ON BOUNDARY POINTS OF MINIMAL CONTINUOUSLY HUTCHINSON INVARIANT SETS

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Abstract. A linear differential operator $T = Q(z)\frac{d}{dz} + P(z)$ with polynomial coefficients defines a continuous family of Hutchinson operators when acting on the space of positive powers of linear forms has a unique minimal Hutchinson-invariant set in the complex plane. Using a geometric interpretation of the boundary of this minimal set in terms of envelops of certain families of rays, we subdivide its boundary points into local arcs (portions of integral curves of the vector field $\frac{Q(z)}{P(z)}\partial_z$), global arcs, and finitely many singular points of different types which we classify.

The latter decomposition of the boundary of the minimal set appears to be largely determined by its intersection with the plane algebraic curve formed by the inflection points of trajectories of the rational vector field $\frac{Q(z)}{P(z)}\partial_z$. We provide an upper bound of the number of the above local arcs in terms of $\deg P$ and $\deg Q$. As an application of our classification, we deduce new global geometric properties of minimal Hutchinson-invariant sets.

Contents

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1. Introduction

Given a linear differential operator

$$T = Q(z)\frac{d}{dz} + P(z) \tag{1.1}$$

where P, Q are polynomials that are not identically vanishing, we say that a closed subset $S \subset \mathbb{C}$ is *continuously Hutchinson invariant* for T (T_{CH}-invariant set for short) if for any $u \in S$ and any arbitrary non-negative number t, the image $T(f)$ of the function $f(z) = (z-u)^t$ either has all roots in S or vanishes identically. In [AHN+22], we have initiated the study of general topological properties of T_{CH}-invariants sets. In particular, the following results have been obtained:

- provided that either P or Q is not a constant polynomial, there is a unique *minimal* continuously Hutchinson invariant set M_{CH}^{T} for a given operator T (in what follows we will always assume that this condition is satisfied);
- the only T_{CH}-invariant set is the whole $\mathbb{C}$ unless $|\deg Q - \deg P| \leq 1$;
- a complete characterization of operators T for which M_{CH}^{T} has an empty interior has been obtained (see Section 2.1 for details).

In this paper, we will focus on operators whose minimal set M_{CH}^{T} has a nonempty interior.

Definition 1.1. For an operator T given by (1.1) with P and Q not vanishing identically, at each point z such that $PQ(z) \neq 0$, we define the *associated ray* $r(z)$ as the half-line $\{z + t\frac{Q(z)}{P(z)} \mid t \in \mathbb{R}^+\}$.

Remarkably, T_{CH}-invariant sets (and, in particular, the minimal one) can be characterized in terms of associated rays.

Theorem 1.2 (Theorem 3.18 in [AHN+22])**.** *A closed subset $S \subseteq \mathbb{C}$ is T_{CH}-invariant if and only if it satisfies the following two conditions:*

(1) S contains the roots of the polynomials P and Q;
(2) for any point $z \notin S$, the associated ray $r(z)$ is disjoint from S.

1.1. **Main results.** In the present paper, using Theorem 1.2, we provide a qualitative description of the boundary of minimal continuously Hutchinson invariant sets, including an exhaustive typology of its singular points. Our classification mainly depends on the intersection of the boundary $\partial \mathrm{M}_{CH}^{T}$ with the curve of inflections $\mathfrak{I}_R$ of the field $R(z)\partial_z = \frac{Q(z)}{P(z)}\partial_z$.

Definition 1.3. The *curve of inflections* $\mathfrak{I}_R$ of the vector field $R(z)\partial_z$ is defined as the closure of the set of points satisfying $\mathrm{Im}(R') = 0$, see [AHN+22]. It is a real plane algebraic curve of degree at most $d = 3\deg P + \deg Q - 1$.

The curve of inflections splits the complex plane into *inflection domains* where the sign of $\mathrm{Im}(R')$ remains the same.

Points of $\partial \mathrm{M}_{CH}^{T}$ outside its intersection with $\mathfrak{I}_R$ are classified with the help of two correspondences Γ and Δ sending the boundary $\partial \mathrm{M}_{CH}^{T}$ to itself and defined as follows:

For a given point z of the boundary $\partial \mathrm{M}_{CH}^{T}$, $\Gamma(z)$ is essentially the intersection of M_{CH}^{T} with the integral curve of the rational field $R(z)\partial_z$ starting at z, where $R(z) = Q(z)/P(z)$. In contrast, Δ is the intersection of $\partial \mathrm{M}_{CH}^{T}$ with the associated ray $r(z)$ (see Definition 4.1 for details). Qualitatively, the boundary $\partial \mathrm{M}_{CH}^{T}$ is made of two kinds of arcs:

- *local arcs* which are integral curves of the field $R(z)\partial_z$ (i.e. $\Delta(z) = \emptyset$ and $\Gamma(z) \neq \emptyset$);
- *global arcs* at each point z of which the associated ray $r(z)$ is tangent to $\partial \mathrm{M}_{CH}^{T}$ elsewhere (i.e. $\Gamma(z) = \emptyset$ and $\Delta(z) \neq \emptyset$).

Local arcs are locally strictly convex real-analytic arcs (see Proposition 4.11). In contrast, global arcs (formed by points of global type) can fail to be C^1.

Local arcs inherit an obvious orientation from the vector field $R(z)\partial_z$. Global arcs also have canonical orientation, but its definition requires some work (see Section 4.4.2).

Local and global arcs connect special singular points of $\partial \mathrm{M}_{CH}^{T}$ which in most of the cases belong to the curve of inflections. The latter decomposes into three loci (singular, tangent and transverse), each determining its own variety of singular points.

Definition 1.4. The *curve of inflections* $\mathfrak{I}_R$ of the field $R(z)\partial_z$ decomposes into:

- the *singular locus* $\mathfrak{S}_R$ formed by the points where several branches of $\mathfrak{I}_R$ intersect;
- the *tangency locus* $\mathfrak{T}_R$ formed by the nonsingular points where the field $R(z)\partial_z$ is tangent to $\mathfrak{I}_R$;
- *transverse locus* $\mathfrak{I}_R^*$ formed by the nonsingular points of $\mathfrak{I}_R$ where the field $R(z)\partial_z$ is transverse to $\mathfrak{I}_R$.

The singular and the tangency loci are given by algebraic conditions. Therefore their intersection with $\partial \mathrm{M}_{CH}^{T}$ is controlled in terms of $\deg P$ and $\deg Q$. On the contrary, many points of the boundary can belong to the transverse locus $\mathfrak{I}_R^*$. We refine the definition of the correspondence Δ according to the value of $\frac{R(z)}{u-z}$ (which, by definition, is a positive number).

Definition 1.5. For any $z \in \mathfrak{I}_R \setminus \mathcal{Z}(PQ)$, we have $\Delta(z) = \Delta^-(z) \cup \Delta^0(z) \cup \Delta^+(z)$ where $u \in \Delta(z)$ belongs to:

- $\Delta^-(z)$ if $R'(z) \leq -\frac{R(z)}{u-z}$;
- $\Delta^0(z)$ if $R'(z) = -\frac{R(z)}{u-z}$;
- $\Delta^+(z)$ if $R'(z) \geq -\frac{R(z)}{u-z}$.

In particular, if $R'(z) > 0$, then $\Delta^-(z) = \emptyset$.

The main result of the present paper is a classification of boundary points of minimal continuously Hutchinson sets.

Theorem 1.6. *For any linear differential operator T given by* (1.1)*, any point z of the boundary $\partial \mathrm{M}_{CH}^{T}$ of its minimal T_{CH}-invariant set belongs to one of the following types:*

- roots of polynomials P *and* Q *(at most* $\deg P + \deg Q$ *of them);*
- singular points of the curve of inflections *(at most* $2d$ *of them);*
- tangency points between the curve of inflections and the field $R(z)\partial_z$*:*
  - *straight segments, half-lines and lines (contained in at most* $\deg P + \deg Q + 1$ *lines);*
  - *at most* $2d^2$ *isolated points;*
- points of the transverse locus $\Im_R^*$ *belonging to one of the four subclasses:*
  - bouncing type*:* $\Delta^+ \neq \emptyset$ *and* $\Gamma \cup \Delta^- \neq \emptyset$*;*
  - switch type*:* $\Delta^+(z) \neq \emptyset$ *and* $\Gamma \cup \Delta^-(z) = \emptyset$*;*
  - C^1-inflection type*:* $\Delta^+ = \emptyset$*,* $\Delta^- \neq \emptyset$ *and* $\Gamma = \emptyset$*;*
  - C^2-inflection type*:* $\Delta^+ = \emptyset$ *and either* $\Delta^- = \emptyset$ *or* $\Gamma \neq \emptyset$*.*
- points not on the curve of inflections *belonging to one of the three subclasses:*
  - local type*:* $\Gamma(z) \neq \emptyset$ *and* $\Delta(z) = \emptyset$*;*
  - global type*:* $\Gamma(z) = \emptyset$ *and* $\Delta(z) \neq \emptyset$*;*
  - extruding type*:* $\Gamma(z) \neq \emptyset$ *and* $\Delta(z) \neq \emptyset$*.*

Here, $d = 3\deg P + \deg Q - 1$*.*

There can be many singular points of bouncing, extruding, C^1-inflection, C^2-inflection and switch types (we do not have a polynomial bound of their number in terms of $\deg P$ and $\deg Q$). An extensive description of their geometric features is given below:

- at *points of extruding type*, the boundary of $\partial \mathrm{M}_{CH}^{T}$ is not convex and it switches from a global to a local arc (see Section 4.5 and Figure ???);
- at *points of bouncing type*, $\partial \mathrm{M}_{CH}^{T}$ hits the curve of inflections, but does not cross it. In a neighborhood of such a point, the boundary $\partial \mathrm{M}_{CH}^{T}$ remains in the closure of the same inflection domain (see Section 5.2 and Figure ???);
- at *points of switch type*, $\partial \mathrm{M}_{CH}^{T}$ is strictly convex, crosses the curve of inflections and the boundary switches from a local to a global arc (see Section 5.5 and Figure ???);
- at points of C^1*-inflection type*, $\partial \mathrm{M}_{CH}^{T}$ crosses the curve of inflections and it switches from a global to another global arc having the opposite orientation. At such a point the curvature of $\partial \mathrm{M}_{CH}^{T}$ is discontinuous (see Section 5.4 and Figure ???);
- at points of C^2*-inflection type*, $\partial \mathrm{M}_{CH}^{T}$ crosses the curve of inflections and the boundary switches from a global arc to a local arc. Besides, the curvature of $\partial \mathrm{M}_{CH}^{T}$ is continuous at such a point (see Section 5.3 and Figure ???).

Our second main result is an upper bound on the number of points of C^1-inflection, C^2-inflection and switch type in terms of $d = 3\deg P + \deg Q - 1$.

Theorem 1.7. *??? For any operator T given by* (1.1)*,* $\partial \mathrm{M}_{CH}^{T} \cap (\Im_R)^c$ *contains at most* Ae^{Bd^2} *local arcs (maximal open arcs of* $\partial \mathrm{M}_{CH}^{T}$ *coinciding with integral curves of field* $R(z)\partial_z$*). They are real-analytic and locally strictly convex. (A and B are explicitly given below.) ???*

In the last section of the paper, we deduce many results about the global geometry of minimal sets from the classification of boundary points. In several cases, an exact description can be given in terms of local and global arcs. In particular, we can prove that in generic case, the minimal T_{CH}-invariant set is connected in $\mathbb{C}$.

Theorem 1.8. *For any linear differential operator T given by* (1.1)*, the minimal continuously Hutchinson invariant set* M^T_{CH} *is a connected subset of* $\mathbb{C}$ *with the possible exception of the case when* $R(z)$ *is of the form* $\lambda + \frac{\mu}{z} + o(z^{-1})$ *with* $\lambda \in \mathbb{C}^*$ *and* $\mu/\lambda \in \mathbb{R}$.

In this later case, (unless both P *and* Q *are constants and then there is no reasonable notion of a minimal set),* M^T_{CH} *is formed by at most* $\frac{1}{2}\deg P + \frac{1}{2}\deg Q$ *connected components.*

1.2. **Organization of the paper.**

- In Section 2, we provide the basic background information on Hutchinson invariant sets developed in [AHN+22], including the results about their asymptotic geometry.
- In Section 3, we describe the local geometry around singular points of the vector field $R(z)\partial_z$ in terms of their local degree and principal value. We also describe the main properties of the curve of inflections defined by the equation $Im(R') = 0$ and we also introduce the notion of horns.
- In Section 4, we describe boundary points in the complement to the curve of inflections, introducing $\Gamma-$ and $\Delta-$ correspondences.
- In Section 5, we classify boundary points in the generic locus of the curve of inflections, proving Theorems 1.6 and 1.7 (in Sections 5.6 and 5.8 respectively).
- In Section 6, we apply the latter results to get precise descriptions of minimal sets in several cases. Theorem 1.8 is proven in Section 6.5.

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2. Preliminary results and basic properties of M^T_{CH}

The following notation will be important throughout this text.

Notation 2.1. Given an operator T as in (1.1), we define $p_\infty, q_\infty \in \mathbb{C}^*$, and $p, q \in \mathbb{N}$ so that

$$P(z) = p_\infty z^p + o(z^p);$$

$$Q(z) = q_\infty z^q + o(z^q).$$

Furthermore, we set $\lambda = \frac{q_\infty}{p_\infty} \in \mathbb{C}^*$ and $\phi_\infty = \arg(\lambda)$.

Similarly, for any point $\alpha \in \mathbb{C}$, we have $R(z) = r_\alpha (z-\alpha)^{m_\alpha} + o(|z-\alpha|^{m_\alpha})$ with $r_\alpha \neq 0$ and $m_\alpha \in \mathbb{Z}$. We denote by ϕ_α the argument of r_α.

Remark 2.2. Observe that frequently used affine changes of the variable z are applied to the vector field $R(z)\partial_z$ and not to the rational function $R(z)$ itself.

2.1. **Regularity of the minimal set.** For an operator T as in (1.1), its minimal set M_{CH}^T can be of three possible types:

- *regular* if M_{CH}^T coincides with the closure of its interior;
- *fully irregular* if M_{CH}^T has empty interior;
- *partially irregular* if M_{CH}^T has nonempty interior but is not regular.

Actually, irregularity is related to specific reality conditions. The characterization of operators for which M_{CH}^T is fully irregular is contained in Theorem 1.15 of [AHN+22].

Theorem 2.3. *For an operator T as in* (1.1)*, the minimal set M_{CH}^T is fully irregular in the following cases:*

- *$R(z) = \lambda$ for some $\lambda \in \mathbb{C}^*$;*
- *$R(z) = \lambda(z-\alpha)$ for some $\lambda \notin \mathbb{R}_{<0}$, $\alpha \in \mathbb{C}$ and $\deg Q = 1$;*
- *$R(z) = \lambda(z-\alpha)$ for some $\lambda \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{C}$ and $\deg Q \geq 2$;*
- *operators satisfying the following conditions (up to an affine change of variable):*
  - *$R(z)$ is real on $\mathbb{R}$;*
  - *roots of P and Q are real, simple and interlacing (i.e. the roots of P and Q alternate along the real axis);*
  - *$|\deg Q - \deg P| \leq 1$;*
  - *if $\deg Q - \deg P = 1$, then $\lambda \in \mathbb{R}_{>0}$.*

In any other case, M_{CH}^T has a nonempty interior.

In this paper, we will always assume that M_{CH}^T has a nonempty interior.

Remark 2.4. If $\deg P + \deg Q \leq 1$, then M_{CH}^T is either totally irregular or coincides with $\mathbb{C}$ (see Theorem 1.15 of [AHN+22]). Therefore, our operators will always satisfy $\deg P + \deg Q \geq 2$.

Referring to the closure of the interior of M_{CH}^T as the *regular locus* and its complement in M_{CH}^T as the *irregular locus*, we observe that the latter is contained in very specific lines of the plane.

Definition 2.5. For a given rational function $R(z)$, a line Λ is called R-invariant if for any $z \in \Lambda$ such that $R(z)$ is defined, we have $z + R(z) \in \Lambda$.

In particular, up to an affine change of variable, we can assume $\Lambda = \mathbb{R}$ and thus $R(z)$ is a real rational function. Besides, a R-invariant line is automatically an irreducible component of the curve of inflections $\mathfrak{I}_R$.

Definition 2.6. For an operator T whose minimal set M_{CH}^T is not fully irregular, a *tail* is a semi-open straight segment $[\alpha, \beta[$ in M_{CH}^T satisfying the following conditions:

- the segment $]\alpha, \beta]$ belongs to an R-invariant line;
- for any $z \in]\alpha, \beta]$, $\frac{\beta-\alpha}{R(z)} \in \mathbb{R}_{>0}$;
- for any $z \in]\alpha, \beta]$, z is disjoint from the regular locus of M_{CH}^T;
- α belongs to the regular locus of M_{CH}^T;
- $\beta \in \mathcal{Z}(PQ)$;
- β is a root of the same multiplicity for both P and Q.

In particular, every tail belongs to a R-invariant line.

The following fact has been proven in Corollary 7.8 of [AHN+22].

Theorem 2.7. *For an operator T whose minimal set M_{CH}^T is not fully irregular, the irregular locus of M_{CH}^T is a (possibly empty) finite union of tails.*

In particular, if P and Q have no common roots, then the minimal set of the corresponding operator is either regular or fully irregular.

2.2. **Extended complex plane.** Following Theorem 1.2, T_{CH}-invariant sets are characterized by the position of the associated rays starting in their complements. Let us introduce a certain compactification[1] of $\mathbb{C}$ which comes very handy in our considerations. We baptise it the *extended complex plane* $\mathbb{C} \cup \mathbb{S}^1 \supset \mathbb{C}$.

The extended complex plane $\mathbb{C} \cup \mathbb{S}^1$ is set-theoretically the disjoint union of $\mathbb{C}$ and $\mathbb{S}^1$ endowed with the topology defined by the following basis of neighborhoods:

- for a point $x \in \mathbb{C}$, we choose the usual open neighborhoods of x in $\mathbb{C}$;
- for a direction $\theta \in \mathbb{S}^1$, we choose open neighborhoods of the form $I \cup C(z, I)$ where I is an open interval of $\mathbb{S}^1$ containing θ and $C(z, I)$ is an open cone with apex $z \in \mathbb{C}$ whose opening (i.e. the interval of directions) is I.

Definition 2.8. Given $R(z)$ as above, let $p \in \mathbb{C}$ be a non-singular point of $R(z)$. We define $\sigma(p)$ as the argument of $R(p)$. We think of $\sigma(p)$ as a point of the circle at infinity.

One can easily see that $\mathbb{S}^1$ of the extended plane $\mathbb{C} \cup \mathbb{S}^1$ can be identified with the above circle at infinity. The extended plane is compact and homeomorphic to a closed disk. In particular, usual straight lines in $\mathbb{C}$ have compact closures in $\mathbb{C} \cup \mathbb{S}^1$. (Below we will make no distinction between a real line in $\mathbb{C}$ and its closure in $\mathbb{C} \cup \mathbb{S}^1$). *Open half-planes* in $\mathbb{C} \cup \mathbb{S}^1$ are, by definition, connected components of the complement to a line.

Given a T_{CH}-invariant set $S \subset \mathbb{C}$, we denote by $\overline{S}$ its closure in the extended plane $\mathbb{C} \cup \mathbb{S}^1$.

The following result has been proved in Lemma 4.4 of [AHN+22].

Lemma 2.9. *Given an T_{CH}-invariant set $S \subset \mathbb{C}$, let $\alpha : [0, 1] \to \mathbb{C}$ be such that:*

- $\forall t \in (0, 1)$, $\alpha_t \in S^c$;
- $\sigma(\alpha_0) \neq \sigma(\alpha_1)$;
- *$\sigma(\alpha)$ is homotopic to the positive arc from $\sigma(\alpha_0)$ to $\sigma(\alpha_1)$ in the circle at infinity via a homotopy $H(t, x) : [0, 1] \times [0, 1]$ such that $H(0, x_0) = \sigma(\alpha(0))$, $H(1, x_0) = \sigma(\alpha(1))$ for all $x_0 \in [0, 1]$.*

If X denotes the connected component containing the interval $]\sigma(\alpha_0), \sigma(\alpha_1)[$ in the complement of $r(\alpha_0) \cup \alpha \cup r(\alpha_1)$, then $X \subset S^c$.

2.3. **Integral curves.** Another result has been proved in Proposition A.2 of [AHN+22].

Proposition 2.10. *Given a T_{CH}-invariant set $S \subset \mathbb{C}$ and some point $z_0 \in S$, if there is a positively oriented integral curve $\gamma : [0, \epsilon[\to \mathbb{C}$ of the vector field $R(z)\partial_z$ such that $\lim_{t \to \epsilon} \gamma(t) = z_0$, then for any $t \in [0, \epsilon]$, $\gamma(t) \in S$.*

2.4. **Root trails.** For any point $u \in \mathbb{C}$, the *root trail* $\mathfrak{tr}_u$ of u is the closure of the set of points z such that the associated ray $r(z)$ contains u. Except for the trivial cases described in Section 3 of [AHN+22], root trails are plane real-analytic curves. By definition, the root trail of any point of M_{CH}^T is also contained in M_{CH}^T. Furthermore, for any fixed $u \in \mathbb{C}$, we defined a *t-trace* (corresponding to u) as any continuous function $\gamma(t)$ such that

$$Q(\gamma(t)) + (\gamma(t) - u)P(\gamma(t)) = 0$$

for all $t \geq 0$. That is, any t-trace $\gamma(t)$ is a concatenation of parts of $\mathfrak{tr}_u$ such that the resulting curve is continuous for any $t \geq 0$.

[1]Notice that the most frequently used compactification of $\mathbb{C}$ is $\bar{\mathbb{C}} = \mathbb{C}P^1$.

Lemma 2.11. *Consider a linear differential operator T given by* (1.1) *and some point $u \in \mathbb{C}$. Assuming that $R(z)$ is not of the form $\lambda(z-u)$, then*

(i) for any point $u \in \mathbb{C}$ and any point $z_0 \notin \mathcal{Z}(PQ)$ such that $z_0 \in \mathfrak{tr}_u$ and $R(z_0)+(u-z_0)R'(z_0) \neq 0$, the root trail $\mathfrak{tr}_u$ has a unique branch passing through z_0 and its tangent slope is the argument of $\frac{R^2(z_0)}{R(z_0)+(u-z_0)R'(z_0)}$ (mod π).

(ii) If $R(z_0)+(u-z_0)R'(z_0)=0$ and $m \geq 2$ is the smallest integer such that $R^{(m)}(z_0) \neq 0$, then $\mathfrak{tr}_u$ has m intersecting branches at z. Their tangent slopes are:

$$\frac{\theta_0}{m}+\frac{k\pi}{m},$$

where θ_0 is the argument of $\frac{R(z_0)}{R^{(m)}(z_0)}$ and $k \in \mathbb{Z}/m\mathbb{Z}$.

Before proving Lemma 2.11 we prove the next two Lemmas.

Lemma 2.12. *If $\gamma(t)$ is smooth planar curve, $\gamma(0)=z_0$, and $\dot\gamma(t)=G(\gamma(t))$ for some function G holomorphic and non-vanishing at z_0 then the sign of the curvature of $\gamma(t)$ at z_0 coincides with the sign of* $\operatorname{Im} G'(0)$.

Indeed, then $\ddot\gamma(t)=G'(\gamma(t))\cdot\dot\gamma(t)$. By definition, the sign of the curvature of $\gamma(t)$ at z_0 coincides with the sign of $\operatorname{Im}\frac{\ddot\gamma(t)}{\dot\gamma(t)}|_{t=0}=\operatorname{Im} G'(0)$.

Lemma 2.13. *Let F be a rational function holomorphic at z_0 with $F(z_0)\in\mathbb{R}$ and let $m=\operatorname{ord}_{z_0}F'+1$. Then the germ of $I_F=\{\operatorname{Im} F=0\}$ at z_0 consists of m smooth branches with tangent slopes $\frac{\theta_0}{m}+\frac{k\pi}{m}$, $k\in\mathbb{Z}/m\mathbb{Z}$, where $\theta_0=-\arg F^{(m)}(z_0)$.*

If $m=1$ and $\gamma(t)$ is a parameterization of I_F such that $F(\gamma(t))\equiv F(z_0)+t$ then the sign of the curvature of $\gamma(t)$ coincides with the sign of $-\operatorname{Im}\left[\frac{F''(z_0)}{(F')^2(z_0)}\right]$.

Proof. Indeed, we have $F(z)=a_0+a_m(z-z_0)^m+\ldots$, $a_0\in\mathbb{R}$, so the branches of I_F are tangent to the m lines satisfying equation $\operatorname{Im} a_m(z-z_0)^m=0$, which have slopes as stated.

For the second claim, note that $\dot\gamma(t)=\frac{1}{F'(\gamma(t))}$, so the claim follows from Lemma 2.12. □

Proof of Lemma 2.11. Note that by definition

$$\mathfrak{tr}_u=\left\{z\in\mathbb{C}\text{ s.t. }\frac{R(z)}{u-z}\in\mathbb{R}_+\right\}\subset\left\{\operatorname{Im}\frac{R(z)}{u-z}=0\right\},$$

and Lemma 2.11 follows from the Lemma 2.13 with $F(z)=\frac{R(z)}{u-z}$ and the fact that $\arg R(z_0)=\arg(u-z_0)$. □

Remark 2.14. The condition $R(z_0)+(u-z_0)R'(z_0)=0$ means that the point $u=z_0-\frac{R(z_0)}{R'(z_0)}$ is obtained as the the first iteration of Newton's method of approximating roots of $R(z)$ with the starting point z_0.

When u is a point at infinity in the extended plane $\mathbb{C}\cup\mathbb{S}^1$, the root trail $\mathfrak{tr}_u$ of u is the closure of the points z where the argument of $R(z)$ coincides with u.

Lemma 2.15. *Consider a linear differential operator T given by* (1.1) *such that $R(z)$ is not constant. For any point u at infinity and any point $z_0\notin\mathcal{Z}(PQ)$ such that $z_0\in\mathfrak{tr}_u$, provided $R'(z_0)\neq 0$, the root trail $\mathfrak{tr}_u$ has a unique branch passing through z_0 and its tangent slope is the argument of $\frac{R(z_0)}{R'(z_0)}$ (mod π).*

If $R'(z_0)=0$ and $m\geq 2$ is the smallest integer such that $R^{(m)}(z_0)\neq 0$, then $\mathfrak{tr}_u$ has m intersecting branches at z. Their tangent slopes are:

$$\frac{\theta_0}{m}+\frac{k\pi}{m},$$

where θ_0 is the argument of $\frac{R(z_0)}{R^{(m)}(z_0)}$ and $k\in\mathbb{Z}/m\mathbb{Z}$.

Proof. In this case $\mathfrak{tr}_u \subset \{\operatorname{Im}(R(z)/R(z_0)) = 0\}$ and the claim follows again from Lemma 2.13 □

Remark 2.16. From Lemmas 2.11 and 2.15 it immediately follows that if a root trail $\mathfrak{tr}_u$ can have $m \geq 2$ branches at some point z_0, then z_0 belongs to the curve of inflections $\mathfrak{I}_R$ (because $R(z_0)$ and $u - z_0$ are real colinear).

Besides, if $m \geq 3$, then $R^{(k)}(z_0) = 0$ for $2 \leq k \leq m-1$ and z_0 is a singular point of $\mathfrak{I}_R$.

2.4.1. *Concavity of root trails.*

Proposition 2.17. *Let u be a point of the extended plane $\mathbb{C} \cup \mathbb{S}^1$ and z_0 be a point of $\mathfrak{tr}_u$ such that $z_0 \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $z_0 \neq u$. We denote by L the tangent line to $\mathfrak{tr}_u$ at z_0. We define $f(z,u)$ to be:*

- $\frac{(u-z_0)R(z_0)[R(z_0)R''(z_0)-2(R'(z_0))^2]-2R^2(z_0)R'(z_0)}{(R(z_0)+(u-z_0)R'(z_0))^2}$ *if $u \in \mathbb{C}$;*
- $\frac{R''(z_0)R(z_0)}{R'(z_0)^2}$ *if u is a point at infinity.*

Then the germ of $\mathfrak{tr}_u$ at z_0 belongs to

(i) the same half-plane bounded by L as the associated ray $r(z_0)$ if $\operatorname{Im}(f)$ and $\operatorname{Im}(R'(z_0))$ have opposite signs.

(ii) They belong to distinct half-planes bounded by L if $\operatorname{Im}(f)$ and $\operatorname{Im}(R'(z_0))$ have the same sign.

Finally, $\mathfrak{tr}_u$ has an inflection point at z_0 if $\operatorname{Im}(f) = 0$.

Alternative Proof of Proposition 2.17. I got something different, please check.
— Mitya

Proof. Let $F_u(z) = \frac{R(z)}{u-z}$ for $u \in \mathcal{C}$ and $F_u(z) = u^{-1}R(z)$ for $u \in \mathbb{S}^1$ so that $\mathfrak{tr}_u = \{\operatorname{Im} F_u(z) = 0\}$. Let $c = F'_u(z_0)$. We have

$$L = \left\{z_0 + c^{-1}\mathbb{R}\right\} = \left\{z \middle| \operatorname{Im}\left(c(z - z_0)\right) = 0\right\}.$$

If $\gamma(t)$ is a local parameterization of $(tr)_u$ at z_0 such that $F_u(\gamma(t)) = F_u(z_0) + t$ then $\dot{\gamma}(0) = c^{-1}$. Moreover, $(tr)_u \subset L_+ = \{\operatorname{Im} c(z - z_0) > 0\}$ if the curvature of $\gamma(t)$ is positive and $(tr)_u \subset L_- = \{\operatorname{Im} c(z - z_0) < 0\}$ otherwise.

The tangent ray $r(z_0) = \{z_0 + R(z_0)\mathbb{R}_+\}$ lies in L_+ if $\operatorname{Im} R(z_0)c > 0$ and in L_- otherwise.

By Lemma 2.13 the sign of curvature of $\gamma(t)$ is opposite to the sign of $\operatorname{Im}\left[\frac{F''_u(z_0)}{(F'_u)^2(z_0)}\right]$.

For $u \in \mathcal{C}$ we have $c = F'_u(z) = \frac{R'(z_0)(u-z_0)+R(z_0)}{(u-z_0)^2}$ and

$$F''_u(z_0) = \frac{R''(z_0)(u-z_0)^2 + 2R'(z_0)(u-z_0) + 2R(z_0)}{(u-z_0)^3},$$

so we are interested in signs of

$$\operatorname{Im} R(z_0)c = \operatorname{Im} R(z_0)\frac{R'(z_0)(u-z_0) + R(z_0)}{(u-z_0)^2} = \operatorname{Im} R'(z_0)$$

(recall that $\frac{R(z_0)}{u-z_0} > 0$) and

$$\operatorname{Im} \frac{F''_u(z_0)}{\left(F'_u\right)^2(z_0)} = \operatorname{Im} \frac{\left(R''(z_0)(u-z_0)^2 + 2R'(z_0)(u-z_0) + 2R(z_0)\right]\left(u-z_0\right)}{\left(R'(z_0)(u-z_0) + R(z_0)\right)^2}. \quad (2.1)$$

For $u \in \mathbb{S}^1$ we have $c = u^{-1}R'(z_0)$ and we are interested in the signs of $\operatorname{Im} R'(z_0)$ and $\operatorname{Im} \frac{R''(z_0)R(z_0)}{(R')^2(z_0)}$.

□

End of alternative proof. RECHECK.

— Mitya

Proof. By Lemma 2.11, the slope of L is the argument of $\frac{R^2(z_0)}{R(z_0)+(u-z_0)R'(z_0)}$ (mod π). Since $\mathrm{Im}(R'(z_0)) \neq 0$, the associated ray $r(z_0)$ is transversal to L.

Firstly we consider the case where u is a point at infinity. We consider some small $\eta = \frac{R(z_0)}{R'(z_0)}(x+yi)$ with $x, y \in \mathbb{R}$ such that $z_0+\eta$ belongs to the root trail of u. We have $\frac{R(z_0+\eta)}{R(z_0)} \in \mathbb{R}$. Unless $R(z)$ is a linear function, we have

$$R(z_0+\eta) = R(z_0) + R'(z_0)\eta + \frac{R''(z_0)}{2}\eta^2 + o(\eta^2).$$

Therefore, the imaginary part of $\frac{R(z_0+\eta)}{R(z_0)}$ is equal to

$$y + \mathrm{Im}(\frac{R''(z_0)R(z_0)}{2(R'(z_0)^2})x^2) + o(x^2).$$

Therefore, the root trail has an inflection point at z_0 if

$$\mathrm{Im}\left(\frac{R''(z_0)R(z_0)}{2(R'(z_0)^2}\right) = 0.$$

Otherwise, the sign of the curvature depends on the sign of the imaginary part of $\frac{R''(z_0)R(z_0)}{R'(z_0)^2}$.

In case when $u \in \mathbb{C}$, it has been proven in [AHN+22] that (germs of) root trails are trajectories of the time-dependent vector field

$$V(z,t)\partial_z = -\frac{R(z)}{1+tR'(z)}\partial_z.$$

For such a trajectory γ_t containing z_0, we have $\gamma_t' = \frac{-R(\gamma_t)}{1+tR'(\gamma_t)}$. It follows that

$$\gamma_t'' = \frac{-\gamma_t' R'(\gamma_t)(1+tR'(\gamma_t)) + R(\gamma_t)(R'(\gamma_t) + t\gamma_t' R''(\gamma_t))}{(1+tR'(\gamma_t))^2}.$$

Consequently, we have

$$\frac{\gamma_t''}{\gamma_t'} = \frac{\gamma_t' R'(\gamma_t)(1+tR'(\gamma_t)) - R(\gamma_t)(R'(\gamma_t) + t\gamma_t' R''(\gamma_t))}{R(\gamma_t)(1+tR'(\gamma_t))}.$$

After simplification, we obtain

$$\frac{\gamma_t''}{\gamma_t'} = -\frac{2R'(\gamma_t) + t\gamma_t' R''(\gamma_t)}{1+tR'(\gamma_t)} = \frac{tR(\gamma_t)R''(\gamma_t) - 2R'(\gamma_t)(1+tR'(\gamma_t))}{(1+tR'(\gamma_t))^2}.$$

Since $\gamma_t = z_0$ and $t = \frac{u-z_0}{R(z_0)}$, we obtain that

$$\frac{\gamma_t''}{\gamma_t'} = \frac{(u-z_0)R^2(z_0)R''(z_0) - 2R(z_0)R'(z_0)(R(z_0) + (u-z_0)R'(z_0))}{(R(z_0)+(u-z_0)R'(z_0))^2}.$$

After simplification, the expression reduces to:

$$\frac{\gamma_t''}{\gamma_t'} = \frac{(u-z_0)R(z_0)[R(z_0)R''(z_0) - 2(R'(z_0))^2] - 2R^2(z_0)R'(z_0)}{(R(z_0)+(u-z_0)R'(z_0))^2}.$$

Just like in the case when u is a point at infinity, the sign of the curvature depends on the sign of $\mathrm{Im}(\gamma_t''/\gamma_t')$. The root trail has a point of inflection if the latter quantity vanishes.

Let us orient the line L according to γ_t'. Since $\gamma_t' = \frac{-R(\gamma_t)}{1+tR'(\gamma_t)}$, we deduce that the associated ray $r(z_0)$ is pointing to the left of L if $\mathrm{Im}(R'(z_0))$ is negative. Similarly, $r(z_0)$ is pointing to the right of L if $\mathrm{Im}(R'(z_0))$ is positive. The claim follows. $\square$

In the transverse locus $\mathfrak{I}^*_R$ of the curve of inflections, the concavity of root trails with respect to the line containing the associated ray depends on the sign of some geometrically meaningful real function.

Proposition 2.18. *Consider a point $z_0 \in \mathfrak{I}^*_R \setminus \mathcal{Z}(PQ)$ and some point $u \in \mathbb{C} \cup \mathbb{S}^1$. Assume that $R(z_0) + R'(z_0)(u - z_0) \neq 0$ (or $R'(z_0) \neq 0$ if u is a point at infinity). Let L be the line containing the associated ray $r(z_0)$.*

The germ of $\mathfrak{tr}_u$ at z_0 and the positive germ $\gamma^+_{z_0}$ of the integral curve of the field $R(z)\partial_z$ starting at z_0 belong to the same open half-plane bounded by L if $R'(z_0) + R(z_0)/(u - z_0)$ is negative ($R'(z_0) < 0$ if u is a point at infinity).

The germ of $\mathfrak{tr}_u$ at z_0 and $\gamma^+_{z_0}$ belong to opposite open half-planes bounded by L if $R'(z_0) + R'(z_0)/(u - z_0)$ is positive ($R'(z_0) > 0$ if u is a point at infinity).

Proof. Without loss of generality, we assume that $z_0 = 0$ and $R(z) = 1 + R'(0)z + (a+bi)z^2 + o(z^2)$ with $R'(0) \in \mathbb{R}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0}$ ($b \neq 0$ because $z_0 = 0$ belongs to the transverse locus of the curve of inflections). Necessarily $u > 0$. Since $b > 0$, γ^+_0 belongs to the upper half-plane.

By Lemma 2.15, $\mathfrak{tr}_u$ has a unique branch at 0 tangent to $\mathbb{R}$. Let $F_u(z) = \frac{R(z)}{u - z_0}$ for $u \in r(z_0)$ and $F_u(z) = R(z)$ for $u \in \mathbb{S}^1$, so $\mathfrak{tr}_u = \{\operatorname{Im} F_u(z) = 0\}$. Choose a parameterization $\gamma(t)$ of this branch in such a way that $F_u(\gamma(t)) = F_u(z_0) + t$. Then

$$\dot\gamma(0) = \frac{1}{F'_u(0)} = \frac{u - z_0}{R'(0) + \frac{R(0)}{u - z_0}} \qquad \text{or} \qquad \dot\gamma(0) = \frac{1}{R'(0)}$$

for $u \in \mathbb{C}$ or $u \in \mathbb{S}^1$ being a point at infinity, respectively. Therefore $\dot\gamma(0) > 0$ if $R'(0) + R(0)/(u - z_0) > 0$ (resp. $R'(0) > 0$) and $\dot\gamma(0) < 0$ otherwise.

By (2.1) the sign of the curvature of $\gamma(t)$ at 0 is opposite to the sign of $\operatorname{Im} R''(0) = \operatorname{Im} b > 0$, i.e. is negative. Thus $\gamma(t)$ lies in the lower half-plane (i.e. not in the same half-plane as γ^+_0) if $R'(0) + R(0)/(u - z_0) > 0$ is positive and in the same half-plane as γ^+_0 if $R'(0) + R(0)/(u - z_0) < 0$ ($R'(0) > 0$ and $R'(0) < 0$ resp. for $u \in \mathbb{S}^1$). Since $R'(z_0) \in \mathbb{R}$, the number $R'(z_0) + R(z_0)/(u - z_0)$ is invariant under the maps $z \mapsto az + b$ and $z \mapsto \bar z$ used for normalization, and the claim follows. $\square$

2.4.2. *Root trails and connected components of the minimal set.* When $\deg Q - \deg P = 0$, root trails provide a bound on the number of connected components of the minimal set (in all other cases, it is known that M^T_{CH} is connected).

Proposition 2.19. *Consider a linear differential operator T given by* (1.1) *and satisfying* $\deg Q - \deg P = 0$. *Any connected component C of M^T_{CH} satisfies the following conditions:*

- *C contains at least one root of P;*
- *C contains at least one root of Q;*
- *the sum of orders of zeros and poles of $R(z)$ in C vanishes.*

Proof. We assume that a connected component C of M^T_{CH} is disjoint from $\mathcal{Z}(P)$. Note that $\deg Q - \deg P = 0$ implies that the union of the zeros of $tQ(z) + P(z)(z-u)$ for any $u \in \mathbb{C}$, $T > 0$ and $t \in [0, T]$ is bounded.

Here I didn't understand the argument. As I understand it is: Take any $u \in \mathrm{M}^T_{CH} \setminus C$. Its trail is a real algebraic curve such that each its component intersects $\mathcal{Z}(P)$ and therefore is disjoint from C

— Mitya

Hence, for any $u \in \mathrm{M}^T_{CH} \setminus C$, the root trail of u is disjoint from C. Since M^T_{CH} coincides with the T_{CH}-extension of any point in M^T_{CH} (see Lemma 2.2 of [AHN+22]), it follows that C cannot belong to the minimal invariant set.

Suppose now that there is a component for which the sums of orders of the zeros of Q does not equal the sums of orders of the zeros of P. Then there is a component C such that the sums of the orders of the zeros of P, say d_0 is strictly greater than the sums of the orders of the zeros of Q, say d_1. Taking $u \in C$ we have that for all t, the zeros of $tQ(z) + P(z)(z-u)$ belonging to C have total degree $d_0 + 1$. However, when sending $t \to 0$, d_1 of these zeros tend to the zeros of Q belonging to C and at most one tend to ∞. This implies that at least $d_0 - d_1 > 0$ of the end points of the root trail of u does not belong to C, a contradiction. □

We prove that the intersection of the interior $(\mathrm{M}_{CH}^T)^\circ$ of the minimal set with any small enough neighborhood of a boundary point that is not a zero nor a pole of $R(z)$ is connected.

Lemma 2.20. *For any linear differential operator T given by* (1.1) *we consider a point α of the boundary $\partial \mathrm{M}_{CH}^T$ that is neither a zero nor a pole of $R(z)$. Then for any open set $U \subset \mathbb{C}$ containing α, there is a neighborhood V of α contained in U such that $V \cap (\mathrm{M}_{CH}^T)^\circ$ is connected or empty.*

Proof. Recall that the forward trajectories of $-R(z)\partial_z$ for any point in M_{CH}^T belong to M_{CH}^T (Proposition 2.10). Besides, we have $R(z) = r_\alpha + o(z-\alpha)$ for some $r_\alpha \in \mathbb{C}^*$.

If α does not belong to the regular locus of M_{CH}^T (the closure of its interior), then $V \cap (\mathrm{M}_{CH}^T)^\circ$ is empty for some neighborhood V of α. If α belongs to the regular locus of M_{CH}^T, then the forward trajectory γ of $-R(z)\partial_z$ starting at α does not belong to a tail (see Theorem 2.7) and therefore belongs to the closure of a connected component $V \cap (\mathrm{M}_{CH}^T)^\circ$ for any small enough neighborhood V of α. As α is not a singularity of the vector field, a continuity argument proves that any trajectory of $-R(z)\partial_z$ starting from a point close enough to α intersects the same connected component. Therefore, $V \cap (\mathrm{M}_{CH}^T)^\circ$ is connected provided V is small enough. □

In the following, we prove that the closure of a connected component of the interior of M_{CH}^T cannot be disjoint from $\mathcal{Z}(P)$.

Lemma 2.21. *For any linear differential operator T given by* (1.1), *one of the following statements holds:*

(1) M_{CH}^T *is fully irregular;*
(2) $\mathrm{M}_{CH}^T = \mathbb{C}$;
(3) the closure of any connected component of the interior $(\mathrm{M}_{CH}^T)^\circ$ of the minimal set contains a root of $P(z)$;
(4) the closure of any connected component of the interior $(\mathrm{M}_{CH}^T)^\circ$ of the minimal set contains an endpoint of a tail.

Proof. We suppose that we are not in the case (1), (2). Besides, we assume the existence of a connected component C of the interior $(\mathrm{M}_{CH}^T)^\circ$ of the minimal set whose closure is disjoint from $\mathcal{Z}(P)$, contradicting statement (3).

We first prove that C cannot be the only connected component of $(\mathrm{M}_{CH}^T)^\circ$. Indeed, roots of $P(z)$ that do not belong to the regular locus of M_{CH}^T (the closure of the interior) belong to tails (see Theorem 2.7) and they are not zeros or poles of $R(z)$. Besides, M_{CH}^T is assumed to be distinct from $\mathbb{C}$. Consequently, we have $|\deg Q - \deg P| \leq 1$. The only case where the regular locus of M_{CH}^T can be disjoint from $\mathcal{Z}(P)$ is when $R(z)$ is of the form λ or $\lambda(z-\alpha)$. In the first case, M_{CH}^T is known to be totally irregular. In the second case, either $\lambda \in \mathbb{R}_{>0}$ (and M_{CH}^T is totally irregular, see Theorem 2.3) or $\lambda \notin \mathbb{R}_{>0}$ and M_{CH}^T has no tails (and $P(z)$ has no root at all). We assume therefore that the interior M_{CH}^T has several connected components.

We denote by A the set of points of C that belongs to the closure of another connected component of $(\mathrm{M}_{CH}^T)^\circ$. We know by hypothesis that these points are not roots of $P(z)$ and Lemma 2.20 proves that each of them is a zero of $R(z)$.

Since M^T_{CH} is minimal, there is a point $u \in \overline{\mathrm{M}^T_{CH} \setminus C}$ and a point $z_0 \in \mathfrak{tr}_u \cap C$. As root trail $\mathfrak{tr}_u$ changes continuously in u, u may be chosen outside A. Since $|\deg Q - \deg P| \leq 1$, the zeros of $tQ(z) + P(z)(z-u)$ as $t \to 0$ tends to $\mathcal{Z}(P) \cup \{u\}$. The minimal set M^T_{CH} contains therefore a continuous path $\gamma(t)$ from an element of $\mathcal{Z}(P) \cup \{u\}$ to z_0 such that $\gamma(t)$ solves $tQ(\gamma(t)) + P(\gamma(t))(\gamma(t) - u) = 0$.

The path γ has to enter the component C and can do so either through a tail or an element of A. The path γ cannot contain any element $\alpha \in A$ because the equations $Q(\alpha) = 0$ and $tQ(\alpha) + P(\alpha)(\alpha - u) = 0$ (for some $t > 0$) imply $P(\alpha) = 0$, contradicting our assumption. Our assumption that neither (1), (2) nor (3) was satisfied this implies (4). □

2.5. **Asymptotic geometry of Hutchinson invariant sets.** Let us recall the results of [AHN+22] concerning minimal Hutchinson invariant sets (see Theorems 1.11 and 1.12 of [AHN+22]).

Theorem 2.22. *For any operator T as in* (1.1) *with a minimal set M^T_{CH} having a nonempty interior, M^T_{CH} is:*

- *a compact contractible subset of $\mathbb{C}$ if $\deg Q - \deg P = 1$, and $Re(\lambda) \geq 0$;*
- *a noncompact non-trivial subset of $\mathbb{C}$ if $\deg Q - \deg P = 0$ or -1;*
- *trivial, i.e. equal to $\mathbb{C}$ otherwise.*

Besides, the closure $\overline{\mathrm{M}^T_{CH}}$ in the extended plane $\mathbb{C} \cup \mathbb{S}^1$ is contractible, connected and compact.

Thus, the only interesting cases for the description of $\partial \mathrm{M}^T_{CH}$ are those for which the values of $\deg Q - \deg P$ are 1, 0 or -1. In the latter two cases, we have more precise results given below.

2.5.1. $\deg Q - \deg P = -1$. The following statement has been proved in Corollary 6.2 of [AHN+22].

Proposition 2.23. *For an operator T as in* (1.1) *such that $\deg Q - \deg P = -1$. Then the complement of its minimal Hutchinson invariant set M^T_{CH} in $\mathbb{C}$ has exactly two connected components X_1, X_2. Each X_i contains infinite cones whose intervals of directions are arbitrarily close to $\left(\frac{\phi_\infty - \pi}{2}, \frac{\phi_\infty + \pi}{2}\right)$ and $\left(\frac{\phi_\infty + \pi}{2}, \frac{\phi_\infty + 3\pi}{2}\right)$ respectively.*

2.5.2. $\deg Q - \deg P = 0$. The following statement has been proven in Corollary 6.4 of [AHN+22].

Proposition 2.24. *Take any operator T as in* (1.1) *such that $\deg Q - \deg P = 0$. Then for any $\epsilon > 0$, there exists an open cone Γ whose interval of directions is arbitrary close to $(\phi_\infty + \pi, \phi_\infty + \pi)$ and such that M^T_{CH} is contained in Γ.*

3. Local analysis of the boundary of M^T_{CH}

We consider an operator T as in (1.1) whose minimal set M^T_{CH} has a nonempty interior.

Notation 3.1. For any point $\alpha \in \partial \mathrm{M}^T_{CH}$, we define $r_\alpha \in \mathbb{C}^*$, $m_\alpha \in \mathbb{Z}$ so that

$$R(z) = \frac{Q(z)}{P(z)} = r_\alpha (z - \alpha)^{m_\alpha} + o(|z - \alpha|^{m_\alpha}). \tag{3.1}$$

We also define $\phi_\alpha = arg(r_\alpha)$ and $d_\alpha : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ where $d_\alpha(\theta) = \phi_\alpha + m_\alpha \theta$.

3.1. **Description of a tangent cone.**

Definition 3.2. For any $\alpha \in \partial \mathrm{M}_{CH}^T$, we define $\mathcal{K}_\alpha$ as the subset of $\mathbb{S}^1$ formed by directions θ such that there is a sequence $(z_n)_{n\in\mathbb{N}}$ satisfying the following conditions:

- for any $n \in \mathbb{N}$, $z_n \in (\mathrm{M}_{CH}^T)^c$;
- $z_n \longrightarrow \alpha$;
- $arg(z_n - \alpha) \longrightarrow \theta$.

We also define $\mathcal{L}_\alpha$ as the subset of $\mathbb{S}^1$ formed by directions θ such that the half-line $\alpha + e^{i\theta}\mathbb{R}^+$ does not intersect the interior of M_{CH}^T.

Lemma 3.3. *For any $\alpha \in \partial \mathrm{M}_{CH}^T$, the following statements hold:*

(1) $\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ are nonempty closed subsets of $\mathbb{S}^1$;
(2) $\mathcal{L}_\alpha \subset \mathcal{K}_\alpha$;
(3) for any $\theta \in \mathcal{K}_\alpha$, $d(\theta) \in \mathcal{L}_\alpha$. In particular, K_α is invariant under d_α;
(4) for any $\theta \in \mathcal{K}_\alpha$, there exists a closed interval $J \subset \mathcal{K}_\alpha$ of length at most π containing both θ and $d_\alpha(\theta)$;
(5) $\mathcal{L}_\alpha \neq \mathbb{S}^1$.

Proof. From Definition 3.2 it immediately follows that $\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ are closed subsets of $\mathbb{S}^1$.

If $\alpha \in \partial \mathrm{M}_{CH}^T$, then we can find a sequence of points in the complement of M_{CH}^T approaching α. By compactness of $\mathbb{S}^1$, we can choose a subsequence for which the arguments converge to some limit. Thus $\mathcal{K}_\alpha$ is nonempty.

Then, for any $\theta \in \mathcal{K}_\alpha$, we have a sequence $(z_n)_{n\in\mathbb{N}}$ in the complement of M_{CH}^T accumulating to α with the limit slope θ. The associated rays $r(z_n)$ accumulate to $\alpha + e^{id_\alpha(\theta)}\mathbb{R}^+$. Since none of them intersects the interior of M_{CH}^T, the half-line $\alpha + e^{id_\alpha(\theta)}\mathbb{R}^+$ does not intersect it either and $d_\alpha(\theta) \in \mathcal{L}_\alpha$.

Besides (up to taking a subsequence of $(z_n)_{n\in\mathbb{N}}$) there is a closed interval $J \subset \mathbb{S}^1$ such that:

- the endpoints of J are θ and $d_\alpha(\theta)$;
- the length of J is at most π;
- for any $\eta \in J$, there is a bound $N(\eta)$ such that for any $n \geq N(\eta)$, the associated ray $r(z_n)$ intersects the half-line $\alpha + e^{i\eta}\mathbb{R}^+$ at some point $P_{\eta,n}$.

Existence of sequences $(P_{\eta,n})_{n\geq N(\eta)}$ proves that for any $\eta \in J$, one has $\eta \in \mathcal{K}_\alpha$.

Finally, $\mathcal{L}_\alpha \neq \mathbb{S}^1$ because in this case, M_{CH}^T would have empty interior. □

Let us deduce local description of $\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ depending on the local invariants of α.

Corollary 3.4. *For any $\alpha \in \partial \mathrm{M}_{CH}^T$, the following statements hold:*

- *if $|m_\alpha| \geq 2$, then $\mathcal{K}_\alpha = \mathcal{L}_\alpha$ and they are contained in the finite set of arguments satisfying $\theta \equiv \frac{\phi_\alpha}{1-m_\alpha}\ [\frac{2\pi}{1-m_\alpha}]$;*
- *if $m_\alpha = 1$, then $\phi_\alpha = 0$ and $\mathcal{K}_\alpha = \mathcal{L}_\alpha$;*
- *if $m_\alpha = 0$, then $\phi_\alpha \in \mathcal{L}_\alpha$;*
- *if $m_\alpha = -1$, then $\mathcal{K}_\alpha = \mathcal{L}_\alpha$ and these sets are formed by at most two intervals, each of length at most π and having their midpoints at $\frac{\phi_\alpha}{2}$ and $\frac{\phi_\alpha}{2} + \pi$.*

Proof. We consider maximal interval J in $\mathcal{K}_\alpha$ (which is non-empty by Lemma 3.3). The images of J under the iterated action of d_α belong to $\mathcal{L}_\alpha$.

If $|m_\alpha| \geq 2$, then J is a singleton since otherwise the union of its iterates would coincide with $\mathbb{S}^1$ (contradicting Lemma 3.3). Thus J has to be a fixed point of the map d_α.

If $m_\alpha = 1$ and $\phi_\alpha \neq 0$, then J coincides with $\mathbb{S}^1$ because no other connected subset of the circle is preserved under the action of nontrivial rotation. Therefore d_α is the identity map.

If $m_\alpha = 0$, then for any $\theta \in \mathcal{K}_\alpha$, $d_\alpha(\theta) = \phi_\alpha$. Therefore $\phi_\alpha \in \mathcal{L}_\alpha$.

If $m_\alpha = -1$, then J is invariant under the action of $\theta \mapsto \phi_\alpha - \theta$. Thus, either $\frac{\phi_\alpha}{2}$ or $\frac{\phi_\alpha}{2} + \pi$ is the bisector of J. If J is of length strictly bigger than π, then Lemma 3.3 shows that its complement (of length strictly smaller than π) is also contained in $\mathcal{L}_\alpha$. Therefore $L_\alpha = \mathbb{S}^1$ which is a contradiction. $\square$

We obtain a bound on the number of petals of M^T_{CH} that can be attached to a boundary point.

Corollary 3.5. *For any linear differential operator T given by* (1.1) *we consider a point α of the boundary $\partial\mathrm{M}^T_{CH}$. Then for any open set $U \subset \mathbb{C}$ containing α, there is a neighborhood V of α contained in U such that $V \cap (\mathrm{M}^T_{CH})^\circ$ has at most:*

- $1 - m_\alpha$ *connected components if $m_\alpha \neq 1$;*
- $\deg P$ *connected components if $m_\alpha = 1$*

where $R(z) = \lambda(z-\alpha)^{m_\alpha} + o((z-\alpha)^{m_\alpha})$ with $\lambda \in \mathbb{C}^*$ and $m_\alpha \in \mathbb{Z}$.

Proof. Following Theorem 2.22, the closure $\overline{\mathrm{M}^T_{CH}}$ in the extended plane $\mathbb{C} \cup \mathbb{S}^1$ is contractible, connected and compact. Therefore, for any connected component C of $(\mathrm{M}^T_{CH})^\circ$, if α belongs to the closure of C, then for any small enough neighborhood V of α, $V \cap C$ is connected.

If α is not a zero or a pole of $R(z)$, then Lemma 2.20 proves the statement. Besides, if $m_\alpha \notin \{0, 1\}$, Corollary 3.4 proves that α is in the closure of at most $1 - m_\alpha$ components.

In the remaining cases, α is a simple zero of $R(z)$. If α is also a root of degree d of P, then it is a root of degree $d+1$ of Q.

We can divide P and Q by $(z-\alpha)^d$ while keeping the same minimal set M^T_{CH} (because in this case $\mathcal{Z}(PQ)$ remains unchanged). Consequently, we can assume that α is not a root of P. Lemma 2.21 proves that for any connected component C of $(\mathrm{M}^T_{CH})^\circ$ such that α is in the closure of C, either some root of $P(z)$ belongs to the closure of C or some tail is attached to C. If α is in the closure of several connected components of $(\mathrm{M}^T_{CH})^\circ$, then a same root of P cannot be in the closure of two of them because $\overline{\mathrm{M}^T_{CH}}$ would fail to be contractible. Similarly a given tail is attached to only one connected component of $(\mathrm{M}^T_{CH})^\circ$ (and contains exactly one root of P). Therefore, α is in the closure of at most $\deg P$ components. $\square$

3.2. **Curve of inflections.** In §A.3 of [AHN+22] we introduced the curve of inflections $\mathfrak{I}_R$ of an analytic vector field $R(z)\partial_z$. By definition, it is the closure of the subset of $\mathbb{C}$ at each point of which the integral curve of the vector field $R(z)\partial_z$ passing through this point has zero curvature. Here we provide some additional information about $\mathfrak{I}_R$.

For an operator T for which $R(z)$ is not of the form λ or $\lambda(z-\alpha)$ for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$, the function $R'(z)$ is a non-constant rational function. Therefore the curve of inflections $\mathfrak{I}_R$ of $R(z)\partial_z$ (which is defined as the closure of the set of points for which $Im(R'(z)) = 0$) is a real plane algebraic curve.

We first characterize the points at which several local branches of the curve of inflections intersect.

Lemma 3.6. *A point $z_0 \in \mathfrak{I}_R$ belongs to exactly $m \geq 2$ local branches of $\mathfrak{I}_R$ in the following cases:*

1. *z_0 is a critical point of $R'(z)$ of order $m-1$ (including zeroes of order m of $R(z)$);*

(2) z_0 is a pole of $R(z)$ of order $m-1$.

The $2m$ limit slopes of the local branches at z_0 form a regular $2m$-gon in $\mathbb{S}^1$.

Proof. This follows immediately from Lemma 2.13. □

Lemma 3.7. *Let $F(z):\mathbb{C}\to\mathbb{C}P^1$ be a non-constant rational function of degree d. Then the real algebraic curve $\Gamma=\{z\in\mathbb{C}|\operatorname{Im}F(z)=0\}\cup\mathcal{P}(F)$ is non-empty, has at most d connected components and has exactly d connected components for generic F.*

Proof. Clearly, as $F^{-1}(x)\neq\emptyset$ for any $x\in\mathbb{R}\setminus\{F(\infty)\}$, $\Gamma\neq\emptyset$ as well.

By open mapping theorem, the map $F:\bar{\Gamma}\to\mathbb{R}P^1$, where $\bar{\Gamma}$ is the closure of Γ in $\mathbb{C}P^1$, is onto on each connected component of $\bar{\Gamma}$. Since F has degree d this means that Γ has at most d components.

Note that the ramification points of $F:\bar{\Gamma}\to\mathbb{R}P^1$ coincide with the ramification points of $F:\mathbb{C}P^1\to\mathbb{C}P^1$ lying on $\bar{\Gamma}$. Thus if the ramification values of F are not in $\mathbb{R}P^1$ then the former map is an unramified cover of degree d, so has exactly d connected components. This means that the bound is sharp. □

Corollary 3.8. *The curve of inflections $\mathfrak{I}_R$ has at most $3\deg P+\deg Q-2$ singular points.*

Proof. There are at most $\deg P$ poles of $R'(z)$ and the critical points of $R'(z)$ are the zeroes of $R''(z)$. □

3.2.1. *Inflection domains.*

Definition 3.9. The curve of inflections $\mathfrak{I}_R$ subdivides $\mathbb{C}$ into two open (not necessarily connected) domains: $\mathfrak{I}^+$ given by $Im(R'(z))>0$ and $\mathfrak{I}^-$ given by $Im(R'(z))<0$.

Observe that in $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$), the integral curves of the vector field $R(z)\partial_z$ are turning counterclockwise (resp. clockwise).

3.2.2. *Circle at infinity.* Consider the closure of the curve of inflections $\mathfrak{I}_R$ in the extended complex plane $\mathbb{C}\cup\mathbb{S}^1$.

Lemma 3.10. *The intersection $\mathfrak{I}_R\cap\mathbb{S}^1$ is:*

- *is empty if $\deg Q-\deg P=1$ and $\lambda\notin\mathbb{R}$;*
- *coincides with the set $\{\frac{\phi_\infty}{2},\frac{\phi_\infty}{2}+\frac{\pi}{2},\frac{\phi_\infty}{2}+\pi,\frac{\phi_\infty}{2}+\frac{3\pi}{2}\}$ if $\deg Q-\deg P=-1$.*

In the remaining two cases:

- $\deg Q-\deg P=1$ *and* $\lambda\in\mathbb{R}$*;*
- $\deg Q-\deg P=0$*; or*
- $\deg Q-\deg P\notin\{-1,0,1\}$

the set $\mathfrak{I}_R\cap\mathbb{S}^1$ consists of $2k$ points forming a regular $2k$-gon for some k satisfying $k\leq\max\{\deg P,\deg Q\}+1$.

Proof. If $k=\deg Q-\deg P\in\mathbb{Z}\setminus\{0,1\}$, then $R'(z)$ has an expansion of the form $k\lambda_k z^{k-1}+o(z^{k-1})$ near ∞ from which the characterization of the infinite branches of the real locus of $R'(z)$ by Lemma 2.13.

If $\deg Q-\deg P=0$, then $R(z)$ has an expansion $\lambda+\frac{A}{z^k}+o(z^{-k})$ for some $A\in\mathbb{C}^*$ and $k\in\mathbb{N}^*$ near ∞. (The case when $R(z)$ is constant is ruled out by the genericity assumptions). Therefore $R'(z)$ has an expansion $-\frac{Ak}{z^{k+1}}+o(z^{-k-1})$. We conclude that $\mathfrak{I}_R$ has $2k$ infinite branches whose limit directions form a regular $2k$-gon.

If $\deg Q - \deg P = 1$, then $R(z)$ has an expansion $\lambda z + A + Bz^{-k} + o(z^{-k})$ for some $A \in \mathbb{C}$, $B \in \mathbb{C}^*$, and $k \in \mathbb{N}^*$. (The case when $R(z)$ is a linear function is ruled out by the genericity assumptions). We obtain that $R'(z)$ is of the form $\lambda - \frac{Bk}{z^{k+1}} + o(z^{-k-1})$. Consequently, unless λ is real, the curve of inflections $\mathfrak{I}_R$ is compact in $\mathbb{C}$. If λ is real, the infinite branches of $\mathfrak{I}_R$ are asymptotically the same as that of the real locus of $-\frac{kB}{z^{k+1}}$. Therefore $\mathfrak{I}_R$ has $2k$ infinite branches whose limit directions form a regular $2k$-gon.

In these last two cases, we have $R'(z) = \frac{M}{z^{k+1}} + o(z^{-k-1})$ for some $M \in \mathbb{C}^*$ and $k \geq 1$. The number k is the ramification index of either $R - \lambda z$ (for $\deg Q - \deg P = 1$) or R (for $\deg Q - \deg P = 0$) at infinity, thus K cannot be bigger than the degree $\max\{\deg P, \deg Q\}$ of R. Therefore $k \leq \max\{\deg P, \deg Q\} + 1$.

□

3.2.3. *Singularities of the vector field.* Next we deduce from Corollary 3.4 a proof of the statement that any root of $P(z)$ or $Q(s)$ belonging to $\partial \mathrm{M}_{CH}^T$ automatically belongs to the curve of inflections.

Corollary 3.11. *Consider an operator T as in* (1.1) *such that M_{CH}^T does not coincide with $\mathbb{C}$ and has a nonempty interior. Let α be a zero or a pole of $R(z)$ such that $\alpha \in \partial \mathrm{M}_{CH}^T$. Then α also belongs to the curve of inflections $\mathfrak{I}_R$. Additionally, the number of local branches of $\mathfrak{I}_R$ at α equals:*

- $a + 1$ *if α is a pole of order $a \geq 1$;*
- $a - 1$ *if α is a zero of order $a \geq 2$;*
- *some integer $b \geq 1$ if α is a simple zero.*

Proof. The statement is proved by direct computation of $Im(R')$ in case of a pole or a zero of order $a \geq 2$. If α is a simple zero of $R(z)$, then we have $R(\alpha + \epsilon) = R'(\alpha)\epsilon + o(\epsilon)$. If $\alpha \in \partial \mathrm{M}_{CH}^T$, then $\phi_\alpha = arg(R'(\alpha)) = 0$ (see Corollary 3.4). Thus $\alpha \in \mathfrak{I}_R$.

Unless $R(z)$ is linear, $R(z)$ is of the form $R'(\alpha)(z-\alpha) + M(z-\alpha)^d + o(|z-\alpha|^d)$ for some $d \geq 2$ and $M \in \mathbb{C}^*$. Thus $R'(z) = R'(\alpha) + Md(z-\alpha)^{d-1} + o(|z-\alpha|^{d-1})$. Consequently, the number of local branches of the equation $Im(R') = 0$ equals $d - 1$.

If $R(z) = \lambda(z - \alpha)$, then $Re(\lambda) \geq 0$ (otherwise $\mathrm{M}_{CH}^T = \mathbb{C}$) and $Im(\lambda) \neq 0$ (otherwise M_{CH}^T is totally irregular). It follows that $Im(R'(z))$ is a non-vanishing constant and the curve of inflections is empty. In this case, M_{CH}^T does not contain any zero or pole of $R(z)$. □

3.2.4. *Tangency locus.*

Definition 3.12. For the rational vector field $R(z)\partial_z$, the *tangency locus* $\mathfrak{T}_R$ is the subset of the curve of inflections $\mathfrak{I}_R$ where $R(z)\partial_z$ is tangent to some branch of $\mathfrak{I}_R$.

Proposition 3.13. *For an operator T as in* (1.1)*, the tangency locus $\mathfrak{T}_R$ is the union of:*

- *at most* $\max\{\deg Q, \deg P\} + 1$ *lines;*
- *at most* $2(3\deg P + \deg Q - 1)^2$ *points.*

Proof. For any point $z \in \mathcal{T}_R$, an immediate computation involving the Taylor expansion of $R'(z)$ proves that z belongs to the intersection of the curve of inflections (given by $\mathrm{Im}(R') = 0$) with a real plane algebraic curve given by the equation $\mathrm{Im}(R''R) = 0$. The degrees of these two curves are respectively $\deg Q + 3\deg P - 1$ and $2\deg Q + 6\deg P - 2$. Therefore, Bézout's theorem implies that $\mathfrak{T}_R \cap \partial \mathrm{M}_{CH}^T$ contains at most $2(\deg Q + 3\deg P - 1)^2$ such points and some irreducible components corresponding to the common factors of the two equations.

By definition of the tangency locus these irreducible components are the integral curves of $R(z)\partial_z$ contained in the curve of inflections. Such integral curves have identically vanishing curvature and therefore they are segments of straight lines. Therefore the relevant irreducible components are straight lines. But $\mathfrak{I}_R$ intersects $\mathbb{S}^1$ at most $2\max\{\deg Q, \deg P\}+2$ points by Lemma 3.10. Thus the number of the lines is at most $\max\{\deg Q, \deg P\}+1$. □

We deduce an estimate on the number of connected components of the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections. Denote $d = 3\deg P + \deg Q - 1 = \deg \mathfrak{I}_R$.

Corollary 3.14. *For an operator T as in* (1.1)*, the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections is formed by at most $2d^2+6d+2$ connected components.*

Proof. A connected component of $\mathfrak{I}_R^*$ is either a smooth closed loop (so a connected component of $\mathfrak{I}_R$) or an arc joining points at infinity, or singular points of $\mathfrak{I}_R^*$ or isolated points of the tangent locus.

Following Proposition 3.13, the tangent locus contains at most $2d^2$ isolated points. Each of them is the endpoint of two arcs of the transverse locus.

Lemma 3.10 proves that at most $2\max\{\deg P, \deg Q\}+2$ arcs of the transverse locus go to infinity.

Lemma 3.6 provides the analog result for the multiple points of the curve of inflections. In the worst case, poles of $R(z)$ and critical points of $R'(z)$ are simple. At most four arcs of the transverse locus are incident to such points. There are at most d such points (see Corollary 3.8) so they are incident to at most $4d$ arcs.

Adding these bounds, we obtain an upper bound $4d^2+10d+4$ on the number of ends of non-compact connected components of the transverse locus, i.e. there are at most $2d^2+5d+2$ non-compact connected components. By Lemma 3.7 the number of the compact connected components (loops) of $\mathfrak{I}_R$ is at most d, which gives the required upper bound.

□

Corollary 3.15. *On each connected component of the transverse locus $\mathfrak{I}_R^*$, the sign of* $\operatorname{Im}(R''R)$ *remains constant. If* $\operatorname{Im}(R''R)$ *is positive (resp. negative), then for any point z of the component, the associated ray $r(z)$ points towards $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$).*

Proof. Any regular point z of the curve of inflections satisfying $\operatorname{Im}(R''(z)R(z)) = 0$ belongs to the tangent locus (see the proof of Proposition 3.13). A direct computation proves the rest of the claim. □

3.3. **Horns.** In this section, we introduce some curvilinear triangles called *horns* and find conditions under which we can conclude that they do not belong to the minimal set $\overline{\mathrm{M}_{CH}^T}$. Our aim is to prove that some parts of the boundary of the minimal sets are portions of integral curves of the vector field $R(z)\partial_z$.

3.3.1. *Definitions.* Recall that $\sigma(q)$ is the argument of $R(q)$, i.e. $\sigma(q) = \operatorname{Im}\log R(q)$ and $r(q) = q + R(q)\mathbb{R}_+$ is the associated ray.

Definition 3.16. Assume that a segment $\gamma_p^{p'}$ of the positive trajectory of $R(z)\partial_z$ starting at $p \notin \mathcal{Z}(PQ)$ and ending at p' doesn't intersect the curve of inflections except possibly at p. Assume that $\gamma_p^{p'}$ rotates by less than π: $\left|\sigma|_p^{p'}\right| < \pi$.

We define the *horn* ${}_p\triangle_{p''}^{p'}$ at p as an open curvilinear triangle formed by $\gamma_p^{p'}$ and tangents to this trajectory at p and p' intersecting at a point p''.

Definition 3.17. A horn ${}_p\triangle_{p''}^{p'}$ is called *small positive* (resp. *small negative*) if

(1) for any point $u \in {}_p\triangle_{p''}^{p'}$, the argument $\sigma(u + tR(u))$ is monotone increasing (resp. decreasing) in the variable t as long as $0 \leq t$ and $u + tR(u) \in {}_p\triangle_{p''}^{p'}$
(2) for any two points $u, v \in {}_p\triangle_{p''}^{p'}$, the scalar product $(R(u), R(v))$ is positive.

A horn ${}_p\triangle_{p''}^{p'}$ is called *small* if it is either small positive or small negative.

Remark 3.18. A small positive horn becomes a small negative one after the conjugation, i.e. after replacing $R(z)$ with $\overline{R(\bar{z})}$. Indeed,

$$(R(u), R(v)) = \operatorname{Re} R(u)\overline{R(v)}$$

remains the same after the conjugation, and

$$\frac{d\sigma(u + tR(u)}{dt}(t) = \operatorname{Im} \frac{R'(u + tR(u))}{R(u + tR(u))} R(u)$$

changes sign.

Lemma 3.19. *The curve of inflections (given by* $\operatorname{Im} R' = 0$*) does not intersect small horns.*

Proof. We have that $\frac{d\sigma(u+tR(u)}{dt}|_{t=0} = \operatorname{Im} R'(u) \geq 0$. Assume that we have the equality at some $u \in {}_p\triangle_{p''}^{p'}$. Since ${}_p\triangle_{p''}^{p'}$ is open and R' is an open map, this assumption will imply that $\frac{d\sigma(u+tR(u)}{dt}|_{t=0}$ changes sign in ${}_p\triangle_{p''}^{p'}$, which contradicts the smallness assumptions. □

We define the cone complementary to ${}_p\triangle_{p''}^{p'}$ (in short, the *complementary cone*) to be the open cone ${}_{p''}\angle$ with the apex p'' bounded by part of the ray $r(p)$ starting at p'' and by the ray extending the segment $p'p''$.

Lemma 3.20. *Consider a point p which neither belongs to $\mathcal{Z}(PQ)$ nor to the interior of M_{CH}^T. Assume that the integral curve γ of the vector field $R(z)\partial_z\partial_z$ containing p is not a straight line. Then there exists a horn ${}_p\triangle_{p''}^{p'}$ such that both ${}_p\triangle_{p''}^{p'}$ and its complementary cone ${}_{p''}\angle$ do not intersect M_{CH}^T.*

Proof. By definition, $r(p) \subset \overline{{\mathrm{M}_{CH}^T}^c}$. Choose some $\delta > 0$ and define $p_0 = p$ and

$$p_{i+1} = p_i + \delta R(p_i) \in r(p_i) \subset {\mathrm{M}_{CH}^T}^c, \quad i = 0, ..., N = N(\delta) = C/\delta.$$

The broken line $\hat{\gamma}_p^\delta = \cup_{i=0}^N [p_i, p_{i+1}] \subset {\mathrm{M}_{CH}^T}^c$ is the Euler approximation to the positive trajectory γ_p of $R\partial_z$ starting from p. Since $\hat{\gamma}_p^\delta \to \gamma_p$ as $\delta \to 0$, we see that $\gamma_p^{p'} \subset \overline{{\mathrm{M}_{CH}^T}^c}$ for sufficiently small $\gamma_p^{p'}$.

Clearly, if $\gamma_p^{p'}$ is in the curve of inflections then it is a straight line, which is excluded by our assumption. Thus we can assume that for p' sufficiently close to p the curve $\gamma_p^{p'}$ intersects the curve of inflections only at p (and therefore is convex) and of angle smaller than π. The claim now follows from

$$\left(\bigcup_{s \in \gamma_p^{p'}} r(s)\right)^\circ = {}_{p''}\angle \bigcup {}_p\triangle_{p''}^{p'} \subset {\mathrm{M}_{CH}^T}^c.$$

□

3.3.2. *Small horns exist.*

Proposition 3.21. *For any point $p \notin \mathcal{Z}(PQ)$ such that the trajectory $\gamma(p)$ of R starting at p is not a straight line, there exists a small horn ${}_p\triangle_{p''}^{p'}$.*

Proof. Using an affine change of variables we can assume that $p = 0$ and $R(0) = 1$. By assumption $R(z)$ is not a real rational function. Let

$$R(u) = 1 + \rho(u) + ibu^m + O(u^{m+1}), \quad b > 0,\ \rho \in \mathbb{R}[u],\ m \geq 1 \tag{3.2}$$

be the Taylor expansion of $R(z)$ at 0 (the case $m = 1$ is covered by Lemma 3.22). Here we can assume that $b > 0$ by replacing $R(z)$ by $\overline{R(\bar{z})}$, if necessary.

First, we consider the case $m = 1$, i.e. $p \notin \mathfrak{I}_R$.

Lemma 3.22. *For every compact set K not intersecting the curve of inflections $\mathfrak{I}_R$, there is a $\delta = \delta(K) > 0$ such that for every $p \in K$, there is a small horn ${}_p\triangle_{p''}^{p'}$ of length $\gamma_p^{p'} \geq \delta$.*

Proof. Indeed, for any $p \in K$ the functions $\operatorname{Re} R(u)\overline{R(v)}$ and $\operatorname{Im} \frac{R'(u)}{R(u)} R(v)$ are both non-zero at $(p, p) \in^2$, so they remain non-zero for all $(u, v) \in \mathbb{C}^2$ such that $\operatorname{dist}((p, p), (u, v)) < \delta = \delta(p)$ by continuity. This means that any ${}_p\triangle_{p''}^{p'} \subset U_{\delta(p)}(p)$ is a small horn. The uniform lower bound follows from the continuity of $\delta(p)$. $\square$

From now on we assume that $m \geq 2$. Our next goal is to find the asymptotics of $\gamma(0)$ and ${}_0\triangle_{p''}^{p'}$.

Lemma 3.23.

$$\gamma_0 = \left\{0 < x < \epsilon, \quad y = \tfrac{b}{m+1} x^{m+1} + O(x^{m+2})\right\} \subset \{y \geq 0\} \tag{3.3}$$

$$ {}_0\triangle_{p''}^{p'} \subset \{0 < x < \epsilon, 0 < y < \gamma_0(x)\}. \tag{3.4}$$

Proof. Note that $\gamma_0 \subset \{\operatorname{Im} F = 0\}$, where $F' = \frac{1}{R}$, $F(0) = 0$.

Now,

$$\frac{1}{R} = \frac{1}{1 + \rho(u)} - \frac{ibu^m}{(1 + \rho(u))^2} + O(u^{m+1}), \tag{3.5}$$

so

$$F(u) = u + \tilde{\rho}(u) - i\tfrac{b}{m+1} u^{m+1} + O(u^{m+2}), \quad \tilde{\rho} \in \mathbb{R}[u].$$

For $u = x + iy$ we get

$$\operatorname{Im} F(u) = y(1 + o(1)) - \tfrac{b}{m+1} x^{m+1} + O(u^{m+2}).$$

Recalling that γ_0 is tangent to the real axis, we have $y = o(x)$. Therefore

$$\gamma_0 \subset \{\operatorname{Im} F = 0\} = \left\{y = \tfrac{b}{m+1} x^{m+1} + O(x^{m+2})\right\} \subset \{y \geq 0\}$$

and the claim of the Lemma follows since $r(0) = \mathbb{R}_+$.

$\square$

We have to check the two conditions in Definition 3.17 for ${}_0\triangle_{p''}^{p'}$ with p' sufficiently close to 0. The second condition is easy: since $R(0) = 1$ then the scalar product $(R(u), R(v))$ is positive for all $u, v \in {}_0\triangle_{p''}^{p'}$ by continuity.

To check the first condition set $u = x_1 + iy_1, v = x_2 + iy_2 = u + tR(u) \in {}_0\triangle_{p''}^{p'}$ with $t > 0$. By the second property of the small horns, we have $x_2 > x_1$. By (3.4) we have $y_i = O\left(x_i^{m+1}\right)$. Combining (3.5) and

$$R'(v) = \rho'(v) + imbv^{m-1} + O(v^m), \tag{3.6}$$

we get

$$\begin{aligned}
\frac{R'(v)}{R(v)} &= \left(\rho'(x_2) + imbx_2^{m-1} + O(x_2^m)\right) \frac{1 + \rho(x_2) - ibx_2^m + O(x_2^{m+1})}{\left(1 + \rho(x_1)\right)^2} \\
&= \frac{\rho'(x_2)\left(1 + \rho(x_2)\right) + imbx_2^{m-1} + O(x_2^m)}{\left(1 + \rho(x_2)\right)^2}.
\end{aligned}$$

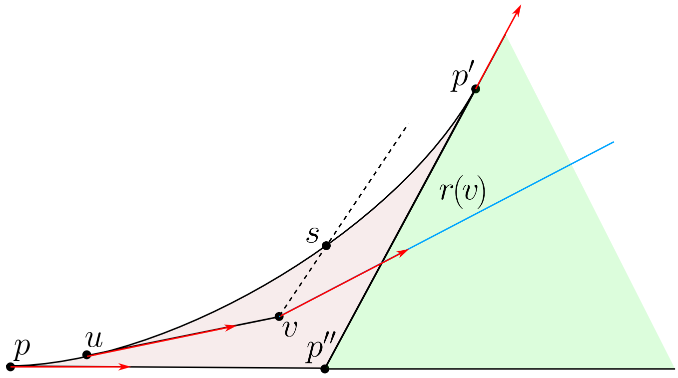

FIGURE 1. Removing small horns.

Thus, using (3.2), we get for $\Phi(u,v) = (1+\rho(x_2))^2 \operatorname{Im} R(u)\frac{R'(v)}{R(v)}$ the equation

$$\begin{aligned}\Phi &= \operatorname{Im}\left(\left[1+\rho(x_1)+ibx_1^m+O(x_1^{m+1})\right]\cdot\left[\rho'(x_2)\,(1+\rho(x_2))+imbx_2^{m-1}+O(x_2^m)\right]\right)\\ &= mbx_2^{m-1}+O(x_2^m) > 0, \qquad (3.7)\end{aligned}$$

where we use $x_1 \le x_2$. This proves the first requirement of Definition 3.17. $\square$

Corollary 3.24. *The germ of $\mathfrak{I}_R$ at p cannot lie between γ_p^+ and $r(p)$.*

Proof. This would mean that this germ lies inside ${}_p\triangle_{p''}^{p'}$ which is impossible by Proposition 3.21 and Lemma 3.19. $\square$

3.3.3. *Removing small horns.* We will use the following general Lemma

Lemma 3.25. *Assume that for some open set $U \subset \mathbb{C}\setminus\mathcal{Z}(PQ)$ and every point $u \in U$, the associated ray $r(u)$ lies in the union $U \cup \left(\mathrm{M}_{CH}^T\right)^c$. Then $\mathrm{M}_{CH}^T \cap U = \emptyset$.*

Indeed, if not then $\mathrm{M}_{CH}^T \setminus U \subsetneq \mathrm{M}_{CH}^T$ will be again invariant, which contradicts minimality of M_{CH}^T.

The crucial property of small horns is the following Lemma.

Lemma 3.26. *For any $v \in {}_p\triangle_{p''}^{p'}$, one has $r(v) \subset {}_p\triangle_{p''}^{p'} \cup {}_{p''}\angle$.*

Proof. We prove the statement assuming that the small horn ${}_p\triangle_{p''}^{p'}$ is positive, the negative case will follow by conjugation.

Let $u \in \gamma_p^{p'}$ be a point such that $v \in r(u)$. By definition of small horns, we have $\sigma(p) < \sigma(u) < \sigma(v)$, see Fig. 1.

The ray r_v does not intersect $\gamma_p^{p'}$. Indeed, assume that the ray $r(v)$ intersects $\gamma_p^{p'}$ at a point s. Then at the intersection point the slope of $\gamma_p^{p'}$ should be smaller than the slope of $r(v)$, i.e. $\sigma(s) < \sigma(v)$ which contradicts to the requirement that the slope is monotone increasing along the segment joining v and s. Also $r(v)$ can not intersect pp'' since $\sigma(v) > \sigma(p)$ and at a possible point of intersection s necessarily $\sigma(s) < \sigma(p)$, again contradicting monotonical increasing of $\sigma alongthesegmentjoiningvands$.

Thus $r(v)$ leaves ${}_p\triangle_{p''}^{p'}$ and enters ${}_{p''}\angle$ at some point of $p''p'$ with the slope $\sigma(p) < \sigma(v) < \sigma(p')$. Thus $r(v)$ never leaves ${}_{p''}\angle$. $\square$

Proposition 3.27. *Assume that ${}_p\triangle_{p''}^{p'}$ is a small horn and ${}_{p''}\angle \subset {\mathrm{M}_{CH}^T}^c$. Then p is not in the interior of M_{CH}^T.*

Proof. Follows from Lemmas 3.25 and 3.26. $\square$

4. Boundary arcs

Recall that we consider an operator T whose minimal set M_{CH}^T is different from $\mathbb{C}$ and has a nonempty interior. We want to describe its boundary in combinatorial and dynamical terms. To do this, we introduce two set-valued functions.

Recall that in our terminology, $\overline{\mathrm{M}_{CH}^T}$ is the closure of M_{CH}^T in the extended plane $\mathbb{C} \cup \mathbb{S}^1$.

4.1. Correspondences Γ and Δ.

Definition 4.1. For any $x \in \partial\mathrm{M}_{CH}^T \setminus \mathcal{Z}(PQ)$, we define:

- $\Gamma(x) = \{y \in \gamma_x^+ \mid y \neq x\} \cap \overline{\mathrm{M}_{CH}^T}$ where γ_x^+ is the positive trajectory of the vector field $R(z)\partial_z$ starting at x;
- $\Delta(x) = \{y \in r(x) \mid y \neq x\} \cap \overline{\mathrm{M}_{CH}^T}$.

Using correspondences Γ and Δ, we split the set of boundary points of M_{CH}^T disjoint from the curve of inflections into the following three types.

Definition 4.2. A point of $\partial\mathrm{M}_{CH}^T \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$ is a point of:

- *local type* if $\Gamma(z) \neq \emptyset$ and $\Delta(z) = \emptyset$;
- *global type* if $\Gamma(z) = \emptyset$ and $\Delta(z) \neq \emptyset$;
- *extruding type* if $\Gamma(z) \neq \emptyset$ and $\Delta(z) \neq \emptyset$.

By Proposition 4.7 these are the only possibilities for points in $\partial\mathrm{M}_{CH}^T \setminus \mathfrak{I}_R$.

4.2. **Support lines.** In this (sub)section We prove that for a given point z, the condition $\Delta(z) \neq \emptyset$ means that the associated ray $r(z)$ is a *support line* of $\overline{\mathrm{M}_{CH}^T}$.

For any oriented support line of $\overline{\mathrm{M}_{CH}^T}$, we define the *co-orientation* of its support in the following way. The support point x is:

- a *direct support point* if the standard orientation of $\partial\mathrm{M}_{CH}^T$ and the orientation of the support line agree at x;
- an *indirect support point* otherwise.

In particular, if the support line is the positively oriented real axis, a support point x is called *direct* if the intersection of M_{CH}^T with a neighborhood of x is contained in the upper half-plane (see Fig. ???).

Definition 4.3. Consider $z \in \mathbb{C}$ such that:

- z does not belong to the tangency locus $\mathcal{T}_R$ of the curve of inflections $\mathfrak{I}_R$;
- z is not a root of P or Q.

Then we say that $z \in \mathfrak{E}^+$ (resp. $\mathfrak{E}^-$) if the associated ray $r(z)$ is pointing inside the inflection domain $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$).

Lemma 4.4. *Consider $z \in \partial\mathrm{M}_{CH}^T \setminus \mathcal{Z}(PQ)$ such that $z \in \mathfrak{E}^+$ (resp. $\mathfrak{E}^-$). If $y \in \Delta(z)$, then y is an indirect support point (resp. a direct support point).*

Proof. Without loss of generality, we can assume that $z \in \mathfrak{E}^+$, $z = 0$, $r(z) = \mathbb{R}_{>0}$ and $y = 1$. This implies that γ_0 lies in the upper half-plane. By Lemma 3.20 there is a neighborhood V of y such that $V \cap \left(\overline{\mathrm{M}_{CH}^T}\right)^\circ$ is contained in the lower half-plane. Therefore y is an indirect support point. $\square$

Lemma 4.5. *Take $x, y \in \partial\mathrm{M}_{CH}^T$ such that:*

- *$x, y \in \mathfrak{E}^- \cup \mathfrak{E}^+$*
- *the associated rays $r(x)$ and $r(y)$ intersect at some point $m \in \mathbb{C}$;*
- *$\sigma(y) \in]\sigma(x) - \pi, \sigma(x)[$.*

Then the open cone Γ with apex m and the interval of directions $]\sigma(y), \sigma(x)[$ is disjoint from $\left(\mathrm{M}_{CH}^T\right)^\circ$ and there are the following subcases:

- *either* $y \in \mathfrak{E}^+$ *or* $\Delta(y) \subset [y, m]$*;*
- *either* $x \in \mathfrak{E}^-$ *or* $\Delta(x) \subset [x, m]$.

Proof. The fact that Γ is disjoint from $\left(\mathrm{M}_{CH}^T\right)^\circ$ follows from Lemma 2.9 applied to a path formed by the concatenation of segments $[x, m]$ and $[m, y]$. Then, we assume by contradiction that $y \in \mathfrak{E}^-$ and some point $z \in \Delta(y)$ does not belong to $[y, m]$. Since Γ is disjoint from $\left(\mathrm{M}_{CH}^T\right)^\circ$, it follows that z is an indirect support point of the line containing $r(y)$ which contradicts Lemma 4.4. Consequently either $\Delta(y) \subset [y, m]$ or $y \notin \mathfrak{E}^-$.

An analogous argument proves that either $x \in \mathfrak{E}^-$ or $\Delta(x) \subset [x, m]$. □

4.3. **Local arcs.** In this section, we prove that local points (see Proposition 4.7) form *local arcs.*

Definition 4.6. A *local arc* of $\partial\mathrm{M}_{CH}^T$ is a maximal open arc of an integral curve of vector field $R(z)\partial_z$ that contains only local points. In particular, it is disjoint from $\mathcal{Z}(PQ)$ and $\mathfrak{I}_R$.

Local arcs are oriented by the vector field $R(z)\partial_z$.

Using the geometry of horns (see Section 3.3), we can show that every local point actually belongs to a local arc of $\partial\mathrm{M}_{CH}^T$.

Proposition 4.7. *Consider a point p belonging to $\partial\mathrm{M}_{CH}^T$ and such that $\Delta(p) = \emptyset$ and $p \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Then, the germ of the integral curve γ_p of $R(z)\partial_z$ passing through p belongs to $\partial\mathrm{M}_{CH}^T$.*

Without loss of generality, we assume that $p = 0$, $r(p) = \mathbb{R}_+$ and $p \in \mathfrak{E}^+$, so $\gamma_p^{p'}$ lies in the upper half-plane. The proof consists of two steps illustrated by Figure 2 and Figure 3 respectively.

Lemma 4.8. M_{CH}^T *lies above the integral curve γ_p of $R(z)\partial_z$ passing through p.*

Proof. By Lemma 3.20 the union ${}_p\triangle_{p''}^{p'} \cup {}_{p''}\angle$ is outside of M_{CH}^T. Let $q' \in \gamma_p^{p'}$, $q' \neq p, p'$, and let $q'' \in \mathbb{R}_+$ be the intersection point of $\mathbb{R}_+$ with the line tangent to $\gamma_p^{p'}$ at q'. Thus, ${}_p\triangle_{q''}^{q'}$ is a horn at p, ${}_p\triangle_{q''}^{q'} \subset {}_p\triangle_{p''}^{p'}$. Clearly, $\sigma(q') < \sigma(p')$.

The condition $\Delta(p) = \emptyset$ implies that $q'' \in {\mathrm{M}_{CH}^T}^c$. Moreover, as $+\infty \notin \Delta(0)$, there is an open sector S with a vertex on $\mathbb{R}$, containing q'' and disjoint from M_{CH}^T. Take a point $\tilde{p}$ close to p, lying below the trajectory of $R(z)\partial_z$ passing through p and in the same inflection domain as $\gamma_p^{p'}$. Consider a horn ${}_{\tilde{p}}\triangle_{\tilde{q}''}^{\tilde{q}'}$ with vertices $\tilde{q}'$ and $\tilde{q}''$ close to q' and q'', respectively. By continuity, the part of $r(\tilde{p})$ starting from $\tilde{q}''$ lies in S. Also, $\tilde{q}'$ lies in the horn ${}_p\triangle_{p''}^{p'}$, so $r(\tilde{q}') \cap \mathrm{M}_{CH}^T = \emptyset$ by Lemma 3.20.

Thus the complementary cone ${}_{\tilde{q}''}\angle$ of $\tilde{p}$ with vertex $\tilde{q}''$ lies outside of M_{CH}^T. Therefore by Proposition 3.27 $\tilde{p} \notin \mathrm{M}_{CH}^T$ and therefore near p the set M_{CH}^T lies above the trajectory of $R(z)\partial_z$ passing through p, see Fig. 2.

□

Lemma 4.9. *The boundary $\partial\mathrm{M}_{CH}^T$ coincides with the integral curve γ_p in a neighborhood of p.*

Proof. Lemma 4.8 implies that p lies on the boundary of a sector S with a vertex $s \in \mathbb{R}$, $s \neq p$, and disjoint from M_{CH}^T. Assume that a point $q \notin \mathrm{M}_{CH}^T$ close to p lies above γ_p. By Lemma 3.20, both ${}_q\triangle_{q''}^{q'}$ and ${}_{q''}\angle$ lie outside of M_{CH}^T. Choosing q sufficiently close to γ_p we can assume by Lemma 3.22 that $q'' \notin \mathrm{M}_{CH}^T$, and, moreover, that this is true for any point on the trajectory γ_q of $R(z)\partial_z$ passing through q sufficiently close to p.

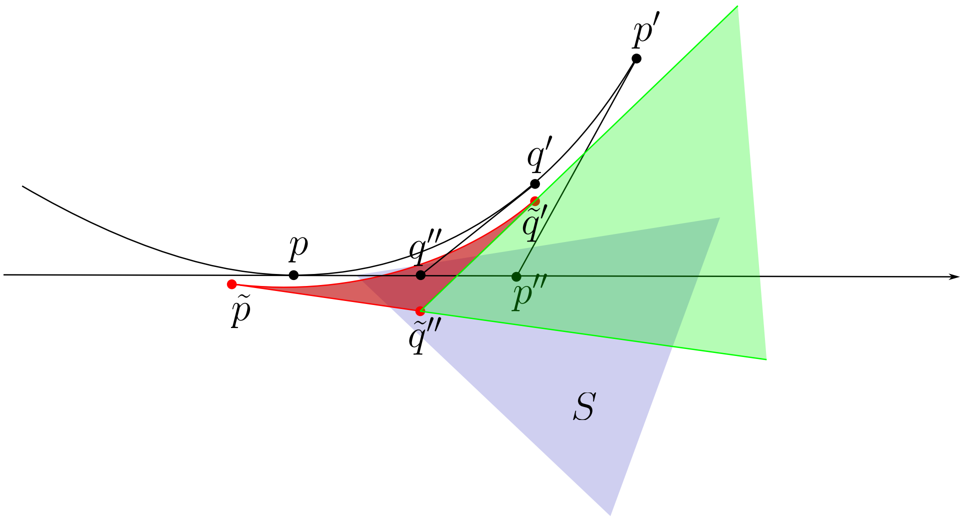

FIGURE 2. M_{CH}^{T} lies above the trajectory $\gamma_p^{p'}$.

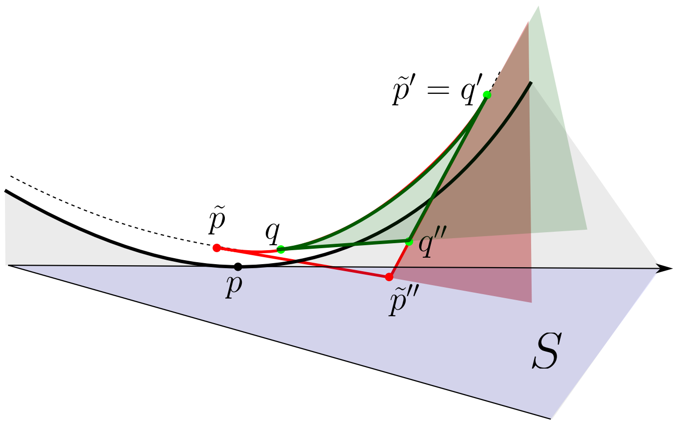

FIGURE 3. $\gamma_p^{p'}$ is boundary of M_{CH}^{T}

Let $\tilde{p}$ be a point on γ_q close to q and in the negative direction from q, take $\tilde{p}' = q'$ and $\tilde{p}''$ be the intersection of $r(\tilde{p})$ and the line $q'q''$. The ray $\tilde{p}''\tilde{p}'$ lies outside of ${\mathrm{M}_{CH}^{T}}^{\circ}$. Moreover, as long as the ray $p'' + R(\tilde{p})\mathbb{R} \subset r(\tilde{p})$ lies outside ${\mathrm{M}_{CH}^{T}}^{\circ}$ we have ${}_{\tilde{p}''}\angle \cap \mathrm{M}_{CH}^{T} = \emptyset$, so $\tilde{p} \notin \mathrm{M}_{CH}^{T}$ by Proposition 3.27. But these arguments work for points $\tilde{p}$ with slope $\sigma(\tilde{p})$ exceeding some negative number, namely the slope of the second side of S. In particular, it includes some neighborhood of p, which means that $p \notin \mathrm{M}_{CH}^{T}$, a contradiction, see Figure 3.

□

Local analysis of horns (see Section 3.3) leads to the following results about the correspondence Γ.

Corollary 4.10. *Consider $z \in \partial \mathrm{M}_{CH}^{T}$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. If $\Gamma(z) \neq \emptyset$ then z is either the starting point or a point of a local arc.*

Proof. We just have to prove that for some $y \in \Gamma(z)$ such that y is close enough to z, the arc α of integral curve between z and y belongs to $\partial \mathrm{M}_{CH}^{T}$. This follows from Lemma 3.20. □

Proposition 4.11. *Any local arc is a locally strictly convex real-analytic curve. Its orientation coincides with the standard topological orientation of $\partial \mathrm{M}_{CH}^{T}$ if it is contained in $\mathfrak{I}^{+}$ (and with the opposite orientation otherwise).*

Proof. As any integral curve of a real-analytic vector field, a local arc is a real-analytic curve in $\mathbb{R}^2$. The arc has to be locally convex because otherwise, the associated ray (which is contained in the tangent line) at some point would cross the interior of M_{CH}^{T}. Besides, direct computation shows that the curvature of an integral curve becomes zero only at points belonging to $\mathfrak{I}_R$. □

Let us check that a local arc of $\partial \mathrm{M}_{CH}^{T}$ can not end inside an inflection domain. It can not be periodic either.

Proposition 4.12. *Every local arc has an endpoint that belongs to $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$.*

Besides, if such an endpoint belongs to $\mathcal{Z}(PQ)$, it is either a regular point or a simple pole of $R(z)$.

Proof. Assume that the local arc γ is periodic and doesn't intersect $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Then following Proposition 4.11, γ is a strictly convex closed loop disjoint from $\mathcal{Z}(PQ)$ and M_{CH}^{T} is a strictly convex compact domain bounded by γ (in particular γ encompasses every point of $\mathcal{Z}(PQ)$). A neighborhood of γ is foliated by periodic integral curves γ_t of the vector field $R(z)\partial_z$ that are also disjoint from $\mathcal{Z}(PQ)$ and $\mathfrak{I}_R$, so strictly convex as well. Each of them cuts out a strictly convex compact domain $\mathcal{D}_t$. For each point z in the complement of some $\mathcal{D}_t$, $r(z)$ remains disjoint from $\mathcal{D}_t$, which by Lemma 3.25 contradicts the minimality of M_{CH}^{T}.

Now, we have to prove that a local arc can not go to infinity. When $|\deg Q - \deg P| \leq 1$, integral curves going to infinity enter the cones disjoint from M_{CH}^{T} and never leave them (see Section 2.5).

In the remaining cases, Poincaré-Bendixson theorem proves that a local arc γ has an ending point $y \in \partial \mathrm{M}_{CH}^{T}$. We assume by contradiction that $y \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. We consider an arc β formed by a portion of the integral curve ending at y and a portion of the associated ray $r(y)$. Provided that β remains in the same inflection domain as y, the family of associated rays starting at the points of the arc β sweeps out a domain containing a cone (see Lemmas 2.9 and 3.20). Therefore, we have $\Delta(y) = \emptyset$. Proposition 4.7 then proves that the local arc can be continued in a neighborhood of y.

If $y \in \mathcal{Z}(PQ)$ and is a zero or a pole of $R(z)$, then $\mathcal{L}_y$ contains an interval of length at least π (see Definition 3.2). Corollary 3.4 proves that y is either a simple pole or a simple zero satisfying $\phi_y = 0$. In the latter case, y is a repelling singular point of $R(z)\partial_z$ and therefore can not be the endpoint of a local arc. □

As we will see in Section 4.5, in contrast with the case of ending points, a local arc can start inside an inflection domain at a point of extruding type.

4.4. Global arcs.

4.4.1. *Additional results about correspondence Δ.*

Lemma 4.13. *Consider $z \in \partial \mathrm{M}_{CH}^{T} \setminus \mathcal{Z}(PQ)$ such that $z \in \mathfrak{E}^{+}$ (resp. $\mathfrak{E}^{-}$). If $y \in \partial \mathrm{M}_{CH}^{T}$ and $y \in \Delta(z)$, then one of the following statements holds:*

- *$y \in \mathcal{Z}(PQ) \cup \mathfrak{I}_R$;*
- *$y \in \mathfrak{I}^{-}$ (resp. $\mathfrak{I}^{+}$).*

Proof. Without loss of generality, we assume that $z = 0$, $r(z) = \mathbb{R}^+$ and $z \in \mathfrak{E}^+$.

We consider $y \in \Delta(z)$ such that $y \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. If $\Delta(y) = \emptyset$, then Proposition 4.7 shows that y belongs to a local arc. Besides, y is an indirect support point of the associated ray $r(z)$ (see Lemma 4.4). If $y \in \mathfrak{I}^+$, then the associated rays starting from a germ of the local arc at y sweep out a neighborhood of z and we get a contradiction. Therefore $y \in \mathfrak{I}^-$.

Now we consider the case where $\Delta(y) \neq \emptyset$ and assume by contradiction that $y \in \mathfrak{I}^+$. If $Im(R(y)) > 0$, then Lemma 4.5 provides an immediate contradiction. If $Im(R(y)) < 0$, then $\mathcal{L}_y$ (see Definition 3.2) contains an interval of length strictly larger than π such that $\sigma(y)$ is one of the ends. It follows from Corollary 3.4 that y is a simple zero of $R(z)$ (and therefore $y \in \mathcal{Z}(PQ)$).

If $r(y) = y + \mathbb{R}^-$, then for some small $\epsilon > 0$, points of the interval $]-\epsilon, \epsilon[$ are disjoint from the interior of M_{CH}^T. Associated rays starting from the points of $]-\epsilon, \epsilon[$ sweep out an open cone containing a neighborhood of y. This contradicts the assumption $y \in \partial\mathrm{M}_{CH}^T$. Therefore, $r(y) = y + \mathbb{R}^+$ and $r(y) \subset r(z)$. In this case, for some small $\epsilon' > 0$, points of $]y - \epsilon', 0[$ are disjoint from the interior of M_{CH}^T and their associated rays will sweep out a neighborhood of y if $y \in \mathfrak{I}^+$. Therefore, in that case we get that $y \in \mathfrak{I}^-$. Similar result holds for $z \in \mathfrak{E}^-$. □

Definition 4.14. For any point $z \in \partial\mathrm{M}_{CH}^T$ such that $s \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$, we define $\Delta^{min}(z)$ (resp. $\Delta^{max}(z)$) as the infimum (resp. the supremum) in $\Delta(z)$ of the order induced by the orientation of the associated ray $r(z)$.

Besides, we define $L_z = |\Delta^{min}(z) - z|$.

Lemma 4.15. *For any $z \in \partial\mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$, we have $\Delta^{min}(z) \neq z$ and $L_z \neq 0$.*

Proof. Without loss of generality, we assume that $z = 0$, $z \in \mathfrak{I}^+$ and $z(x) = \mathbb{R}_{>0}$. For any small enough real positive ϵ, we have $Re(R(\epsilon)), Im(R(\epsilon)) > 0$ and $\epsilon \in \mathfrak{I}^+$. If such an ϵ belongs to $\Delta(z)$, then it contradicts Lemma 4.13. □

Since $\overline{\mathrm{M}_{CH}^T}$ is compact in $\mathbb{C} \cup \mathbb{S}^1$, it follows immediately that for any z, $\Delta^{min}(z)$ is actually a point of $\partial\overline{\mathrm{M}_{CH}^T}$.

Definition 4.16. For any point $z \in \partial\mathrm{M}_{CH}^T$ such that $s \notin \mathcal{Z}PQ \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$, we define $\mathcal{U}(z)$ as the connected component of $(\mathrm{M}_{CH}^T)^c \setminus [z, \Delta^{min}(z)]$ incident to:

- the right side of $[z, \Delta^{max}(z)]$ if $z \in \mathfrak{I}^+$;
- the left side of $[z, \Delta^{max}(z)]$ if $z \in \mathfrak{I}^-$.

i.e. in the half-plane bounded by $r(z)$ different to that containing the germ of the trajectory of $R\partial_z$ starting at z.

Lemma 4.17. *Consider $z \in \partial\mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$, $\Delta(z) \neq \emptyset$ and $z \in \mathfrak{I}^+$ (resp. $\mathfrak{I}^-$). For any $y \in \partial\mathrm{M}_{CH}^T \cap \partial\mathcal{U}(z)$ such that $y \in \mathfrak{I}^+$ (resp. $\mathfrak{I}^-$) and $\Delta(y) \neq \emptyset$, we have $\mathcal{U}(y) \subset \mathcal{U}(z)$.*

Besides, if $y \neq z$, we have $\mathcal{U}(y) \subsetneq \mathcal{U}(z)$.

Proof. Applying Lemma 4.5 to the associated rays $r(z)$ and $r(y)$, we see that $\Delta(y) \subset \partial\mathcal{U}(z)$. Thus $\mathcal{U}(y) \subset \mathcal{U}(z)$.

In case when $\mathcal{U}(y) = \mathcal{U}(z)$, the associated ray $r(y)$ has to coincide with $r(z)$ (with the same orientation since y and z belong to the same inflection domain). It follows that $y = z$. □

4.4.2. *Orientation of global arcs.*

Definition 4.18. *A global arc* in $\partial\mathrm{M}_{CH}^T$ is a maximal open connected arc formed by points of global type.

We have a geometrically meaningful way to define orientation on global arcs.

Lemma 4.19. *Any global arc $(\alpha_t)_{t\in I}$ can be oriented in such a way that for $t' > t$, we have:*

- $\alpha_{t'} \in \partial \mathrm{M}^T_{CH} \cap \partial\mathcal{U}(\alpha_t)$;
- $\mathcal{U}(\alpha_t) \subset \mathcal{U}(\alpha_{t'})$.

In particular, in $\mathfrak{I}^+$, the orientation of global arcs coincides with the standard topological orientation of $\partial \mathrm{M}^T_{CH}$ (it coincides with the opposite orientation in $\mathfrak{I}^-$).

Besides, a global arc is an interval, i.e. it can not be a closed loop.

Proof. Following Lemma 4.13, $\Delta^{min}(\alpha_t)$ either belongs to $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ or to an inflection domain different from α_t. It follows that global arc α does not contain a boundary arc from α_t to $\Delta^{min}(\alpha_t)$. Removal of α_t from α cuts the arc into two pieces, one of which is contained in $\partial\mathcal{U}(\alpha_t)$. Lemma 4.17 then proves the inclusion of the sets of the form $\mathcal{U}(\alpha_t)$ as t sweeps out the interval I which provides a meaningful orientation on the global arc. □

Lemma 4.20. *Along a global arc α, the function $\sigma(z) = \arg(R(z))$ is a monotone mapping of α to a closed half-circle of $\mathbb{S}^1$.*

Besides, if $\sigma(\alpha_t) = \sigma(\alpha_{t'})$ for some $t > t'$, then $\Delta(\alpha_t)$ coincides with the point at infinity $\sigma(\alpha_t) = \sigma(\alpha_{t'})$ that also belongs to $\Delta(\alpha_{t'})$.

Proof. Consider two points α_t and $\alpha_{t'}$ of a global arc satisfying $t > t'$ for the canonical orientation. By Lemma 4.19, $\alpha_{t'} \in \partial\mathcal{U}(\alpha_t)$.

Without loss of generality, we assume that α is contained in $\mathfrak{I}^+$, $\alpha_t = 0$ and $r(\alpha_t) = \mathbb{R}^+$. If $\sigma(\alpha_{t'}) \in [-\pi, 0[$, any associated ray starting in a small enough neighborhood of $\alpha_{t'}$ will cross M^T_{CH}. If $\sigma(\alpha_{t'}) = 0$, then the interior of the strip bounded by $r(\alpha_t), r(\alpha_{t'})$ and the portion of global arc between α_t and $\alpha_{t'}$ is disjoint from M^T_{CH}. It follows that $\Delta(\alpha_t)$ contains only the point at infinity. In the remaining case, we have $\sigma(\alpha_{t'}) \in]0, \pi[$. □

Proposition 4.21. *Consider $z \in \partial \mathrm{M}^T_{CH}$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$. Then, z is either the endpoint or a point of a global arc.*

Proof. We consider an arbitrarily small open arc α of $\partial \mathrm{M}^T_{CH} \cap \mathcal{U}(z)$ ending at z. By assumptions, α is disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. If some point $y \in \alpha$ satisfies $\Gamma(y) \neq \emptyset$, then α partially coincides with a local arc. Since the ending point of any local arc belongs to $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ (see Proposition 4.12), comparison of the orientation of local arcs and the orientation of $\partial \mathrm{M}^T_{CH}$ in a given inflection domain (see Lemma 4.11) proves that z also belongs to this local arc. This is a contradiction. Therefore, any point y in arc α satisfies $\Gamma(y) = \emptyset$. Proposition 4.7 then proves by contradiction that each point of arc α satisfies $\Delta(y) \neq \emptyset$ and is thus a point of global type. Therefore, z is either the endpoint or a point of a global arc containing α. □

Proposition 4.22. *If a point $z \in \partial \mathrm{M}^T_{CH}$ satisfies:*

- $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$;
- $\Delta(z) \neq \emptyset$;
- $\Gamma(z) = \emptyset$;

then z belongs to a global arc.

Proof. Following Proposition 4.21, z is either the ending point or a point of a global arc. We consider a connected neighborhood V of z in $\partial \mathrm{M}^T_{CH}$ that is disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Without loss of generality, we assume that V belongs to $\mathfrak{I}^+$.

We consider a point $y \in V$ such that the oriented arc from y to z in $\partial \mathrm{M}^T_{CH}$ has the same orientation as the standard topological orientation of the boundary. If $\Gamma(y) \neq \emptyset$, then y is either a point or the starting point of a local arc (Corollary 4.10) that can be continued til z (see Proposition 4.12) since V is disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Therefore, $\Gamma(y) = \emptyset$ and it follows then from Proposition 4.7 that $\Delta(y) = \emptyset$. Thus, any such point y is a global point belonging to global arc α.

Then, we consider points $y \in V$ such that the oriented arc from y to z in $\partial \mathrm{M}_{CH}^T$ has the opposite orientation as the standard topological orientation of the boundary. If such point y satisfies $\Gamma(y) \neq \emptyset$, then it is a point or the starting point of a local arc. Since V is connected and disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$, it contains at most one local arc starting at a point of extruding type. The complement of the closure of this local arc in V coincides with global arc β. By hypothesis, z is not a point of extruding type so it belongs to a global arc β. $\square$

Proposition 4.23. *If $z_0 \in \mathbb{C}$ is the endpoint of a global arc α and is neither a zero nor a pole of $R(z)$, then $\Delta(z_0) \neq \emptyset$.*

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Proof. We assume that α is parameterized by the interval $]0, 1[$ (with the correct orientation) and $\alpha(t) \to z_0$ as $t \to 1$. For any $n \geq 2$, we pick a point $\beta_n \in \Delta(\alpha(1 - 1/n))$. Since $\mathbb{C} \cup \mathbb{S}^1$ is compact, the sequence $(\beta_n)_{n\geq 2}$ has an accumulation point β. Since z_0 is the endpoint of α, the point β can not coincide with z_0 (see Lemma 4.19). It follows that a family of associated rays accumulates on a half-line starting at z_0 and containing β. Since $\arg(R(z))$ is continuous in a neighborhood of z_0, we get that this half-line coincides with the associated ray $r(z_0)$. $\square$

If at some point z_0, $\Delta(z_0)$ contains more than one point, then $\partial \mathrm{M}_{CH}^T$ cannot be smooth at z_0.

Proposition 4.24. *At a point z such that $\Delta(z)$ contains at least two points, the boundary $\partial \mathrm{M}_{CH}^T$ has a nonconvexity point. There is a cone $\mathcal{C}$ at z of angle strictly bigger than π and a neighborhood V of z such that $\mathcal{C} \cap V \subset \mathrm{M}_{CH}^T$.*

Proof. First assume that $\Delta(z)$ contains two points u, v both of which are not points at infinity. Lemma 2.11 proves that z belongs to two distinct root trails. Assuming that $R(z) + (u - z)R'(z)$ and $R(z) + (v - z)R'(z)$ are nonzero, the tangent slopes of these root trails at z are determined by the argument of $\frac{R^2(z)}{R(z)+(u-z)R'(z)}$ and $\frac{R^2(z)}{R(z)+(v-z)R'(z)}$. Since $Im(R'(z)) \neq 0$, these two branches intersect transversely at z and the claim follows. If $R(z) + (u - z)R'(z) = 0$, then two branches of the root trail intersect transversely either.

In the remaining case, $\Delta(z)$ contains exactly one point u satisfying the condition $R(z) + (u - z)R'(z) \neq 0$ and a point $\sigma(z)$ at infinity. Then the root trail of u at z has a slope given by the argument of $\frac{R^2(z)}{R(z)+(u-z)R'(z)}$ (or $\frac{R(z)}{R'(z)}$ if u is at infinity, see Lemma 2.15). Similarly, $R'(z) \notin \mathbb{R}$ so these curves intersect transversely at z. Summarizing we see that in all possible cases, the boundary $\partial \mathrm{M}_{CH}^T$ has a nonconvexity point. $\square$

For a point z_0 for which $\Delta(z_0)$ consists of a single point u satisfying the condition $R(z_0) + (u - z_0)R'(z_0) \neq 0$, Lemma 2.11 proves that: (i) the root trail $\mathfrak{tr}_u$ has a unique branch at z_0, (ii) it is contained in M_{CH}^T, and (iii) its tangent slope is the argument of $\frac{R^2(z_0)}{R(z_0)+(u-z_0)R'(z_0)}$ (mod π).

4.5. **Points of extruding type.** Outside the local and the global arcs, the only singular boundary points in the complement of $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ which can occur are points of extruding type.

Proposition 4.25. *Let z be a point of extruding type in $\partial \mathrm{M}_{CH}^{T}$. Then z is both the ending point of a global arc and the starting point of a local arc.*

The boundary $\partial \mathrm{M}_{CH}^{T}$ is not C^1 at z and z is a point of nonconvexity. There exist a neighborhood V of z and a cone $\mathcal{C}$ at z of angle strictly bigger than π such that $\mathrm{M}_{CH}^{T} \cap V$ contains $\mathcal{C} \cap V$, see Figure ???.

Proof. By definition of the correspondence Γ, z is the starting point of a local arc. Propositions 4.21 and 4.22 show that z is the ending point of a global arc.

For any point $u \in \Delta(z)$, the root trail $\mathfrak{tr}_u$ has a unique local branch at z and its tangent direction is the argument of $\frac{R^2(z)}{R(z)+(u-z)R'(z)}$ (mod π), see Lemma 2.11. Indeed, $R(z) + (u-z)R'(z) \neq 0$ because $u - z$ is real collinear to $R(z)$ while $\mathrm{Im}(R'(z)) \neq 0$. Since $z \notin \mathfrak{I}_R$, this branch transversely intersects the integral curve of $R(z)\partial_z$ containing z. Both of these branches are (semi)analytic curves contained in M_{CH}^{T}. The claim follows. □

4.6. Boundary arcs in inflection domains.

Proposition 4.26. *For any connected component $\mathcal{D}$ of the complement to the curve of inflections $\mathfrak{I}_R$, $\partial \mathrm{M}_{CH}^{T} \cap \mathcal{D}$ is a union of disjoint topological arcs. In each of them, local and global arcs have the same orientation. If $\mathrm{Im}(R')$ is positive (resp. negative) in $\mathcal{D}$ then the latter orientation coincides with (is opposite to) the topological orientation of $\partial \mathrm{M}_{CH}^{T}$.*

Proof. The statement about orientation follows from Proposition 4.11 and Lemma 4.19. Proposition 4.25 shows that a point of extruding type is incident to a local and a global arcs. It remains to prove that any point of $\mathcal{Z}(PQ) \cap \mathcal{D}$ is incident to exactly two arcs.

Such a point z_0 is neither a zero nor a pole of $R(z)$, see Corollary 3.11. Therefore Lemma 2.20 proves that for a small enough neighborhood V of z_0, the intersection of V with the interior of M_{CH}^{T} is connected. Thus at most two arcs are incident to z_0. □

5. Singular boundary points on the curve of inflections

At points belonging to the curve of inflections the boundary $\partial \mathrm{M}_{CH}^{T}$ can display a more complicated behaviour. In this section, we classify boundary points that belong to the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections (see Definition 1.4).

Definition 5.1. A point of $\partial \mathrm{M}_{CH}^{T} \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}_R^*$ is a point of:

- *bouncing type* if $\Delta^+ \neq \emptyset$ and $\Gamma \cup \Delta^- \neq \emptyset$;
- *switch type* if $\Delta^+(z) \neq \emptyset$ and $\Gamma \cup \Delta^-(z) = \emptyset$;
- *C^1-inflection type* if $\Delta^+ = \emptyset$, $\Delta^- \neq \emptyset$ and $\Gamma = \emptyset$;
- *C^2-inflection type* if $\Delta^+ \cup \Delta^0 = \emptyset$ and either $\Delta^- = \emptyset$ or $\Gamma \neq \emptyset$.

I added Δ^0, this should be rechecked

— Mitya

5.1. Horns at points of the transverse locus.

At a point p of $\mathfrak{I}_R^*$, the curve of inflections is smooth and the vector field is transversal to it. This means that by (3.2) we have

$$R(u) = 1 + \rho u + (a + ib)u^2 + \ldots \tag{5.1}$$

where we assumed that $p = 0$. The condition $\mathrm{Im}\, R'(0) = 0$ means $\rho \in \mathbb{R}$, and the transversality condition is equivalent to $b \neq 0$. Without loss of generality we can assume that $b > 0$. In other words, $m = 2$ in (3.2) which implies that the integral curves locally look like cubic curves with inflections at these points.

We define the diameter of a horn ${}_p\triangle^{p'}_{p''}$ to be the smallest $t > 0$ such that $u + t'R(u) \in {}_p\triangle^{p'}_{p''}$ for any $u \in {}_p\triangle^{p'}_{p''}$ and $t \in [0, t]$.

Lemma 5.2. *Let $p \in \mathfrak{I}^*_R$. Then there exists a sufficiently small neighborhood U of p and $\epsilon > 0$ such that for all points in $\Omega_+ = \mathfrak{I}^+ \cap U$ there exists a small horn of diameter greater than ϵ.*

Proof. Consider the function

$$T(u,t) = \frac{d\sigma(u + tR(u))}{dt}(t) = \operatorname{Im}\left[\frac{R'(u + tR(u))}{R(u + tR(u))}R(u)\right]$$

defined in $\mathbb{C}_u \times \mathbb{R}_t$. Since $T(0, 0, t) = 2bt + O(t^2)$, we have $\frac{\partial T}{\partial t}(0, 0) = 2b > 0$. Therefore $\frac{\partial}{\partial t}T(u,t) > b > 0$ in some sufficiently small neighborhood $U \times (-\epsilon, \epsilon)$ of $(0, 0)$. Moreover, by definition of Ω_+ we have $\Omega_+ \times \{0\} \subset \{T > 0\}$. Taken together, this implies that $\Omega_+ \times [0, \epsilon] \subset \{T > 0\}$ for some $\epsilon > 0$. This means that the argument of $R(u + tR(u))$ is monotone increasing for $0 < t < \epsilon$ and for every $u \in \Omega_+$. Thus for every $u \in \Omega_+$, any horn of diameter less that ϵ is a small horn. □

5.2. **Points of bouncing type.**

Proposition 5.3. *Let z be a point of bouncing type in $\partial \mathrm{M}^T_{CH}$. Then z is the ending point of a global arc and also the starting point of another arc which can either be local or global.*

The boundary $\partial \mathrm{M}^T_{CH}$ is not C^1 at z and z is a point of nonconvexity. There exist a neighborhood V of z and a cone $\mathcal{C}$ at z of angle strictly bigger than π such that $\mathrm{M}^T_{CH} \cap V$ contains $\mathcal{C} \cap V$.

Besides, for small enough V, one has $V \cap \partial \mathrm{M}^T_{CH} \cap \mathfrak{I}_R = \{z\}$.

Proof. Without loss of generality, we can assume that $z = 0$, $R(0) = 1$, and $R'(0) \in \mathbb{R}$. We have $R'(\epsilon) = R'(0) + R''(0)\epsilon + o(\epsilon)$. Since z belongs to $\mathfrak{I}^*_R$, we get $R''(0) = a + bi$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$. Additionally and without loss of generality, let us assume that $b > 0$.

Recall that by Lemma 3.20 a germ of the domain bounded by the integral curve of R starting at z and $r(z)$ lies in the complement to M^T_{CH}.

Let us first consider the case when $\Delta^0(z) = \emptyset$. Since $\Delta^+(z) \neq \emptyset$, we have that for $u \in \Delta^+(z)$, the germ of $\mathfrak{tr}_u$ at z is strictly convex and contained in the lower half-plane (see Proposition 2.18). Then, since $\Gamma(z) \cup \Delta^-(z)$ is nonempty there exists an arc (portion of a root trail or an integral curve) starting at 0 with an horizontal tangent, contained in the upper half-plane, and belonging to M^T_{CH}.

Denoting by α the union of the latter arc with the root trail of u, there exists a neighborhood V of z such that the α cuts V into two parts. The part of V to the left of α is entirely contained in M^T_{CH}. This domain contains the intersection of V with a cone with vertex at z and of angle strictly larger than π. It also contains all the intersection of V with $\mathfrak{I}_R$ excepted for the point z.

If $\Delta^0(z)$ is nonempty, then it contains the unique point $u = -\frac{1}{R'(0)}$. In such a case, the root trail $\mathfrak{tr}_u$ has exactly two branches intersecting at z (see Lemmas 2.11, 2.15 and Remark 2.16). The tangent slopes of these branches (which intersect orthogonally) are $\frac{\theta_0}{2}$ (mod $\pi/2$) where θ_0 is the argument of $\frac{1}{R''(0)}$. In contrast, the tangent slope of $\mathfrak{I}_R$ at z is θ_0 (mod π). Since $R''(0) \notin \mathbb{R}$, we have that $\theta_0 \notin \pi\mathbb{Z}$ and these branches intersect transversely. Consequently, there is a neighborhood V of z such that the intersection of V with M^T_{CH} contains the intersection of V with a cone of angle arbitrarily close to $\frac{3\pi}{2}$. In particular, it contains all the intersection of V with $\mathfrak{I}_R$ except for z itself.

It remains to prove that in a neighborhood of z, $\partial \mathrm{M}_{CH}^T$ is formed by exactly two arcs.

Lemma 5.4. *Assume that $\Gamma(z) = \emptyset$ and $\Delta^- \neq \emptyset$. Then z is a starting point of a global arc.*

Proof. We assume that $z = 0$, $R(0) = 1$ and γ_z lies in the upper half-plane. By Lemma 3.20 the points lying below γ_z are not in M_{CH}^T. Let q be a point lying slightly above γ_z, $q \notin {\mathrm{M}_{CH}^T}^\circ$. Denote by $\tilde{p} = \tilde{p}(q)$ the first point on γ_q such that $\Delta(\tilde{p}) \neq \emptyset$, in particular $\tilde{p} \in \mathrm{M}_{CH}^T$. Using the arguments and notations of Lemma 4.9 (and Lemma 5.2 instead of Lemma 3.22) we see that the arc $\gamma_{\tilde{p}}^q \cap {\mathrm{M}_{CH}^T}^\circ = \emptyset$, thus $\tilde{p} \in \partial \mathrm{M}_{CH}^T$. The points $\tilde{p}(q)$ form a global arc of $\partial \mathrm{M}_{CH}^T$ starting at z and lying between $\mathfrak{tr}_u$ and γ_z in the upper half-plane. □

If $\Gamma(z) \neq \emptyset$, then a local arc whose germ is contained in the upper half-plane starts at z.

In both cases the portion of boundary of $\partial \mathrm{M}_{CH}^T$ in a neighborhood of z in the lower half-plane is a global arc ending at z. Indeed, assume $q \in {}_z\triangle_{z''}^{z'} \cup {}_{z''}\angle \subset \mathrm{M}_{CH}^T$ lies slightly below γ_z, and denote again by $\tilde{p} = \tilde{p}(q)$ the first point on γ_q such that $\Delta(\tilde{p}) \neq \emptyset$, in particular $\tilde{p} \in \mathrm{M}_{CH}^T$. Then repeating the arguments of Lemma 4.8 we see that $\gamma_{\tilde{p}}^q \cap {\mathrm{M}_{CH}^T}^\circ = \emptyset$. Thus the points $\tilde{p}$ form a global arc of $\partial \mathrm{M}_{CH}^T$ lying in the lower half-plane and ending at z.

□

5.3. **Points of C^2-inflection type.**

Proposition 5.5. *Consider a point p of $\partial \mathrm{M}_{CH}^T \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}_R^*$. If $\Delta(p) = \emptyset$, then $\Gamma(p) \neq \emptyset$ and p is the starting point of a local arc.*

Proof. We keep the previous normalization $p = 0$, $R(0) = 1$, $\operatorname{Im} R''(0) = b > 0$. The positive trajectory γ_0 cuts $\mathfrak{I}^+$ into two parts, one containing the convex hull of γ_0 (denoted by Ω_{++}) and another one denoted by Ω_{+-}.

Repeating the arguments of Lemma 4.8 and using Lemma 5.2 (see Figure 2) for points in Ω_{+-}, we can conclude that M_{CH}^T does not intersect Ω_{+-}. Together with $\Delta(0) = \emptyset$ this implies that some small sector $S_- = \{-\epsilon < \arg z < 0\}$ doesn't intersect M_{CH}^T.

Now, assume by contradiction, that there exists some $q \in \Omega_{++} \setminus \mathrm{M}_{CH}^T$. By decreasing Ω_{++} if needed and repeating the arguments of Lemma 4.9 (see Figure 3), we obtain that $\Omega_{++} \cap \mathrm{M}_{CH}^T = \emptyset$. Therefore there is a neighbourhood U_+ of 0 in $\mathfrak{I}^+$ which is disjoint from M_{CH}^T. We can assume that U_+ is an intersection of $\mathfrak{I}^+$ with a small disk centered at 0.

For sufficiently small $\epsilon > 0$ the set $U = U_+ \cup \{\epsilon < \arg z < \epsilon\}$ is disjoint from M_{CH}^T: the part lying in the lower half-plane is in $U_+ \cup S_-$ and the part lying in the upper half-plane is in $U_+ \cup {}_0\triangle_{p''}^{p'} \cup {}_{p''}\angle$.

Now, take a small neighborhood U_- of 0 in Ω_- bounded by a convex curve transversal to R. For any $u \in U$, the ray $r(u)$, being close to R_+, lies inside $U \cup U_-$. By Lemma 3.25, this implies that $U_- \subset {\mathrm{M}_{CH}^T}^c$, and therefore $0 \notin \mathrm{M}_{CH}^T$, a contradiction. Thus $\Omega_{++} \subset \mathrm{M}_{CH}^T$ and $\gamma_0 \subset \partial \mathrm{M}_{CH}^T$. □

Proposition 5.6. *Consider a point p of C^2-inflection type. Then there is a neighborhood V of p in which $\partial \mathrm{M}_{CH}^T$ is formed by:*

- *a portion of local arc γ parameterized by an interval $[0, \epsilon[$, $\epsilon > 0$ with $\gamma(0) = p$;*
- *a portion of global arc α parameterized by $[0, \epsilon[$ and such that $\alpha(0) = p$ and $\Delta(\alpha(t)) = \{\gamma(t)\}$.*

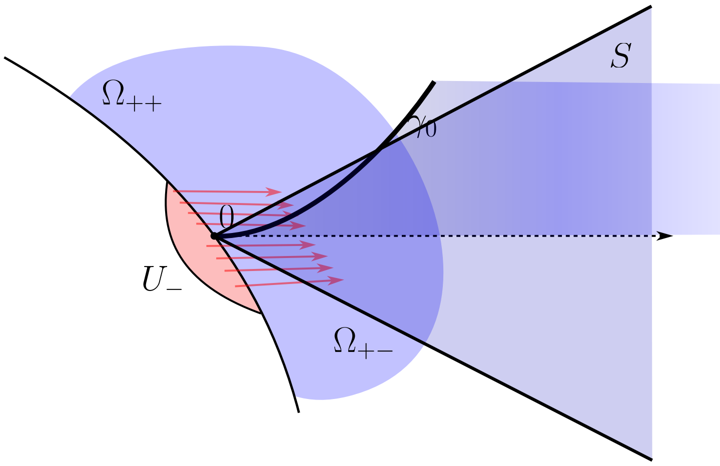

FIGURE 4. A point $p \in \partial \mathrm{M}_{CH}^T \cap \mathfrak{I}_R^*$ with $\Delta(p) = \emptyset$ is a starting point of a local arc.

In particular, p is simultaneously the starting point of a local arc and the starting point of a global arc.

Proof. We again assume that $p = 0$ and $R(0) = 1$. Following the definition of a point of C^2-inflection type (see Theorem 1.6), we have that $\Delta^+(0) \cup \Delta^0(0) = \emptyset$ and either $\Delta^-(0) = \emptyset$ or $\Gamma(0) \neq \emptyset$. By Proposition 5.5, if $\Delta^-(0) = \emptyset$, $\Gamma(0)$ is also nonempty. Therefore, we will assume that 0 is the starting point of a local arc γ_0 and deduce the shape of the boundary close to 0 from the assumption $\Delta^+(0) \cup \Delta^0(0) = \emptyset$.

The curve γ_0 divides the domain $\mathfrak{I}^+$ into two parts, and as before we denote the one containing the horn of 0 by Ω_{+-}.

Lemma 5.7. $\Omega_{+-} \cap \mathrm{M}_{CH}^T = \emptyset$.

Proof. Consider the case $R'(0) < 0$ first. Denote $I_- = \partial \Omega_{+-} \cap \mathfrak{I}_R \setminus \{0\}$. We claim that $r(u) \cap \mathrm{M}_{CH}^T = \emptyset$ for all $u \in I_-$ sufficiently close to 0.

Denote $\rho = -(R'(0))^{-1}$. By assumption $\mathrm{M}_{CH}^T \cap \mathbb{R}_+ = \Delta(0) = \Delta^-(0)$ is a compact subset of $(\rho, +\infty]$, so $\mathrm{M}_{CH}^T \cap \mathbb{R}_+ \subset [\rho', +\infty]$, $\rho' > \rho$. Again, by compactness of M_{CH}^T and by Lemma 3.20 we can assume that

$$\mathrm{M}_{CH}^T \cap \{|\operatorname{Im} z| < \delta'\} \subset \{\operatorname{Re} z > \rho' - \epsilon, \operatorname{Im} z \leq 0\} \cup \{\operatorname{Re} z < \epsilon\}. \tag{5.2}$$

For all $u \in I_-$, $u \neq 0$, the slope $\sigma(u)$ is positive:

$$\operatorname{Im} R(u) = R'(0) \operatorname{Im} u + O(u^2) > 0,$$

as $\operatorname{Im} u < 0$ and $\operatorname{Re} u = O(\operatorname{Im} u)$ by transversality of $\mathfrak{I}_R$ and $r(0)$. Thus $\operatorname{Im}(u + tR(u)) = 0$ for $t = \rho + O(u)$, and $r(u) \cap \mathbb{R}_+ = \rho + O(u)$. Therefore for any $\epsilon > 0$ for any point $u \in I_-$ sufficiently close to 0 the ray $r(u)$ has arbitrarily small slope and $r(u) \cap \mathbb{R}_+ \in (\rho - \epsilon, \rho + \epsilon)$. Therefore

$$r_+(u) = r(u) \cap \{\operatorname{Im} z > 0\} \subset {}_0\triangle_{p''}^{p'} \cup {}_{p''}\angle \subset {\mathrm{M}_{CH}^T}^c \tag{5.3}$$

for all $u \in I_-$ sufficiently close to 0.

Let $u \in I_-$, $|\operatorname{Im} u| < \delta'$, and take $u'' \in r(u)$ with $\operatorname{Re} u'' = \epsilon$. If ϵ is sufficiently small then by Lemma 5.2 we can assume that the horn ${}_u\triangle_{u''}^{u'}$ is small.

The horn ${}_{u''}\angle$ lies above $r(u)$ and to the right of $\{\operatorname{Re} z > \epsilon\}$. Thus

$${}_{u''}\angle \cap \{\operatorname{Im} z < 0\} \subset \{\epsilon < \operatorname{Re} z < -\rho^{-1} + \epsilon, \operatorname{Im} z > -\delta\}$$

and therefore ${}_{u''}\angle \cap \{\operatorname{Im} z \leq 0\} \subset {\mathrm{M}_{CH}^{T}}^{c}$. Repeating the arguments of Lemma 4.8 we conclude that ${}_{u''}\angle \cap \{\operatorname{Im} z > 0\} \subset {\mathrm{M}_{CH}^{T}}^{c}$ as well, which implies that $u \in {\mathrm{M}_{CH}^{T}}^{c}$ and therefore $\Omega_{+-} \subset {\mathrm{M}_{CH}^{T}}^{c}$ as well.

If $R'(0) > 0$ then $\Delta^{-}(0) = \Delta^{0}(0) = \emptyset$, so $\Delta(0) = \emptyset$. Then the arguments of Lemma 4.8 are applicable for all $\tilde{p} \in \Omega_{+-}$, which proves the claim in this case as well.

□

Let $\tilde{\gamma}_0$ be the curve of points in $\mathfrak{I}_{-}$ whose associated rays are tangent to the positive trajectory γ_0 of $R\partial_z$ starting at 0.

Lemma 5.8. *For R as in* (5.1) *the local arc γ_0 is described by equation $y(x) = \frac{b}{3}x^3 + o(x^3)$, $x \geq 0$ and curve $\tilde{\gamma}_0$ is described by $y(x) = 5\frac{b}{3}x^3 + o(x^3)$, $x \leq 0$.*

Proof. Using (3.3) for (5.1) we see that

$$\gamma_0(t) = t + o(t) + i\left(\frac{b}{3}t^3 + O(t^4)\right)$$

with $a \in \mathbb{R}$ and therefore the slope $\sigma(\gamma_0(t)) = bt^2 + O(t^3)$.

The point $u \in \tilde{\gamma}_0$ whose associated ray $r(u)$ is tangent to γ_0 at $\gamma_0(t)$ has the form

$$u = \gamma_0(t) - sR(0) = t - s + o(t) + i\left(-sbt^2 + \frac{b}{3}t^3 + O(t^4)\right)$$

with the condition $\sigma(u) = \sigma(\gamma_0(t)) = bt^2 + O(t^3)$. The latter condition means $s = 2t + o(t)$ and therefore

$$u = -t + o(t) - i\left(\frac{5}{3}bt^3 + O(t^4)\right).$$

□

Lemma 5.9. *For any $\alpha \in \tilde{\gamma}_0$ the ray $r(\alpha)$ doesn't intersect $\tilde{\gamma}_0 \cup \gamma_0$ between α and the point of tangency $z = z(\alpha)$ of $r(\alpha)$ and γ_0.*

Proof. By Lemma 5.8 $\gamma_0 \cup \tilde{\gamma}_0 = \{y = \gamma(x)\}$, with γ'' being continuous, monotonic and vanishing at $x = 0$ on the interval $[\operatorname{Re}\alpha, \operatorname{Re} z]$. Let $r(\alpha) = \{y = kx + b\}$. By construction, $\hat{\gamma} = \gamma(x) - kx - b$ vanishes at $\operatorname{Re}\alpha$ and has a double zero at $\operatorname{Re} z$. Any other point of intersection of $r(\alpha)$ and $\tilde{\gamma}_0 \cup \gamma_0$ will mean existence of another zero of $\hat{\gamma}$ on $[\operatorname{Re}\alpha, \operatorname{Re} z]$. By Rolle Theorem this will imply existence of two zeros of $\hat{\gamma}'' = \gamma''$ on $[\operatorname{Re}\alpha, \operatorname{Re} z]$, which contradicts monotonicity of γ''. □

Corollary 5.10. *Let $\alpha_+ = r(\alpha) \cap \mathfrak{I}_R$ and let $r_+(\alpha) = r(\alpha) \setminus \Omega_{--} = \alpha_+ + \sigma(\alpha)\mathbb{R}_+$. Then $r_+(\alpha) \cap {\mathrm{M}_{CH}^{T}}^{\circ} = \emptyset$.*

Proof. Indeed, the piece of $r_+(\alpha)$ between α_+ and the point of tangency $z = z(\alpha)$ of $r(\alpha)$ and γ_0 lies in Ω_{+-} by Lemma 5.9, and the remaining piece coincides with $r(z)$. Thus the claim follows from Lemma 5.7 and Lemma 3.20. □

The curve $\tilde{\gamma}_0$ divides Ω_- into two parts. Denote by Ω_{--} the part consisting of points whose associated ray does not intersect γ_0 and let Ω_{-+} denotes the second part. Clearly $\Omega_{-+} \subset \mathrm{M}_{CH}^{T}$.

Lemma 5.11. *For any $u \in \Omega_{--}$*

(1) $r(u) \cap \tilde{\gamma}_0 = \emptyset$,
(2) $r(u) \setminus \Omega_{--} \subset {\mathrm{M}_{CH}^{T}}^{c}$.

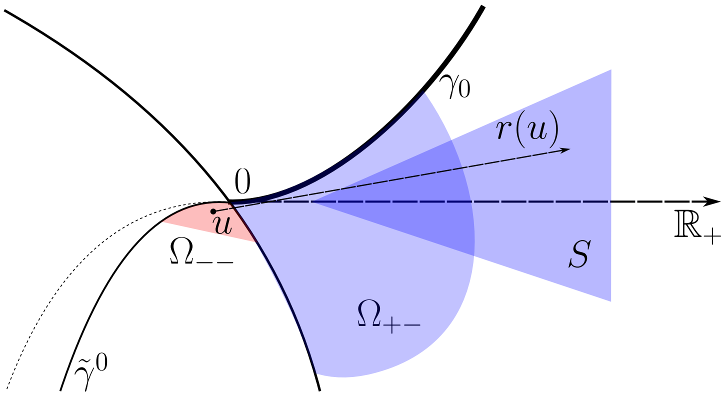

FIGURE 5. A point in the transverse locus of $\partial \mathrm{M}_{CH}^T \cap \mathfrak{I}_R$ with empty Δ correspondence is the starting point of a global arc.

Proof. Let γ_α^ω be the piece of trajectory of $R\partial_z$ containing u and with ends $\alpha = \alpha(u) \in \tilde{\gamma}_0$ and $\omega = \omega(u) \in (I)_R$: by Lemma 5.8 $\tilde{\gamma}_0$ is transversal to the trajectories of $R\partial_z$ near 0 except γ_0. Let $D(u) = \cup_{z \in \gamma_\alpha^\omega} r(z)$ be the domain sweeped by rays tangent to γ_α^ω. As γ_α^ω is convex, $\partial D(u) = r(\alpha) \cup \gamma_\alpha^\omega \cup r(\beta)$. By Lemma 5.9, Lemma 5.8 and Lemma 5.7 $\partial D(u) \cap \tilde{\gamma}_0 = \emptyset$, so $r(u) \cap \tilde{\gamma}_0 = \emptyset$ as well

The boundary of $D_+(u) = D(u) \setminus \Omega_{--}$ consists of $r(\beta)$, the piece of $\mathfrak{I}_R$ lying between β and the point α_+ and $r_+(\alpha)$ which do not intersect ${\mathrm{M}_{CH}^T}^\circ$ by Lemma 5.7 and Corollary 5.10. This implies the second claim of the Lemma. □

Proof of Proposition 5.6. Take some $u \in \Omega_{--}$ and let $\Omega_{--}(u)$ be the curvilinear triangle bounded by γ_α^β, $\tilde{\gamma}_0$ and $\mathfrak{I}$. The ray $r(u)$ doesn't cross γ_α^β by convexity and doesn't cross $\tilde{\gamma}_0$ by Lemma 5.11, so it leaves $\Omega_{--}(u)$ through $\mathfrak{I}$, with $r(u) \setminus \Omega_{--}(u) \subset {\mathrm{M}_{CH}^T}^c$ Lemma 5.11. As $\Omega_{--}(u') \subset \Omega_{--}(u)$ for any $u' \in \Omega_{--}(u)$, this means that $r(u') \subset \Omega_{--}(u) \cup {\mathrm{M}_{CH}^T}^c$ for all $u' \in \Omega_{--}(u)$, and the claim follows by Lemma 3.25. □

□

5.4. Points of C^1-inflection type.

Proposition 5.12. *A point $p \in \partial \mathrm{M}_{CH}^T \cap \mathfrak{I}_R^*$ of C^1-inflection type is the starting point of two global arcs (one in each of the incident inflection domains).*

Proof. We assume that $p = 0$, $R(0) = 1$ and $\gamma_z \subset \mathfrak{I}^+$ lies in the upper half-plane. We use $z = x + iy$ notations.

Exactly as in the case of bouncing type, the conditions $\Delta^- \neq \emptyset$ and $\Gamma = \emptyset$ imply that z is a starting point of a global arc, see Lemma 5.4. Denote this arc by $\eta = \{y = \xi(x)\}$. Arguments of Lemma 5.4 show that η lies between $\mathfrak{tr}_\infty$ and $\mathfrak{tr}_{u_\infty}$, where $u_\infty = \sup \Delta^-$.

We repeat the arguments of C^2-inflection case above. The Lemma 5.7 can be repeated verbatim, thus we get $\Omega_{+-} \subset \mathrm{M}_{CH}^T$.

Let $\tilde{\eta}$ be the curve of points in $\mathfrak{I}^-$ bounding (the germ at 0 of) the domain Ω_{--} consisting of points of $\mathfrak{I}^-$ whose associated rays do not intersect η. In particular, $r(\alpha)$ is tangent to η for all $\alpha \in \tilde{\eta}$. If γ_α^β is a trajectory of $R\partial_z$ starting at $\alpha \in \tilde{\eta}$ and ending at $\beta \in \mathfrak{I}_R$ then $\gamma_\alpha^\beta \subset \Omega_{--}$ by convexity of γ_α^β, as in Lemma 5.11.

The same arguments as in Lemma 5.11 and in the proof of Proposition 5.6 now show that Ω_{--} satisfies the conditions of Lemma 3.25 and is therefore disjoint from M_{CH}^T.

WRONG: why $r(u)$ doesn't intersect $\tilde{\eta}$? We need an analogue of Lemma 5.9. Lemma below (check it: is the reparameterization trick kosher?) shows that $\tilde{\eta}$ is C^2, but I do not know a proof right now

— Mitya

□

Lemma 5.13. *Let $\gamma_1(s)$ be an arc supported by a C^k-smooth arc $\gamma_2(s)$: the ray $r(\gamma_1(s))$ is tangent to γ_2 at point $\gamma_2(s)$. Then $\gamma_1(s)$ is C^{k+1}-smooth.*

Proof. Let $p = \gamma_1(s)$ and take a point $p' = \gamma_1(s')$ close to p. Necessarily $r(p)$ and $r(p')$ should intersect near the arc joining $q = \gamma_2(s)$ and $q' = \gamma_1(s)$. This means that the map

$$(s,t) \to w(s,t) = \gamma_1(s) + tR(\gamma_1(s)) \tag{5.4}$$

has critical locus at points $\left(s, t = t(s) = \frac{\gamma_2(s)-\gamma_1(s)}{R(\gamma_1(s))}\right)$ (note that $w(s,t(s)) = \gamma_2(s)$ by definition). In other words, the vectors

$$\frac{\partial w}{\partial t}(s,t(s)) = R(\gamma_1(s)), \qquad \frac{\partial w}{\partial s}(s,t(s)) = [1 + t(s)R'(\gamma_1(s))]\,\dot{\gamma}_1(s), \tag{5.5}$$

should be $\mathbb{R}$-collinear, $\frac{\partial w}{\partial s}(s,t(s)) = \alpha\frac{\partial w}{\partial t}(s,t(s))$ for some $\alpha = \alpha(s) \in \mathbb{R}\setminus\{0\}$.

If $\alpha = 0$, then $1 + t(s)R' = 0$ and $R' \in \mathbb{R}\setminus\{0\}$ (as $t(s) \in \mathbb{R}$), so $\gamma_1(s) \in \mathfrak{I}_R$.

Otherwise, by reparameterisation of $\gamma_1(s)$ we can assume $\alpha = 1$, and we get that $\gamma_1(s)$ (up to reparameterization of γ_1, γ_2) satisfies an ordinary differential equation

$$\dot{\gamma}_1(s) = \frac{R^2(\gamma_t(s))}{R(\gamma_t(s)) + (\gamma_2(s) - \gamma_1(s))R'(\gamma_1(s))} \tag{5.6}$$

with continuous RHS. Thus if $\gamma_2(s)$ was C^k-smooth, then $\gamma_1(s)$ will be C^{k+1}-smooth.

□

5.5. **Points of switch type.** Recall that a point p of $\partial \mathrm{M}^T_{CH} \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}^*_R$ is called a point of switch type if $\Delta^+(p) \neq \emptyset$ and $\Gamma(p) \cup \Delta^-(p) = \emptyset$.

Proposition 5.14. *The negative part $\gamma_0^-(t)$ of the integral curve $\gamma_0(t)$ is part of the boundary of M^T_{CH}.*

As before, we assume that $p = 0$, $R(0) = 1$ and $\operatorname{Im} R''(0) > 0$. The trajectory $\gamma_0(t)$ and the curve $\mathfrak{I}_R$ divide U into four domains $\Omega_{\pm,\pm}$, with Ω_{+-} containing the (germ of) H_0.

Lemma 5.15. $\Omega_{++} \subset {\mathrm{M}^T_{CH}}^c$.

Proof. Recall that by Lemma 3.20 the union $H(0) = {}_0\triangle^{p'}_{p''} \cup {}_{p''}\angle$ doesn't intersect M^T_{CH}.

We assume first that $R'(0) < 0$ and denote $\rho = -(R'(0))^{-1} \in \mathbb{R}_+$. As $\Delta(0) \subset (0,\rho)$ is compact and $+\infty \notin \Delta(0)$ there exists a small open sector S with the ray $(\rho', +\infty)$ as bisector, for some $\rho' < \rho$, and disjoint from M^T_{CH}.

Let U be a small neighborhood of 0 such that the slope $\sigma(z)$ is smaller than the slope of sides of S for all $z \in U$. Moreover, we assume that U is so small that the rays $r(z)$, $z \in U$, do not intersect the interval $[0, \rho']$: trails of these points lie in the lower half-plane. Thus

- for any $z \in U$, $\operatorname{Im} z > 0$, the ray $r(z)$ intersects the boundary of $H(0) \cup S$ only once at $\gamma_0^+(t)$.

The same conclusion holds in the case $R'(0) > 0$: in this case $\sigma(z) > 0$ for all $z \in U$, $\operatorname{Im} z > 0$.

Since $\Gamma(0) = \emptyset$ there is a point $q \in \Omega_{++} \setminus \mathrm{M}^T_{CH}$. The proof now follows from arguments of Lemma 4.9. □

Proposition 5.16. *Let $p = \gamma(t)$ for some $t < 0$ sufficiently close to 0. The curvilinear triangle $H(p) \subset \Omega_{-+}$ bounded by $r(p)$, the curve $\gamma^-(t)$ and the inflection curve lies outside M_{CH}^T.*

We write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

Lemma 5.17. *$\frac{\partial}{\partial x}\sigma(x+iy) < 0$ as long as $z = x + iy$ lies in a sufficiently small sector $\{|z| < \delta, x < 0, |y| < -\epsilon x\}$ for some $\epsilon, \delta > 0$ depending on R only.*

Proof. We recall that $R(z) = 1 + az + bz^2 + ..$, with $\operatorname{Im} b > 0$. Then $\log R = az + \left(b - \frac{a^2}{2}\right) z^2 + ...$ and $(\log R)' = a + (2b - a^2)z + ...$ Therefore

$$\frac{\partial}{\partial x}\sigma(x+iy) = \operatorname{Im}(\log R)' = \operatorname{Im}\big((2b-a^2)z + ...\big) < 0$$

for z satisfying the conditions above, as $0 < \arg(2b - a^2) < \pi$. □

Lemma 5.18. *$r(z)$ doesn't intersect $\gamma_0^-(t)$ for all $z \in H(p)$.*

Proof. Consider first the case $\operatorname{Im} z < 0$. As $\gamma_0^-(t)$ is tangent to $\mathbb{R}$, we can assume that this part of $H(p)$ satisfies the conditions of Lemma 5.17, so $\sigma(z) > \sigma(z_+)$, where $z_+ = z + t(z) \in \gamma_0^-(t)$. Thus the ray $r(z)$ lies in the half-plane bounded by the line tangent to $\gamma_0^-(t)$ at z_+ and containing z. Therefore $r(z)$ doesn't intersect $\gamma_0^-(t)$ by convexity of $\gamma_0^-(t)$.

Now, assume $\operatorname{Im} z > 0$. Recall that we chose U so small that for any $z \in U$, $\operatorname{Im} z > 0$, the intersection $r(z) \cap \mathbb{R} \subset (\rho', +\infty) \subset \mathbb{R}_+$. Thus $r(z) \cap \{\operatorname{Im} w < 0\} \subset \{\operatorname{Re} z > \rho' > 0\}$ which is disjoint from γ^-. □

Lemma 5.19. *$r(z)$ doesn't intersect $r(p)$ for all $z \in H(p)$.*

Proof. Let γ_z^- be a part of an integral curve of R ending at z and starting at $z'' \in \gamma_z^-$ with $\operatorname{Im} z'' = \operatorname{Im} p$. Necessarily $z'' \notin H(p)$. Moreover the trajectory γ_z^- cannot intersect the trajectory γ_0^- by uniqueness of solutions of ODE, so necessarily $\operatorname{Re} z'' < \operatorname{Re} z$ and therefore by Lemma 5.17 we have $\sigma(z'') > \sigma(z)$.

Let $z' = \gamma_z^- \cap r(p)$ be the point where γ_z^- enters $H(p)$. This point necessarily lies on $r(p)$ as γ_z^- doesn't intersect neither the inflection point nor γ^-. Thus $\sigma(z') < \sigma(p)$.

Assume now that $r(z)$ intersects $r(p)$. Then $\sigma(z) > \sigma(p)$. Therefore the slope $\sigma(w)$, $w \in \gamma_z^-$, is not monotonic, so γ_z^- has inflection point, which is impossible since γ_z^- doesn't intersect the inflection curve. □

Proof of Proposition 5.16. By minimality, it is enough to prove that $r(z) \subset H(p) \cup {\mathrm{M}_{CH}^T}^c$ for any $z \in H(p)$. By Lemmas 5.18, 5.19 the ray $r(z)$ doesn't intersect $r(p)$ and $\gamma_0^-(t)$. Thus $r(z)$ leaves $H(p)$ through the inflection curve $\mathfrak{I}_R$ with a small slope.

If the slope is positive then $r(z) \subset U_{++} \cup H(0)$. In particular, this is the case for all $z \in H(p)$, $\operatorname{Im} z < 0$ by Lemma 5.17.

Assume now that $\sigma(z) < 0$ (and therefore $\operatorname{Im} z > 0$). Recall that we chose U so small that for any $z \in U$, $\operatorname{Im} z > 0$, the intersection $r(z) \cap \mathbb{R} \subset (s', +\infty) \subset \mathbb{R}_+$. Thus $r(z) \setminus H(p) \subset U_{++} \cup H(0) \cup S$.

□

Proposition 5.20. *Consider a point p of switch type in $\partial \mathrm{M}_{CH}^T$. Then there is a neighborhood V of p such that $\mathrm{M}_{CH}^T \cap V$ is contained in a half-disk centered at p. Besides, no neighborhood of p in M_{CH}^T can be contained in a cone centered at p with an angle strictly smaller than π. A point of switch type is the ending point of both a local arc and a global arc.*

Proof. We essentially repeat the arguments of Lemma 4.8. Take $z' \in H(0)$ slightly below $\gamma_0^+(t)$ and let $z \in \gamma_{z'}^-$ be the first point such that $r(z) \cap \mathrm{M}_{CH}^T \neq \emptyset$. Clearly $z \in \partial \mathrm{M}_{CH}^T$ and such points form a global arc ending at 0 by condition $\Delta(0) \neq \emptyset$. □

5.6. **Classification of boundary points.** Here, we summarize several results obtained in previous sections to obtain a complete classification of boundary points.

Proof of Theorem 1.6. It follows from Corollary 3.8 that there are at most $2d$ singular points in the curve of inflections ($d = 3\deg P + \deg Q - 1$). Proposition 3.13 proves that the tangency locus $\mathfrak{T}_R$ is formed by at most $2d^2$ isolated points and d lines.

The classification of points in $\mathfrak{I}_R^*$ is trivial. If Δ^+ is nonempty, then a point is of bouncing or switch type depending whether $\Gamma \cup \Delta^-$ is empty or not. If Δ^+ is empty, then a point is of C^1-inflection or C^1-inflection type depending whether the conjunction of $\Delta^- \neq \emptyset$ and $\Gamma = \emptyset$ is satisfied or not.

Finally, for points that lie outside $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$, we just have to check that Γ and Δ can not both be empty. This is proved in Proposition 4.7. □

5.7. **Estimates concerning local and global arcs.** We prove that the number of points of switch type is an estimate of the number of local arcs (up to an error depending only on $\deg P$ and $\deg Q$).

Lemma 5.21. *In the boundary of M_{CH}^T, the ending point of every local arc (excepted at most $d(2d+1)$ of them) is a point of switch type ($d = 3\deg P + \deg Q - 1$). Conversely every point of switch type is the endpoint of some local arc.*

Proof. Proposition **??** proves that every point of switch type is the endpoint of some local arc. It remains to list every possible endpoint for a local arc.

Following Proposition 4.12, every local arc has an endpoint that belongs to $\mathcal{Z}(PQ)$ or $\mathfrak{I}_R$. For any point α that is the endpoint of a local arc, $\mathcal{L}_\alpha$ contains an interval of length at most π. It follows then from Corollary 3.4 that such a point is either a simple pole of $R(z)$ or a point that is neither a zero or a pole of $R(z)$. Only two local arcs can have the same simple pole as an endpoint. In other point is the endpoint of at most one local. Consequently, at most $3\deg P + \deg Q$ local arcs have an endpoint in $\mathcal{Z}(PQ)$.

It remains to count local arcs whose endpoint belongs to $\mathfrak{I}_R \setminus \mathcal{Z}(PQ)$. Any such point is incident to a unique integral curve so it can be the endpoint of only one local arc. If such a point belong to the transverse locus of the curve of inflections, then it is a point of switch type (see Proposition **??**). There are $|\mathcal{S}|$ of them. Following Proposition 3.13, the tangency locus of $\mathfrak{I}_R$ is formed by at most $2d^2$ points and d lines (where $d = 3\deg P + \deg Q - 1$). On these lines, vectors of field $R(z)\partial_z$ are contained in the line so a local arc ending in one of their points would already belong to them. It follows that $|\mathcal{L}| \leq |\mathcal{S}| + d(2d+1)$. □

Similarly, we prove an estimate on the number of global arcs that do not start at a point of the transverse locus of the curve of inflections.

Lemma 5.22. *In the boundary of M_{CH}^T, the starting point of every local arc (excepted at most $12d + 5d^2$ of them) is a point of C^1-inflection, C^2-inflection or bouncing type.*

Proof. We list every possible starting point for a global arc (Lemma 4.19 proves that global arcs cannot be closed loops).

Since points of extruding type are not starting points of global arcs (see Proposition 4.25), every global arc either starts at a point at infinity or starts at a point of $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$.

We first count the number of global arcs that can start at infinity. If $\deg Q - \deg P = 1$, we know from Theorem 2.22 that M^T_{CH} is compact. If $\deg Q - \deg P = -1$, then Proposition 6.9 proves that M^T_{CH} has only one connected component while its clement has two connected components. Therefore, we have at most four infinite global arcs in this case. If $\deg Q - \deg P = 0$, the complement of M^T_{CH} is connected so each connected component has at most two infinite global arcs. Following Proposition 2.19, M^T_{CH} has at most $\deg P + \deg Q$ connected components so the number of global arcs starting at infinity is at most $2\deg P + 2\deg Q$. In the only case where $4 > 2\deg P + 2\deg Q$ while $\deg Q - \deg P = -1$, $Q(z)$ is constant while $\deg P = 1$. In this case, M^T_{CH} is a straight line (see Proposition 6.7).

Now we consider points of the transverse locus of $\mathfrak{I}_R$. Each point of C^1-inflection type is the starting point of exactly two global arcs (see Proposition 5.12). Each point of bouncing or C^2-inflection type is the starting point of exactly one global arc (see Propositions 5.3 and 5.6). No global arc starts at a point of switch type (see Proposition 5.20).

Now we consider the tangent locus of $\mathfrak{I}_R$. It is formed by at most $2d^2$ points and $\deg P + \deg Q + 1$ R-invariant lines. Each line contains the starting point of at most four global arcs (because the line has two sides and rays have two possible directions). Otherwise, rays starting from these global arcs intersect some of the other global arcs. Using Lemma 2.20, we prove that each of the $2d^2$ remaining points of the tangent locus is the starting point of at most two global arcs. For the same reasons, each singular point of $\mathfrak{I}_R$ that does not belong to $\mathcal{Z}(PQ)$ is the starting point of at most two global arcs. There are $2d$ such points (see Corollary 3.8).

It remains to estimate the number of global arcs that start at a root α of $P(z)$ or $Q(z)$ in terms of the local degree m_α of $R(z)$ in α. Corollary 3.5 proves that α is the starting point of at most:

- two arcs if $m_\alpha = 0$;
- $2(1 - m_\alpha)$ arcs if $m_\alpha \leq -1$;
- $2\deg P$ arcs if $m_\alpha \geq 1$.

Consequently, in the worst case, roots of $P(z)$ and $Q(z)$ are simple and disjoint so at most $2\deg P(\deg P + 2)$ can start at these points.

Therefore, the number of global arc whose starting point does not belong to $\mathfrak{I}^*_R$ is at most $(2\deg P + 2\deg Q) + 4(\deg P + \deg Q + 1) + 4d^2 + 4d + 2\deg P(\deg P + 2)$.

If $\deg P = 0$, then $\deg Q = 1$ (otherwise M^T_{CH} is trivial) and M^T_{CH} is fully irregular (and has therefore no global arc) so we can replace the obtained bound by the slightly weaker (but more practical) upper bound $12d + 5d^2$. □

5.8. **Bounding the number of intersection points between $\partial\mathrm{M}^T_{CH}$ and the transverse locus of $\mathfrak{I}_R$.** In order to prove Theorem 1.7, we introduce a new decomposition of the boundary $\partial\mathrm{M}^T_{CH}$.

Lemma 5.23. *For any linear differential operator T given by* (1.1)*, we define the long arcs as the connected components of $\partial\mathrm{M}^T_{CH}$ in the complement of the union of:*

- *the roots $\mathcal{Z}(PQ)$;*
- *the singular locus $\mathfrak{S}_R$ of the curve of inflections;*
- *the tangency locus $\mathfrak{T}_R$ of the curve of inflections.*

Besides, one of the following statements hold:

1. *a long arc is a closed loop and coincides with $\partial\mathrm{M}^T_{CH}$;*
2. *each long arc is an interval and there are at most ??? of them (where $d = 3\deg P + \deg Q - 1$).*

Proof. Theorem 2.22 proves that $\overline{\mathrm{M}_{CH}^T}$ is connected and contractible in the extended plane. It follows that if a long arc is a closed loop, then it coincides with $\partial\mathrm{M}_{CH}^T$. We will then assume that each long arc is an interval and investigate its endpoints.

A long arc is the union of local and global arcs, glued along points of bouncing, C^1-inflection, C^2-inflection and switch type (see Sections 5.2 to 5.5 for a description of these points, classified in Theorem 1.6).

EST-CE QU'ON A BESOIN DE CA ???

ON AURAIT BESOIN D'UN LEMME

□

1) Definition of a long arc 2) Estimate on the number of long arcs 3) Word and connected components of ???

Definition 5.24. On each edge E of the curve of inflections $\mathfrak{I}_R$, the direction of vector field $R(z)$ defines a same co-orientation between the two inflection domains bounded by edge E. In the exceptional case where $R(z)$ is real on the real axis (up to an affine change of variables), the co-orientation of each edge of the real axis is defined to be trivial.

OBSERVATION: AT POINTS OF INFLECTION AND SWITCH TYPE, $\partial\mathrm{M}_{CH}^T$ intersects transversely $\mathfrak{I}_R$.

OBSERVATION: AN ASSOCIATED RAY INTERSECTS THE CURVE OF INFLECTION AT MOST d times.

Proof of Theorem 1.7. ??? □

6. Global geometry of minimal sets

At present we do not know a general recipe how to describe non-trivial M_{CH}^T. Nevertheless we can prove some general statements about their global geometry and provide some illuminating examples.

Recall that M_{CH}^T can be nontrivial if and only if $\deg Q - \deg P \in \{-1, 0, 1\}$.

In some cases, description of the convex hull $Conv(\mathrm{M}_{CH}^T)$ is easier to obtain. The following has been proved as Corollary 5.16 in [AHN+22].

Proposition 6.1. *Consider a linear differential operator T given by* (1.1)*.Then the intersection of all convex Hutchinson invariant set coincides with the convex hull $Conv(\mathrm{M}_{CH}^T)$ of the minimal set M_{CH}^T.*

The local analysis of boundary points carried on in the previous sections provides interesting partial results towards a characterization of points where $\partial\mathrm{M}_{CH}^T$ is locally convex.

6.1. Local convexity of the boundary.

Local analysis in terms of correspondences Γ and Δ shows that corner points of M_{CH}^T have to satisfy very specific conditions.

Corollary 6.2. *For a linear differential operator T given by* (1.1)*, consider a point α which is a corner point of the boundary $\partial\mathrm{M}_{CH}^T$. In other words, there is a neighborhood V of α such that $V \cap \mathrm{M}_{CH}^T$ is contained in a cone with apex α and with the opening strictly smaller than π. Then one of the following statements holds:*

- *α is a simple zero of $R(z)$ satisfying $\phi_\alpha = 0$ (see 3.1);*
- *α is a common root of $P(z)$ and $Q(z)$ of the same multiplicity (i.e. α is neither a zero nor a pole of $R(z)$).*

Besides, if α is a cusp (neighborhoods of α in M_{CH}^T can be included in cones of arbitrarily small opening angle), then one of the following statements holds:

- *α is a common root of $P(z)$ and $Q(z)$ of the same multiplicity;*
- *M_{CH}^T is totally irregular and contained in a half-line.*

Proof. Corollary 3.4 immediately implies that α can not be a pole or a multiple zero of $R(z)$. Besides, if α is a simple zero, it has to satisfy the condition $\phi_\alpha = 0$. Now we assume that α is neither a zero nor a pole of $R(z)$. It remains to prove that $\alpha \in \mathcal{Z}(PQ)$.

Assume that $\alpha \notin \mathcal{Z}(PQ)$. In this case if some point of the forward trajectory of $R(z)\partial_z$ starting at α belongs to M_{CH}^T, then a germ of the integral curve starting at α is contained in M_{CH}^T (see Proposition 2.10) and α can not be a corner point. We conclude that $\Gamma(\alpha) = \emptyset$.

If $\Delta(\alpha)$ contains some point y, then a branch of the root trail $\mathfrak{tr}_y$ containing α belongs to M_{CH}^T (see Lemmas 2.11 and 2.15). Thus in this case α can not be a corner point and we get $\Delta(\alpha) = 0$.

Now we take a cone $\mathcal{C}$ with apex at α, of angle at least π which is locally disjoint from M_{CH}^T. If $r(\alpha)$ is contained in $\mathcal{C}$, but is not one of the two limit rays, we can freely remove a neighborhood of α from M_{CH}^T and still get an invariant set. In any other case, we can find an arc contained in a neighborhood of α and the complement of M_{CH}^T whose associated rays sweep out a domain containing α. Thus we get a contradiction in this case as well which implies that α has to be in $\mathcal{Z}(PQ)$.

Finally if α is a simple zero of $R(z)$ and a cusp, then we have $\mathcal{L}_\alpha = \mathbb{S}^1$ (see Definition 3.2). It follows that M_{CH}^T has empty interior. All such cases have been completely classified in [AHN+22] (see Section 7 of loc. cit). □

Further local analysis provides necessary conditions under which boundary points belong to locally convex part of $\partial\mathrm{M}_{CH}^T$.

Proposition 6.3. *For a linear differential operator T given by* (1.1)*, consider a point α such that there is a neighborhood V of α with the property that $V \cap \mathrm{M}_{CH}^T$ is contained in a half-plane whose boundary contains α.*

If $\alpha \in \mathcal{Z}(PQ)$, then one of the following statements holds:

- *α is a simple pole of $R(z)$;*
- *α is a simple zero of $R(z)$ satisfying $\phi_\alpha = 0$ (see 3.1);*
- *α is a common root of $P(z)$ and $Q(z)$ of the same multiplicity (i.e. α is neither a zero nor a pole of $R(z)$).*

If $\alpha \in \mathfrak{I}_R^* \setminus \mathcal{Z}(PQ)$, then α is a point of switch type.

If $\alpha \notin \mathfrak{I}_R \cup \mathcal{Z}(PQ)$, then one of the following statements holds:

- *α is a point of local type;*
- *α is a point of global type and for any $u \in \Delta(\alpha)$, we have that $\mathrm{Im}(f(u, \alpha))$ and $\mathrm{Im}(R'(\alpha))$ have opposite signs (for f defined as in Proposition 2.17).*

Proof. The case $\alpha \in \mathcal{Z}(PQ)$ follows from Corollary 3.4. If $\alpha \in \mathfrak{I}_R^* \setminus \mathcal{Z}(PQ)$ and $\Delta^-(\alpha) \neq \emptyset$, then M_{CH}^T contains both the germ of an integral curve of the field $-R(z)\partial_z$ at α and the germ of the root trail $\mathfrak{tr}_u$ for some $u \in \Delta^-(\alpha)$. Proposition 2.18 implies that M_{CH}^T can not be convex at α. Besides, if $\Gamma(\alpha) \neq \emptyset$, then M_{CH}^T can not be convex in α either because a germ of an integral curve having an inflection point at α is contained in M_{CH}^T In the remaining cases, we have $\Gamma(\alpha) \cup \Delta^-(\alpha) = \emptyset$. If $\Delta^+(\alpha) \neq \emptyset$, this characterizes points of switch type (see Theorem 1.6). If $\Delta^-(\alpha) = \emptyset$, then we obtain a point of C^2-inflection type, α is the starting point of a local arc and $\Gamma(\alpha)$ is therefore nonempty (see Proposition 5.5).

Now we consider the case $\alpha \notin \mathfrak{I}_R \cup \mathcal{Z}(PQ)$. If $\Gamma(\alpha) \neq \emptyset$ and α is a point of local type (α can not be a point of extruding type because of Proposition 4.25).

If $\Gamma(\alpha) = \emptyset$, then it follows from Proposition 4.7 that $\Delta(\alpha) \neq \emptyset$. Proposition 2.17 then provides the necessary condition. □

6.2. **Case** $\deg Q - \deg P = -1$**.** We have a rational vector field $R(z)\partial_z$ satisfying $R(z) = \frac{\lambda}{z} + \frac{\mu}{z^2} + o(1/z^2)$ with $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$.

6.2.1. *Horizontal locus and special line.* We define the following loci.

Definition 6.4. The *horizontal locus* $\mathcal{H}_R$ is the closure in $\mathbb{C}$ of the set formed by points $z \notin \mathcal{Z}(PQ)$, for which $\sigma(z) = \frac{\arg(\lambda)\pm\pi}{2}$.

We also denote by $\mathcal{L}_R$ the *special line* formed by points z given by the equation $Im(z/\lambda) = Im(\mu/\lambda^2)$.

For the sake of simplicity, the vector field $R(z)\partial_z$ is normalized by an affine change of variable as $R(z) = -\frac{1}{z} + o(z^{-2})$ ($\lambda = -1$ and $\mu = 0$). The line $\mathcal{H}_R$ then coincides with the real axis $\mathbb{R}$.

Lemma 6.5. *$\mathcal{H}_R$ is a real plane algebraic curve of degree at most $\deg P + \deg Q$. It has two asymptotic infinite branches. The line $\mathcal{L}_R$ is the asymptotic line for both of them.*

Proof. Curve $\mathcal{H}_R$ can be seen as the pull back of the real axis under the mapping $R(z) : \mathbb{CP}^1 \to \mathbb{CP}^1$. We have $R(\infty) = 0$ and ∞ is a simple root of $R(z)$. Therefore, $\mathcal{H}_R$ is smooth near ∞.

It remains to show that the tangent line to $\mathcal{H}_R$ at infinity coincides with the real axis. Actually the tangent line is the line at which the linearization of $R(z)$ at ∞ attains real zeroes. Since this linearization is exactly $-\frac{1}{z}$, the result follows. □

Corollary 6.6. *The closure $\overline{\mathrm{M}_{CH}^T}$ of the minimal set in the extended plane contains asymptotic directions 0 and π. Besides, curve $\mathcal{H}_R$ is contained in the minimal set M_{CH}^T.*

Proof. Looking at separatrices of the vector field $R(z)\partial_z$ and using Proposition 2.10 we get that the closure $\overline{\mathrm{M}_{CH}^T}$ in the extended plane contains asymptotic directions 0 and π. The associated rays of points of $\mathcal{H}_R$ are thus asymptotically tangent to M_{CH}^T and $\mathcal{H}_R$ is contained in the minimal set. □

Proposition 6.7. *Consider a linear differential operator T given by* (1.1) *such that $\deg Q - \deg P = -1$. Then the minimal convex Hutchinson invariant set $Conv(\mathrm{M}_{CH}^T)$ is a bi-infinite strip (domain bounded by two parallel lines).*

More precisely, $Conv(\mathrm{M}_{CH}^T)$ is the smallest strip containing $\mathcal{H}_R \cup \mathcal{Z}(PQ)$.

Proof. The minimal convex Hutchinson invariant set $Conv(\mathrm{M}_{CH}^T)$ is the complement of the union of every open half-plane disjoint from M_{CH}^T. Since $\mathcal{H}_R$ is contained in M_{CH}^T (Corollary 6.6), these open half-planes have to be disjoint from $\mathcal{H}_R$. Conversely, any open half-plane H disjoint from $\mathcal{H}_R$ is such that $Im(R(z))$ is either positive or negative for every $z \in H$. Therefore, provided H does not contain any zero or pole of $R(z)$, one can conclude that it can be removed from any Hutchinson invariant set. In other words, $Conv(\mathrm{M}_{CH}^T)$ is the complement to the union of all half-planes disjoint from $\mathcal{H}_R \cup \mathcal{Z}(PQ)$. Since $\mathcal{H}_R$ has asymptotically horizontal infinite branches, the boundary line of every half-plane disjoint from $\mathcal{H}_R$ has to be horizontal.

It remains to prove that such half-planes exist. It follows from the asymptotic description of $\mathcal{H}_R$ in Lemma 6.5 that $|Im(z)|$ is bounded on $\mathcal{H}_R$. Therefore we can find two (disjoint) open half-planes that are also disjoint from $\mathcal{H}_R$. These half-planes contain half-planes which, in addition, are disjoint from $\mathcal{Z}(PQ)$. □

6.2.2. *Asymptotic geometry of the minimal set.* Following Proposition 6.7, $Conv(\mathrm{M}_{CH}^T)$ is the smallest horizontal strip containing the curve $\mathcal{H}_R \cup \mathcal{Z}(PQ)$. The projection of $Conv(\mathrm{M}_{CH}^T)$ on the vertical axis is an interval $[y^-, y^+]$ where $y^- \leq 0 \leq y^+$.

Lemma 6.8. *For $0 < y < y_0$, denote by M_t the intersection point between the associated ray $r(t+iy)$ and the horizontal line $Im(z) = y_0$. Then the following statements hold:*

- *for $t \longrightarrow +\infty$, $\mathrm{Re}(M_t) \longrightarrow -\infty$ if $y < \frac{y_0}{2}$;*
- *for $t \longrightarrow +\infty$, $\mathrm{Re}(M_t) \longrightarrow +\infty$ if $\frac{y_0}{2} < y < y_0$.*

Analogous statements hold for $t \longrightarrow -\infty$ or $y_0 < y < 0$.

Proof. For large values of t, we have $\mathrm{Re}(R(z)) = -\frac{1}{t} + o(t^{-1})$ and $\mathrm{Im}(R(z)) = \frac{y}{t^2} + o(t^{-2})$. Provided t is large enough, $\mathrm{Im}(R(z))$ is positive and the associated ray $r(z)$ intersects the line $Im(z) = y_0$. Then the real part of the intersection point equals $t - (y_0 - y)\frac{t}{y} + o(t)$. After simplification, we obtain $\frac{(2y-y_0)t}{y} + o(t)$. The sign of the main term is then determined by the sign of $2y - y_0$. □

Proposition 6.9. *The minimal set M_{CH}^T is connected in $\mathbb{C}$.*

Proof. Plane curve $\mathcal{H}_R$ (contained in M_{CH}^T by Corollary 6.6) splits $\mathbb{C}$ into connected domains in which $Im(R(z))$ is either positive or negative. Following the proof of Proposition 6.7, two such domains contain a half-plane. These half-planes belong to the complement $(\mathrm{M}_{CH}^T)^c$ of M_{CH}^T in $\mathbb{C}$. Thus we know that the complement $(\mathrm{M}_{CH}^T)^c$ has at least two connected components. Since each such connected component contains associated rays, they have at least one topological end.

Proving that M_{CH}^T is connected in $\mathbb{C}$ amounts to showing that the two ends of $(\mathrm{M}_{CH}^T)^c$ containing the asymptotic directions $]0, \pi[$ and $]\pi, 0[$ are the only ends of $(\mathrm{M}_{CH}^T)^c$. We will call them the *main ends*. We assume by contradiction that there is another topological end κ.

For any sequence $\{z_n\}$ of points in $(\mathrm{M}_{CH}^T)^c$ approaching κ, we have (up to taking a subsequence) the sequence $\{\arg(z_n)\}$ converging either to 0 or to π (since otherwise, κ would not be distinct from the two main ends). Let's assume without loss of generality that it is 0. Again, we can assume that $\{Im(z_n)\}$ converges to some value $y_e \in [y^-, y^+]$.

If $y_e > 0$, then Lemma 6.8, shows that for any horizontal line L_f with $y_f \in]y_e, 2y_e[$, the associated rays of the points in ${\mathrm{M}_{CH}^T}^c$ converging to the end κ sweep out points of L_f whose real part is arbitrarily close to $+\infty$. Assuming that y_e is the maximal possible limit value, we deduce that no infinite component of M_{CH}^T can separate κ from the upper main end containing asymptotic directions of $]0, \pi[$. A similar statement holds for $y_e < 0$.

Now, for a sequence $\{z_n\}$ of points in $(\mathrm{M}_{CH}^T)^c$ approaching κ, the only accumulation value of $\{Im(z_n)\}$ is 0. In this case, the associated rays $r(z_n)$ accumulate onto the $\mathbb{R}$-axis which is therefore disjoint from the interior of M_{CH}^T. Since for any $x \in \mathbb{R}$, the associated ray $r(x)$ can not cross the interior of M_{CH}^T, we get that for every $x \in \mathbb{R}$, $Im(R(x))$ keeps the same sign or is equal to zero. Without loss of generality, we will assume that $Im(R(x)) \geq 0$ for any $x \in \mathbb{R}$. It is immediate that no point of $\mathcal{H}_R$ is contained in the open upper half-plane. Therefore, the open upper half-plane is contained in $(\mathrm{M}_{CH}^T)^c$ and contains the associated rays $r(z_n)$ accumulating to the $\mathbb{R}$-axis. In this situation, no infinite connected component of M_{CH}^T can separate κ from the upper main end containing the asymptotic directions in $]0, \pi[$. □

Corollary 6.10. *For any $\epsilon < \frac{\min(-y^-, y^+)}{2}$, denote by S_ϵ the horizontal strip characterized by $y^- + \epsilon < 2\,\mathrm{Im}(z) < y^+ - \epsilon$. Then there exists a compact set K_ϵ such that $S_\epsilon \cap K_\epsilon^c \subset \mathrm{M}_{CH}^T \cap K_\epsilon^c$.*

Proof. We argue by contradiction. Consider a path γ joining a point of M_{CH}^T with the y-coordinate equal to y^+ to the end of asymptotic direction π (inside M_{CH}^T). Following Lemma 6.8, for any $\epsilon > 0$, there is constant M_ϵ such that for any z satisfying $|t| > M_\epsilon$ and $0 < y < \frac{y^+ - \epsilon}{2}$, associated ray $r(z)$ crosses γ (Proposition 6.9 proves that the associated ray can not be below γ all the time).

A similar reasoning works for the negative values of y, the case $y = 0$ follows from the fact that M_{CH}^T is a closed set. □

Proposition 6.11. *There is a compact set K and a positive constant $B > 0$ such that the intersection $\mathrm{M}_{CH}^T \cap K^c$ is contained in the domain bounded by the hyperbolas given by $y = \frac{y^\pm}{2}(1 + \frac{B}{\pm x})$.*

Proof. By Lemma 6.5 for any y in $J = [y^- \frac{y^-}{2}[\cup]\frac{y^+}{2}, y^+]$, there is a positive constant $A > 0$ the union of the two semi-infinite horizontal strips characterized by $\mathrm{Im}(z) \in J$ and $|\mathrm{Re}(z)| > A$ is disjoint from $\mathcal{H}_R$.

Consider some positive number $B > A$ and introduce the domain D_B characterized by the inequalities:

- $\mathrm{Im}(z) > y^+$ if $\mathrm{Re}(z) \in [-B, B]$;
- $\mathrm{Im}(z) > g(t)$ where $g(t) = \frac{y^+}{2}\frac{|t|+B}{|t|}$ if $t = \mathrm{Re}(z) \notin [-B, B]$.

For any point z such that $\mathrm{Im}(z) > y^+$, the associated ray $r(z)$ remains in D_B. Now we assume that $z = t + iy$ satisfies the conditions

$$|t| > B \quad \text{and} \quad \frac{y^+}{2}\frac{|t| + B}{|t|} < |y| \leq y^+.$$

Without loss of generality, we assume that $t < -B$.

In order to prove that the associated ray $r(z)$ remains in D_B, we have to show that for any $t < -B$ and any $s \in [t, -B]$, we get $\frac{\mathrm{Im}(R(z))}{\mathrm{Re}(R(z))} > \frac{g(s)-g(t)}{s-t} \geq \frac{By^+}{2st} \geq -\frac{y^+}{2t}$.

In our case $\mathrm{Re}(R(z)) = -\frac{1}{t} + o(t^{-2})$ and $\mathrm{Im}(R(z)) = \frac{y}{t^2} + o(t^{-3})$ implying that

$$\frac{\mathrm{Im}(R(z))}{\mathrm{Re}(R(z))} = -\frac{y}{t} + o(t^{-2}).$$

Since $y - \frac{y^+}{2} > \frac{B}{t}$, the inequality holds provided B is large enough.

By replacing y^+ by y^-, we get an analogous result for the lower part of the complement to M_{CH}^T. □

Proposition 6.12. *The complement $\mathbb{C} \setminus \mathrm{M}_{CH}^T$ has two connected components, each bounded by a unique curve contained in $\partial\mathrm{M}_{CH}^T$. Unless such a curve coincides with the special line, it contains exactly two infinite global arcs.*

Proof. It follows from the topology of M_{CH}^T that its complement has two connected components, each of them having a unique boundary component (see Propositions 6.7 and 6.9).

Assuming that one of these boundary components does not coincide with the special line, Theorem 1.6 proves that it is formed by finitely many arcs. Each of such curves is a local arc, a global arc, or a portion of straight line contained in the curve of inflections. There are two infinite arcs (unless they coincide) and they are asymptotically horizontal. We consider one such infinite arc α.

The curve of inflections has only two infinite branches. If the asymptotically horizontal infinite branch contains a straight segment, then this branch has to be a horizontal line. Moreover near infinity it coincides with $\mathcal{H}_R$ and thus has to be the special line $\mathbb{R}$. This case has already been ruled out. Therefore α is either a global or a local arc. UNCLEAR???

If α is a local arc, then it has to start at infinity. Close to infinity, the integral curves look like hyperbolas and therefore their curvature has the wrong sign to form a convex boundary of M_{CH}^T (see Proposition 4.11). Therefore α has to be a global arc.

The two infinite global arcs have the associated rays that form the direct and indirect support lines for M_{CH}^T. Thus they do not belong to the same inflection domain. Instead, they are two distinct infinite arcs. □

6.2.3. *Examples.* Consider a family of operators of the form $T_\alpha = Q(z)\frac{d}{dz} + P(z)$ where $Q(z) = (z-\alpha)^k$ and $P(z) = z(\alpha - z)^k$ with the common root $\alpha \in \mathbb{C}$ of degree $k \in \mathbb{N}^*$.

The family T_α provides a rich assortment of examples. We have $R(z) = -\frac{1}{z}$. The special line is the real axis $\mathbb{R}$ which coincides with the horizontal locus $\mathcal{H}_R$. Besides, the integral curves of $R(z)\partial_z$ are hyperbolas (level sets of xy).

Proposition 6.13. *If $\alpha \in \mathbb{R}$, then the minimal set M_{CH}^T of operator T_α coincides with the real axis $\mathbb{R}$.*

Proof. Follows immediately from Proposition 6.7. □

If α does not belong to the real axis, we get different pictures depending on whether or not α belongs to the imaginary axis. Without loss of generality, we will assume that $Im(\alpha) > 0$.

Proposition 6.14. *If α is of the form $y_0 i$ with $y_0 > 0$, then the minimal set M_{CH}^T is the union of the segment $[\frac{y_0}{2}i, y_0 i]$ with the horizontal strip formed by points z satisfying $0 \leq Im(z) \leq \frac{y_0}{2}$.*

Proof. From Proposition 6.7 it follows immediately that the convex hull of M_{CH}^T is contained in the strip bounded by $\mathbb{R}$ and the horizontal line $Im(z) = Im(y_0)$. For any point of segment $[0, y_0 i]$, the associated ray contains α so $[0, y_0 i] \subset \mathrm{M}_{CH}^T$.

For any point of the horizontal strip given by the inequalities $0 \leq Im(z) \leq \frac{y_0}{2}$, a simple computation proves that its associated ray intersects the segment $[0, y_0 i]$.

Finally, for any point z such that $Im(z) > \frac{y_0}{2}$ and $Re(z) \neq 0$, the associated ray is disjoint from the segment $[0, y_0 i]$. This completely characterizes the minimal set. □

The latter case provides an example of a partially irregular minimal set whose irregularity locus is contained in a R-invariant line (the imaginary axis in this case).

In the general case, the boundary of M_{CH}^T is more complicated. Up to conjugation, we can restrict us to the case when $\mathrm{Re}(\alpha), \mathrm{Im}(\alpha) > 0$.

Proposition 6.15. *If α is of the form $x_0 + y_0 i$ with $x_0, y_0 > 0$, then the minimal set M_{CH}^T of T_α is bounded by the following arcs:*

- *the real $\mathbb{R}$-axis ;*
- *global arc $(t, f_1(t))$ where $f_1(t) = \frac{y_0 t}{2t - x_0}$ for $t \in [x_0, +\infty[$;*
- *local arc $(t, f_2(t))$ where $f_2(t) = \frac{x_0 y_0}{t}$ for $t \in [x_0, x_e]$;*
- *global arc $(t, f_3(t)$ where $f_3(t) = \frac{x_0 y_0 t}{(2\sqrt{x_0 t} + x_0)^2}$ for $t \in [0, x_e]$;*
- *global arc $(t, f_4(t))$ where $f_4(t) = \frac{y_0 t}{2t - x_0}$ for $t \in]-\infty, 0]$.*

Here, (x_e, y_e) is a point of extruding type. Its coordinates are $x_e = (3 + 2\sqrt{2})x_0$ and $y_e = \frac{y_0}{3 + 2\sqrt{2}}$.

Proof. The convex hull of M_{CH}^T is contained in the strip bounded by $\mathbb{R}$ and the horizontal line $Im(z) = Im(y_0)$, see Proposition 6.7. The arcs $(t, f_1(t))$ and $(t, f_4(t))$ are characterized by the fact that the associated rays starting from their points contain $x_0 + iy_0$ (this can be checked by a direct computation). In particular, they belong to two distinct branches of the same hyperbola. Besides, the domain $\mathcal{D}$ between $\mathbb{R}^-$ and arc $(t, f_4(t))$ is automatically contained in M_{CH}^T.

Following Proposition 2.10, the backward trajectory of the vector field $R(z)\partial_z$ starting at $x_0 + y_0 i$ is contained in M_{CH}^T. The domain between this portion of the integral curve and the arc $(t, f_1(t))$ is also contained in M_{CH}^T.

We denote by $\mathcal{D}'$ the domain in the open right upper quadrant where the associated ray intersects the domain $\mathcal{D}$. At each point $(t, \gamma(t))$ of the upper boundary of $\mathcal{D}'$, the associated ray is tangent to the branch of hyperbola $(s, f_4(s))$ for some $s \leq 0$. Since $R(z) = -\frac{1}{z}$, the argument of $t + i\gamma(t)$ equals the negative of the slope of $(s, f_4(s))$ at s. Since $\frac{df_4}{ds}(s) = -\frac{x_0 y_0}{(2s-x_0)^2}$, we get

$$\frac{\gamma(t)}{t} = \frac{x_0 y_0}{(2s - x_0)^2}.$$

Since the tangent line has to intersect the imaginary axis at $2\gamma(t)i$, we obtain the following equation:

$$\frac{f_4(s) - 2\gamma(t)}{s} = -\frac{x_0 y_0}{(2s - x_0)^2}.$$

Replacing $\gamma(t)$ by $\frac{x_0 y_0 t}{(2s-z_0)^2}$, we get $t = \frac{s^2}{x_0}$.

Since s is the negative square root of $x_0 t$, we deduce that $\gamma(t) = \frac{x_0 y_0 t}{(2\sqrt{x_0 t} + x_0)^2}$. In particular, for $s = -x_0$, we get $t = x_0$ and $\gamma(x_0) = \frac{y_0}{9}$.

The arc γ and the backward trajectory starting at $x_0 + iy_0$ (which is a branch of hyperbola) intersect each other at some point $x_e + iy_e$. From a computation, we obtain $x_e = (3 + 2\sqrt{2})x_0$ and therefore $y_e = \frac{y_0}{3+2\sqrt{2}}$.

It is then geometrically clear that for any point z above the curve formed by arcs defined by functions f_1, f_2, f_3, f_4, the associated ray can not intersect any of these arcs. $\square$

The latter example provides a remarkable illustration of a point of extruding type. Since the boundary arcs are explicit algebraic curves, we can obtain the exact picture shown in Figure 6.

6.3. **Case $\deg Q - \deg P = 0$.** For the sake of simplicity, we normalize the vector field $R(z)\partial_z$ by an affine change of variable so that $R(z) = 1 + \frac{\mu}{z^\kappa} + o(z^{-\kappa-1})$ for some $\mu \in \mathbb{C}^*$ and $\kappa \geq 1$. (The case of a constant vector field is already treated in Section 2.3 of [AHN+22].)

Remark 6.16. For $\deg Q - \deg P = 0$, the starting point of a local arc can be at infinity (opposite to the case $\deg Q - \deg P = 1$ when the minimal set is compact or the case $\deg Q - \deg P = -1$, see Proposition 6.12). In this situation the point at infinity has to be of the form $\phi_\infty + \pi$ (π in our current normalization). Indeed, any other point is ruled out by Proposition 2.24).

Under the genericity assumptions $\mathrm{Im}(\mu) \neq 0$ and $\kappa = 1$, we are going to prove that the minimal set M_{CH}^T is connected. Firstly we show that M_{CH}^T is regular and disjoint from the curve of inflections $\mathfrak{I}_R$ outside a compact set.

Lemma 6.17. *Assuming that* $\mathrm{Im}(\mu)(-1)^\kappa > 0$, *there is a cone* $\mathcal{C}$ *and a compact set* K *such that:*

- *for any* $z \in \mathcal{C}$, $\mathrm{Im}(R(z)) > 0$ *and* $\mathrm{Im}(R'(z)) > 0$*;*
- $\mathrm{M}_{CH}^T \subset \mathcal{C} \cap K$.

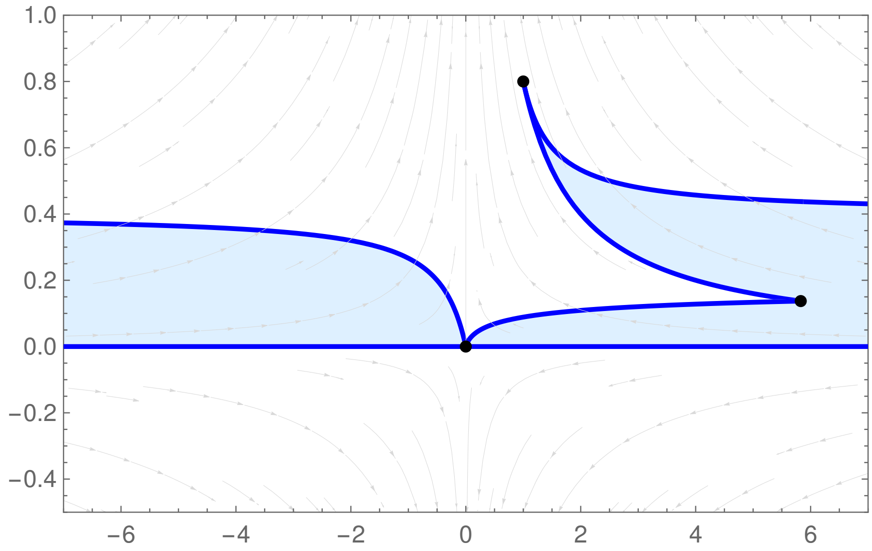

FIGURE 6. The case when $\alpha = 1 + 0.8i$.

Besides, M^T_{CH} *is a regular subset of* $\mathbb{C}$.

Proof. It follows from Proposition 2.24 and the fact that asymptotic directions of infinite branches of algebraic curves defined by equations $\mathrm{Im}(R) = 0$ and $\mathrm{Im}(R') = 0$ are not horizontal. UNCLEAR WHAT?? Computing $R(t)$ and $R'(t)$ for a negative real number t, we obtain that $R(t) = 1 + \frac{\mu}{t^\kappa} + o(t^{-\kappa} - 1)$ and thus $\mathrm{Im}(R(t)) \sim \mathrm{Im}(\mu)t^{-\kappa}$. The sign of the latter is the sign of $\mathrm{Im}(\mu)(-1)^\kappa$. Similarly we obtain that it is also the sign of $\mathrm{Im}(R'(t))$ for t close enough to infinity.

Any R-invariant line (see Definition 2.5) has to be horizontal and therefore it intersects the cone $\mathcal{C}$. Thus some points of any R-invariant line Λ have the associated rays that are not contained in Λ. Therefore, there are no R-invariant lines for such a vector field $R(z)\partial_z$. The minimal set M^T_{CH} has no tails and Theorem 2.22 guarantees that M^T_{CH} is regular. □

Corollary 6.18. *Assuming that* $\mathrm{Im}(\mu)(-1)^\kappa > 0$, *consider a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *of points of* $\partial\mathrm{M}^T_{CH}$ *such that* $|\alpha_n| \to +\infty$ *and* $\Delta(\alpha_n) \neq \emptyset$ *for any* $n \in \mathbb{N}$. *Then there exists a subsequence* $(\alpha_{f(n)})_{n\in\mathbb{N}}$ *such that:*

- *there is a line* $\mathcal{L}_{y_0}$ *given by* $\mathrm{Im}(z) = y_0$ *which is the indirect support of* M^T_{CH} *at some point;*
- *the line* $\mathcal{L}_{y_0}$ *is disjoint from the interior of* M^T_{CH}*;*
- $\mathrm{Re}(\alpha_{f(n)}) \to -\infty$*;*
- $\mathrm{Im}(\alpha_{f(n)}) \leq y_0$ *for any* $n \in \mathbb{N}$.

Proof. Up to taking a subsequence, we can also assume that every a_n belongs to the cone $\mathcal{C}$ defined in Lemma 6.17. Lemma 4.13 implies that for any n, points of $\Delta(\alpha(t_n))$ belong to $\mathfrak{I}^-$, $\mathfrak{I}_R$ or $\mathcal{Z}(PQ)$. Therefore, following Lemma 6.17, points of $\Delta(\alpha(n))$ accumulate in a compact set as $n \to \infty$. We denote by z_0 one of their accumulation points and by L_{y_0} the horizontal line containing z_0 (here $y_0 = \mathrm{Im}(z_0)$).

Thus, (up to taking a subsequence of α), we get a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to y$ and $y_n \in \Delta(\alpha_n)$ for any $n \in \mathbb{N}$. Therefore the associated rays $r(\alpha_n)$ accumulate on L_{y_0}. Thus the line L_{y_0} is disjoint from the interior of M^T_{CH}. Besides, since $\alpha_n \in \mathcal{C}$ for any $n \in \mathbb{N}$, $Im(R(\alpha_n)) > 0$ and therefore $\mathrm{Im}(\alpha_n) \leq y_0$. □

Lemma 6.19. *Provided thatt* $\mathrm{Im}(\mu) \neq 0$ *and* $\kappa = 1$*, no integral curve has a horizontal asymptotic line at infinity.*

Proof. Since $R(z) = 1 + \frac{\mu}{z} + o(z^{-2})$, the integral curve $\gamma(t)$ satisfies $\mathrm{Re}(\gamma(t)) \sim t$ as $t \to \pm\infty$. Then $Im(\gamma'(t)) = \frac{Im(\mu)}{t} + o(t^{-1})$. We obtain that $Im(\gamma(t))$ has logarithmic growth as $t \to \pm\infty$ and therefore the integral curve has no asymptotic lines at infinity. □

Corollary 6.20. *Provided that* $\mathrm{Im}(\mu) \neq 0$ *and* $\kappa = 1$*, the minimal set* M_{CH}^T *is connected in* $\mathbb{C}$*. Besides,* $\partial\mathrm{M}_{CH}^T$ *has exactly two infinite arcs: one is a local arc starting at infinity while the other is a global arc ending at infinity.*

Proof. Without loss of generality, we can assume that $\mathrm{Im}(\mu) > 0$. Proposition 2.19 shows that there are finitely many connected components of M_{CH}^T. Moreover they are attached to the point $\pi \in \mathbb{S}^1$ at infinity in some cyclic order. We refer to these components of M_{CH}^T as $X_1, \ldots, X_k$ where X_1 is the lowest component while X_k is the highest component. Besides the boundary ∂X_i of any component X_i has exactly two topological ends. We call them the lower end ∂X_i^- and the upper end ∂X_i^+.

Since $\partial\mathrm{M}_{CH}^T \cap \mathfrak{I}_R$ is contained in a compact set (see Lemma 6.17), points of ∂X that are close enough to infinity are either of local, global or of extruding types. Besides, we know from Proposition 4.11 that in $\mathfrak{I}^+$ local arcs have the same orientation as $\partial\mathrm{M}_{CH}^T$. Proposition 4.12 proves that every local arc has an endpoint in $\mathfrak{I}_R \cup \mathcal{Z}(PQ)$. Thus the ends of ∂X are represented either by a local arc starting at infinity or by a global arc. Besides, the orientation constraint shows that only the upper end is represented by a local arc if $\mathrm{Im}(\mu) > 0$ (the lower end if $\mathrm{Im}(\mu) < 0$ respectively). Otherwise, the local arc would have the point at infinity as its endpoint. For the same reason, any lower end ∂X_i^- has to be represented by an infinite global arc ending at infinity (see Lemma 4.19).

For any component X_i, the lower end ∂X_i^- of its boundary is arbitrary close to the points that are not of local type. Applying Corollary 6.18 to a sequence of such points we prove the existence of a horizontal line L_i lying below the component X_i and disjoint from the interior of M_{CH}^T. Thus no component of M_{CH}^T lying below the line L_i can contain an infinite local arc because the latter has no asymptotic line at infinity (see Lemma 6.19). Consequently, among the ends of $\partial\mathrm{M}_{CH}^T$, only ∂X_k^+ can be represented by a local arc.

It remains to prove that M_{CH}^T has only one connected component. Assuming there are several of them, we consider the upper end ∂X_1^+. We already know that it can be approached by points $(\alpha_n)_{n\in\mathbb{N}}$ which are not of local type. Applying Corollary 6.18, we prove the existence of a line L disjoint from the interior of M_{CH}^T, direct support of M_{CH}^T at some point z_0 and such that points of $(\alpha_n)_{n\in\mathbb{N}}$ lie above the line L. Since z_0 has to belong to some component X_i, line L automatically intersects the interior of component X_i. This is a contradiction, so M_{CH}^T is connected. Its upper end is a local arc while its lower end is a global arc. □

6.4. **Case** $\deg Q - \deg P = 1$**.** In [AHN+22] we found that necessary and sufficient condition for the compactness of M_{CH}^T in case $\deg Q - \deg P = 1$ is $\mathrm{Re}(\lambda) \geq 0$. (One has to exclude a rather trivial case $\deg P = 0,\ \deg Q = 1$.) Moreover in case $\mathrm{Re}(\lambda) < 0$, we get $\mathrm{M}_{CH}^T = \mathbb{C}$.

We will describe M_{CH}^T for $\mathrm{Re}(\lambda) = 0$. Unfortunately, in the most interesting situation $\mathrm{Re}(\lambda) > 0$, we do not have a general description of M_{CH}^T, but we provide a number of partial results, observations and examples.

6.4.1. $\mathrm{Re}(\lambda) = 0$. In this case a complete characterization of $\partial\mathrm{M}_{CH}^T$ can be carried out.

Theorem 6.21. *Consider a linear differential operator T given by* (1.1) *such that* $\deg Q - \deg P = 1$ *and* $\mathrm{Re}(\lambda) = 0$. *In this case, the neighborhood of infinity is foliated by a family $\mathcal{C}$ of closed integral curves of the vector field $R(z)\partial_z$.*

The boundary $\partial \mathrm{M}^T_{CH}$ of the minimal set of T is described as the first closed leaf of the family $\mathcal{C}$ containing a point of $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$.

If the latter leaf γ contains a point of the curve of inflections $\mathfrak{I}_R$, then the latter point is a tangency point between γ and $\mathfrak{I}_R$. Moreover it is the first leaf that is non-strictly convex (the curvature at the tangency point vanishes).

In particular, $\partial \mathrm{M}^T_{CH}$ is formed by finitely many local arcs. It is real-analytic and convex (but can fail to be strictly convex). It contains neither zeros nor poles of $R(z)\partial_z$.

Proof. It follows from $\lambda \in \mathbb{C}^*$ and $\mathrm{Re}(\lambda) = 0$ that $\mathrm{Im}(\lambda) \neq 0$. The curve of inflections $\mathfrak{I}_R$ is therefore compact. The neighborhood of infinity is foliated by a family $\mathcal{C}$ of integral curves of vector field $R(z)\partial_z$. The orientation of these integral curves depends on the sign of $\mathrm{Im}(\lambda)$. By compactness of $\mathfrak{I}_R$ a possibly smaller neighborhood $\mathcal{C}'$ of infinity is foliated by strictly convex integral curves (the curvature of integral curves vanishes precisely on $\mathfrak{I}_R$).

We first consider the case when some point α of $\mathcal{Z}(PQ)$ belongs to $\mathcal{C}'$. Denoting by γ the periodic leaf α belongs to, we deduce from Proposition 2.10 that γ belongs to M^T_{CH} and bounds a strictly convex domain $\mathcal{D}$. Provided the complement of $\mathcal{D}$ does not contain any other point of $\mathcal{Z}(PQ)$, we obtain that $\mathcal{D}$ coincides with M^T_{CH}. Since α is disjoint from $\mathfrak{I}_R$, it follows from Corollary 3.11 that it can not be a zero or a pole of $R(z)$ (α is a root of both P and Q of the same multiplicity).

In the remaining cases, we can assume that $\mathcal{Z}(PQ)$ is disjoint from $\mathcal{C}'$. The cylinder $\mathcal{C}$ is bounded by a singular curve formed by separatrices (integral curves connecting singularities of $R(z)\partial_z$). We denote by Σ the union of these separatrices and by $\mathcal{S}$ the smallest simply connected subset containing Σ. By Proposition 2.10, Σ and $\mathcal{S}$ are contained in M^T_{CH}. For the same reason, a point z of cylinder $\mathcal{C}$ is contained in M^T_{CH} if and only if the periodic integral curve containing z belongs entirely to M^T_{CH}. Therefore, the boundary of M^T_{CH} coincides with some periodic integral curve of the cylinder $\mathcal{C}$.

Since the associated rays can not cross the interior of M^T_{CH}, its boundary $\partial \mathrm{M}^T_{CH}$ (which is a periodic integral curve) has to be convex. Therefore, it is contained in the domain of inflection of infinity (or in its boundary). Since the domain $\mathcal{C}'$ does not belong to the interior of M^T_{CH} (its complement is clearly a T_{CH}-invariant set), these conditions characterize the boundary γ of $\mathcal{C}'$ as the boundary of M^T_{CH}.

The curve γ can not cross the curve of inflections because it is convex, not strictly convex. Thus γ contains has a tangency point with $\mathcal{I}_R$. At this point, the curvature of γ vanishes.

The boundary $\partial \mathrm{M}^T_{CH}$ is formed by local arcs joining points of $\mathcal{Z}(PQ)$ (with the same multiplicity of P and Q) and some points of the tangency locus. By Proposition 4.7, there arcs are strictly convex and real-analytic. □

6.4.2. $\mathrm{Re}(\lambda) > 0$. As we mentioned above, we do not have a general description of M^T_{CH}, but only a number of interesting examples. Observe that in this case ∞ is a sink of $R(z)\partial_z$).

A qualitative description of the convex hull $Conv(\mathrm{M}^T_{CH})$ is the best that we can obtain with our current knowledge.

Proposition 6.22. *Consider a linear differential operator T given by* (1.1) *with* $\deg Q - \deg P = 1$. *The boundary $\partial Conv(\mathrm{M}^T_{CH})$ of the convex hull $Conv(\mathrm{M}^T_{CH})$ of the minimal set is formed by:*

- *finitely many straight segments;*

- *finitely many portions of integral curves of vector field $R(z)\partial_z$.*

In particular, the latter are strictly convex and belong to local arcs of $\partial \mathrm{M}_{CH}^T$. In particular, $\partial Conv(\mathrm{M}_{CH}^T)$ is piecewise-analytic.

Proof. We denote by $\mathcal{S}$ the set of points where the boundary $\partial Conv(\mathrm{M}_{CH}^T)$ is strictly convex. They also belong to $\partial \mathrm{M}_{CH}^T$ (these points belong to the support of the hull). It follows from Theorem 1.6 that outside finitely many points, $\mathcal{S}$ is formed by either local or global arcs of $\partial \mathrm{M}_{CH}^T$. If such a point z belongs to a global arc, then the line containing the associated ray $r(z)$ is a support line of $Conv(\mathrm{M}_{CH}^T)$ at z and every point of $\Delta(z)$. It follows that $[z, \Delta^{max}(z)]$ is a straight segment contained in $\partial Conv(\mathrm{M}_{CH}^T)$. Consequently any arc of $\mathcal{S}$ has to be a portion of local arc.

We know that $\partial Conv(\mathrm{M}_{CH}^T)$ is formed by straight segments and portions of local arcs. It remains to prove that there are finitely many of them. We consider an arc α of $\partial Conv(\mathrm{M}_{CH}^T)$ contained in a local arc γ of $\partial \mathrm{M}_{CH}^T$. The endpoint of α (with the orientation defined by $R(z)\partial_z$) has at the same time to be the endpoint of γ (since otherwise the associated rays starting at points of α would intersect M_{CH}^T). Therefore, the endpoint of every such arc α in $\partial Conv(\mathrm{M}_{CH}^T)$ belongs to $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ (see Proposition 4.12). Since there are finitely many such points in $\mathcal{S}$, there are finitely many such arcs in $\partial Conv(\mathrm{M}_{CH}^T)$.

If the boundary of the convex hull is not formed by finitely many straight segments and portions of integral curves, then there are infinitely many corner points of angle smaller than π between the pairs of consecutive straight segments of the boundary. It follows from Corollary 6.2 that these points belong to $\mathcal{Z}(PQ)$. Therefore, we have finitely many corner points and finitely many straight segments. $\square$

In the examples below (including a very interesting family of operators in which $Q(z)$ has simple roots and $P(z) = Q'(z)$), $Conv(\mathrm{M}_{CH}^T)$ is a polygon.

Proposition 6.23. *Consider a linear differential operator T given by* (1.1)*, such that every root α of $Q(z)$ is simple and satisfies $P(\alpha) \neq 0$ and $\phi_\alpha = 0$.*

Then, $Conv(\mathrm{M}_{CH}^T)$ coincides with the convex hull of $\mathcal{Z}(Q)$.

Proof. The argument is similar to the one used in the proof of the classical Gauss–Lucas theorem (see [Mor]). If the differential form $\frac{P(z)dz}{Q(z)}$ has all positive residues, then the roots of $P(z)$ are contained in the convex hull of $\mathcal{Z}(Q)$.

The proof is based on consideration of the electrostatic force F created by the system of point charges placed at the poles of $\frac{P(z)dz}{Q(z)}$ where the value of each charge equals the residue at the corresponding pole. This electrostatic force F equals the conjugate of $\frac{P(z)dz}{Q(z)}$ and one can show that if we take any line L not intersecting the convex hull of $\mathcal{Z}(Q)$ then at any point $p \in L$, F points inside the half-plane of $\mathbb{C} \setminus L$ not containing $\mathcal{Z}(Q)$. Now recall that the associated ray has the same direction as the conjugate of P/Q. Thus, the associated ray $r(p)$ does not intersect the convex hull of $\mathcal{Z}(Q)$. $\square$

6.4.3. *The first family of examples.* Consider a family of operators of the form $T_\lambda = Q(z)\frac{d}{dz} + P(z)$ where $Q(z) = \lambda(z-1)^k z$ and $P(z) = (z-1)^k$ for some principal coefficient $\lambda \in \mathbb{C}^*$ and some degree $k \in \mathbb{N}^*$.

Integral curves of the vector field $R(z)\partial_z$ are logarithmic spirals parametrized by $\gamma(t) = \gamma(0)e^{\lambda t}$. In particular, they are concentric circles for $Re(\lambda) = 0$.

Depending on the value of λ, the shape of the minimal set M_{CH}^T can change drastically. Namely,

- if $Re(\lambda) < 0$, then $\mathrm{M}_{CH}^T = \mathbb{C}$ (see Theorem 1.11 of [AHN+22]);
- if $Re(\lambda) = 0$, then M_{CH}^T is the closed unit disk (see Theorem 6.21);

- if $Re(\lambda) > 0$ and $Im(\lambda) = 0$, then M_{CH}^T is segment $[0, 1]$.

When $Re(\lambda) > 0$ and $Im(\lambda) \neq 0$, M_{CH}^T has a more complicated shape we describe below in terms of local and global arcs. Up to conjugation, we will assume that $\mathrm{Im}(\lambda) > 0$.

Proposition 6.24. *If λ satisfies $\mathrm{Re}(\lambda), \mathrm{Im}(\lambda) > 0$, then the minimal set M_{CH}^T of operator T_λ is bounded by the following arcs:*

- *local arc γ where $\gamma(t) = e^{-\lambda t}$ and $t \in]0, t_0[$;*
- *global arc α where $\alpha(t) = \frac{1}{1+\lambda t}$ and $t \in]0, t_1[$.*

These two arcs intersect at 1 and the point $\gamma(t_0) = \alpha(t_1)$ of extruding type characterized as the first intersection point between α and γ defined on $\mathbb{R}_{>0}$.

Proof. The backward trajectory of the vector field $R(z)\partial_z$ starting at 1 is parametrized by $\gamma(t) = e^{-\lambda t}$ and $t \in [0, \infty)$. Proposition 2.10 shows that this arc is entirely contained in M_{CH}^T.

Points z for which the associated ray contains 1 are characterized by the condition $\frac{1-z}{\lambda z} \in \mathbb{R}_{>0}$. They form an arc parametrized by $\alpha(t) = \frac{1}{1+\lambda t}$ for $t \in [0, +\infty[$. This arc is also contained in M_{CH}^T.

Since $R(z) = \lambda z$, it is geometrically clear that these two arcs bound M_{CH}^T. The boundary $\partial \mathrm{M}_{CH}^T$ is formed by a portion of each of them with two singular points at 1 (when $t = 0$) and the first intersection point in the parametrization. There are different ways to see that such an intersection occurs. One of them is to note that $\lim_{t\to\infty} \alpha(t) = 0$ and $\lim_{t\to\infty} \arg(\alpha'(t)) = \lim_{t\to\infty} = \arg(\frac{-\lambda}{(1+\lambda t)^2})$ exists. Since the vector field $R(z)$ has residue with positive real and imaginary parts, it follows that $\alpha(t)$ and $\gamma(t)$ intersect infinitely many times. The endpoint distinct from 1 common to α and γ is the first intersection point between the two parametrized arcs defined on $\mathbb{R}_{>0}$. □

6.4.4. *The second family of examples.* Consider the family $T = z(z^k-1)\frac{d}{dz}+(z^k+1)$, where k is a positive integer. We are going to prove that for any k, the minimal set M_{CH}^T is the unit disk.

Lemma 6.25. *Set $f(z) = z + t\frac{z(z^k-1)}{z^k+1}$, with $t > 0$. Then $|f(z)| > 1$ whenever $|z| > 1$.*

Proof. We substitute $z = r^{\frac{1}{k}} e^{i\theta}$ with $r > 1$. After some algebraic manipulations, we find that

$$\frac{|f(z)|^2}{|z|^2} = \frac{|f(r^{\frac{1}{k}} e^{i\theta})|^2}{r^{2/k}} = 1 + t\frac{2r^2 - 2 + r^2 t - 2\cos(\theta k)rt + t}{r^2 + 2\cos(\theta k)r + 1}. \tag{6.1}$$

Setting $c := \cos k\theta$ and rewriting further, we get

$$\frac{|f(r^{\frac{1}{k}} e^{i\theta})|^2}{r^{2/k}} = 1 + t\frac{2(r^2-1) + t\left((r-c)^2 + (1-c^2)\right)}{(r+c)^2 + (1-c^2)}. \tag{6.2}$$

Since $-1 \le c \le 1$, it follows that

$$\frac{|f(r^{\frac{1}{k}} e^{i\theta})|^2}{r^{2/k}} > 1 + t\frac{r^2-1}{(r+1)^2} > 1.$$

Consequently, $|f(z)| > |z|$ whenever $|z| > 1$ and the statement follows. □

Lemma 6.26. *The separatrices of the vector field $R(z)\partial_z = \frac{z(z^k-1)}{z^k+1}\partial_z$ are the arcs of the unit circle, connecting roots of $P(z)$ with roots of $Q(z)$.*

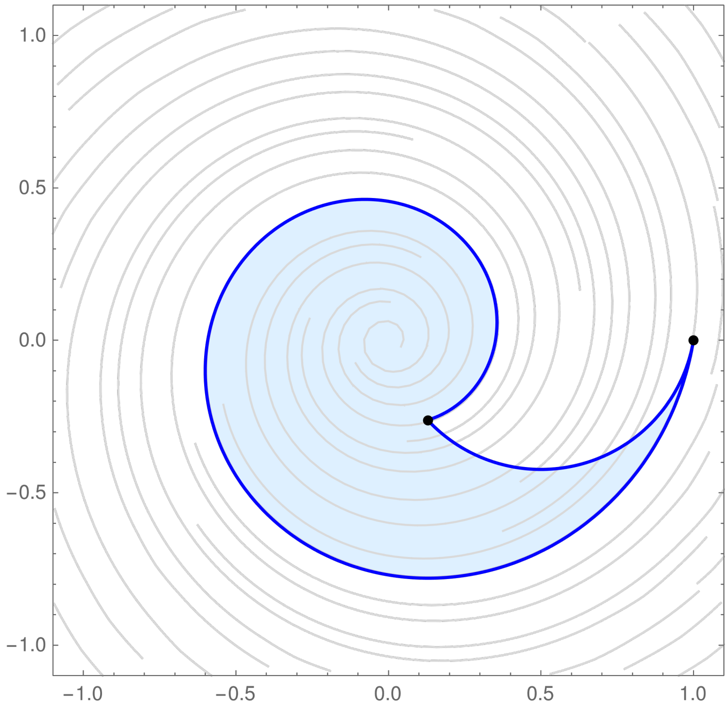

FIGURE 7. Illustration of the boundary of the minimal set when $\lambda = 1 + 6i$ in Prop. 6.24.

Proof. Assuming that z is not a root of Q, we have that

$$\int \frac{z^k+1}{z(z^k-1)}dz = k^{-1}\log\left(\frac{(1-z^k)^2}{z^k}\right).$$

Now for $z = e^{i\theta}$, we find that

$$\operatorname{Im}\log((1-z^k)^2/z^k) = \arg((1-z^k)^2/z^k) = \arg(-2+e^{ik\theta}+e^{-ik\theta}) = \pi.$$

Hence, the unit circle consists of the integral trajectories of $R(z)\partial_z$. Since the roots of P lie on the unit circle, these integral trajectories must be separatrices. □

Corollary 6.27. *For $T = z(z^k-1)\frac{d}{dz} + (z^k+1)$, the minimal set M_{CH}^T coincides with the unit disk.*

Proof. By Lemma 6.25, we have that all the associated rays for points lying outside the unit disk never intersect the unit disk. Therefore, M_{CH}^T is contained in the unit disk. Since the unit circle consists of separatrices of $-R(z)\partial_z$ (see Lemma 6.26), it follows that M_{CH}^T contains the unit circle. The associated ray of any point (distinct from 0) of the open unit disk intersects the unit circle so M_{CH}^T coincides with the unit disk. □

Example 6.28. Here we provide a construction for an invariant set in the case $\operatorname{Re}(\lambda) > 0, \operatorname{Im}(\lambda) \neq 0$. In this situation, ∞ is a sink of the vector field $R\frac{d}{dz}$ with non-real residue. Hence, for any point sufficiently close to ∞, its forward trajectory circulates around and towards ∞. Let S be the set of all points for which its forward trajectory ends in ∞. Take any $z \in S$ and let $\phi(t)$ be the integral curve of $R\frac{d}{dz}$ passing through z, defined on its maximal interval of definition. Consider a line going through $\phi(t_0)$ perpendicular to the associated ray of $\phi(t_0)$. For sufficiently large t_0, on one side of $\phi(t_0)$, the line intersects only $\phi(t)$ for $t > t_0$. Denote the smallest such t by t_1 and call the closed line segment between these two points L_{t_0}, and denote by M_{ϕ,t_0} the minimal simply connected set containing L_{t_0} and $\phi([t_0,t_1])$. Then for sufficiently large t_0, M_{ϕ,t_0} is invariant since M_{ϕ,t_0} contains the zeros of P and Q and all associated rays of points in $\partial M_{\phi,t_0}$ intersect M_{ϕ,t_0} only in $\partial M_{\phi,t_0}$, since $\Im_R$ is compact in $\mathbb{C}$. This remains true for all $t < t_0$ such that $\phi(s)$ does not intersect $\Im_R$, $R(z)$ is not tangent to L_s for some $z \in L_s$ and L_s does not contain a zero of PQ for $s \in [t,t_0]$. We denote by t_ϕ the maximal t such that $\phi(t)$ intersects $\Im_R$, $R(z)$ is tangent to L_s for some $z \in L_s$ or L_s contains a zero of PQ for $s \in [t,t_0]$ a zero of PQ, and obtain the invariant set M_{ϕ,t_ϕ}. This set is indeed invariant as it may be written as the intersection of invariant sets. We do this construction for each integral curve ϕ contained in S, and obtain invariant sets M_{ϕ,t_ϕ}. As the intersection of invariant sets is again invariant, we find the smallest invariant set using this construction

$$M = \bigcap_{\phi\subset S} M_{\phi,t_\phi}$$

6.5. **Connected components of minimal sets.** Putting together partial results for the different values of $\deg Q - \deg P$, we are able to state a bound on the number of connected components of M^T_{CH} in $\mathbb{C}$. It is already known that the closure of M^T_{CH} in the extended plane $\mathbb{C}\cup\mathbb{S}^1$ is always connected.

Proof of Theorem 1.8. For any operator T satisfying $|\deg Q - \deg P| > 1$, it has been proved in Theorem 1.11 of [AHN+22] that $\mathrm{M}^T_{CH} = \mathbb{C}$. Besides, when $\deg Q - \deg P = 1$, Section 6.3 and Corollary 5.20 of the same paper proves that M^T_{CH} is connected and contractible. For $\deg Q - \deg P = -1$, it follows from Proposition 6.9.

The only case where there could be several connected components is $\deg Q - \deg P = 0$. If $R(z)$ is constant, then there are two situations. If P, Q are both constant, then there is no meaningful notion of minimal set (see Section 2.3.1 in [AHN+22]). Otherwise, M^T_{CH} is formed by parallel half-lines starting at points of $\mathcal{Z}(PQ)$. Since every point of $\mathcal{Z}(PQ)$ is a common root of P and Q (otherwise $R(z)$ would not be constant) we get that there are at most $\frac{1}{2}\deg P + \frac{1}{2}\deg Q$ such half-lines.

If $R(z)$ is not constant, then we have $R(z) = \lambda + \frac{\mu}{z^\kappa} + o(z^{-\kappa})$ for some $\lambda, \mu \in \mathbb{C}^*$ and $\kappa \in \mathbb{N}^*$. If $\kappa = 1$ and $\mathrm{Im}(\mu/\lambda) \neq 0$, then Corollaries 6.20 proves that M^T_{CH} is connected. Otherwise, Proposition 2.19 provides an upper bound $\frac{1}{2}\deg P + \frac{1}{2}\deg Q$. □

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