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The M-property of flag varieties

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Abstract

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We consider arbitrary "generic" arrangements of high-dimensional Schubert cells in the series of real flag varieties PT^*P^n and prove the coincidence of the sum of Betti numbers of these arrangements with the sum of Betti numbers of their complexifications. We give an example of the violation of this property in the case of arrangements in $G_{2,4}$. We also prove degeneration of the Mayer–Vietoris spectral sequence in the E_1 term for some class of configurational spaces.

Keywords: Flag variety, Schubert cell decomposition, arrangements in general position, M-property, Mayer–Vietoris spectral sequence.

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1. Introduction

The well-known Smith inequality implies that for an arbitrary real algebraic variety $X^{\mathbb{R}}$ we have $\sum b_i(X^{\mathbb{R}}) \leqslant \sum b_i(X^{\mathbb{C}})$, where b_i denotes the ith Betti number with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and $X^{\mathbb{C}}$ denotes the complexification of $X^{\mathbb{R}}$ (see for example [2]).

In the particular case of a planar real algebraic curve this inequality is called Harnack's inequality and the planar curves for which Harnack's inequality is in fact the equality are called M-*curves*. M-curves were studied by several authors (see [10, 8, 5, 7]).

1.1. *The main definition*

A real algebraic variety $X^{\mathbb{R}}$ (the set of real points of $X^{\mathbb{C}}$) is called an M-*manifold* if $\sum b_i(X^{\mathbb{R}}) = \sum b_i(X^{\mathbb{C}})$. (We shall also say that in this case $X^{\mathbb{R}}$ has the M-property.)

There are several articles by authors from the Leningrad and Gorky schools about M-surfaces (see [4, 12]).

In the beginning of 70's it was found that several configuration spaces have the M-property. For example in the paper [6] by Orlik and Solomon it was proved that the complement of an arbitrary arrangement of real hyperplanes has the M-property (see also [6, 11]).

In this paper we will consider other configuration spaces and establish the M-property for another series of manifolds. In this sense our work is the development of [6].

1.2. The main results

Definition. Two (incomplete, in general) flags in P^n are called *transversal* if the intersection of any pair of their subspaces has the minimal possible dimension.

Let PT^*P^n denote the manifold of all flags in P^n consisting of a hyperplane and a distinguished point in it. (Notice that two flags belonging to PT^*P^n are transversal if it holds for both that a distinguished point of one does not belong to the hypersurface of the other.)

Hereinafter the term "*the set of flags in general position*" means that it belongs to some open dense domain in the space of all sets.

Theorem A. *The locus of all flags from* PT^*P^n *which are transversal to each flag from a given set in general position has the* M-*property* (*see Section* 2).

Theorem B. *There exists an open set of* 4-*tuples of real lines in* P^3 (*flags in* F^4) *such that the corresponding locus of all lines* (*flags*) *transversal to all lines* (*flags*) *from the given set violates the* M-*property.*

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2. The M-property of PT^*P^n

This section contains the proof of Theorem A.

Definition 2.1. The train Tn_f of a flag f is the locus of all flags nontransversal to f. If $\hat{f}$ is an arbitrary set $\hat{f}=\{f_1,\dots,f_k\}$ of flags in P^n, then we denote

$$\mathrm{Tn}_{\hat{f}}=\bigcup_{f_i\in\hat{f}}\mathrm{Tn}_{f_i}.$$

Definition 2.2. Let V_1, V_2 be two n-dimensional vector spaces. The subspace $L\subset V_1\oplus V_2$ is called *decomposable* if $L=L_1\oplus L_2$, where $L_1\subset V_1$, $L_2\subset V_2$. We denote by $p_1: V_1\oplus V_2\to V_1$; by $p_2: V_1\oplus V_2\to V_2$ projections onto the first and the second summand, respectively. Let $\Phi: V_1\times V_2\to\mathbb{R}$ $(\mathbb{C})$ be a pairing (possibly degenerate). With the pairing $\Phi: V_1 \times V_2 \to \mathbb{R}\,(\mathbb{C})$ we associate a quadratic form $\bar{\Phi}(V_1 \oplus V_2) \to \mathbb{R}\,(\mathbb{C})$ such that $\bar{\Phi}(v) = \Phi(p_1 v, p_2 v)$.

Lemma 2.3. *Let V_1, V_2 be n-dimensional linear spaces; $\Phi: V_1 \times V_2 \to \mathbb{R}$ a pairing and $\bar{\Phi}(V_1 \oplus V_2) \to \mathbb{R}$ the associated quadratic form. If L is decomposable, then the restriction $\bar{\Phi}|_L$ has zero signature (the number of positive squares minus the number of negative squares vanishes).*

Proof. For any pairing Φ we can choose appropriate coordinates x_i on V_1 and y_i on V_2 so that for any

$$v = (x_1, \ldots, x_n), \qquad w = (y_1, \ldots, y_n),$$

$$\Phi(v, w) = \sum_{i=1}^{r} x_i y_i,$$

where $r = \operatorname{rank} \Phi \leq n$.

The signature of the quadratic form $\hat{\Phi} = x_1 y_1 + \cdots + x_r y_r$ equals zero. For any decomposable vector space L the restriction $\Phi|_L$ is also a pairing and $\bar{\Phi}|_L = \overline{(\Phi|_L)}$. □

Definition 2.4. Let $\hat{\mathbb{R}}^n$ ($\hat{\mathbb{C}}^n$) denote the one-point compactification of $\mathbb{R}^n$ ($\mathbb{C}^n$), Φ an arbitrary real or complex valued function on $\mathbb{R}^n$ ($\mathbb{C}^n$) and V the variety of solutions $\{\Phi(x) = 0\}$.

The subvariety $\hat{V} = V \cup \infty$ in $\hat{\mathbb{R}}^n$ ($\hat{\mathbb{C}}^n$) is the one-point compactification of V.

The aim of this section is to study the relative mod ∞ homology of the variety $\hat{V}$ for the nonhomogeneous quadratic form $\bar{\Phi}$ associated with a pairing Φ (see Definitions 2.1 and 2.2 and Lemma 2.3). The following lemma is obvious.

Lemma 2.5. *The nonhomogeneous quadric function associated with a pairing $\Phi: V_1 \times V_2 \to \mathbb{R}\,(\mathbb{C})$ can be transformed by the* Aff$(V_1) \times$ Aff(V_2)*-action into one of the following:*

(A) $\bar{\Phi} = l(x_i, y_i)$,

(1) $\bar{\Phi} = l(x_i, y_i) + \sum_2^m (x_i \cdot y_i), \quad m \leq n,$ (*)

(B) $\bar{\Phi} = 0$,

(2) $\bar{\Phi} = \sum_1^m (x_i \cdot y_i) - 1, \quad m \leq n,$ (**)

(C) $\bar{\Phi} = 1$,

(3) $\bar{\Phi} = \sum_1^m (x_i \cdot y_i), \quad m \leq n,$ (***)

where $l(x_i, y_i)$ is a nonconstant linear function.

Table 1

	(A)	(B)	(C)	(1)	(2)	(3)
R	t^{2n-1}	t^{2n}	0	t^{2n-1}	$t^{2n-1}+t^{2n-m}$	$t^{2n-1}+2t$
C	t^{4n-2}	t^{4n}	0	t^{4n-2}	$t^{4n-2}+t^{4n-2m-1}$	$t^{4n-2}+t^{4n-2m}+t$

Lemma 2.6. *The Poincare polynomials for the homology of the pair* $(\hat{V}, \mathrm{mod}\,\infty)$ *are listed in Table* 1.

Proof. We start with the space $\mathbf{R}^{2n}$. The cases (A), (B), (C), (1) are obvious, e.g.

(1) $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to $(S^{2n-1}, \mathrm{mod(point)})$;

(2) $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the Thom space of a $(2n-m)$-dimensional vector bundle over S^{m-1}.

(3) The third case splits into two subcases:

(3a) $m=n$;

(3b) $m<n$.

Case (3a). Since V is homogeneous $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the suspension mod(point) over its intersection with the unit sphere.

This intersection is given by the system:

$$\begin{cases} \sum_{i=1}^{n} x_i^2 = 1/2, \\ \\ \sum_{i=1}^{n} y_i^2 = 1/2. \end{cases}$$

Hence the intersection is homeomorphic to $S^{n-1}\times S^{n-1}$ and $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the suspension mod(point) over $S^{n-1}\times S^{n-1}$.

Case (3b). $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the Thom space mod(point) of the $2(n-m)$-dimensional bundle over $S^{m-1}\times S^{m-1}$ whose fiber over one point is glued by a cell.

Now let us pass to the space $\hat{\mathbf{C}}_{2n}$.

(1) $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the sphere S^{4n-2} (mod(point)).

To study the other cases we must decompose the left sides of (**) and (***) into real and imaginary parts:

$$\begin{cases} \sum ((\mathrm{Re}\, x_i)^2 + (\mathrm{Im}\, y_i)^2 - (\mathrm{Im}\, x_i)^2 - (\mathrm{Re}\, y_i)^2), \\ 2\sum ((\mathrm{Re}\, x_i)(\mathrm{Im}\, x_i) - (\mathrm{Re}\, y_i)(\mathrm{Im}\, y_i). \end{cases}$$

Substituting $\bar{y}$ instead of y we obtain:

$$\begin{cases} \sum ((\mathrm{Re}\, x_i)^2 + (\mathrm{Im}\, y_i)^2 - (\mathrm{Im}\, x_i)^2 - (\mathrm{Re}\, y_i)^2), \\ 2\sum ((\mathrm{Re}\, x_i)(\mathrm{Im}\, x_i) + (\mathrm{Re}\, y_i)(\mathrm{Im}\, y_i)). \end{cases}$$

Thus in case (2) we see that $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the Thom space (mod(point)) of the $(4n-2m)$-dimensional bundle over S^{2m-1}.

Case (3) splits, as before, into two subcases.

Case (3a). $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the suspension mod(point) over the spheric bundle STS^{2n-1} associated with TS^{2n-1}.

Case (3b). $(\hat{V}, \mathrm{mod}\,\infty)$ is homeomorphic to the Thom space mod(point) of the $4(n-m)$-dimensional bundle over suspension over STS^{2m-1} one fiber of which is glued by a cell.

The proof of the remaining statements of the lemma is the calculation of the suspension homology and the Thom space homology in terms of the homology of their base spaces. □

Now we shall go ahead with the proof of Theorem A.

Definition 2.7. A set of flags from PT^*P^n is called *minimally degenerate* if the space spanned by the points of these flags intersects nontrivially with the intersection of the hyperplanes of the flags from the considered set.

Definition 2.8. The set of flags from PT^*P^n is called *degenerate* if it contains a minimally degenerate subset, and *nondegenerate* otherwise.

Now let us give the precise formulation of Theorem A.

Theorem A. *The complement to the union of the trains for a nondegenerate set of flags from PT^*P^n possesses the* M*-property.*

Proof. Embed the space PT^*P^n into $P^n \times P^{n*}$ as a quadric given by the equation: $\sum_{i=0}^n x_iy_i = 0$, where $x_0:x_1:\cdots:x_n$; $y_0:y_1:\cdots:y_n$ are homogeneous coordinates in P^n and P^{n*} respectively. We call a variety $L \subset P^n \times P^{n*}$ a "projective subspace" if $L = L_1 \times L_2$ where $L_1 \subset P^n$ and $L_2 \subset P^{n*}$ are projective subspaces. If the codimension of $L \subset P^n \times P^{n*}$ equals 1, then we call L a "hyperplane".

Having introduced these definitions we can notice that the train of any flag coincides with the intersection of Q with the union of two "hyperplanes". The complement in $P^n \times P^{n*}$ to the union of these "hyperplanes" coincides with $A^n \times A^{n*}$ and thus the complement in PT^*P^n to the train of some flag can be realized as an affine quadric Q_A in $A^n \times A^{n*}$ given by the equation $x_0 + y_0 + \sum_{i=2}^n x_iy_i = 0$.

The union (intersection) of the irreducible components of the trains for various flags is the union (intersection) of Q with the corresponding "hyperplanes". Thus any intersection of the irreducible components is the intersection of Q with some "projective subspaces". If we pass from the projective spaces to the affine ones, then the inverse image of the considered "projective subspace" is identified with the linear decomposable subspace since the restriction of the quadratic form $\sum x_iy_i$ onto this subspace has zero signature according to Lemma 2.5. Hence the equation of the irreducible components in any affine chart will be given by one of the equations

(*), (**) or (***), i.e., any intersection of irreducible components mod ∞ is the M-manifold considered in Lemmas 2.5 and 2.6.

To finish the proof we will show that all differentials are 0 in the term E_1 of the relative Mayer–Vietoris spectral sequence (see [1]) for the union of trains for a nondegenerate set $\hat{f}=\{f_0,\ldots,f_m\}$ of flags from $\mathbf{PT}^*P^n$.

Consider the space

$$(\bigcup \mathrm{Tn}_{\hat{f}}, \bmod \mathrm{Tn}_{f_0}) = \Big(\bigcup_{j=1,2;\, i=1,m} N_i^j, \bmod (N_0^1 \cup N_0^2)\Big)$$
$$= \Big(\bigcup_{j=1,2;\, i=1,m} \hat{N}_i^j, \bmod \infty\Big),$$

where $N_i^{1(2)}$ is the first (second) irreducible component of the ith flag train. (The flags from the first component satisfy the condition that the hyperplane of the considered flag contains the point of the ith flag; conversely, the points of flags belonging to the second component belong to the hyperplane of the ith flag.) Let $\hat{N}_i^j$ as before denote the one-point compactification of N_i^j. Hereafter we shall work in the chart for which $N_0^1 \cup N_0^2$ belongs to the hyperplane at infinity. Under these assumptions "hyperplanes" are hyperplanes and "projective subspaces" are affine.

Let W denote the intersection

$$W = \bigcap_{s=1}^{k_0} (N_{l_s}^1) \cap \Big(\bigcap_{p=1}^{k_1} N_{m_p}^1\Big) \cap \Big(\bigcap_{q=1}^{k_2} N_{n_q}^2\Big),$$

where l_s, m_p, n_q are pairwise different.

Then:

(i) for $k_1 \neq 0$ or $k_2 \neq 0$, W is defined by the type (*) equation;

(ii) for $k_0 \geq 2$, $k_1 = 0$, $k_2 = 0$, W is defined by the type (**) equation;

(iii) for $k_0 = 1$, $k_1 = 0$, $k_2 = 0$, W is defined by the type (***) equation.

This follows from the fact that the intersection of the irreducible components of the pairwise different flags is nonsingular. The structure of the term E_1 in the considered relative Mayer–Vietoris spectral sequence is shown on Fig. 1, where asterisks mark the nontrivial places and arrows indicate the possible nontrivial differentials.

The proof of the triviality of the considered differentials is similar to the analogous proof for the complex case (see Section 4). □

Definition 2.9. Given the above quadric Q and a set $\mathcal{L} = \{L_1,\ldots,L_k\}$ of the affine hyperplanes in $\mathbb{R}^n$ we will say that the set $\mathcal{L}$ is *h-singular* if there exists a subset of indices $i_1,\ldots,i_h$ such that either the intersection $L_{i_1} \cap \cdots \cap L_{i_h} \cap Q$ is singular or the set of hyperplanes $L_{i_1},\ldots,L_{i_h}$ is not in general position (i.e., there exists a nontransversal intersection); and *h-nonsingular* otherwise.

Let $\alpha = \{\alpha_j\} = \{i_1^j,\ldots,i_h^j\}$ be the set of multiindices, where $1 \leq j \leq \nu$ and $\mathcal{L}_\alpha = \bigcup_{j=1}^{\nu} (L_{i_1}^j \cap \cdots \cap L_{i_h}^j)$. Any continuous deformation $\{L_1,\ldots,L_k\}$ in the locus of all h-nonsingular sets can be extended to a continuous deformation of $\mathcal{L}_\alpha \cap Q$.

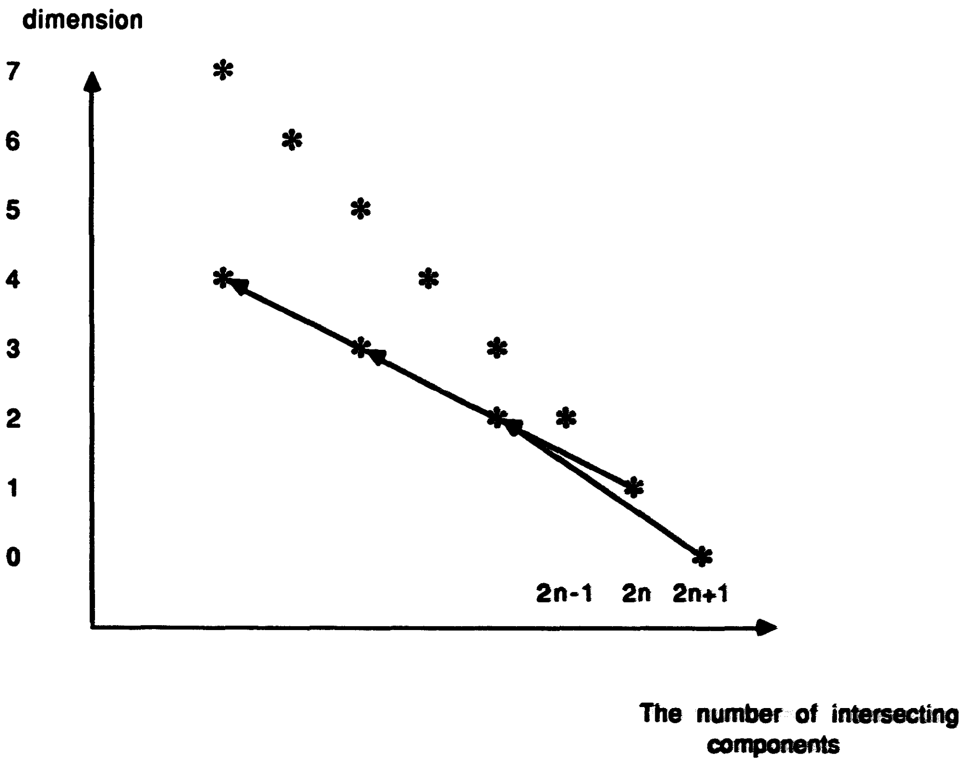

Fig. 1. The structure of the term E_1.

Hence, if we manage to prove that within the considered quadric we can nondegenerately deform an arbitrary nonsingular set of hyperplanes in a neighborhood of infinity, then the triviality of all differentials d_t for $t \geqslant 1$ will follow. As shown on Fig. 1 the "suspicious" differentials exist for $t \leqslant 3$ and this will imply that they can "strike" only in the middle-dimensional homology, i.e., the homology of the intersections defined by (**) and (***).

Let us start with (***). The hypersurface $Q=0$ is a cone. (For the nondegenerate set of flags this can happen only for the intersection with the train of no more than one flag.) Consider the intersection of Q with an affine subspace L which in its turn is the intersection of several irreducible components of the other flag trains. In the case of the nondegenerate flag set L cannot contain the cone's vertex (the origin). If we make a homothetic transformation we obtain the necessary nondegenerate deformation.

Now consider (**). The quadric Q is given by the equation $\sum_{i=1}^{n} x_i y_i = 1$. Intersect Q with the hyperplane

$$\alpha_0 + \sum \alpha_i x_i = 0.$$

The conditions of the singularity of this intersection are as follows. There exist x_i^0, y_i^0:

$$\alpha_0 + \sum \alpha_i x_i^0 = 0, \tag{1}$$

$$\sum x_i^0 y_i^0 = 1, \tag{2}$$

$$\operatorname{rk}\begin{pmatrix} y_1^0 & \cdots & y_n^0 & x_1^0 & \cdots & x_n^0 \\ \alpha_1 & \cdots & \alpha_n & 0 & \cdots & 0 \end{pmatrix} = 1. \tag{3}$$

Conditions (2) and (3) contradict to each other. Hence any intersection of the considered quadric with one hyperplane is nonsingular. Consequently, $d_1 \equiv 0$.

Analogously the intersection of the quadric with a subspace given by the system:

$$\begin{cases} \alpha_0^1 + \sum_{i=1}^{n} \alpha_i^1 x_i = 0, \\ \alpha_0^2 + \sum_{i=1}^{n} \alpha_i^2 x_i = 0 \end{cases}$$

is nonsingular if and only if the vectors $(\alpha_i)^1$ and $(\alpha_i)^2$ are linear independent. Similarly, the intersection of these hyperplanes with Q is nonsingular if the set of hyperplanes is nondegenerate and all the pairwise intersections are nonsingular. Consequently, $d_2 \equiv 0$.

Thus for $t \leqslant 3$ we need only to define when the intersection $L_1 \cap L_2 \cap Q$ is singular, where $L_1 \cap L_2$ is given by:

$$\begin{cases} \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i = 0, \\ \beta_0 + \sum_{i=1}^{n} \beta_i y_i = 0. \end{cases}$$

The conditions of degeneracy are as follows: There exist x_i^0, y_i^0 such that:

$$\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i^0 = 0, \tag{1'}$$

$$\beta_0 + \sum_{i=1}^{n} \beta_i y_i^0 = 0, \tag{2'}$$

$$\sum x_i^0 y_i^0 = 1, \tag{3'}$$

$$\operatorname{rk}\begin{pmatrix} y_1^0 & \cdots & y_n^0 & x_1^0 & \cdots & x_n^0 \\ \alpha_1 & \cdots & \alpha_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1 & \cdots & \beta_n \end{pmatrix} = 2. \tag{4'}$$

(4') implies that

$$y_1^0/\alpha_1 = \cdots = y_n^0/\alpha_n = \alpha;$$

$$x_1^0/\beta_1 = \cdots = x_n^0/\beta_n = \beta,$$

(1) and (2) imply

$$\alpha = -\alpha_0/\sum \alpha_i\beta_i;$$

$$\beta = -\beta_0/\sum \alpha_i\beta_i,$$

and finally (3) gives the singularity condition:

$$\alpha_0\beta_0 = \sum \alpha_i\beta_i.$$

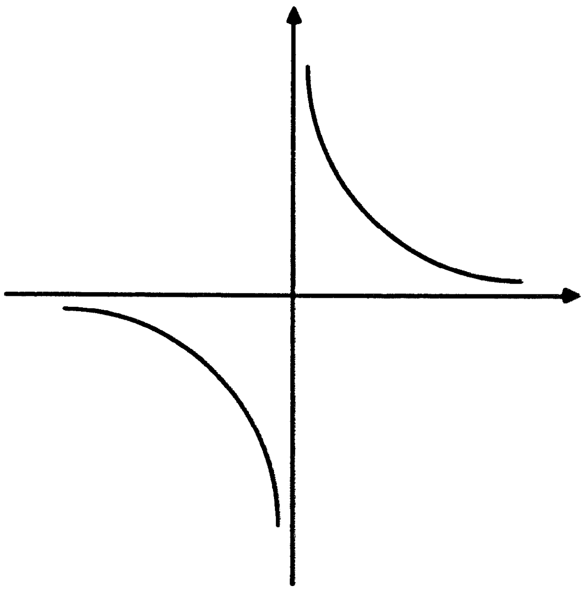

Fig. 2. The graph of train's section.

For fixed α_i, β_i let us depict on the (α_0, β_0)-plane the graph $\alpha_0\beta_0 = \text{const}$ (Fig. 2). This graph does not bound any compact connected component on the plane. Consequently any nondegenerate pair of hyperplanes can be nonsingularly deformed in a neighborhood of the infinity.

Thus we see that a nondegenerate set of hyperplanes can be nonsingularly deformed in a neighborhood of the infinity.

This finishes the proof of the degeneracy for the spectral sequence in the term E_1 in the real case.

In the complex case the same arguments prove the degeneracy of the spectral sequence (see Section 4).

3. Example of violation of the M-principle

In this section we will consider an example of the flag variety $\mathcal{M}$ and the set of flags for which the complement to the union of their trains violates the M-property.

As before $\mathcal{M}$ denotes the flag space and $\mathbf{Tn}_f$ denotes the flag train. Let $\hat{f} = \{f_0, \ldots, f_d\}$ be a set of flags,

$$\mathbf{Tn}_{\hat{f}} = \bigcup_{i=0}^{d} \mathbf{Tn}_{f_i}$$

and

$$\mathcal{M}_{\hat{f}} = \mathcal{M} \setminus \mathbf{Tn}_{\hat{f}}.$$

Theorem B. *There exists an open nonempty set U of 4-tuples of lines in $\mathbf{R}P^3$ such that for each $\hat{f} \in U$*

$$\sum b_i(\mathcal{M}_{\hat{f}}^{\mathbb{C}}) > \sum b_i(\mathcal{M}_{\hat{f}}^{\mathbb{R}})$$

where $\mathcal{M}^{\mathbb{C}}$ ($\mathcal{M}^{\mathbb{R}}$) is the complement in the complex (real) Grassmann manifold $G_{2,4}$ to the union of the trains of these lines.

Proof. The group PGL_4 acts transitively on 3-tuples of pairwise nonintersecting lines in P^3. There exists a transformation that maps the three lines under consideration onto the surface of the standard hyperboloid of one sheet $x_0^2 = x_1^2 + x_2^2 + x_3^2$.

A hyperboloid of one sheet has two families of lines or rulings (denoted as family 1 and 2 respectively) such that the lines from one family do not intersect, while any pair of lines from different families has a one-point intersection. Since the three lines in consideration do not intersect, therefore they belong to one of these families (say, family 1). Consequently the intersection of their trains is one-dimensional and coincides with the other family of rulings (family 2) (see [9]).

The union of the points belonging to any of these families coincides with the whole hyperboloid. Each point of the hyperboloid belongs to a unique line from the family 1. Given an arbitrary fourth line we thus obtain that the intersection of the trains of four lines is the set of lines from the family 2 passing through the points of intersection of the fourth line with the hyperboloid (see Fig. 3).

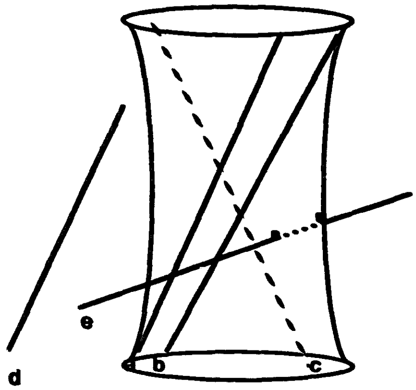

Fig. 3. The one-sheeted hyperboloid representing the stratum of lines nontransversal to the lines *a*, *b* and *c* in the general position; lines *d* and *e* give examples of two different arrangements of the fourth line.

In the complex case for the set of 4-tuples of lines "in general position" the number of the intersection points of 4-tuples of trains equals 2; in the real case there exists an open nonempty subset U in the space of 4-tuples of lines for which the intersection of their trains is empty.

The Smith inequality implies that $\dim E_1^{\mathbb{C}} \geqslant \dim E_1^{\mathbb{R}}$. Moreover, Theorem C from Section 4 implies that for a set of lines "in general position"

$$\dim H_*(\bigcup \mathrm{Tn}_f^{\mathbb{C}}, \bmod \mathrm{Tn}_{f_0}^{\mathbb{C}}) = \dim E_1^{\mathbb{C}} \geqslant \dim E_1^{\mathbb{R}}$$

$$\geqslant \dim H_*(\bigcup \mathrm{Tn}_f^{\mathbb{R}}, \bmod \mathrm{Tn}_{f_0}^{\mathbb{R}}).$$

Thus the complement to a union of trains possesses the M-property if at least $\dim E_1^{\mathbb{C}} = \dim E_1^{\mathbb{R}}$, i.e., M-property is fulfilled for the intersection of arbitrary subset of trains belonging to $\hat{f}$. Since, as we have seen above for the 4-tuples of lines belonging to U

$$2 = H^*\left(\bigcap_{i=1}^{4} \mathrm{Tn}_{f_i}^{\mathbb{C}}\right) > H^*\left(\bigcap_{i=1}^{4} \mathrm{Tn}_{f_i}^{\mathbb{R}}\right) = 0.$$

Then $\dim E_1^{\mathbb{C}}(\cdots) > \dim E_1^{\mathbb{R}}$ and the M-property is violated. □

Corollary 3.1. *There exists an open nonempty set $\mathcal{V}$ of 4-tuples of complete flags in $\mathbb{R}^4$ for which $\sum b_i(\mathcal{M}_{\hat{f}}^{\mathbb{C}}) > \sum b_i(\mathcal{M}_{\hat{f}}^{\mathbb{R}})$ for each $\hat{f} \in \mathcal{V}$.*

Proof. Consider the 4-tuples of flags containing the 2-dimensional subspaces belonging to the domain $\mathcal{U}$ from the previous theorem. For them the M-property is also violated, since in this case $\dim E_1^{\mathbb{C}} > \dim E_1^{\mathbb{R}}$. □

4. The degeneracy of the Mayer–Vietoris spectral sequence

Consider a cellular complex K which is represented as the union of cellular subcomplexes $K = \bigcup_i K_i$. Then there exists a spectral sequence converging to the homology of K (see [1]).

Lemma 4.1. *Consider the following diagram:*

$$\begin{array}{ccc} E & \xrightarrow[\pi]{} & C \\ \big\downarrow{\scriptstyle s} & & \\ A & & \end{array}$$

where A, E, C are projective varieties, C is irreducible and s has the following property. Let E_c denote the inverse image $\pi^{-1}(c)$ for any $c \in C$ and A_c denote the image $s(E_c)$. Assume that s restricted to E_c is an embedding.

Then there exists an open dense set $U \in C$ such that for arbitrary c_1, c_2 from U the set A_{c_1} is homotopic to A_{c_2} in A.

Lemma 4.2. *For any $c_0 \in C$ there exists a nonempty open $U \subset C^d$ such that for any set $(c_1, \ldots, c_d) \in U$ the map $I_* : (\bigcup A_{c_j}, \bmod A_{c_0}) \to (A, \bmod A_{c_0})$ is homologically trivial (i.e., $I_* = 0$ on the homological level).*

Proof. Since A_c is an algebraic variety then it is a neighborhood retract in A. Let Ω be its retracting neighborhood. There exists a nonempty open and hence dense locus $U \subset C$ such that for any $\{u_1, \ldots, u_d\}$, $\{v_1, \ldots, v_d\} \subset U$ the union $\bigcup A_{u_j}$ is homotopic to $\bigcup A_{v_j}$ by the above lemma. Choose $\{u_1, \ldots, u_d\}$ belonging to the ξ-neighborhood of the point $\{c_0, \ldots, c_0\} \cap U$.

Thus $\bigcup A_{u_j} \subset \Omega$. Retracting Ω to A_{c_0} we contract $\bigcup A_{u_j}$ to A_{c_0}. Thus the following composition:

$$\bigcup A_{v_j} \xrightarrow{\text{homotopy}} \bigcup A_{u_j} \xrightarrow{\text{retraction}} A_{c_0}$$

gives us a chain whose boundary coincides with $\bigcup A_{v_i}$ mod A_{c_0}. □

Now we are able to prove the following crucial result.

Theorem C. *For any $c \in C$ there exists a nonempty subset $U \subset C^d$ such that for any $c_1, \ldots, c_d \in U$ the relative Mayer–Vietoris sequence for $\bigcup A_{c_j}$ mod A_{c_0} degenerates in the E_1-term.*

Proof. The action of all differentials d_s for $s \geq 1$ is induced by the sequence of embeddings. Let $\{\tau_1, \ldots, \tau_k\} \subset \{c_1, \ldots, c_d\}$ be a subset. Then

$$A_{\tau_1} \cap A_{\tau_2} \cap \cdots \cap A_{\tau_k} \hookrightarrow A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_m} \cap \cdots \cap A_{\tau_k};$$

$$(A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_p} \cap \cdots \cap A_{\tau_k}) \cup (A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_g} \cap \cdots \cap A_{\tau_k})$$
$$\hookrightarrow A_{\tau_1} \cap \cdots \cap \hat{A}_{\tau_p} \cap \cdots \cap \hat{A}_{\tau_g} \cap \cdots \cap A_{\tau_k};$$

$$(A_{\tau_1} \cap \cdots \cap A_{\tau_{k-s}} \cap A_{\tau_{k-s+1}}) \cup \cdots \cup (A_{\tau_1} \cap \cdots \cap A_{\tau_{k-s}} \cap A_{\tau_k})$$
$$\hookrightarrow A_{\tau_1} \cap \cdots \cap A_{\tau_{k-s}}$$

where ^ means the omission of the corresponding term. On the homological level the maps induced by these embeddings are trivial by Lemma 4.2. □

5. Example: Case PT^*P^2

In this section we shall give an explicit illustration of Theorem A in the simplest case of flags on $\mathbb{R}P^2$. The main results are:

Theorem 5.1. *For a set of flags $\hat{f} = \{f_1, \ldots, f_k\}$ on $\mathbb{R}P^2$ "in general position" the variety $\mathcal{M}_{\hat{f}}^{\mathbb{R}}$ of all flags transversal to all flags belonging to $\hat{f}$ is homeomorphic to the disjoint union of $k^3 - k + k$ three-dimensional cells.*

Theorem 5.2. *For a set of flags $\hat{f} = \{f_1, \ldots, f_k\}$ on $\mathbb{C}P^2$ "in general position" the $\mathbb{Z}$-homology of the variety $\mathcal{M}_{\hat{f}}^{\mathbb{C}}$ of all flags transversal to all flags belonging to $\hat{f}$ are torsion free and its Betti numbers are $b_0 = 1$, $b_1 = 2(k-1)$, $b_2 = 2(k-1)^2$, $b_3 = (k-1)^3$, $b_i = 0$, $i \geq 4$.*

Each flag in $\mathbb{R}P^2$ consists of a line and its point.

Definition 5.3. *Bifurcation lines* are lines passing through points of different flags from $\hat{f}$.

Definition 5.4. A set $\hat{f}$ of flags on $\mathbb{R}P^2$ is called a set "*in general position*" if:

(a) flags are mutually transversal;
(b) none of the 3-tuples of the flag lines intersects in one point;
(c) none of the 3-tuples of the flag points belongs to one line;
(d) no bifurcation line passes through the intersection point of the flag lines.

Under these assumptions we prove Theorem 5.1.

5.1. Proof of Theorem 5.1

Consider a set $\hat{f}=\{f_1,\ldots,f_k\}$ of k flags "in general position". The lines of these flags divide $\mathbb{R}P^2$ into $N_1=k(k-1)/2$ open polygons (see condition (b)). Let M denote one of these polygons and $\Omega(M)$ denote the locus of flags transversal to $\hat{f}=\{f_1,\ldots,f_k\}$ whose points belong to M. $\Omega(M)$ is a one-dimensional bundle the fiber of which is an interval and the base B identifies with the set of lines crossing M and not passing through the points of flags $\{f_1,\ldots,f_k\}$. Let $l_1,\ldots,l_k$ denote the lines on $\mathbb{R}P^{2^*}$ dual to the points of flags $\hat{f}=\{f_1,\ldots,f_k\}$. By condition (c) the lines $l_1,\ldots,l_k$ are "in general position".

Definition 5.5. We call a polygon $M^*\subset\mathbb{R}P^{2^*}$ *dual* to a given polygon $M\subset\mathbb{R}P^2$ if the points of M^* are dual to all lines on $\mathbb{R}P^2$ nonintersecting M. If M is affine (i.e., contained in some affine chart) and convex, then M^* is also affine and convex. If M is open, then M^* is closed and vice versa. The base B of the bundle $\Omega(B)$ equals $\mathbb{R}P^{2^*}\backslash(M^*\cup l_1\cup\cdots\cup l_k)$. All the lines $l_1,\ldots,l_k$ intersect the closure of M^*.

Lemma 5.6. *B consists of $k+q(M)$ open two-dimensional cells, where $q(M)$ denotes the number of bifurcation lines intersecting M.*

Corollary 5.7. *$\Omega(M)$ consists of $k+q(M)$ open three-dimensional cells.*

The proof of Lemma 5.6 is an immediate consequence of the following more general fact.

Proposition 5.8. *Let $\mathscr{A}$ be a nonstrictly convex domain of $\mathbb{R}P^2$. Then the number of connected components on which $\mathscr{A}^-=\mathbb{R}P^2\backslash\mathscr{A}$ is separated by k lines "in general position" each of them intersecting $\mathscr{A}^-$ equals $k+r$, where r is the number of pairwise intersections of lines belonging to $\mathscr{A}$. All the components on which $\mathscr{A}^-$ is separated are contractible.*

Proof. For $k=1$ the statement is true. Suppose that it is proved for $k-1$ and δ is a segment (or point) of the intersection of $\mathscr{A}^-$ with the kth line. Then the points of intersection of l_k with the other lines which belong to $\mathscr{A}^-$ divide $l_k \backslash \delta$ into $\nu+1$ intervals, where ν is the number of these points. Each of these intervals is a simple path connecting two boundary points of the contractible (by inductive hypothesis) domain. Consequently, the inclusion of a new line l_k increases the number of domains on which $\mathscr{A}^-$ is divided by $\nu+1$. Their contractability is obvious. □

The total number of connected components of $\mathscr{M}^{\mathbb{R}}$ is equal to the sum of the number of connected components of $\Omega(M)$ over all the polygons M. Thus it equals $N_3=kN_1+N_2$, where N_2 is the total number of domains on which bifurcation lines are divided by the set of flag lines. By conditions (a) and (d) N_2 equals $k^2(k-1)/2$. Consequently $N_3=k^3-k^2+k$. Theorem 5.1 is proved.

5.2. Proof of Theorem 5.2

Definition 5.9. The *train* $\mathbf{Tn}_f$ of a complete flag f is the set of all complete flags nontransversal to f.

Lemma 5.10. *The restriction of the train of a complete flag f given in $\mathbb{R}^n$ ($\mathbb{C}^n$) on the complement in F_n (where F_n is the manifold of all complete flags) to the train of some flag g transversal to f is diffeomorphic to the surface given by the equation:*

$$\Delta \cdot \Delta_1 \cdots \Delta_{n-1} = o, \tag{5}$$

where Δ_i are the determinants of $(i \times i)$-minors formed by the first i rows and the last i columns of the upper triangular $(n \times n)$-matrix with the units on the main diagonal.

Proof. Let $e_1, \ldots, e_n$ be a basis in $\mathbb{R}^n$ ($\mathbb{C}^n$). We construct two flags f_d and f_q such that the i-dimensional subspaces of f_d are spanned by $e_n, \ldots, e_i$ and the i-dimensional subspaces of f_q are spanned by $e_1, \ldots, e_i$. Since the group $\mathbf{GL}_n$ acts on the pairs of transversal flags transitively, then it suffices to consider only the pair (f_d, f_q). Take the bundle $\pi_n : \mathbf{GL}_n \to F_n$, where π_n maps each nondegenerate matrix to the complete flag whose i-dimensional subspace is spanned by the first i rows of the matrix. $\pi_n^{-1}(F_n \backslash \mathbf{Tn}_{f_q})$ consists of matrices whose principal minors do not vanish. This implies that we can construct a section in $\pi_n^{-1}(F_n \backslash \mathbf{Tn}_{f_q})$ choosing in each fiber an upper triangular matrix with the units on the main diagonal. Thus the group of upper triangular matrices with the units on the main diagonal is the natural affine chart for the cell $(F_n \backslash \mathbf{Tn}_{f_q})$. Under this identification the condition of nontransversality of k-dimensional subspace of the flag corresponding to some upper triangular matrix to the $(n-k)$-dimensional subspace of flag f coincides with the vanishing of the determinant of the $((n-k) * (n-k))$-minor formed by its first rows and last columns. □

Remark 5.11. For $\mathbf{PT}^*P^2$ the local equations of two train components are:

$$z=0; \quad \text{and} \quad z=xy. \tag{6}$$

Definition 5.12. Let $\hat{f}=\{f_1,\dots,f_k\}$ be an arbitrary set of complete flags in $\mathbf{C}^n$. As before, $\widehat{\mathbf{Tn}_{\hat{f}}}$ denotes the one-point compactification of

$$\mathbf{Tn}_{\hat{f}}=\bigcup_{i=1}^{k-1}\mathbf{Tn}_{f_i}\cap(F_n\backslash\mathbf{Tn}_{f_k})$$

and

$$\mathcal{M}_{\hat{f}}=F_n\backslash\bigcup_{=1}^{k}\mathbf{Tn}_{f_i}.$$

Lemma 5.13. $\tilde{H}_{i-1}(\mathcal{M}_{\hat{f}})=\tilde{H}^{n(n-1)-i}(\widehat{\mathbf{Tn}_{\hat{f}}})$.

Proof. $F_n\backslash\mathbf{Tn}_{f_k}$ is diffeomorphic to the $n(n-1)$-dimensional cell. The Alexander duality implies (see [3]):

$$\tilde{H}_{i-1}(\mathcal{M}_{\hat{f}})=H_c^{n(n-i)-i}\left(\bigcap_{i=1}^{k}(F_n\backslash\mathbf{Tn}_{f_k})\right)=\tilde{H}_c^{n(n-1)-i}(\widehat{\mathbf{Tn}_{\hat{f}}})$$

where $\tilde{H}_j$ and $\hat{H}_c^j$ denote the reduced homology and the compact support homology respectively. Consequently:

$$\tilde{H}_c^{n(n-1)-i}(\mathbf{Tn}_{\hat{f}})=\tilde{H}^{n(n-1)-i}(\widehat{\mathbf{Tn}_{\hat{f}}},\infty)=\tilde{H}^{n(n-1)-i}(\widehat{\mathbf{Tn}_{\hat{f}}}),$$

where ∞, as before, denotes the compactifying point. To compute $\tilde{H}^*(\widehat{\mathbf{Tn}_{\hat{f}}})$ we use (according to the described machinery) the Mayer–Vietoris spectral sequence. □

Lemma 5.14. *Each of two irreducible components of* $\mathbf{Tn}_{\hat{f}}$ *in* $\mathbf{C}^3$ *is biholomorphically equivalent to* $\mathbf{C}^2$. *These components intersect in a pair of complex lines, which in their turn intersect in a point.*

Proof. See formula (6). □

Corollary 5.15. *The cohomology groups of a compactified train have no torsion and their nontrivial Betti numbers are equal to* $1,0,1,2,2$.

Proof. Use the Mayer–Vietoris exact sequence. □

Lemma 5.16. *The one-point compactification of the intersection* $\mathbf{Tn}_{f_1}\cap\mathbf{Tn}_{f_2}$ *is homeomorphic to the complex depicted on Fig. 4 and is homotopically equivalent to the bouquet of four spheres* S^2 *and six circles.*

Proof. The geometrical description of $\mathbf{Tn}_{f_1}\cap\mathbf{Tn}_{f_2}$ before its compactification is as follows. It consists of four components A, B, C, D:

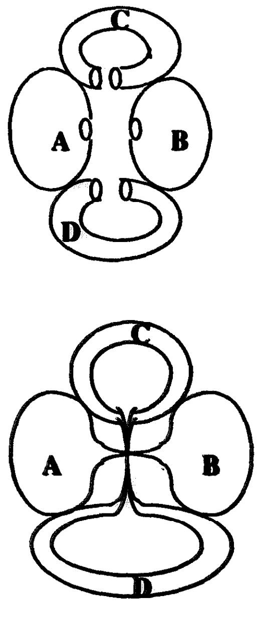

Fig. 4. Noncompactified and one-point compactified $\mathrm{Tn}_{f_1} \cap \mathrm{Tn}_{f_2}$.

- A corresponds to the case when the flag line passes through the points of f_1 and f_2;
- B corresponds to the case when the flag line passes through the point of f_1 and its point lies on the line of f_2;
- C, the flag point coincides with the intersection of f_1- and f_2-lines;
- D, the flag line passes through the point of while its point lies on the f_1-line.

Components A and C are diffeomorphic to $S^2 \setminus \{\text{point}\}$, while B and D are diffeomorphic to $S^2 \setminus \{\text{two points}\}$.

The following pairs of components have one-point intersections (which are pairwise different): (A, B), (A, D), (B, C), (C, D). On Fig. 4 there are shown noncompactified and compactified intersections $\mathrm{Tn}_{f_1} \cap \mathrm{Tn}_{f_2}$. From this figure one can easily obtain the necessary facts. □

Corollary 5.17. *The cohomology of* $\mathrm{Tn}_{f_1} \widehat{\cap} \mathrm{Tn}_{f_2}$ *is torsion-free and its nontrivial Betti numbers are* $1, 6, 4$.

Lemma 5.18. *Any* 3-*tuple of flags* f_1, f_2, f_3 *on* $\mathbb{C}P^2$ *"in general position" has six flags nontransversal to each of them. The lines of three flags pass through the pair of points of* f_i *and* f_j *while their points lie on the line of* f_k; *the lines of three others connect the intersection point of* f_i- *and* f_j-*lines with the point of* f_k, *where* (i, j, k) *is an arbitrary permutation of* 1, 2 *and* 3.

The term E_1 of the Mayer–Vietoris spectral sequence for the reduced cohomology of $\widehat{\mathrm{Tn}}_f$ is shown on Fig. 5. The degeneracy results proven in the previous part of this paper finish the proof of Theorem 5.2.

5			
4	$\underbrace{Z+\cdots+Z}_{2(k-1)\text{ times}}$		
3	$\underbrace{Z+\cdots+Z}_{2(k-1)\text{ times}}$		
2	$\underbrace{Z+\cdots+Z}_{(k-1)\text{ times}}$	$\underbrace{Z+\cdots+Z}_{4\binom{k-1}{2}\text{ times}}$	
1		$\underbrace{Z+\cdots+Z}_{6\binom{k-1}{2}\text{ times}}$	
0			$\underbrace{Z+\cdots+Z}_{\binom{k-1}{3}\text{ times}}$
	0	1	2

Fig. 5. Structure of the term E_1.

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