# Moment modules of polytopal measures and multisymmetric invariants

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*To Hermann Weyl, a mathematical polymath.*

**Abstract**

For a compactly supported measure in $\mathbf{R}^d$ one can package all moments into its normalized Fantappiè transform. When the measure is the standard measure of a polytope with vertex set $X = \{x_1, \dots, x_n\}$, this transform is a rational function whose denominator divides

$$D_X(\mathbf{u}) = \prod_{j=1}^{n} (1 - \langle x_j, \mathbf{u} \rangle).$$

The Taylor coefficients, viewed as polynomial functions of the vertex coordinates, generate the moment ring of the polytope. We show that the coefficient module of this ring is a finite module over the ring of diagonal invariants, equivalently over the ring of multisymmetric polynomials in the vertex coordinates. More precisely, if

$$\mathcal{F}_P(\mathbf{u}) = \frac{N_P(\mathbf{u}, X)}{D_X(\mathbf{u})}, \qquad \deg_{\mathbf{u}} N_P \le n - d - 1,$$

and if $N_P(\mathbf{u}, X) = \sum p_K(X)\mathbf{u}^K$, then the $\Lambda_{n,d}$-module generated by all moment coefficients is exactly the $\Lambda_{n,d}$-module generated by the finitely many numerator coefficients $p_K(X)$. Consequently all sufficiently high moments satisfy explicit universal linear recurrence relations with coefficients in the diagonal invariant ring. The paper also formulates a universal simplex-sum model for polytopal measures, gives explicit planar polygon formulas including a flip identity showing triangulation independence, and records the relation with multisymmetric functions and diagonal coinvariants.

## 1 Introduction

This paper continues the study initiated in [5] of moments of measures supported on polytopes and of their normalized moment generating functions. The aim is to organize the moment data of a polytope into an algebraic object, called here its *moment ring*, and to relate this object to the classical theory of diagonal invariants.

Throughout the paper, $\mathbf{R}^d$ is endowed with a fixed Euclidean coordinate system

$$(x_1, \dots, x_d)$$

and with the standard scalar product $\langle \cdot, \cdot \rangle$.

Let $\mu$ be a finite compactly supported complex Borel measure on $\mathbf{R}^d$. For a multiindex

$$I = (i_1, \dots, i_d) \in \mathbf{Z}^d_{\ge 0},$$

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write

$$|I| = i_1 + \cdots + i_d, \qquad \mathbf{x}^I = x_1^{i_1} \cdots x_d^{i_d}.$$

The *moment* of $\mu$ corresponding to $I$ is

$$m_I(\mu) = \int_{\mathbf{R}^d} \mathbf{x}^I \, d\mu(\mathbf{x}). \tag{1}$$

Following [5], define the *normalized moment generating function* by

$$\mathcal{F}_\mu(\mathbf{u}) = \sum_{I \geq 0} \frac{(|I| + d)!}{i_1! \cdots i_d!} \, m_I(\mu) \, \mathbf{u}^I, \qquad \mathbf{u}^I = u_1^{i_1} \cdots u_d^{i_d}. \tag{2}$$

Equivalently,

$$\mathcal{F}_\mu(\mathbf{u}) = d! \int_{\mathbf{R}^d} \frac{d\mu(\mathbf{x})}{(1 - \langle \mathbf{x}, \mathbf{u} \rangle)^{d+1}}, \tag{3}$$

a special case of the Fantappiè transform; see [1, 5].

If $P \subset \mathbf{R}^d$ is a bounded measurable set, we denote by

$$\mu_P = \chi_P \, dx_1 \cdots dx_d$$

its standard measure and write $\mathcal{F}_P = \mathcal{F}_{\mu_P}$. For a $d$-simplex $\Delta$ with vertex set $\mathrm{Vert}(\Delta) = \{\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}\}$, the basic formula is the following.

**Proposition 1.1.** *For any $d$-simplex $\Delta \subset \mathbf{R}^d$,*

$$\mathcal{F}_\Delta(\mathbf{u}) = \frac{d! \, \mathrm{Vol}(\Delta)}{\prod\limits_{\mathbf{v} \in \mathrm{Vert}(\Delta)} (1 - \langle \mathbf{v}, \mathbf{u} \rangle)}. \tag{4}$$

Thus, for a simplex, the normalized moment generating function is a rational function whose denominator is determined by the vertices. A central point of the present paper is that this phenomenon persists after passing from a simplex to a polytope, and that the resulting rational form leads naturally to finite modules over rings of diagonal invariants.

## Motivation

A polytope with finitely many vertices depends on finitely many parameters, while its sequence of moments is infinite. It is therefore natural to expect algebraic relations among the moments. Describing these relations is relevant both structurally and for inverse moment problems, where one attempts to reconstruct a body or a measure from finitely many moments.

The simplest nontrivial illustration is the family of triangles in $\mathbf{R}^2$. Let

$$\Delta(X, Y) = \mathrm{conv}\big((x_1, y_1), (x_2, y_2), (x_3, y_3)\big),$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. For $I = (i, j) \in \mathbf{Z}^2_{\geq 0}$ put

$$m_I(X, Y) = \int_{\Delta(X,Y)} x^i y^j \, dx \, dy.$$

The triangle depends on six parameters, and its moments cannot be algebraically independent. Already the lowest moments satisfy nontrivial polynomial relations. One such relation is

$$\begin{aligned} m_{00}^5 + 8m_{02}m_{10}^2 - 8m_{01}m_{10}m_{11} + 3m_{00}m_{11}^2 \\ + 8m_{01}^2 m_{20} - 12m_{00}m_{02}m_{20} = 0. \end{aligned} \tag{5}$$

This example suggests the general problem of understanding the algebra generated by all moment coordinates.

## Moment rings

Let $P \subset \mathbf{R}^d$ be a polytope with labeled vertices

$$X = (x_1, \dots, x_n), \qquad x_j = (x_{j1}, \dots, x_{jd}).$$

Whenever $P$ is deformed within a fixed combinatorial type, each moment may be regarded locally as a function of the vertex coordinates. If $P$ admits a triangulation by simplices with vertices among $x_1, \dots, x_n$, Proposition 1.1 and additivity of the standard measure imply that these functions are polynomial in the vertex coordinates, provided oriented simplex volumes are used.

**Definition 1.2.** Let $P$ be a polytope, or more generally a polytopal measure, with labeled vertex set $X = (x_1, \dots, x_n)$. Write

$$\mathcal{F}_P(\mathbf{u}) = \sum_{I \geq 0} f_I(X)\mathbf{u}^I.$$

The *moment ring* of $P$ is the $\mathbf{Q}$-subalgebra

$$\mathcal{R}(P) = \mathbf{Q}[f_I(X) : I \in \mathbf{Z}^d_{\geq 0}] \subseteq \mathbf{Q}[x_{jr} : 1 \leq j \leq n,\ 1 \leq r \leq d].$$

The associated *moment coefficient module* over a coefficient ring $A$ is the $A$-module generated by the coefficients $f_I(X)$.

Different labelings of the vertices produce canonically isomorphic objects, because relabeling acts by the symmetric group on the vertex variables.

The basic problem is the following.

**Problem 1.3.** Given a polytope $P$, describe generators and relations of $\mathcal{R}(P)$ and of the corresponding moment coefficient modules.

## Main results

Let

$$R_{n,d} = \mathbf{Q}[x_{jr} : 1 \leq j \leq n,\ 1 \leq r \leq d]$$

and let $\mathfrak{S}_n$ act diagonally by permuting the first index $j$. Denote the invariant ring by

$$\Lambda_{n,d} = R_{n,d}^{\mathfrak{S}_n}.$$

This is the ring of multisymmetric polynomials, or equivalently the ring of diagonal invariants for the action of $\mathfrak{S}_n$ on $n$ vectors in $\mathbf{R}^d$.

For a labeled configuration $X = (x_1, \dots, x_n)$ put

$$D_X(\mathbf{u}) = \prod_{j=1}^{n} (1 - \langle x_j, \mathbf{u} \rangle). \tag{6}$$

For a convex polytope $P$ with vertex set contained in $X$, a triangulation using the vertices of $P$ gives

$$\mathcal{F}_P(\mathbf{u}) = \frac{N_P(\mathbf{u}, X)}{D_X(\mathbf{u})}, \qquad \deg_{\mathbf{u}} N_P \leq n - d - 1. \tag{7}$$

The same formula holds for any finite linear combination of simplex measures with vertices in $X$.

The main theorem below strengthens the finite generation statement implicit in (7). It shows that the infinite family of moment coefficients is controlled, over $\Lambda_{n,d}$, by the finite family of numerator coefficients.

**Theorem 1.4.** *Let $P$ be a polytopal measure with vertices contained in $X = (x_1, \dots, x_n)$ and suppose that*

$$\mathcal{F}_P(\mathbf{u}) = \sum_{I \geq 0} f_I(X)\mathbf{u}^I = \frac{N_P(\mathbf{u}, X)}{D_X(\mathbf{u})}, \qquad N_P(\mathbf{u}, X) = \sum_{|K| \leq n-d-1} p_K(X)\mathbf{u}^K.$$

*Then*

$$\sum_{I \geq 0} \Lambda_{n,d} f_I(X) = \sum_{|K| \leq n-d-1} \Lambda_{n,d} p_K(X). \tag{8}$$

*In particular, the moment coefficient module is a finitely generated $\Lambda_{n,d}$-module, generated by at most*

$$\binom{n-1}{d}$$

*elements.*

The next consequence gives a concrete recurrence for all high moments.

**Corollary 1.5.** *Write*

$$D_X(\mathbf{u}) = \sum_{A \geq 0} (-1)^{|A|} e_A(X)\mathbf{u}^A, \qquad e_0 = 1, \tag{9}$$

*where $e_A(X) = 0$ for $|A| > n$. Then for every multiindex $I$ with $|I| > n - d - 1$ one has*

$$f_I(X) = \sum_{0 < A \leq I} (-1)^{|A|+1} e_A(X) f_{I-A}(X). \tag{10}$$

*Thus all sufficiently high moment coefficients are obtained from lower ones by universal linear recurrences with coefficients in the diagonal invariant ring $\Lambda_{n,d}$.*

This makes the finite nature of the moment data explicit. The recurrence depends only on the vertex configuration through the elementary multisymmetric functions $e_A(X)$, while the initial data are encoded by the numerator $N_P$.

### Organization

Section 2 recalls polytopal measures and the universal simplex-sum model. Section 3 reviews the necessary facts about multisymmetric functions and proves Theorem 1.4. Section 4 records recurrence and finite-determination consequences, including planar quadrilaterals, an edge-flip identity, and the general convex planar $n$-gon. Section 5 discusses diagonal coinvariant shadows and lists several open problems.

# 2 Polytopal measures and universal simplex sums

## 2.1 Simplex measures

We first prove the simplex formula used throughout the paper.

*Proof of Proposition 1.1.* By (3),

$$\mathcal{F}_\Delta(\mathbf{u}) = d! \int_\Delta \frac{d\mathbf{x}}{(1 - \langle \mathbf{x}, \mathbf{u} \rangle)^{d+1}}.$$

Write every point of $\Delta$ in barycentric coordinates,

$$\mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_{d+1}\mathbf{v}_{d+1}, \qquad t_i \geq 0, \quad \sum_{i=1}^{d+1} t_i = 1.$$

The Jacobian is $d!\,\mathrm{Vol}(\Delta)$, and

$$1 - \langle \mathbf{x}, \mathbf{u} \rangle = \sum_{i=1}^{d+1} t_i (1 - \langle \mathbf{v}_i, \mathbf{u} \rangle).$$

The standard Dirichlet integral

$$\int_{t_i \geq 0,\, \sum t_i = 1} \frac{dt_1 \cdots dt_d}{\left(\sum_{i=1}^{d+1} a_i t_i\right)^{d+1}} = \frac{1}{d!\, a_1 \cdots a_{d+1}}$$

then gives (4). □

## 2.2 Universal simplex sums

Fix integers $d \geq 1$ and $n \geq d+1$, and let

$$X = (x_1, \dots, x_n), \qquad x_j \in \mathbf{R}^d.$$

For every $(d+1)$-element subset $T = \{i_0, \dots, i_d\} \subset \{1, \dots, n\}$, denote by

$$\Delta_T = \mathrm{conv}(x_{i_0}, \dots, x_{i_d})$$

the corresponding oriented simplex and by $\mathrm{Vol}_T(X)$ its oriented Euclidean volume. Introduce coefficients $\lambda_T$ and set

$$\Phi_{X,\lambda}(\mathbf{u}) = \sum_{|T|=d+1} \lambda_T \frac{d!\,\mathrm{Vol}_T(X)}{\prod\limits_{i \in T} (1 - \langle x_i, \mathbf{u} \rangle)}. \tag{11}$$

This is the universal simplex-sum model associated with $n$ labeled vertices in $\mathbf{R}^d$.

**Proposition 2.1.** *The function (11) has the common-denominator form*

$$\Phi_{X,\lambda}(\mathbf{u}) = \frac{N_{X,\lambda}(\mathbf{u})}{D_X(\mathbf{u})}, \qquad D_X(\mathbf{u}) = \prod_{j=1}^{n} (1 - \langle x_j, \mathbf{u} \rangle), \tag{12}$$

*where $N_{X,\lambda}(\mathbf{u})$ is a polynomial in $\mathbf{u}$ of degree at most $n-d-1$.*

*Proof.* Multiplying each simplex term in (11) by the missing factors in the common denominator gives

$$N_{X,\lambda}(\mathbf{u}) = \sum_{|T|=d+1} \lambda_T d!\,\mathrm{Vol}_T(X) \prod_{j \notin T} (1 - \langle x_j, \mathbf{u} \rangle).$$

Each summand has degree $n-d-1$ in $\mathbf{u}$, proving the claim. □

**Proposition 2.2** (Explicit numerator coefficients)**.** *With notation as in Proposition 2.1, write*

$$N_{X,\lambda}(\mathbf{u}) = \sum_{|K| \leq n-d-1} p_K(X, \lambda) \mathbf{u}^K.$$

*For a subset $T \subset \{1, \dots, n\}$ of cardinality $d+1$, put*

$$D_{X \setminus T}(\mathbf{u}) = \prod_{j \notin T} (1 - \langle x_j, \mathbf{u} \rangle) = \sum_{K \geq 0} (-1)^{|K|} e_K(X \setminus T) \mathbf{u}^K.$$

*Then*

$$p_K(X, \lambda) = (-1)^{|K|} \sum_{|T|=d+1} \lambda_T d!\,\mathrm{Vol}_T(X) e_K(X \setminus T). \tag{13}$$

*Thus every numerator coefficient is an explicit polynomial expression obtained from oriented simplex volumes and elementary multisymmetric functions in the complementary vertices.*

*Proof.* This is just the coefficient extraction from

$$N_{X,\lambda}(\mathbf{u}) = \sum_{|T|=d+1} \lambda_T d! \operatorname{Vol}_T(X) \prod_{j \notin T} (1 - \langle x_j, \mathbf{u} \rangle),$$

which was obtained in the proof of Proposition 2.1. □

**Corollary 2.3.** *Let $P \subset \mathbf{R}^d$ be a convex polytope with vertex set contained in $X = (x_1, \ldots, x_n)$. Then $\mathcal{F}_P$ has the form (12). In particular, the degree of its numerator is at most $n - d - 1$.*

*Proof.* Every convex polytope admits a triangulation using only its vertices. Additivity of the standard measure and Proposition 1.1 express $\mathcal{F}_P$ as a sum of simplex transforms of the form (11). Proposition 2.1 applies. □

*Remark* 2.4. For generalized, possibly nonconvex, polytopes with vertex set contained in $X$, the same conclusion holds whenever the standard measure is a finite linear combination of standard simplex measures with vertices in $X$. The conjecture below from [5] asserts that this happens in much greater generality.

## 2.3 Polytopal measures

Given a finite spanning set $S \subset \mathbf{R}^d$, let $\mathcal{P}(S)$ be the set of generalized $d$-dimensional polytopes all of whose vertices belong to $S$. Let

$$\mathcal{M}(S) = \operatorname{span}\{\mu_P : P \in \mathcal{P}(S)\}$$

be the vector space of polytopal measures supported on $\operatorname{conv}(S)$, and let $\mathcal{M}^{\Delta}(S)$ be the subspace spanned by the standard measures of $d$-simplices with vertices in $S$.

**Conjecture 2.5** (Gravin–Pasechnik–Shapiro–Shapiro)**.** *For any finite spanning set $S \subset \mathbf{R}^d$,*

$$\mathcal{M}(S) = \mathcal{M}^{\Delta}(S).$$

*Equivalently, the standard measure of every generalized polytope with vertices in $S$ is a linear combination of standard measures of simplices with vertices in $S$.*

Conjecture 2.5 would imply that the universal simplex-sum model (11) controls the moment generating functions of all generalized polytopes with prescribed vertex set. The results below do not require the conjecture; they apply to every polytopal measure whose transform has the rational form (12).

*Remark* 2.6 (Vertex formula for simple polytopes). For a simple convex polytope $P$, there is also a vertex formula for $\mathcal{F}_P$ in terms of tangent cones. If $w_1(v), \ldots, w_d(v)$ are edge vectors spanning the tangent cone at a vertex $v$, then Theorem 2 and Corollary 3 of [5] give

$$\mathcal{F}_P(\mathbf{u}) = (-1)^d \sum_{v \in \operatorname{Vert}(P)} \frac{\langle v, \mathbf{u} \rangle^d |\det(w_1(v), \ldots, w_d(v))|}{\prod_{j=1}^d \langle w_j(v), \mathbf{u} \rangle} \frac{1}{1 - \langle v, \mathbf{u} \rangle}.$$

This formula is useful for computations but will not be needed in the proof of the main module theorem.

# 3 Multisymmetric functions and moment modules

Let

$$R_{n,d} = \mathbf{Q}[x_{jr} : 1 \le j \le n,\ 1 \le r \le d]$$

and let $\mathfrak{S}_n$ act by

$$\sigma(x_{jr}) = x_{\sigma(j),r}.$$

The invariant ring

$$\Lambda_{n,d} = R_{n,d}^{\mathfrak{S}_n}$$

is the ring of multisymmetric polynomials. It is the ring of diagonal invariants of $\mathfrak{S}_n$ acting on $n$ ordered vectors in $\mathbf{R}^d$. This ring goes back to classical invariant theory, in particular to Weyl's point of view on invariants [8], and has been studied extensively in modern treatments of multisymmetric functions; see for instance [3, 7, 2].

Set

$$\ell_j(\mathbf{u}) = \langle x_j, \mathbf{u}\rangle = x_{j1}u_1 + \cdots + x_{jd}u_d.$$

Then

$$D_X(\mathbf{u}) = \prod_{j=1}^{n}(1-\ell_j(\mathbf{u})) = \sum_{A\geq 0}(-1)^{|A|}e_A(X)\mathbf{u}^A, \tag{14}$$

where $e_0 = 1$ and $e_A = 0$ for $|A| > n$. The $e_A(X)$ are the elementary multisymmetric functions, and they generate $\Lambda_{n,d}$.

Define the complete multisymmetric functions $h_I(X)$ by

$$H_X(\mathbf{u}) = \frac{1}{D_X(\mathbf{u})} = \prod_{j=1}^{n}(1-\ell_j(\mathbf{u}))^{-1} = \sum_{I\geq 0} h_I(X)\mathbf{u}^I. \tag{15}$$

**Lemma 3.1.** *The functions $h_I(X)$, $I \in \mathbf{Z}^d_{\geq 0}$, generate the ring $\Lambda_{n,d}$.*

*Proof.* Since $D_X(\mathbf{u})H_X(\mathbf{u}) = 1$, comparison of coefficients gives

$$\sum_{0\leq A\leq K}(-1)^{|A|}e_A(X)h_{K-A}(X) = \begin{cases} 1, & K = 0,\\ 0, & K \neq 0.\end{cases} \tag{16}$$

For $K \neq 0$, this relation expresses $e_K$ as a polynomial in the $h_J$ with $|J| \leq |K|$. Conversely, it expresses $h_K$ as a polynomial in the $e_J$ with $|J| \leq |K|$. Since the elementary multisymmetric functions $e_A$ generate $\Lambda_{n,d}$, so do the $h_I$. $\square$

Now let $P$ be a polytopal measure with vertices contained in $X$, and assume that

$$\mathcal{F}_P(\mathbf{u}) = \sum_{I\geq 0} f_I(X)\mathbf{u}^I = \frac{N_P(\mathbf{u}, X)}{D_X(\mathbf{u})}, \qquad N_P(\mathbf{u}, X) = \sum_{|K|\leq m} p_K(X)\mathbf{u}^K, \tag{17}$$

where $m \leq n-d-1$. In the universal simplex-sum situation, the coefficients $p_K(X)$ are explicit linear combinations of oriented simplex volumes multiplied by elementary multisymmetric functions in the complementary vertices.

Multiplying (17) by (15) yields the fundamental convolution formula

$$f_I(X) = \sum_{0\leq K\leq I} p_K(X)h_{I-K}(X), \tag{18}$$

where $p_K = 0$ if $|K| > m$.

**Theorem 3.2.** *With notation as in (17), the $\Lambda_{n,d}$-submodule of $R_{n,d}$ generated by all moment coefficients $f_I(X)$ equals the $\Lambda_{n,d}$-submodule generated by the numerator coefficients $p_K(X)$:*

$$\sum_{I\geq 0}\Lambda_{n,d}f_I(X) = \sum_{|K|\leq m}\Lambda_{n,d}p_K(X). \tag{19}$$

*Consequently, if $m \leq n-d-1$, this module is generated by at most $\binom{n-1}{d}$ elements.*

*Proof.* Formula (18) and Lemma 3.1 immediately imply

$$\sum_{I\geq 0}\Lambda_{n,d}f_I(X) \subseteq \sum_{|K|\leq m}\Lambda_{n,d}p_K(X).$$

For the reverse inclusion, use induction on $|K|$. Since $h_0 = 1$, (18) gives

$$f_K(X) = p_K(X) + \sum_{0 \le L < K} p_L(X) h_{K-L}(X),$$

where $L < K$ means $0 \le L \le K$ and $L \ne K$. Each such $L$ satisfies $|L| < |K|$. By induction, all $p_L$ with $|L| < |K|$ belong to the $\Lambda_{n,d}$-module generated by the $f_I$. Hence $p_K$ also belongs to that module. This proves equality.

Finally, the number of multiindices $K \in \mathbf{Z}_{\ge 0}^d$ with $|K| \le n - d - 1$ is

$$\binom{(n-d-1)+d}{d} = \binom{n-1}{d},$$

which gives the stated bound on the number of module generators. $\square$

**Corollary 3.3.** *The moment ring satisfies*

$$\mathcal{R}(P) \subseteq \mathbf{Q}\big[\Lambda_{n,d},\, p_K(X) : |K| \le m\big].$$

*In particular, it is contained in a finitely generated* $\mathbf{Q}$*-algebra obtained from the diagonal invariant ring by adjoining finitely many geometric numerator coefficients.*

*Proof.* Every generator $f_I$ of $\mathcal{R}(P)$ is a $\Lambda_{n,d}$-linear combination of the finitely many $p_K$ by (18). This proves the inclusion. $\square$

# 4 Consequences and recurrences

The preceding theorem has a useful recurrence interpretation. Write the denominator as in (14). Since

$$D_X(\mathbf{u})\mathcal{F}_P(\mathbf{u}) = N_P(\mathbf{u}, X),$$

comparison of coefficients gives

$$\sum_{0 \le A \le I} (-1)^{|A|} e_A(X) f_{I-A}(X) = p_I(X), \tag{20}$$

where $p_I = 0$ for $|I| > m$.

**Corollary 4.1** (Universal moment recurrences)**.** *For every multiindex $I$ with $|I| > m$, one has*

$$f_I(X) = \sum_{0 < A \le I} (-1)^{|A|+1} e_A(X) f_{I-A}(X). \tag{21}$$

*In particular, if $P$ has $n$ vertices, then all coefficients with $|I| > n - d - 1$ are determined recursively by lower coefficients and by the elementary multisymmetric functions of the vertex configuration.*

*Proof.* For $|I| > m$, the right-hand side of (20) is zero. The term with $A = 0$ is $f_I$, and moving all other terms to the other side gives (21). $\square$

**Corollary 4.2** (Recovery of numerator coefficients)**.** *The numerator coefficients are recovered from the moments by*

$$p_I(X) = \sum_{0 \le A \le I} (-1)^{|A|} e_A(X) f_{I-A}(X), \qquad |I| \le m. \tag{22}$$

*Thus the finite numerator $N_P$ is precisely the obstruction to the moment sequence satisfying the homogeneous recurrence from the beginning.*

**Corollary 4.3** (Finite determination with fixed denominator)**.** *Assume that the denominator $D_X(\mathbf{u})$ is fixed and that $\deg_{\mathbf{u}} N_P \le m$. Then the finitely many coefficients*

$$\{f_I(X) : |I| \le m\}$$

*determine the whole formal power series $\mathcal{F}_P(\mathbf{u})$. In particular, for a polytopal measure with $n$ vertices in $\mathbf{R}^d$, at most*

$$\binom{n-1}{d}$$

*initial normalized moment coefficients determine all the remaining normalized moment coefficients once $D_X$ is known.*

*Proof.* Formula (22) recovers all coefficients of the numerator from the coefficients $f_I$ with $|I| \le m$. The identity $\mathcal{F}_P = N_P/D_X$ then determines the whole series. Equivalently, one may use the recurrence (21) inductively in total degree. □

**Example 4.4** (The simplex case)**.** If $P = \Delta$ is a $d$-simplex, then $n = d+1$ and hence $m = n - d - 1 = 0$. The numerator consists of a single coefficient

$$p_0 = d!\,\mathrm{Vol}(\Delta).$$

Theorem 3.2 says that all moment coefficients of the simplex are contained in the principal $\Lambda_{d+1,d}$-module generated by $d!\,\mathrm{Vol}(\Delta)$. This is exactly the content of the explicit formula

$$\mathcal{F}_\Delta(\mathbf{u}) = d!\,\mathrm{Vol}(\Delta) H_X(\mathbf{u}).$$

**Example 4.5** (A convex quadrilateral)**.** Let $d = 2$ and $n = 4$, and let

$$Q = \mathrm{conv}(q_1, q_2, q_3, q_4), \qquad q_j = (x_j, y_j),$$

be a convex quadrilateral whose vertices are cyclically ordered. Put

$$L_j(u,v) = 1 - x_j u - y_j v, \qquad \delta_{ijk} = \det(q_j - q_i, q_k - q_i).$$

With the chosen cyclic ordering, $\delta_{123}$ and $\delta_{134}$ are positive and equal to twice the Euclidean areas of the triangles $(q_1, q_2, q_3)$ and $(q_1, q_3, q_4)$, respectively. The triangulation along the diagonal $q_1q_3$ gives

$$\mathcal{F}_Q(u,v) = \frac{\delta_{123}}{L_1L_2L_3} + \frac{\delta_{134}}{L_1L_3L_4} = \frac{\delta_{123}L_4 + \delta_{134}L_2}{L_1L_2L_3L_4}. \tag{23}$$

Thus the numerator has degree at most one:

$$N_Q(u,v) = p_{00} + p_{10}u + p_{01}v,$$

where

$$\begin{aligned} p_{00} &= \delta_{123} + \delta_{134} = 2\,\mathrm{Area}(Q), \\ p_{10} &= -(\delta_{123}x_4 + \delta_{134}x_2), \\ p_{01} &= -(\delta_{123}y_4 + \delta_{134}y_2). \end{aligned} \tag{24}$$

Theorem 3.2 says that the whole moment coefficient module of a quadrilateral is generated over $\Lambda_{4,2}$ by these three explicit coefficients:

$$\sum_{a,b\ge 0} \Lambda_{4,2} f_{a,b}(X) = \Lambda_{4,2}p_{00} + \Lambda_{4,2}p_{10} + \Lambda_{4,2}p_{01}.$$

Equivalently, after the denominator

$$D_X(u,v) = \prod_{j=1}^{4}(1 - x_j u - y_j v) = \sum_{r,s\ge 0} (-1)^{r+s} e_{r,s}(X) u^r v^s$$

is fixed, the three normalized moment coefficients $f_{00}, f_{10}, f_{01}$ determine all higher ones. For instance,

$$\begin{aligned} f_{20} &= e_{10}f_{10} - e_{20}f_{00}, \\ f_{11} &= e_{10}f_{01} + e_{01}f_{10} - e_{11}f_{00}, \\ f_{02} &= e_{01}f_{01} - e_{02}f_{00}. \end{aligned}$$

This is the first planar case in which the numerator is nonconstant, and it makes the finite-module theorem completely explicit.

**Proposition 4.6** (Planar flip identity)**.** *Let*

$$Q = \mathrm{conv}(q_1, q_2, q_3, q_4)$$

*be a convex quadrilateral with cyclically ordered vertices, and put*

$$L_j(u,v) = 1 - x_j u - y_j v, \qquad \delta_{ijk} = \det(q_j - q_i, q_k - q_i).$$

*Then*

$$\delta_{123}L_4 + \delta_{134}L_2 = \delta_{124}L_3 + \delta_{234}L_1. \tag{25}$$

*Consequently the common numerator of the Fantappiè transform of a convex planar polygon is independent of the triangulation used to compute it.*

*Proof.* The left-hand side of (25) is the numerator obtained from the triangulation of $Q$ by the diagonal $q_1q_3$, as in Example 4.5. The right-hand side is the numerator obtained from the triangulation by the diagonal $q_2q_4$:

$$\mathcal{F}_Q(u,v) = \frac{\delta_{124}}{L_1L_2L_4} + \frac{\delta_{234}}{L_2L_3L_4} = \frac{\delta_{124}L_3 + \delta_{234}L_1}{L_1L_2L_3L_4}.$$

Both triangulations represent the same standard measure on $Q$, so the two rational functions agree. Multiplying by $L_1L_2L_3L_4$ gives (25). Finally, any two triangulations of a convex polygon are connected by a sequence of diagonal flips, and each flip is exactly the quadrilateral identity above. □

**Example 4.7** (A convex planar $n$-gon)**.** Let $d = 2$, and let

$$P = \mathrm{conv}(q_1, \ldots, q_n), \qquad q_j = (x_j, y_j),$$

be a convex $n$-gon whose vertices are cyclically ordered. As above, put

$$L_j(u,v) = 1 - x_j u - y_j v, \qquad D_X(u,v) = \prod_{j=1}^{n} L_j(u,v).$$

The following formula uses the fan triangulation from $q_1$. Proposition 4.6 shows that the resulting numerator is independent of this choice. Triangulate $P$ by the fan from $q_1$:

$$P = \bigcup_{i=2}^{n-1} \mathrm{conv}(q_1, q_i, q_{i+1}),$$

with disjoint interiors. Set

$$\delta_i = \det(q_i - q_1, q_{i+1} - q_1), \qquad 2 \le i \le n-1.$$

For the chosen cyclic ordering all $\delta_i$ are positive and equal to twice the Euclidean areas of the fan triangles. Proposition 1.1 and additivity give

$$\mathcal{F}_P(u,v) = \sum_{i=2}^{n-1} \frac{\delta_i}{L_1L_iL_{i+1}} = \frac{N_P(u,v)}{D_X(u,v)}, \tag{26}$$

where

$$N_P(u,v) = \sum_{i=2}^{n-1} \delta_i \prod_{j\notin\{1,i,i+1\}} L_j(u,v). \tag{27}$$

Thus $\deg N_P \le n-3$, as predicted by the general bound $n-d-1$.

For any subset $Y \subseteq \{q_1,\dots,q_n\}$ define $e_{ab}(Y)$ by

$$\prod_{q_j\in Y}(1-x_ju-y_jv) = \sum_{a,b\ge 0}(-1)^{a+b}e_{ab}(Y)u^av^b.$$

Writing

$$N_P(u,v) = \sum_{a+b\le n-3} p_{ab}u^av^b,$$

we obtain the explicit formula

$$p_{ab} = (-1)^{a+b}\sum_{i=2}^{n-1}\delta_i\, e_{ab}\big(\{q_1,\dots,q_n\}\setminus\{q_1,q_i,q_{i+1}\}\big), \qquad a+b\le n-3. \tag{28}$$

In particular,

$$p_{00} = \sum_{i=2}^{n-1}\delta_i = 2\,\mathrm{Area}(P).$$

More generally, if $\mathcal{T}$ is any triangulation of the same cyclically ordered polygon and if $\delta_{ijk}$ denotes twice the oriented area of the triangle $(q_i,q_j,q_k)$, then the same numerator can be written in the triangulation-independent form

$$N_P(u,v) = \sum_{\{i,j,k\}\in\mathcal{T}} \delta_{ijk} \prod_{\ell\notin\{i,j,k\}} L_\ell(u,v), \tag{29}$$

where the vertices of each triangle are taken in cyclic order. Independence of $\mathcal{T}$ follows from Proposition 4.6.

Theorem 3.2 therefore gives the concrete planar statement

$$\sum_{a,b\ge 0}\Lambda_{n,2}f_{ab}(X) = \sum_{a+b\le n-3}\Lambda_{n,2}p_{ab}(X), \tag{30}$$

so the moment coefficient module of a convex $n$-gon is generated over the multisymmetric invariant ring by at most

$$\#\{(a,b): a+b\le n-3\} = \binom{n-1}{2}$$

explicit numerator coefficients. Equivalently, after the denominator $D_X$ is fixed, the coefficients

$$\{f_{ab} : a+b\le n-3\}$$

determine the entire normalized moment series. All higher coefficients satisfy

$$f_{ab} = \sum_{\substack{0\le r\le a,\ 0\le s\le b\\ (r,s)\ne(0,0)}} (-1)^{r+s+1}e_{rs}(X)f_{a-r,b-s}, \qquad a+b>n-3. \tag{31}$$

For $n=4$, formulae (26)–(28) reduce to the preceding quadrilateral example.

**Example 4.8** (One-dimensional polytopes)**.** For $d=1$, a polytope is a finite union of intervals with endpoints among $x_1,\dots,x_n$. The denominator is

$$D_X(u) = \prod_{j=1}^{n}(1-x_ju),$$

and the recurrence (21) becomes the usual linear recurrence for moment sequences with characteristic coefficients given by the elementary symmetric polynomials in the endpoints. Thus the theorem recovers, in dimension one, the standard finite recurrence for power sums.

*Remark* 4.9. The bound $\binom{n-1}{d}$ is an a priori bound. It is not expected to be sharp in general. Cancellations in the numerator, affine dependencies among the vertices, symmetries of $P$, or relations among simplex volumes may reduce the minimal number of module generators.

# 5 Diagonal coinvariants and open problems

The invariant ring $\Lambda_{n,d}$ is the coordinate ring of the quotient of $n$ ordered points in affine $d$-space by the diagonal action of $\mathfrak{S}_n$. This should not be confused with the finite-dimensional diagonal harmonic modules themselves. Rather, diagonal harmonics enter after one passes from $\Lambda_{n,d}$ to its positive-degree ideal and to the corresponding coinvariant quotient. Theorem 3.2 gives a geometric source of finite modules over $\Lambda_{n,d}$: the moment coefficient modules of polytopal measures.

A natural next step is to pass from such a module to its diagonal coinvariant shadow by tensoring with

$$\mathbf{Q} = \Lambda_{n,d}/\Lambda_{n,d,+},$$

where $\Lambda_{n,d,+}$ is the ideal of positive-degree invariants. This produces a finite-dimensional space measuring the part of the moment data not generated by diagonal invariant coefficients. Understanding this quotient could lead to a more direct connection with diagonal harmonics. In the two-alphabet case this is the setting of the Garsia–Haiman theory and Haiman's Hilbert-scheme approach to diagonal harmonics [4, 6].

We close with several problems suggested by the preceding results.

**Problem 5.1.** For a given combinatorial type of polytope $P$, determine the minimal number of generators of the $\Lambda_{n,d}$-module

$$\sum_I \Lambda_{n,d} f_I(X).$$

How does this number depend on the face lattice, on affine dependencies among the vertices, and on symmetries of $P$?

**Problem 5.2.** Describe the syzygies among the numerator coefficients $p_K(X)$ as a module over $\Lambda_{n,d}$. Equivalently, describe the relations among all moment coefficients after the universal recurrences (21) have been imposed.

**Problem 5.3.** Determine to what extent the finite numerator $N_P(\mathbf{u}, X)$, together with the denominator $D_X(\mathbf{u})$, determines the polytope or the corresponding polytopal measure. In particular, characterize pairs of noncongruent polytopes with the same moment coefficient module.

**Problem 5.4.** Study the diagonal coinvariant quotient of the moment coefficient module,

$$\left(\sum_I \Lambda_{n,d} f_I(X)\right) \otimes_{\Lambda_{n,d}} \Lambda_{n,d}/\Lambda_{n,d,+},$$

and compare it with known modules of diagonal harmonics, especially for $d = 2$.

**Problem 5.5.** Clarify the relation between Conjecture 2.5 and the universal simplex-sum model. If Conjecture 2.5 holds for a finite set $S$, then every generalized polytope with vertices in $S$ gives a point in the universal family (11). What are the precise linear relations among the corresponding simplex measures?

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