovals*. For the same reasons as above, no two ovals intersect, and each oval lies entirely inside one face. Observe that each face lies entirely in one of the two hemispheres $\bar{\mathbb{C}} \setminus \bar{\mathbb{R}}$; moreover, the image of a face under the standard involution is a face as well, and all the ovals lying inside the former face are mapped bijectively to the ovals lying inside the latter face.

To construct $G(f)$ we start from a planar chord diagram of order $2l(\Sigma)$. The vertices of the diagram correspond to the critical points with real critical values, and the chords correspond to the arcs of $S(f)$ lying in the upper hemisphere; thus, the faces of the diagram correspond to the faces of $S(f)$ lying in the upper hemisphere. The orientation of the edges is induced by the orientation of $\bar{\mathbb{R}}$ in the image. To define the labeling of the chord diagram, consider the natural linear order $<$ on Σ_R (if ∞ belongs to Σ_R, we assume that it is the biggest critical value). The label of a critical point equals the number of the corresponding critical value under this order. To define the weights, consider an arbitrary point $x \in \bar{\mathbb{R}} \setminus \Sigma_R$ and for any given arc (or oval) define $w(x)$ as the number of preimages of x lying on this arc (oval). The weight of the arc (oval) is then defined as the minimum of $w(x)$ over all $x \in \bar{\mathbb{R}} \setminus \Sigma_R$.

To construct the rooted tree corresponding to a given face we proceed inductively. The root of the tree corresponds to the boundary of the face; the inner vertices correspond to the ovals contained in the face under consideration. If there are no inner ovals, the tree consists only of its root. Otherwise, given an oval, the subtree rooted at the corresponding inner vertex contains exactly the vertices whose ovals lie inside the given oval. The weight of an inner vertex is equal to the weight of the corresponding oval. An example of a face and the corresponding rooted tree is given on Fig. 2.

Observe that if g belongs to the same equivalence class of generic real rational functions as f, then the garden constructed for g coincides with the one constructed for f.

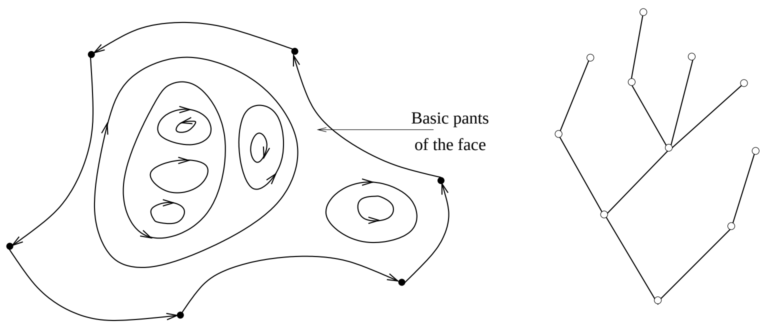

FIG. 2. A FACE AND THE CORRESPONDING ROOTED TREE

It is easy to see that the weight of a node coincides with the multiplicity of f restricted to the corresponding oval (or to the boundary of the corresponding face). Since the total preimage of $\bar{\mathbb{R}}$ under a chosen $f \in \mathbb{R}\mathcal{H}_{0,n}$ coincides with $S(f)$, the total weight of its garden $G(f)$ coincides with its degree and is therefore equal to n.

Given an abstract garden G we can substitute each of its trees by the appropriate system of weighted ovals, see Fig. 2. Such a garden will be called *represented*. In what follows we will freely use both abstract and represented gardens. The connected components of the complement to a represented garden are called *pants* (as before, we disregard the outer face). Each pants is a Riemann surface with a boundary consisting of a single outer boundary component and some number of inner boundary components. Note that we have assigned a certain weight to each connected component of the boundary of each pants. Pants with weights on each boundary component are called *weighted pants*. The chosen direction of the chord diagram of a garden G extends in a unique way to directions of all ovals such that every pants becomes either positive or negative, i.e. lie either to the left (if the pants are positive) or to the right (if the pants are negative) when we traverse any component of the boundary of these pants. The set of all weighted pants of a given garden G is called the *weighted pants collection* and denoted by $\Pi(G)$.

3. Realization Theorem and Connected Components of $\mathbb{R}\mathcal{H}_{0,n}$

The Main Theorem is obviously equivalent to the following pair of statements.

Theorem 1. *Let Σ be an arbitrary set of $2n-2$ distinct complex numbers invariant under the standard involution, of which exactly $2l$ are real. Any garden of order $2l$ and total weight n is isomorphic to the garden $G(f)$ for some real meromorphic function $f \in \mathbb{R}\mathcal{H}_{0,n}$ such that $\Sigma(f) = \Sigma$.*

Theorem 2. *Two rational functions belong to the same connected component of $\mathbb{R}\mathcal{H}_{0,n}$ if and only if they have equivalent gardens.*

Both proofs require a number of additional statements. The idea of the proof of Theorem 1 is to construct a real topological covering $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with a given garden and then, as usual in this field, to transform it into a holomorphic covering inducing the holomorphic structure on the preimage $\bar{\mathbb{C}}$ from that on the image $\bar{\mathbb{C}}$. The topological covering will be glued using branched coverings of a hemisphere by pants (we develop the appropriate technique below). The proof of Theorem 2 relies on the connectivity of the moduli spaces for the above branched coverings, cp. [N2].

3.1. On the space of branched covering of a hemisphere by a Riemann surface with a boundary. We start with some constructions. Denote by Λ^+ the upper hemisphere $\{z \in \bar{\mathbb{C}} \mid \operatorname{Im} z \geqslant 0\}$, and by P a genus g topological surface with a boundary consisting of k connected components. Consider the set $\mathcal{H}^k_{g,m}$ of all generic degree m branched coverings of the form $f \colon P \to \Lambda^+$. Let $a_1, \dots, a_k$ be all the distinct connected components of ∂P. Given a partition $(m_1, ..., m_k) \vdash m$, denote by $\mathcal{H}^k_{g,m}(m_1, \dots, m_k) \subset \mathcal{H}^k_{g,m}$ the subset of maps $f \colon P \to \Lambda^+$ such that $\deg f|_{a_i} = m_i$ for $i = 1, \dots, k$. Obviously,

$$\mathcal{H}^k_{g,m} = \bigcup_{(m_1,\dots,m_k)\vdash m} \mathcal{H}^k_{g,m}(m_1, \dots, m_k).$$

Let $\widetilde{P}$ be a compact genus g topological surface. Consider, in parallel, the set $\widetilde{\mathcal{H}}^k_{g,m}$ of all degree m branched coverings $\tilde{f} \colon \widetilde{P} \to \bar{\mathbb{C}}$ satisfying the additional condition that all the ramification points are concentrated on $\Lambda^+ \bigcup (-\mathfrak{i})$, where $\mathfrak{i}$ is the imaginary unit. Consider the obvious restriction map $\Psi \colon \widetilde{\mathcal{H}}^k_{g,m} \to \mathcal{H}^k_{g,m}$, i.e. $\Psi(\tilde{f} \colon \widetilde{P} \to \bar{C}) = (f \colon P \to \Lambda^+)$, where $P = \tilde{f}^{-1}(\Lambda^+)$ and $f = \tilde{f}|_P$. According to [N2, Theorem 4.1], there exist unique complex structures on P and $\tilde{P}$ for which the above mentioned branching coverings are holomorphic. Thus we can consider $\mathcal{H}^k_{g,m}$ and $\widetilde{\mathcal{H}}^k_{g,m}$ as spaces of meromorphic functions.

Lemma 1. *The map Ψ is a bijection; moreover, two maps f_1 and f_2 in $\mathcal{H}^k_{g,m}$ are equivalent if and only if their images $\Psi(f_1)$ and $\Psi(f_2)$ are equivalent.*

Proof. Denote $\Lambda^- = \{z \in \bar{C} \mid \operatorname{Im} z \leqslant 0\}$ and fix a holomorphic degree j map $\xi_j \colon \Lambda^- \to \Lambda^-$ preserving $-\mathfrak{i}$ and having no other ramification points on Λ^- (such a ξ_j obviously exists). Take now an arbitrary function $f \in \mathcal{H}^k_{g,m}$. It is always possible to identify each a_i with $\partial\Lambda^-$ in such a way that $\xi_{m_i}|_{\partial\Lambda^-} = f|_{a_i}$. Glueing copies of Λ^- to all holes in P gives a surface $\widetilde{P}$ without a boundary. At the same time, glueing f and ξ_{m_i}'s together gives a new function $\tilde{f} \in \widetilde{\mathcal{H}}^k_{g,m}$. Obviously, $\Psi(\tilde{f}) = f$, and moreover, this construction sends equivalent functions to equivalent functions. $\square$

Lemma 2. *For any partition $(m_1, ..., m_k) \vdash m$, the space $\mathcal{H}^k_{g,m}(m_1, \dots, m_k)$ is connected.*

Proof. Define the corresponding set $\widetilde{\mathcal{H}}^k_{g,m}(m_1, \dots, m_k) \subset \widetilde{\mathcal{H}}^k_{g,m}$ with $\tilde{f}^{-1}(-\mathfrak{i})$ consisting of a k-tuple of critical points $a_1, \dots, a_k$ with $\deg \tilde{f}|_{a_i} = m_i$. According to [N1, N2], the set $\widetilde{\mathcal{H}}^k_{g,m}(m_1, \dots, m_k)$ is connected. Therefore, by Lemma 1 one gets that the set $\mathcal{H}^k_{g,m}(m_1, \dots, m_k) \subset \mathcal{H}^k_{g,m}$ is connected as well. (The simplest case of this result is called the Lüroth–Clebsch theorem, see e.g. [Hu, Kl].) $\square$

Lemma 3. *With the above notation, any $f \in \mathcal{H}^k_{g,m}(m_1, \dots, m_k)$ has exactly $m + k + 2g - 2$ simple ramification points on $\Lambda^+ \setminus \partial\Lambda^+$.*

Proof. Follows directly from the Riemann–Hurwitz formula

$$\chi(P) + \sharp_f = \chi(\Lambda^+) \deg f,$$

where $\sharp_f$ is the number of simple ramification points of f, $\deg f = m$ is the degree of f, $\chi(P) = 2 - 2g - k$ is the Euler characteristic of P, and $\chi(\Lambda^+) = 1$ is the Euler characteristic of Λ^+. $\square$

Lemma 4. *Given a genus 0 surface P with k boundary components and a partition $(m_1, ..., m_k) \vdash m \geqslant 3$, the Hurwitz number of $\mathcal{H}^k_{0,m}(m_1, \dots, m_k)$ equals*

$$\frac{m^{k-3}(m+k-2)! m_1^{m_1} \dots m_k^{m_k}}{m_1! \dots m_k! s(m_1, \dots, m_k)},$$

where $s(m_1, \dots, m_k)$ is the number of symmetries of the set $\{m_1, \dots, m_k\}$.

Proof. It follows immediately from Lemma 1 that the Hurwitz numbers for the spaces $\widetilde{\mathcal{H}}^k_{0,m}(m_1, \dots, m_k)$ and $\mathcal{H}^k_{0,m}(m_1, \dots, m_k)$ coincide. For the former space, this number equals $\frac{m^{k-3}(m+k-2)! m_1^{m_1} \dots m_k^{m_k}}{m_1! \dots m_k! s(m_1, \dots, m_k)}$ and was apparently known to Hurwitz, see [GJ, St]. $\square$

Remark. It is easy to see that for $m = 2$ and $k = 1, 2$ the Hurwitz number of $\mathcal{H}^k_{0,m}(m_1, \dots, m_k)$ equals 1, while the expression in Lemma 4 gives 1/2. On the other hand, for $m = k = 1$, the expression gives 1, which is the correct answer.

3.2. Proof of Theorem 1. Given a set Σ of $2n-2$ distinct complex numbers of which exactly $2l$ are real, and a garden G of order $2l$ with total weight n, we want to construct a topological branched covering $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ invariant under complex conjugation whose set of ramification points coincides with Σ, and whose garden is isomorphic to G. This will prove the Theorem, since by [N2, Theorem 4.1], there exists a unique complex structure on $\bar{\mathbb{C}}$ for which this topological covering is holomorphic.

Consider G as a represented garden, and let $\Pi(G) = \bigcup_{i=1}^q P_i$ denote the weighted pants collection of G, see §2. In order to construct a required topological branched covering $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$, we perform the following four steps.

Step 1. Distribute $2l$ real numbers from Σ_R between the vertices of G, as described in the construction of $G(f)$ in §2.2.

Step 2. Distribute $n-l-1$ complex conjugate pairs of numbers from Σ_I between all pants in $\Pi(G)$. Let $P_i \in \Pi(G)$ be pants with c_i boundary components, and let m_{ij} be the weight of the jth boundary component. According to Lemma 3, we assign to P_i exactly $\sum_{j=1}^{c_i} m_{ij} + c_i - 2$ pairs from Σ_I.

Step 3. For any pants $P_i \in \Pi(G)$ build a map $f_i \colon P_i \to \Lambda^{\pm}$ with prescribed ramification points. If P_i are positive, then f_i belongs to the space $\mathcal{H}^{c_i}_{0,\mu_i}(m_{i,1}, ..., m_{i,c_i})$, where $\mu_i = \sum_j m_{ij}$; it maps P_i to Λ^+, and the ramification points are chosen as follows: from the each conjugate pair assigned to P_i on the previous step we take the point belonging to Λ^+. If P_i are negative, then f_i maps it to Λ^-, and the ramification points are chosen in a similar way in Λ^-.

STEP 4. Glue all f_i's together to get a map of the hemisphere containing G to $\bar{\mathbb{C}}$ and, finally, glue the latter map with its complex conjugate copy along the boundary of the hemisphere to get the actual branched covering $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$.

Let us explain the fourth step in detail. Consider first the case $l = 0$. Taking the unique pants, called *basic*, whose outer boundary is the circle of G (identified with $\bar{\mathbb{R}}$ in the preimage), we glue to its map the maps of all its neighboring pants by identifying these maps along their common boundary ovals. Since by our construction the multiplicities of two maps having a common oval coincide on this oval, the glueing process is possible (a similar procedure is used in the proof of Lemma 1). Having glued the maps of all the neighbors to that of the basic pants, we continue with the neighbors of the neighbors, etc.

In the general case, notice that each face r, $r = 1, \dots, l+1$, contains the unique pants (called *basic* for r) whose boundary coincides with that of the face r, see Fig 2. We can first glue together the maps of all basic pants and then continue as above. The maps of a neighboring pair of basic pants are glued together along their unique common arc which should be mapped to the prescribed segment of $\bar{\mathbb{R}}$ in the image $\bar{\mathbb{C}}$ between the corresponding real ramification points, i.e. those labeling the endpoints of the arc under consideration (the labels are obtained on Step 1). Observe that these ramification points are regular points for each of f_i's, and become critical points only after glueing basic pants together. The weight of the arc defines the number of complete turns which this arc should do around $\bar{\mathbb{R}}$ in the image, and its direction shows the orientation of the image of the arc. Thus the image of the arc is completely determined by G.

Having glued all f_i's together, we get a map f from the disc containing G (identified with the upper hemisphere) to $\bar{\mathbb{C}}$. We take another copy of this disc (identified with the lower hemisphere) with the conjugate map $\bar{f}$, and glue two hemispheres along $\bar{\mathbb{R}}$ into a sphere $\bar{\mathbb{C}}$ with the final map $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ consisting of f and $\bar{f}$.

One can easily see that the final map is the topological branched covering with all properties required by Theorem 1, and we are done.

3.3. Connected components of $\mathbb{R}\mathcal{H}_{0,n}$. The following statement is crucial for the proof of Theorem 2.

Lemma 5. *Two generic real rational functions $f_i \colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $i = 0, 1$, are equivalent if and only if*

a) their gardens $G(f_0)$ and $G(f_1)$ are isomorphic, i.e. there exists a bijection of the vertex sets of $G(f_0)$ and $G(f_1)$ that preserves chords, their orientation, labels, rooted trees, and their weights;

b) the restrictions of f_0 and f_1 to each pair of pants identified by the above isomorphism of gardens are equivalent. In particular, the sets of complex critical values assigned to each pair of pants identified by the above isomorphism of gardens coincide.

Proof. Obviously, if f_0 and f_1 are equivalent then a)-b) are automatically satisfied. On the other hand, using condition b) we can construct, for each pair of pants identified by the above isomorphism of gardens, a homeomorphism making the restrictions of f_0 and f_1 to these pants equivalent. Now using a) we can glue together these homeomorphisms defined on pairs of pants into a global homeomorphism $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ making f_0 and f_1 equivalent. As usual, the constructed homeomorphism provides a biholomorphic map by inducing the complex structure on the preimage $\bar{\mathbb{C}}$ from that on the image $\bar{\mathbb{C}}$. □

Proof of Theorem 2. Let us show the easy implication first. Assume that two rational functions f_0 and f_1 belong to the same connected component of $\mathbb{R}\mathcal{H}_{0,n}$. Let us show that $G(f_0)$ is equivalent to $G(f_1)$. Take some path $f_t \subset \mathbb{R}\mathcal{H}_{0,n}$, $t \in [0,1]$, connecting f_0 and f_1. The only data related to the gardens of real rational functions which can vary along f_t (up to a diffeomorphism of $\bar{\mathbb{C}}$ invariant under complex conjugation) are the values of ramification points. But since they never collide, one gets that the complex ramification points remain complex, the real ramification points remain real, and can only experience a cyclic shift. Thus, $G(f_0)$ is equivalent to $G(f_1)$.

Conversely, take two functions f_0 and f_1 in $\mathbb{R}\mathcal{H}_{0,n}$ whose gardens $G_0 = G(f_0)$ and $G_1 = G(f_1)$ are equivalent. Notice that the equivalence of G_0 and G_1 implies the 1-1-correspondence between the sets $\Pi(G_0)$ and $\Pi(G_1)$ of the weighted pants collections, i.e. the existence of a 1-1-correspondence between pants for f_0 and f_1.

Let Σ_0 and Σ_1 denote the sets of ramification points of f_0 and f_1, respectively.

The equivalence of G_0 and G_1 implies that Σ_0 and Σ_1 belong to the same connected component of $\mathbb{R}Q_{0,n}$. Moreover, we can connect Σ_0 and Σ_1 by a path Σ_t, $t = [0,1]$, in this component in such a way that for any i, the subset of Σ_0 corresponding to the ith pants in $\Pi(G_0)$ will be transformed along Σ_t into the subset of Σ_1 corresponding to the ith pants in $\Pi(G_1)$. Using the covering homotopy property of $\mathbb{R}\Phi_{0,n}$ (see [N2]) over the path Σ_t, we get another rational map $\tilde{f}_1$ which lies in the same connected component of $\mathbb{R}\mathcal{H}_{0,n}$ as f_0; therefore, the garden $\widetilde{G}_1$ of $\tilde{f}_1$ is equivalent to G_0 (and hence to G_1) by the first part of this proof. Moreover, the set of ramification points of $\tilde{f}_1$ coincides with Σ_1, and the distributions of ramification points among pants for f_1 and $\tilde{f}_1$ are identical.

Now for each pants from $P_i(\widetilde{G}_1)$ we can, using the connectivity of the space of maps proved in Lemma 2, find a path between the restriction of $\tilde{f}_1$ to these pants and the restriction of f_1 to the corresponding pants from $\Pi(G_1)$ that keeps the restrictions of f_1 and $\tilde{f}_1$ to all other pants unchanged. Doing this procedure for every pants we connect $\tilde{f}_1$ with a map $\bar{f}_1$, which together with f_1 satisfies all the conditions of Lemma 5. Therefore, $\bar{f}_1$ is equivalent to f_1, and is connected with f_0 by a path in $\mathbb{R}\mathcal{H}_{0,n}$, hence f_0 and f_1 belong to the same connected component of $\mathbb{R}\mathcal{H}_{0,n}$. □

4. Hurwitz Numbers

To find out the number of nonequivalent functions corresponding to the same garden, consider a weighted rooted tree T. The node set of T is $\{0, 1, \dots, k\}$ for some $k \geqslant 0$, and 0 is the root of T. The weight of node i is denoted w_i. Let i be an arbitrary node, and $i_1, \dots, i_c$ be all of its children. The number of children is called the *degree* of the node i and is denoted c_i (see Figure 3).

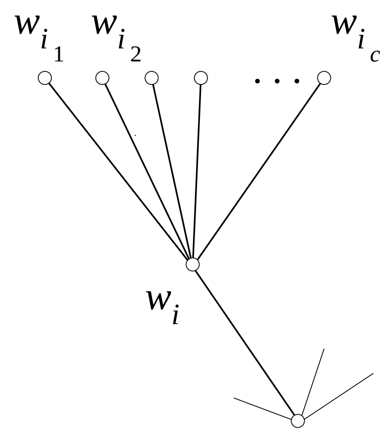

FIG. 3. A NODE AND ITS CHILDREN

Besides, we define the *modified weight* $\widetilde{w}_i$ as the sum of the weight of i and the weights of all of its children. The *total weight* w_T of the tree T is equal to the sum of the modified weights of all of the nodes, including the root. Finally, we define the *symmetry factor* s_i as the number of automorphisms of the set $\{w_i, w_{i_1}, \dots, w_{i_{c_i}}\}$. The *Hurwitz number* H_T of the tree T is defined by

$$H_T = \frac{(w_T - 1)! w_0! 2^{e_T}}{w_0^{w_0}} \prod_{i=0}^{k} \frac{w_i^{2w_i} \widetilde{w}_i^{c_i - 2}}{(w_i!)^2 s_i},$$

where e_T is the number of nodes in T whose modified weight equals 2.

Assume first that the set Σ does not contain real numbers, thus $l(\Sigma) = 0$. The garden $G(f)$ for an arbitrary function f such that $\Sigma(f) = \Sigma$ has order 0 and consists of a trivial chordless chord diagram and a single tree. Such a garden we call an *imaginary* garden. Observe that the total weight of an imaginary garden equals the total weight of its single tree.

Theorem 6. *Let Σ be an arbitrary set of $2n-2$ distinct complex numbers invariant under the standard involution and containing no real numbers. Let G be an imaginary garden of total weight n and T be its single tree. The number of topologically nonequivalent functions $f \in \mathbb{R}\mathcal{H}_{0,n}$ such that $\Sigma(f) = \Sigma$ and $G(f) = G$ is equal to the Hurwitz number H_T.*

Proof. By Lemma 5, we have to calculate the number of ways to execute Steps 2 and 3 in the proof of Theorem 1. First, we distribute the elements of Σ over the pants defined by T. By Lemma 3, the number of ramification points corresponding to the pants P_i equals $\widetilde{w}_i + c_i - 1$. The total number of points to be distributed equals

$$\sum_{i=0}^{k}(\widetilde{w}_i + c_i - 1) = w_T + \sum_{i=0}^{k} c_i - k - 1 = n - 1,$$

since the total weight of T equals n and $\sum_{i=0}^{k} c_i$ is the number of inner nodes in T. Therefore, the total number of the ways to execute Step 2 equals

$$\frac{(n-1)!}{\prod_{i=0}^{k}(\widetilde{w}_i + c_i - 1)}.$$

The number of ways to execute Step 3 is described in Lemma 4 and the Remark following the lemma. Therefore, the total number in question equals

$$\frac{(n-1)!}{\prod_{i=0}^{k}(\widetilde{w}_i + c_i - 1)} \prod_{i=0}^{k} \frac{\widetilde{w}_i^{c_i-2}(\widetilde{w}_i + c_i - 1)! w_i^{w_i} w_{i_1}^{w_{i_1}} \dots w_{i_c}^{w_{i_c}}}{s_i w_i! w_{i_1}! \dots w_{i_{c_i}}!} \cdot 2^{e_T}.$$

The expression $w_i^{w_i}/w_i!$ for a given inner node i enters the latter product twice: once when the node itself is considered, and once more when the node appears as a child of its parent node. After cancellations we get the desired result. $\square$

In the general case, let G be a garden of order $2l$ and total weight w, and $T_1, \dots, T_{l+1}$ be its trees. The Hurwitz number of the garden is defined by

$$\begin{aligned} H_G &= (w - l - 1)! \prod_{i=1}^{l+1} \frac{H_{T_i}}{(w_{T_i} - 1)!} \\ &= 2^{e_G}(w - l - 1)! \prod_{i \in R_G} \frac{w_i!}{w_i^{w_i}} \prod_{i \in N_G} \frac{w_i^{2w_i} \widetilde{w}_i^{c_i-2}}{(w_i!)^2 s_i}, \end{aligned}$$

where N_G is the set of the nodes of the trees $T_1, \dots, T_{l+1}$, R_G is the set of the roots of these trees, and e_G is the number of nodes in N_G whose modified weight equals 2.

The following proposition follows easily from Lemma 5 similarly to Theorem 6.

Theorem 7. *Let Σ be an arbitrary set of $2n-2$ distinct complex numbers invariant under the standard involution, of which exactly $2l(\Sigma)$ are real, and let G be a garden of order $2l(\Sigma)$ and total weight n. The number of topologically nonequivalent functions $f \in \mathbb{R}\mathcal{H}_{0,n}$ such that $\Sigma(f) = \Sigma$ and $G(f) = G$ is equal to the Hurwitz number H_G.*

5. Final Remarks

In this note we assigned to each connected component in the space $R\mathcal{H}_{0,n}$ a combinatorial object called a garden. Unfortunately, to count the total number of all gardens of a given weight n seems to be a difficult problem. Two cases look more accessible, namely, the *elliptic* case when no critical values are real, and the *hyperbolic* case when all critical values are real.

Problem. Count the number of connected components in the space of all hyperbolic and elliptic generic functions of degree n.

In the hyperbolic case, the major combinatorial difficulty is to count the total number of admissible labelings of a given planar chord diagram with $2n-2$ vertices, see [SV2]. In the elliptic case, one should count the total number of nonisomorphic planar trees with total weight n.

References

[Ar] V. Arnold, *Topological classification of real trigonometric polynomials and cyclic serpents polyhedron*, The Arnold-Gelfand mathematical seminars, Birkhäuser, Boston, MA, 1997, pp. 101–106.

[Ba] S. Barannikov, *The space of real polynomials without multiple critical values*, Functional Anal. Appl. **26** (1992), 84–90.

[BC] I. Bauer and F. Catanese, *Generic lemniscates of algebraic functions*, Math. Ann **307** (1997), 417–444.

[CP] F. Catanese and M. Paluszny, *Polynomial–lemniscates, trees and braids*, Topology **30** (1991), 623–640.

[CMR] S. Cordes, G. Moore, and S. Ramgoolam, *Large N $2D$ Yang–Mills theory and topological string theory*, Comm. Math. Phys. **185** (1997), 543–619.

[CT] M. Crescimanno and W. Taylor, *Large N phases of chiral QCD_2*, Nuclear Phys. B **437** (1995), 3–24.

[ELSV] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), 297–327.

[GJ] I. Goulden, and D. Jackson, *Transitive factorizations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc **125** (1997), no. 1, 51–60.

[GJV] I. Goulden, D. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Comb. **4** (2000), 27–46.

[GL] V. Goryunov and S. K. Lando, *On enumeration of meromorphic functions on the line*, The Arnoldfest, Fields Inst. Commun., vol. 24, AMS, Providence, RI, 1999, pp. 209–223.

[Hu] A. Hurwitz, *Uber Riemannsche Flächen mit gegeben Verzweigunspuncten*, Math. Ann. **38** (1891), 1–61.

[Kl] P. Kluitmann, *Hurwitz action and finite quotients of braid groups*, Braids (Santa Cruz, CA, 1986), Contemp. Math, vol. 78, 1988, pp. 299–325.

[N1] S. Natanzon, *Spaces of real meromorphic functions on real algebraic curves*, Soviet Math. Dokl. **30** (1984), 724–726.

[N2] S. Natanzon, *Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves*, Selecta Math. Sovietica **12** (1993), 251–291.

[Sh] B. Shapiro, *On the number of components of the space of trigonometric polynomials of degree n with $2n$ distinct critical values*, Math. Notes **62** (1997), 529–534.

[SV1] B. Shapiro and A. Vainshtein, *On the number of components in the space of M-polynomials in hyperbolic functions*, Proc. 13th Conf. Formal Power Series and Algebraic Combinatorics (FPSAC), 2001, pp. 453–460.

[SV2] B. Shapiro and A. Vainshtein, *Counting rational functions with real critical values*, (in preparation).

[SSV] B. Shapiro, M. Shapiro, and A. Vainshtein, *Ramified coverings of S^2 with one degenerate branching point and enumeration of edge-ordered graphs*, Advances in Mathematical Sciences, vol. 34, AMS, Providence, RI, 1997, pp. 219–228.

[St] V. Strehl, *Minimal transitive products of transpositions—the reconstruction of a proof of A. Hurwitz*, Sem. Lothar. Combin. **37** (1996), S37c.