IN SEARCH OF A HIGHER BOCHNER THEOREM

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To Salomon Bochner, a mathematical hero

Abstract. We initiate the study of a natural generalisation of the classical Bochner-Krall problem asking which linear ordinary differential operators possess sequences of eigenpolynomials satisfying linear recurrence relations of finite length; the classical case corresponds to the 3-term recurrence relations with real coefficients subject to some extra restrictions. We formulate a general conjecture and prove it in the first non-trivial case of operators of order 3.

1. Introduction

1.1. On the classical problem. In 1929 S. Bochner published a short paper [Bo] dealing with orthogonal polynomials and Sturm-Liouville problem. Although after writing [Bo], he left this area for good, his importance for the theory of orthogonal polynomials is difficult to overestimate; at the moment [Bo] has been cited 453 times.

Namely, the following classification problem was formulated by S. Bochner for differential operators of order 2, and by H. L. Krall for differential operators of general order.

Problem A ([Bo], [Kr1]). Describe all linear differential operators with polynomial coefficients of the form:

\[ L = L(x, \partial) = \sum_{i=1}^{k} a_i(x) \partial^i, \]

such that a) \( \deg a_i(x) \leq i \); b) there exists a positive integer \( i_0 \leq k \) with \( \deg a_{i_0}(x) = i_0 \), satisfying the condition that the set of polynomial solutions of the formal spectral problem

\[ Lf(x) = \lambda f(x), \quad \lambda \in \mathbb{R}, \]

is a sequence of polynomials orthogonal with respect to some real bilinear form. (Here \( \partial := \frac{d}{dx} \).)

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Salomon Bochner made substantial contributions to harmonic analysis, probability theory, differential geometry as well as history of mathematics. Several notions and results such as the Bochner integral, Bochner theorem on Fourier transforms, Bochner-Riesz means, Bochner-Martinelli formula bear at present his name. His mathematical production consists of 6 books on various subjects including history of mathematics and about 140 research papers. Among these papers 32 were published in Proc. Nat. Acad. USA, 46 in Annals of Mathematics, 5 in Acta Math., and 4 in Duke Math. J. He belonged to a sizeable group of European mathematicians of Jewish origin who moved to US before or during the WWII and contributed to an enormous development of mathematics in their new motherland.
Following the terminology used in physics, we call linear differential operators given by (1.1) **exactly solvable**, see e.g. \[ Tu, RuTu \]. Observe that every exactly solvable operator has a unique eigenpolynomial of any sufficiently large degree which makes Problem A well-posed.

Let us denote by \( \{ P_L^L(x) \} \) the sequence of eigenpolynomials of an exactly solvable operator \( L \). (We assume that \( \deg P_L^L(x) = n \) where \( n \) runs from some positive integer to \( +\infty \).) An exactly solvable operator which solves Problem A will be called a **Bochner-Krall operator**.

S. Bochner stated and solved Problem A for order two differential operators, see \[ Bo \] and Theorem A below. The order two classification contains four families, corresponding to the classical Hermite, Laguerre, Jacobi, and Bessel polynomials. Eleven years after that H. L. Krall settled the order four case, see \[ Kr2 \]. The order four classification contains seven families: the four classical families corresponding to the case when an order four operator is a function of an operator of order two, and three new families which are polynomial eigenfunctions of differential operators that do not reduce to operators of order two, see \[ Kr1 \] and Theorem B below.

An assortment of families corresponding to order six operators has been found in the ensuing decades, see e.g. \[ KrJun, Chapter XVI \]. W. Hahn showed that the four classical families are the only orthogonal polynomial sequences \( \{ P_n(x) \}_{n=0}^{\infty} \) for which \( \{ \frac{d}{dx} P_n(x) \}_{n=0}^{\infty} \) is also an orthogonal polynomial sequence, see \[ Ha \]. An analog of this was proved for the order 4 case by K. H. Kwon, L. L. Littlejohn, J. K. Lee, and B. H. Yoo \[ KKLY \]. The most general form of Problem A is still open for operators of order six or higher, but K. H. Kwon and J. K. Lee have found a satisfactory solution if the polynomials are required to be orthogonal with respect to a compactly supported positive measure, see \[ KL \]. (A weaker result in the same direction was somewhat earlier obtained in \[ BRSh \].)

In Problem A one may equivalently seek a sequence of moments \( \{ \mu_j \}_{j=0}^{\infty} \) which permits construction of the orthogonal sequence of polynomials, see \[ KrJun, p. 223 \]. Let \( \langle \cdot, \cdot \rangle \) be a candidate bilinear form, and define a weight on the set of polynomials with respect to the inner product by requiring that \( \langle w(x), x^n \rangle = \mu_n \). (Note that such a weight \( w(x) \) is not necessarily positive or unique.) One can show that solutions to Problem A exist only if the product \( wL \) is equal to its formal adjoint when acting on polynomials, and this immediately implies that the order of \( L \) must be even, see \[ KrJun, p. 228 \].

Recall that by a trivial generalisation of Favard’s theorem, every sequence of monic orthogonal polynomials satisfies a 3-term recurrence relation of the form:

\[
(1.2) \quad xP_n(x) = P_{n+1}(x) + u(n)P_n(x) + v(n)P_{n-1}(x),
\]

where \( u(n) \) and \( v(n) \) are real numbers. Observe that the weight function \( w(x) \) is non-negative in \( \mathbb{R} \) if and only if \( v(n) > 0 \) for all \( n > 0 \).

1.2. **Algebraisation and generalisation.** The following natural algebraic version of the classical Bochner-Krall problem was studied in substantial details in \[ GHH, GY \].

**Problem 1** (algebraic Bochner-Krall problem). Describe all linear differential operators (1.1) such that their sequence of monic eigenpolynomials satisfies (1.2) with some complex-valued \( u(n) \) and \( v(n) \).
Observe that contrary to the case of the classical Bochner-Krall problem, Problem 1 is purely algebraic and (hopefully) has an easier solution compared to the classical one, see Conjecture 1.7 below. On the other hand, the connection between the algebraic Bochner-Krall problem and the classical one is rather straightforward.

**Proposition 1.1.** A differential operator solves the classical Bochner-Krall problem with a positive weight function \( w(x) \) if and only if it solves Problem 2 and all its eigenpolynomials are real-rooted.

**Remark 1.2.** Which additional conditions on an exactly solvable operator can guarantee the real-rootedness of its eigenpolynomials is unclear at the moment, but this could be related to the major results in the Pólya-Schur theory, see e.g. [BB].

**Remark 1.3.** Observe that if we allow coefficients of the operator to depend on \( n \), then the whole problem trivializes, i.e., more or less any sequence of polynomials can be obtained in such a way.

In what follows, we also discuss the following natural generalisation of Problem 1.

**Problem 2** (generalised algebraic Bochner-Krall problem). Describe the set of linear differential operators (1.1) such that their sequence of eigenpolynomials satisfies a finite recurrence relation of the form:

\[
xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{d} b_j(n)P_{n-j}(x) = \Lambda(n)P_n,
\]

where \( d \) is independent of \( n \) and the coefficients \( b_j(n) \) are independent of \( x \). Here

\[
\Lambda(n) = T + \sum_{j=0}^{d} b_j(n)T^{-j}
\]

is a difference operator, and \( T f(n) := f(n+1) \) is the shift operator.

**Remark 1.4.** By a theorem of Maroni [Ma], the latter condition means that the sequence \( \{P_n(x)\} \) of polynomials is \( d \)-orthogonal, see e.g., [VI]. The study of \( d \)-orthogonal and multiple orthogonal polynomials has being a popular area of research during the last 3 decades, see e.g., [ApKu].

**Remark 1.5.** It is clear that if an operator \( L \) solves Problems 1 or 2 then for any univariate polynomial \( s \) of positive degree, the operator \( s(L) \) also solves the same problem with the same sequence of eigenpolynomials \( \{P_n^L(x)\} \) and the sequence of eigenvalues coinciding with \( \{s(\lambda(n))\} \). Therefore in what follows, we will always assume that \( L \) is the operator of the lowest order with a given sequence of eigenpolynomials.

**Remark 1.6.** Another way to produce new solutions to Problems 1 and 2 from the existing ones is to apply a bispectral Darboux transformation to the difference operator \( \Lambda \). This will produce a new difference operator \( \hat{\Lambda} \) and a new differential operator \( \hat{L} \). However the order of \( \hat{L} \) could increase and, in fact, this is what happens in all known examples. We will call a differential operator \( L \) irreducible if it cannot be obtained by applying a bispectral Darboux transformation to a differential operator of a lower order.
Most of the above situations allow application of Darboux transformations on the side of the difference operator. More precisely, let us present the operator $\Lambda$ as a matrix, acting on the vector $(P_0, P_1, \ldots)$. Then in certain situations there exists a factorisation of $\Lambda$ as

$$\Lambda = D_1 \circ D_2,$$

where $D_1$ and $D_2$ are rational functions of $n$, and, additionally, $D_1$ is an upper triangular while $D_2$ is a lower triangular matrices. Set

$$\hat{\Lambda} = D_2 \circ D_1$$

and define a new sequence of polynomials by using

$$\hat{P}_n(x) = D_2 P_n(x).$$

It is easy to see that $\deg \hat{P}_n(x) = n$ and also

$$x \hat{P}_n(x) = \hat{\Lambda} \hat{P}_n(x).$$

In such a situation a general procedure described in \cite{BHY1} allows us to obtain a differential operator $\hat{L}$, such that

$$\hat{L} \hat{P}_n(x) = \mu_n \hat{P}_n(x).$$

To carry out this procedure in concrete cases is a nontrivial problem. However in \cite{GHH, GY} it was been successfully performed in the situations when one starts either from the Laguerre or from the Jacobi polynomials. In particular, using a 1-step Darboux transformation one obtains Krall’s polynomials. Analogously, many-step Darboux transformations lead to differential operators of higher order, but they do not increase the order of the difference operator.

In this language, the above mentioned result of S. Bochner \cite{Bo} can be stated as follows.

**Theorem A.** Every orthogonal polynomial system \{$P_n(x)$\} obtained as a sequence of eigenpolynomials of a linear differential operator of order 2 coincides (up to the action of the affine group on the variable $x$ and scaling) with one of the sequences of Hermite, Laguerre, Jacobi or Bessel orthogonal polynomials, i.e., with one of the classical orthogonal polynomial systems.

Similarly, the above mentioned result of H. Krall \cite{Kr2} claims the following.

**Theorem B.** Every linear differential operator of order 4 solving the classical Bochner-Krall problem and which can not be obtained as a function of a linear differential operator of order 2 is given either by

- (a) a 1-step Darboux transformation applied to an operator of order 2 corresponding to a system of Jacobi polynomials with at least one of the parameters $\alpha, \beta$ being an integer, or by
- (b) a 1-step Darboux transformation applied to an operator corresponding to a system of Laguerre polynomials with an integer value of its parameter $\alpha$.

Let us now present our novel conjectures and results related to Problems 1 and 2. Concerning Problem 1 we propose the following general guess supported by the results of \cite{GHH, GY}.
Conjecture 1.7. Any sequence \( \{P_n\} \) of monic polynomials obtained as a sequence of eigenpolynomials of a linear differential operator solving Problem 1 belongs to one of the following 2 classes:

1) A classical sequence of orthogonal polynomials with, in general, complex-valued parameters, i.e.,
   (a) Hermite polynomials \( H_n \);
   (b) Laguerre polynomials \( L_n^{(\alpha)} \);
   (c) Jacobi polynomials \( P_n^{(\alpha,\beta)} \);
   (d) Bessel polynomials \( Y_n \).

2) A sequence of polynomials which can be obtained from
   (e) Laguerre polynomials \( L_n^{(\alpha)} \) with the positive integer values of \( \alpha \) by applying a finite number of Darboux transformations to the difference operator
   \( \Lambda(n) := T + u(n)I + v(n)T^{-1} \)
   occurring in the right-hand side of (1.2);
   (f) Jacobi polynomials \( P_n^{(\alpha,\beta)} \) with the positive integer values of either \( \alpha \), \( \beta \) or both \( \alpha \) and \( \beta \) by applying a finite number of Darboux transformations to the operator \( \Lambda(n) \).

Remark 1.8. The fact that the families (a) - (d) solve Problem 1 is classical. Analogous statement for the families (e)-(f) has been proven in [GHH] and [GY]. The major difficulty in settling Conjecture 1.7 is to show that the above list of cases is exhaustive which is possible in principle, but computationally is quite a challenging problem.

Concerning Problem 2, we have the following two conjectural claims.

Conjecture 1.9. For any irreducible differential operator \( L \) of order \( k \) solving Problem 2, the order of the corresponding difference operator \( \Lambda \) is also \( k \).

The next claim similar to Conjecture 1.7 gives a description of all irreducible operators solving Problem 2.

Conjecture 1.10. For any positive integer \( k \), the irreducible differential operators of order \( k \) solving Problem 2 belong to one of the following two types:

1) \[ L = \sum_{j=1}^{k} a_j x^{j-1} \partial^j + x \partial, \quad a_j \in \mathbb{C}, \ a_k \neq 0, \]
   generating the so-called \((k-1)\)-orthogonal polynomials, see [BChD].
   For any divisor \( \ell \) of \( k \), set \( G := (\sum_{m=0}^{\ell-1} a_m (x \partial)^m) \partial, \quad a_m \in \mathbb{C}, \ a_{\ell-1} \neq 0. \) Let \( q(t) \) be any complex polynomial of degree \( k/\ell \) without a constant term. Then the operator
   \[ L = q'(G)G + x \partial \]
   is an irreducible operator of order \( k \).

Remark 1.11. Both types of operators appearing in Conjecture 1.10 together with their properties were discussed in some detail in recent papers [Ho, Ho1].
Remark 1.12. Notice that the operators
\[ L = \sum_{j=1}^{k} a_j \partial_j + x \partial \]
generating the Appell polynomials \([\text{App}]\) are included in Type 2 with \( \ell = 1 \), see [Ho1].

Example 1.13. Conjecturally, all the irreducible operators of order 4 solving Problem 2 are given by:

1) \[ L = \sum_{j=1}^{4} a_j x^{j-1} \partial^j + x \partial, \quad a_j \in \mathbb{C}, a_4 \neq 0 \]
generating the so-called 3-orthogonal polynomials;

2') \[ L = \sum_{j=1}^{4} a_j \partial^j + x \partial, \quad a_j \in \mathbb{C}, a_4 \neq 0 \]
generating the Appell polynomials, see \([\text{App}], \text{[GH]}\);

2'') \[ L = b_2 G^2 + b_1 G + x \partial, \quad b_2 \neq 0, \]
where \( G = (a_1 x \partial + a_0) \partial, \quad a_1 \neq 0. \)

The difference operator \( \Lambda \) in all these cases is of order 4, i.e., \( d = 3. \)

The main result of the present paper is a proof of Conjecture 1.10 in the first non-trivial case of differential operators \( L \) of order 3.

Theorem 1.14. All irreducible differential operators \( L \) of order 3 solving Problem 2 whose corresponding difference operators \( \Lambda \) also have order 3 are given by:

1) \[ L = \sum_{j=1}^{3} a_j x^{j-1} \partial^j + x \partial, \quad a_j \in \mathbb{C}, a_3 \neq 0, \]
generating the 2-orthogonal polynomials, see \([\text{BChD}]\). They satisfy the 4-term recurrence relation:
\[ xP_n(x) = P_{n+1}(x) - (a_1 + 2(n-1)a_2 + 3(n-1)(n-2)a_3)P_n(x) + (n-1)(a_2 + (3n-6)a_3) \]
\[ (a_1+(n-2)a_2+(n-2)(n-3)a_3)P_{n-1}(x) - (n-1)(n-2)a_3(a_1+(n-3)a_2+(n-3)(n-4)a_3) \]
\[ (a_1 + (n - 2)a_2 + (n - 2)(n - 3)a_3)P_{n-2}(x), \]
with the standard initial conditions: \( P_{-2}(x) = P_{-1}(x) = 0, P_0(x) = 1; \)

2) \[ L = \sum_{j=1}^{3} a_j \partial^j + x \partial, \quad a_j \in \mathbb{C}, a_3 \neq 0 \]
generating the Appell polynomials, see \([\text{App}], \text{[GH]}\). They satisfy the 4-term recurrence relation:
\[ xP_n(x) = P_{n+1}(x) - a_1 P_n(x) - a_2(n-1)P_{n-1}(x) - a_3(n-1)(n-2)P_{n-2}(x), \]
with the standard initial conditions: \( P_{-2}(x) = P_{-1}(x) = 0, P_0(x) = 1. \)
Remark 1.15. Theorem 1.14 will follow from the detailed study of three special cases presented in § 3, § 4, and § 5 respectively. Its proof contains several new ideas, but also a substantial amount of explicit calculations which we were able to carry out by hands in § 3 and § 5. In § 4 we had to use Mathematica package since these calculations were too heavy. It is of course highly desirable to find an alternative proof avoiding such an amount of explicit calculations and which might be applicable for a possible attack on our general Conjecture 1.10 but at the moment we do not see such a possibility.

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2. Preliminary facts

Proof of Proposition 1.1. Notice that a solution to the classical Bochner-Krall problem with a positive weight function on the real line produces a sequence \( \{P_n\} \) of (orthogonal) polynomials which satisfy a 3-term recurrence relation (1.2) with real coefficients \( u(n) \) and \( v(n) > 0 \). Under this conditions, each \( P_n \) can be taken as a real and real-rooted polynomial. On the other hand, requiring that all polynomials satisfying (1.2) are real-rooted will imply that they can be taken real which means that \( u(n) \) and \( v(n) \) are real. Additionally, the real-rootedness of all \( P_n \) implies that \( v(n) > 0 \) for all \( n \). °

Problem 2 asks to describe all exactly solvable linear differential operators \( L \) that have a sequence of polynomial eigenfunctions \( \{P_n(x)\} \) with eigenvalues \( \lambda(n) \), i.e.,

(2.4) \[ LP_n(x) = \lambda(n)P_n(x) \] and which at the same time satisfy a difference equation of the form

(2.5) \[ \Lambda P_n(x) = xP_n(x). \]

In the next lemma we deduce a simple, but very useful necessary condition for the solvability of the latter problem for differential operators of order 3. Let us assume that a polynomial system \( \{P_n(x)\} \) satisfying (2.4) and (2.5) exists. Define \( \text{ad}_x L := [x, L] \) and \( \text{ad}_x^{j+1} L := \text{ad}_x(\text{ad}_x^j L) \).

It is obvious that \( \text{ad}_x L \) is a differential operator of order less than the order of \( L \). Below we assume that the order of \( L \) equals 3, i.e.,

(2.6) \[ L = a_3(x)\partial^3 + a_2(x)\partial^2 + a_1(x)\partial. \]

Then it is obvious that

\[ \text{ad}_x^3 L = 0. \]

The following result obtained in [DG] will be crucial in all our calculations. We state it in a form suitable for the present paper.
Lemma 2.1. The eigenvalues $\lambda(n)$ satisfy the condition
\begin{equation}
\text{ad}^4_{\Lambda} \lambda(n) = 0.
\end{equation}

Proof. The expression $[x, L]P_n(x)$ can be expanded as
\[ [x, L]P_n(x) = xLP_n(x) - LxP_n(x) = x\lambda(n)P_n(x) - L\lambda P_n(x). \]

Using the fact that both $\lambda(n)$ and $\Lambda$ do not depend on $x$, we can rewrite the above as
\[ x\lambda(n)P_n(x) - L\lambda P_n(x). \]

Applying the operators $x$ and $L$ to the right-hand side of the latter formula, we obtain
\[ [x, L]P_n(x) = \lambda(n)\Lambda P_n(x) - \Lambda\lambda(n)P_n(x) = [\lambda(n), \Lambda]P_n(x), \]
i.e., $\text{ad}_{\Lambda} \lambda = -\text{ad}_x L$. The fact that $\text{ad}_x^2 L = (-1)^2 \text{ad}_x^2 \lambda(n)$ implies that $\text{ad}^4_{\Lambda} \lambda(n) = 0$. □

Equation (2.7) will be referred to as the ad-condition and will be our main tool.

Notation. Below we will impose the restriction that $\Lambda$ is a monic difference operator of order 3, i.e.,
\begin{equation}
\Lambda = T + \sum_{i=0}^{2} b_i(n) T^{-i},
\end{equation}
comp. Conjecture 1.9.

In what follows we will use the following notation for the coefficients of the polynomials $a_3(x), a_2(x)$ and $a_1(x)$ respectively:
\begin{align*}
a_3(x) &= a_{33}x^3 + a_{32}x^2 + a_{31}x + a_{30}; \\
a_2(x) &= a_{22}x^2 + a_{21}x + a_{20}; \\
a_1(x) &= a_{11}x + a_{10}.
\end{align*}

2.1. The ad-condition. Notice that $\lambda(n)$ is a polynomial of degree at most 3 in the variable $n$. Indeed, the entries of $\lambda(n)$ come from the differentiation of the term $x^n$ in $P_n(x)$. More exactly,
\[ \lambda(n) = (n)_3 \cdot a_{33} + (n)_2 \cdot a_{22} + n \cdot a_{11}. \]

We will now derive a polynomial form of $\text{ad}^3_{\Lambda} \lambda(n)$.

Lemma 2.2. The operator $\text{ad}^3_{\Lambda} \lambda(n)$ is of the form:
\begin{equation}
\text{ad}^3_{\Lambda} \lambda(n) = \alpha \Lambda^3 + \beta \Lambda^2 + \gamma \Lambda + \delta I = 6a_3(\Lambda),
\end{equation}
where the coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are independent of $n$.

Proof. Put $f(x, n) := P_n(x)$. From the condition $\text{ad}^4_{\Lambda} \lambda(n) = 0$ it follows that $\text{ad}^3_{\Lambda} \lambda(n)$ is a polynomial in $\Lambda$ which we denote by $q(\Lambda)$. The degree of $q(\Lambda)$ is at most 3 since it contains $T$ in at most the third degree. Also notice that
\[ \text{ad}^3_{\Lambda} L = (-1)^3 6a_3(x). \]

Having in mind that
\[ \text{ad}^3_{\Lambda} L(P_n) = (-1)^3 \text{ad}^3_{\Lambda} \lambda(P_n), \]
we obtain
\[ (-1)^3 6a_3(\Lambda) P_n = (-1)^3 \text{ad}^3_{\Lambda} \lambda P_n, \]
i.e., $q(x) = 6a_3(x)$. □
Remark 2.3. Formula \(2.9\) says that the leading polynomial coefficient \(a_3(x)\) coincides with \(\frac{1}{6}(\alpha x^3 + \beta x^2 + \gamma x + \delta)\), i.e., in the above notation \(a_{33} = \alpha/6, a_{32} = \beta/6, a_{31} = \gamma/6, a_{30} = \delta/6\) which we will use below.

3. Case when \(\lambda\) is a linear polynomial

As we mentioned above, \(\lambda(n)\) is a polynomial in \(n\) of degree at most 3. Here we will discuss the simplest case when \(\lambda(n)\) is linear. In terms of the differential operator \(L\) given by (2.6), the linearity of \(\lambda\) means that \(\deg a_3(x) < 3\) (which is equivalent to \(a_{33} = \alpha/6 = 0\)), \(\deg a_2(x) < 2\) (which is equivalent to \(a_{22} = 0\)) and \(\deg a_1(x) = 1\) (which is equivalent to \(a_{11} \neq 0\)). Additionally, observe that in this case we can without loss of generality assume that \(\lambda(n) \equiv n\). Indeed, if \(\lambda(n) = \mu n + \nu\), then we can consider \(L - \nu\) instead of \(L\). Finally, we can divide both sides by \(\mu\), which results in \(\lambda(n) \equiv n\).

Assuming that \(\lambda(n) \equiv n\), we will separately study the following 3 (sub)cases:

(i) \(a_3(x)\) is a quadratic polynomial, i.e., \(a_{33} = 0; a_{32} = \beta/6 \neq 0\);
(ii) \(a_3(x)\) is a linear polynomial, i.e., \(a_{33} = a_{32} = 0, a_{31} = \gamma/6 \neq 0\);
(iii) \(a_3(x)\) is a non-vanishing constant, i.e., \(a_{33} = a_{32} = a_{31} = \gamma/6 = 0, a_{30} = \delta/6 \neq 0\).

3.1. Case \(\deg a_3(x) = 2 \Leftrightarrow \beta \neq 0\). Since \(\lambda(n) \equiv n\), we get

\[
L = (a_{32}x^2 + a_{31}x + a_{30})\partial_x^2 + (a_{21}x + a_{20})\partial^2 + (x + a_{10})\partial_x,
\]

where \(a_{32} \neq 0\). Rescaling \(L\) we can achieve \(a_{32} = 1\). Shifting \(x\) we can additionally assume that \(a_{10} = 0\). (Both changes only result in somewhat shorter formulas, but otherwise are not essential.) Thus without loss of generality, we can assume that the operator \(L\) is of the form

\[
L = (x^2 + a_{31}x + a_{30})\partial_x^2 + (a_{21}x + a_{20})\partial^2 + x\partial_x.
\]

To prove the required result, we do the following. Expanding the polynomials \(P_n(x)\) as

\[
P_n(x) = x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots,
\]

we are going to compute the coefficients \(p_k(n)\) for the first few values of \(k\). Then we compute the coefficients \(b_j(n)\) of the recurrence relation \(\Lambda\). The condition that the polynomials \(P_n(x)\) satisfy a 4-term relation implies that \(b_3(n) \equiv 0\). We will use the following identities:

\[
\begin{align*}
p_1(n) &= p_1(n + 1) + b_0(n); \\
p_2(n) &= p_2(n + 1) + b_0(n)p_1(n) + b_1(n); \\
p_3(n) &= p_3(n + 1) + b_0(n)p_2(n) + b_1(n)p_1(n) + b_2(n); \\
p_4(n) &= p_4(n + 1) + b_0(n)p_3(n) + b_1(n)p_2(n) + b_2(n)p_1(n) + b_3(n).
\end{align*}
\]

These relations imply the following expressions for \(b_j(n)\):

\[
\begin{align*}
b_0(n) &= p_1(n) - p_1(n + 1); \\
b_1(n) &= p_2(n) - p_2(n + 1) - b_0(n)p_1(n); \\
b_2(n) &= p_3(n) - p_3(n + 1) - b_0(n)p_2(n) - b_1(n)p_1(n); \\
b_3(n) &= p_4(n) - p_4(n + 1) - b_0(n)p_3(n) - b_1(n)p_2(n) - b_2(n)p_1(n).
\end{align*}
\]

We want to conclude that under our assumptions, we get \(a_{31} = a_{30} = 0\). The following statement being the major result of this section claims exactly this.
Proposition 3.1. If a differential operator \( L \) of the form
\[
L = (x^2 + a_{31}x + a_{30})\partial_x^3 + (a_{21}x + a_{20})\partial_x^2 + x\partial_x
\]
has a sequence of polynomial eigenfunctions \( \{P_n(x)\} \), \( n = 0, 1, \ldots \) which satisfies a 4-term recurrence relation, then \( a_{31} = a_{30} = a_{20} = 0 \). The operator \( L \) will have the form
\[
L = \sum_{j=1}^{3} a_j x^{j-1}\partial_x^j + x\partial_x.
\]

Proof. We will actually show that if \( b_3(n) \equiv 0 \) (which is a necessary condition for the sequence \( \{P_n(x)\} \) to satisfy a 4-term recurrence), then all three parameters \( a_{31}, a_{30}, a_{20} \) vanish. To do this, we will not need the explicit form of \( b_3(n) \) which is very cumbersome. It would be enough to obtain polynomial coefficients at \( a_{31}, a_{30} \) and \( a_{20} \) in the presentation of \( b_3(n) \).

The reason for the latter sufficiency is as follows. It has been shown in [Ho1] that if \( a_{31} = a_{30} = a_{20} = 0 \), then \( b_3(n) \equiv 0 \). Below we will refer to \( a_{31}, a_{30}, a_{20} \) as parameters.

In the general case, we have
\[
b_3(n) \sim a_{31}n^{m_1} + a_{30}n^{m_2} + a_{20}n^{m_3},
\]
where the numbers \( m_j \) are some nonnegative integers and the relation itself means that \( b_3(n) \) is equal to the r.h.s. modulo some terms that contain higher powers of parameters. Here \( n^{m_j} \) is the leading power of \( n \) of the corresponding parameter.

To compute these leading terms, we first calculate the corresponding terms in \( 4p_4(n) \).

Set
\[
4p_4(n) = cn^{q_0} + a_{31}n^{q_1} + a_{30}n^{q_2} + a_{20}n^{q_3},
\]
where \( c \in \mathbb{C} \) does not depend on the parameters. (Later we are going to use a similar expansion for \( b_3(n) \) itself as well.)

We will use the formulas (3.12). Applying the operator (3.10) to \( P_n(x) \), we get
\[
\begin{align*}
L(P_n) &= (x^2 + a_{31}x + a_{30})(n)_3 \cdot x^{n-3} + p_1(n)(n-1)_2 \cdot x^{n-4} + \ldots + p_k(n)(n-k)_3 \cdot x^{n-k-3} + \ldots \\
&+ (a_{21}x + a_{20})(n)_2 \cdot x^{n-2} + p_1(n)(n-1)_2 \cdot x^{n-3} + \ldots + p_k(n)(n-k)_2 \cdot x^{n-k-2} + \ldots \\
&+ x(n \cdot x^{n-1} + p_1(n)(n-1) \cdot x^{n-2} + \ldots + p_k(n)(n-k) \cdot x^{n-k-1} + \ldots ) \\
&- n(x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots ).
\end{align*}
\]

Comparing the coefficients at \( x^{n-1} \) in both sides of the latter equality, we obtain
\[
p_1(n) = (n)_3 + (n)_2 \cdot a_{21}.
\]

From (3.12) we obtain
\[
b_0(n) = -\Delta(p_1(n)) = -3(n)_2 - 2n \cdot a_{21}.
\]

We see that both \( p_1(n) \) and \( b_0(n) \) do not contain our parameters. We observe that in order to calculate \( b_3(n) \), we need first to compute \( b_1(n) \) and \( b_2(n) \).

Let us start with \( p_2(n) \). We have
\[
2p_2(n) = (n)_3 \cdot a_{31} + (n)_2 \cdot a_{20} + p_1(n-1)p_1(n).
\]

For later use, notice that
\[
p_2(n) \sim \frac{1}{2} \left[ n^3 \cdot a_{31} + n^2 \cdot a_{20} + n^6 \right].
\]
Using \([3, 12]\), we obtain
\[
b_1(n) = -\Delta(p_2(n)) - b_0(n)p_1(n)
\]
\[
= -\frac{3(n_2b}{2} \cdot a_{31} - n \cdot a_{20} + \frac{1}{2} p_1(n)\Delta^2(p_1(n - 1))
\]
which gives
\[
b_1(n) \sim -\frac{3n^2}{2} \cdot a_{31} - n \cdot a_{20} + 3n^4.
\]

Let us apply a similar procedure to find \(p_3(n)\). From the above expression for \(L(P_n(x))\) we get
\[
3p_3(n) = (n_3 \cdot a_{30} + p_1(n)(n - 2)\cdot a_{31} + p_1(n)(n - 1)\cdot a_{20} + p_1(n - 2)p_2(n)
\]
\[
= \frac{1}{2} p_1(n - 2)[(n_3 \cdot a_{31} + (n_2 \cdot a_{20} + p_1(n - 1)p_1(n)]
\]
\[
+ (n_3 \cdot a_{30} + p_1(n)(n - 1)\cdot a_{31} + p_1(n)(n - 1)\cdot a_{20}.
\]
The latter relation implies that
\[
p_3(n) \sim \frac{n^3}{3} \cdot a_{30} + \frac{n^6}{2} \cdot a_{31} + \frac{n^5}{2} \cdot a_{20} + \frac{1}{3!} n^9.
\]

For \(b_2(n)\), we get
\[
b_2(n) = -\Delta(p_3(n)) - b_0(n)p_2(n) - b_1(n)p_1(n)
\]
which implies that
\[
b_2(n) \sim n^4 \cdot a_{31} - n^2 \cdot a_{30} + \frac{2n^3}{3} \cdot a_{20}.
\]

Further, computing \(p_4(n)\), we get
\[
4p_4(n) = p_2(n)(n - 2)\cdot a_{31} + p_1(n)(n - 1)\cdot a_{30} + p_2(n)(n - 2)\cdot a_{20} + p_1(n - 3) \cdot p_3(n)
\]
which implies that
\[
p_4(n) \sim \frac{n^9}{4} \cdot a_{31} + \frac{n^6}{3} \cdot a_{30} + \frac{n^8}{4} \cdot a_{20} + \frac{1}{4!} n^{12}.
\]

Finally, we obtain
\[
b_3(n) = -\Delta(p_4(n)) - b_0(n)p_3(n) - b_1(n)p_2(n) - b_2(n)p_1(n)
\]
by plugging the expressions for \(p_j(n), b_j(n)\) from the above formulas. This gives
\[
b_3(n) \sim \frac{9n^7}{2} \cdot a_{31} + \frac{9n^4}{2} \cdot a_{30} + 3n^6 \cdot a_{20}.
\]

As the consequence of the latter expansion, we see that if we assume that \(b_3(n) \equiv 0\), then, in particular, we get that \(a_{31} = a_{30} = a_{20} = 0\). \(\square\)
3.2. **Case** \( \deg a_3(x) \leq 1 \iff a_{32} = 0 \). Our goal in this subsection is to prove the following statement.

**Proposition 3.2.** The case \( \deg a_3(x) \leq 1 \) is possible, if and only if \( a_{31} = a_{21} = 0 \) and \( a_{30} \neq 0 \). In this case we obtain the Appell polynomials.

**Proof.** We use the same approach as in Subsection 3.1. The operator \( L \) is of the form

\[
L = (a_{31}x + a_{30})\frac{\partial^3}{\partial^3 x} + (a_{21}x + a_{20})\frac{\partial^2}{\partial^2 x} + (x + a_{10})\frac{\partial}{\partial x}.
\]

By translation of \( x \) we can make \( a_{10} = 0 \) and work with

\[
L = (a_{31}x + a_{30})\frac{\partial^3}{\partial^3 x} + (a_{21}x + a_{20})\frac{\partial^2}{\partial^2 x} + x\frac{\partial}{\partial x}.
\]

(Again such change of variables only results in somewhat shorter formulas but otherwise is not essential.)

Expanding

\[
P_n(x) = x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots,
\]

let us apply \( L \) to the polynomial \( P_n(x) \). Straightforward calculation gives for \( L(P_n) \) the expression

\[
L(P_n) = (a_{31}x + a_{30})(n_3 \cdot x^{n-3} + p_1(n)(n-1)_3 \cdot x^{n-4} + \ldots + p_k(n)(n-k)_3 \cdot x^{n-k-3} + \ldots)
+ (a_{21}x + a_{20})(n_2 \cdot x^{n-2} + p_1(n)(n-1)_2 \cdot x^{n-3} + \ldots + p_k(n)(n-k)_2 \cdot x^{n-k-2} + \ldots)
+ x(nx^{n-1} + p_1(n)(n-1)x^{n-2} + \ldots + p_k(n)(n-k)x^{n-k-1} + \ldots)
= n(x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots).
\]

Comparing the coefficients at \( x^{n-1} \) in both sides, we find

\[
p_1(n) = (n)_2 \cdot a_{21}.
\]

From the expansion of \( \Lambda \) we conclude that

\[
b_0(n) = -2n \cdot a_{21}.
\]

From the coefficients at \( x^{n-2} \) we get

\[
p_2(n) = \frac{1}{2} \left( [(n)_3 \cdot a_{31} + (n)_2 \cdot a_{20} + p_1(n)p_1(n-1)] \right).
\]

Again from the expansion of \( \Lambda \), we obtain

\[
b_1(n) = \frac{-3(n)_2 \cdot a_{31} - 2n \cdot a_{20} + 4(n)_2 \cdot a_{21}^2}{2} = \frac{(n)_2 \cdot (2a_{21}^2 - 3a_{31}) - 2n \cdot a_{20}}{2}.
\]

On the other hand, we can use the formula

\[
ad_A^3 \lambda = 6a_3(\Lambda) = 6(a_{31}\Lambda + a_{30})
\]

following from Lemma 2.2. (Observe that \( \alpha = \beta = 0 \).)

We need only the terms containing \( T^1, T^0 = Id \) and \( T^{-1} \) in the latter equation. They can be computed by hand using the appropriate expressions for \( ad_A^1 \lambda, \, ad_A^2 \lambda \) and \( ad_A^3 \lambda \) which we provide in the next lemma.

**Lemma 3.3.** The expressions for \( ad_A^j \lambda, \, j = 1, 2, 3 \) are given by:

\[
ad_A^1(n) = T - b_1(n)T^{-1} + \ldots
\]

\[
ad_A^2(n) = 2a_{21}T - 2\Delta b_1(n)T^0 + \{-3\Delta b_2(n) + b_1(n)2a_{21}\}T^{-1} + \ldots
\]
(3.18) \[ \text{ad}_\Lambda^2(n) = \{4a_{21}^2 - 2\Delta b_1(n)\}T^1 - 3\Delta^2 b_2(n)T^0 + \{b_1(n)[\Delta^2 b_1(n-1) - 4a_{21}^2] - 6\Delta b_2(n)\}T^{-1} + \ldots \]

where by \ldots we denote all terms containing \( T^j \) for \( j < -1 \).

**Proof.** Direct computation using the equality \( \Delta b_0(n) = -2a_{21} \).

Lemma 3.3 leads to the following system of equations:

\[
\begin{cases}
4a_{21}^2 - 2\Delta^2(b_1(n)) = 6a_{31}; \\
-3\Delta^2(b_2(n)) = 6\gamma b_0(n) + 6a_{30}; \\
b_1(n)6\gamma + 6\Delta b_2(n) = 6a_{30}b_1(n).
\end{cases}
\]  

(3.19)

Observe that the first equation implies the trivial identity

\[ 4a_{21}^2 - 2(2a_{21}^2 - 3a_{31}) = 6a_{31} \]

which additionally confirms the correctness of our calculations.

From the second equation, using (3.14) and the relation \( \gamma = 6a_{31} \), we obtain

\[ -3\Delta^2(b_2(n)) = 6\gamma b_0(n) + 6a_{30} = -12n \cdot a_{21}a_{31} + 6a_{30} \]

from which we see that \( b_2(n) \) is at most cubic in \( n \). Expanding

\[ b_2(n) = A_3 \cdot (n)_3 + A_2 \cdot (n)_2 + \ldots, \]

we obtain

\[ 3A_3 = 2a_{21}a_{31}; \quad A_2 = -a_{30}. \]  

(3.20)

Using the expressions for \( b_0(n) \) and \( b_1(n) \), we transform the third equation into

\[ b_1(n)[\gamma + a_{30}] = \Delta(b_2(n)). \]  

(3.21)

To move further, we apply the obvious fact that the coefficient at \( T^{-5} \) in \( \text{ad}_\Lambda^3 = \gamma \Lambda + \delta I \) vanishes. Indeed, observe that terms containing \( T^{-5} \) come from the lowest terms in \( [A, \text{ad}_\Lambda^2] \) which implies that the operators \( b_2(n)T^{-2} \) and \( \{b_2(n)b_1(n-2) - b_1(n)b_2(n - 1)\}T^{-3} \) commute.

Assume now that both operators are not identically zero. Then they can be expressed as univariate polynomials of one and the same operator which in the case under consideration can be written as \( s(n)T^{-1} \). Thus we obtain

\[
\begin{cases}
b_2(n)T^{-2} = (s(n)T^{-1})^2; \\
\{b_2(n)b_1(n-2) - b_1(n)b_2(n - 1)\}T^{-3} = \ell \cdot (s(n)T^{-1})^3, \quad 0 \neq \ell \in \mathbb{C}.
\end{cases}
\]  

(3.22)

The first equation gives \( b_2(n) = s(n)s(n-1) \). Such relation is only possible if \( b_2(n) \) is a polynomial of degree 2 in \( n \) which implies that the coefficient \( A_3 \) vanishes. Due to the relation \( 3A_3 = 2a_{31}a_{21} \), we get either \( a_{21} = 0 \) or \( a_{31} = 0 \).

Let us first consider the case when \( a_{21} = 0 \). Then we obtain that \( b_0 = 0 \) and

\[ b_1 = \frac{(n)_2 \cdot (-3a_{31}) - 2n \cdot a_{20}}{2}. \]

However if \( a_{31} \neq 0 \), we conclude that \( \gamma + a_{30} \) must vanish since otherwise \( b_2 \) will be of degree 3 in \( n \). But if \( \gamma + a_{30} = 0 \), we get that \( b_2 \) is a constant. The second equation from (3.19) together with the fact that \( b_0 = 0 \) gives that \( a_{30} = 0 \). Using that \( \gamma + a_{30} = 0 \),
we come to the conclusion that $\gamma = 6a_{31} = 0$. This is a contradiction since under such assumptions the order of $L$ drops from 3 to 2.

We see that only the case $a_{31} = 0$ is possible. The third equation from (3.19) gives that $\Delta b_2 = a_{30}b_1$. In this case the formula for $b_1$ gives

$$b_1(n) = (n)_{2} \cdot a_{21}^2 - n \cdot a_{20}$$

which is only possible if $a_{21} = 0$ since $a_{30} \neq 0$. \hfill $\square$

4. Case when $\lambda$ is a quadratic polynomial

This case corresponds to $\deg a_3(x) < 3$, $\deg a_2(x) = 2$, and $\deg a_1(x) \leq 1$. We want to show that in this situation there are no differential operators whose eigenpolynomials satisfy a finite recurrence relation.

**Proposition 4.1.** The case when $\lambda(n)$ is quadratic is impossible, i.e., no linear differential operator $L$ of order 3 satisfying this condition can solve Problem 2.

*Proof.* Our arguments are partially computer-aided due to the complexity of calculations. (The corresponding Mathematica code and its results can be requested from the third author.) The algorithm of what we are doing is presented below.

We are going to compute the coefficients of the 4-term recurrence in terms of the coefficients of the operator $L$. In the case under consideration, we can write the operator $L$ in the form

$$L = (a_{32}x^2 + a_{31}x + a_{30})\partial^3 + (x^2 + a_{21}x + a_{20})\partial^2 + (a_{11}x + a_{10})\partial.$$  

We can also assume that $a_{21} = 0$, which can be achieved by a translation of $x$. (By a slight abuse of notation we use the same letters for the coefficients of $L$.) From the above form of $L$ we get that

$$\lambda(n) = n(n - 1) + \nu n.$$  

As above, introduce

$$P_n(x) = x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots.$$  

We are going to compute the coefficients $p_k(n)$ of $P_n(x)$ by considering $L(P_n(x))$ and using the formulas (3.11) and (3.12). $L(P_n(x))$ satisfies the relation

$$L(P_n) = (a_{32}x^2 + a_{31}x + a_{30})((n)_{3}x^{n-3} + p_1(n)(n - 1)_{3}x^{n-4} + \ldots + p_k(n)(n - k)_{3}x^{n-k-3} + \ldots)$$

$$+ \ (x^2 + a_{20})((n)_{2}x^{n-2} + p_1(n)(n - 1)_{2}x^{n-3} + \ldots + p_k(n)(n - k)_{2}x^{n-k-2} + \ldots)$$

$$+ \ (a_{11}x + a_{10})nx^{n-1} + p_1(n)(n - 1)x^{n-2} + \ldots + p_k(n)(n - k)x^{n-k-1} + \ldots$$

$$= \ (n(n - 1) + \nu n)(x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots).$$

We now equalize the coefficients at the same powers of $n$ in the right-hand and the left-hand sides of the latter equation. For our purposes it would be enough to find expressions for $p_1(n), p_2(n), p_3(n)$, and $p_4(n)$ only. Knowing $p_1(n), p_2(n), p_3(n), p_4(n)$ and using the above formulas, we can express $b_0(n), b_1(n), b_2(n)$ and $b_3(n)$. As before, to obtain a recurrence relation of order at most 4 for the eigenpolynomials, we need to ensure that $b_3(n) \equiv 0$.

We will normalize the above expression for $L$ using the action of the affine group on $x$. First, let us consider the case when $a_{32} \neq 0$. By rescaling we can make $a_{32} = 1$. Then the leading of $b_3(n)$ equals $11184n^{13}$, see (7.25) which implies that $b_3(n)$ can not vanish identically.
If $a_{3,2} = 0$ but $a_{3,1} \neq 0$ by rescaling we can assume that $a_{3,1} = 1$. In this case the leading coefficient of $b_3(n)$ equals $4224n^{11}$ which again implies that $b_3(n)$ can not vanish identically.

Next assume that $a_{3,2} = a_{3,1} = 0$. Let us rescale $x$ to obtain $a_{3,0} = 1$. We still need to find if there exist values of $a_{2,0}$, $a_{1,1}$, $a_{1,0}$ for which $m_9 = m_8 = m_7 = m_6 = 0$, where $m_i$ is the coefficient at $n^i$ in the expression \((7.27)\) for $b_3(n)$.

First, we get that $m_9 = 8(96a_{20}^2 + 288a_{10})$. Assuming that $m_9 = 0$ we obtain $a_{10} = -\frac{a_{20}^2}{3}$. Inserting the latter expression for $a_{10}$ in $b_3(n)$, and using $m_8 = 0$, we get that either $a_{11} = 2$ or $a_{20} = 0$. In the former case, setting $a_{10} = -\frac{a_{20}^2}{3}$ and $a_{11} = 2$ in the expressions for $m_7$ and $m_6$, we get

\[
\begin{cases}
  m_6 = -\frac{64}{3} a_{20}^2 (27 + 4a_{20}^3); \\
  m_7 = \frac{64}{3} a_{20}^2 (9 + 2a_{20}^3).
\end{cases}
\]

Thus, if $a_{20} \neq 0$, then the equations for $m_6 = 0$ and $m_7 = 0$ are obviously incompatible, i.e., have no common solutions.

Now let us consider the case when $a_{32} = a_{31} = a_{21} = a_{20} = 0$. We will show that $b_3 = 0$, which however is not enough to claim that all $b_j = 0$ for $j > 2$. For this reason we are going to compute them up to $j = 5$. We know that in this case also $a_{10} = 0$. Again applying the operator $L$ to the polynomials $P_{n}(x)$, we obtain

\[
L(P_n) = (n)3 \cdot x^{n-3} + p_1(n)(n-1)3 \cdot x^{n-4} \ldots + p_k(n)(n-k)3 \cdot x^{n-k-3} + \ldots \\
+ x^2((n)2 \cdot x^{n-2} + p_1(n)(n-1)2 \cdot x^{n-3} \ldots + p_k(n)(n-k)2 \cdot x^{n-k-2} + \ldots) \\
+ a_{11}x(nx^{n-1} + p_1(n)(n-1)x^{n-2} \ldots + p_k(n)(n-k)x^{n-k-1} + \ldots) \\
= (n(n-1) + a_{11}n)(x^n + p_1(n)x^{n-1} + \ldots + p_k(n)x^{n-k} + \ldots).
\]

Then, for $k > 3$, we find the following formulas for $p_j(n)$:

\[
\begin{cases}
p_1(n)(2 - 2n - a_{11}) = 0; \\
p_2(n)(6 - 4n - 2a_{11}) = 0; \\
p_3(n)(12 - 6n - 3a_{11}) = -(n)3; \\
\vdots \\
p_k(n)(k(k + 1) - 2kn - ka_{11}) = -(n - k + 3)3 \cdot p_{k-3}(n).
\end{cases}
\]

By induction, we see that $p_{3j} \neq 0$ while $p_i = 0$ for $i \neq 3j$. In particular,

\[p_6(n)(42 - 12n - 6a_{11}) = -(n - 3)3 \cdot p_3(n)\]

i.e.,

\[p_6(n) = \frac{(n)_5}{18(2n + a_{11} - 4)(2n + a_{11} - 7)}.
\]
From this we easily compute the several first $b_j$’s getting
\[
\begin{align*}
  b_0 &= 0; \\
  b_1 &= 0; \\
  b_2 &= -\Delta p_3(n) \neq 0; \\
  b_3 &= -\Delta p_4(n) - b_0 p_3 - b_1 p_2 - b_2 p_1 = 0; \\
  b_4 &= -\Delta p_5(n) - b_0 p_4 - b_1 p_3 - b_2 p_2 - b_3 p_1 = 0; \\
  b_5 &= -\Delta p_6(n) - b_0 p_5 - b_1 p_4 - b_2 p_3 - b_3 p_2 - b_4 p_1.
\end{align*}
\]

In the expression for $b_5$ there are two non-vanishing terms: $\Delta p_6(n)$ and $b_2 p_3$. Both terms are of order $n^3$. We only need to show that their sum is not identically zero. Straight-forward calculations give
\[
b_{2p3} = \frac{1}{9} \frac{(n)_3 \cdot (n)_2 \cdot (4n - a_{11})}{(2n + a_{11} - 2)(2n + a_{11} - 4)^2}
\]
and
\[
\Delta p_6(n) = \frac{(n)_4 \cdot R(n)}{(2n + a_{11} - 2)(2n + a_{11} - 4)(2n + a_{11} - 5)(2n + a_{11} - 7)},
\]
where $R(n)$ is a polynomial of degree at most 2, whose explicit expression is irrelevant for our purposes. We see that the sum of $b_{2p3}$ and $\Delta p_6(n)$ cannot vanish since they contain different factors.

In fact, the explicit expression for $b_5$ can be obtained even without computer. Our symbolic computations give
\[
(4.23) \quad b_5(n) = \frac{n(n-1)(n-2)(n-3)(n-4)(3a_{1,1} + 5n - 16)}{9(a_{1,1} + 2n - 2)(a_{1,1} + 2n - 7)(a_{1,1} + 2n - 5)(a_{1,1} + 2n - 4)}. \quad \square
\]

5. Case when $\lambda$ is a cubic polynomial

In case when $\lambda(n)$ is a cubic polynomial in $n$ using translation and scaling, we can without loss of generality assume that
\[
\lambda(n) = n(n - 1)(n - 2) + \nu n(n - 1) + \mu n.
\]
As before, one can easily observe that $\nu = a_{22}$ and $\mu = a_{11}$.

Using this ansatz, we first formulate some important preliminary statements about the coefficients of $\Lambda$ and $L$.

**Lemma 5.1.** The coefficient $\alpha$ in (2.9) equals 6.

**Proof.** To prove this, we compute the term of the highest degree in $T$ in the two expressions for $ad^3_{\lambda} \lambda(n)$ which we presented earlier. On one hand, from (2.9) we obtain that $ad^3_{\lambda} \lambda(n) = \alpha T^3 + \ldots$. Let us compute the term containing $T^3$ in $ad^3_{\lambda} \lambda(n)$. Simple computation shows that
\[
\begin{align*}
  ad^1_{\lambda}(\lambda) &= \Delta(\lambda(n)) \cdot T + \ldots; \\
  ad^2_{\lambda}(\lambda) &= \Delta^2(\lambda(n)) \cdot T^2 + \ldots; \\
  ad^3_{\lambda}(\lambda) &= \Delta^3(\lambda(n)) \cdot T^3 + \ldots.
\end{align*}
\]
Since $\Delta^3(\lambda(n)) = 6$, the result follows. \quad \square
**Lemma 5.2.** The coefficient $b_2(n)$ in the expansion (2.8) of $\Lambda$ vanishes identically.

*Proof.* Suppose that $b_2(n) \neq 0$. Then again we can use the above approach and compute the term in $\text{ad}_A^3 \lambda(n)$ containing $T^{-6}$ in two different ways. On one hand, if taken from $6\Lambda^3$, this term is given by

$$\alpha \cdot b_2(n)b_2(n-2)b_2(n-4)T^{-6} = 6b_2(n)b_2(n-2)b_2(n-4)T^{-6}.$$

On the other hand, the computation of the same term from $\text{ad}_A^3 \lambda(n)$ shows that it coincides with

$$b_2(n)b_2(n-2)b_2(n-4)(-\lambda(n-6)+3\lambda(n-4)-3\lambda(n-2)+\lambda(n))T^{-6}.$$

However, $-\lambda(n-6)+3\lambda(n-4)-3\lambda(n-2)+\lambda(n) = 48$ which is only possible if $b_2(n) \equiv 0$. □

**Lemma 5.3.** Under our current assumptions, the coefficient $b_1(n)$ in the expansion (2.8) of $\Lambda$ vanishes identically.

*Proof.* The argument is similar to the above computation of $b_2(n)$. Assume that $b_1(n) \neq 0$. Notice that by Lemma 5.2

$$\Lambda = T + b_0(n)T^0 + b_1(n)T^{-1}.$$  

The coefficient of $T^{-3}$ in the expression for $6\Lambda^3$ is given by $6b_1(n)b_1(n-1)b_1(n-2)$.

From the expression for $\text{ad}_A^3 \lambda(n)$, we find that the same term coincides with

$$b_1(n)b_1(n-1)b_1(n-2)(\lambda(n-3)-3\lambda(n-2)+3\lambda(n-1)-\lambda(n)) = -6b_1(n)b_1(n-1)b_1(n-2).$$

Again we get a contradiction. □

**Lemma 5.4.** The coefficient $b_0(n)$ in the expansion (2.8) of $\Lambda$ is an arbitrary constant.

*Proof.* By Lemmas 5.2 and 5.3 $\Lambda = T + b_0(n)$ which makes the computation of both $\Lambda^j$ and $\text{ad}_A^j \lambda(n)$ very easy. Moreover, we only need the coefficient at $T^0 = \text{Id}$. We have

$$\Lambda^j = \ldots + b_0^j(n)T^0, \quad j \in \mathbb{N}.$$ 

For $\text{ad}_A^j \lambda(n)$, $j \in \mathbb{N}$, it is obvious that it does not contain $T^0$. Comparing the coefficients at $T^0$, we obtain

$$\alpha b_0^3(n) + \beta b_0^2(n) + \gamma b_0(n) + \delta = 0.$$ 

However this is only possible if $b_0(n)$ is a constant. □

Denoting this above constant by $p$ we get the following claim.

**Proposition 5.5.** The only linear differential operators of order 3 solving Problem 2 which have polynomial eigenfunctions with $\lambda(n) = n(n-1)(n-2)+\nu n(n-1)+\mu n$ are of the form

$$L = 6(x-p)^3\partial_x^3 + \nu(x-p)^2\partial_x^2 + \mu(x-p)\partial.$$

Hence any such operator $L$ is reducible.

*Proof.* By the last lemma we have

$$xp_n(x) = p_{n+1}(x) + PP_n(x).$$ 

Hence $P_{n+1} = (x-p)^n$. The corresponding differential operator of order 3 is as announced. □
6. Final Remarks

I. To the best of our knowledge, Conjectures \[1.7\] and \[1.10\] containing a complete conjectural description of the set of all solutions to the algebraic version of the classical Bochner-Krall Problem \[1\] and its generalisation Problem \[2\] have never previously appeared in the literature although Bochner’s original paper is almost 90 years old and there have been many attempts to solve the classical problem. However the way our conjectures are formulated, it is difficult to verify them for operators of somewhat high order unless one finds an alternative description. On the other hand, the case of operators of order 4 seems to be doable using the methods of the present paper.

II. One additional restriction of the validity of the results obtained in the present paper comes from the fact that we are using the assumptions of Conjecture \[1.9\]. To actually claim that we have solve Problem \[2\] for all operators of order 3, we need for settle this conjecture at least in this case. We plan to return to this project in a future publication.

III. Our proof of the main Theorem \[1.14\] is based on consideration of a large number of special subcases and is partially computer-aided. To be able to approach our general conjectures such method is not very effective and a more conceptual understanding of our proof is needed.

7. Appendix. Computer-aided formulas relevant for §4

\[ L_3 = (a_{32}x^2 + a_{31}x + a_{30})\partial^3 + (x^2 + a_{21}x + a_{20})\partial^2 + (a_{11}x + a_{10})\partial. \]

\[ (7.24) \quad xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{n} b_j(n)P_{n-j}(x). \]

Putting \(a_{3,2} = 1\) we get

\[ (7.25) \quad b_3(n) = \frac{(11184n^{13} + O(n^{12}))}{24(a_{1,1} + 2n - 5)(a_{1,1} + 2n - 4)(a_{1,1} + 2n - 3)^2(a_{1,1} + 2n - 2)^4(a_{1,1} + 2n - 1)}. \]

With \(a_{3,2} = 0\) and \(a_{3,1} = 1\) we get

\[ (7.26) \quad b_3(n) = \frac{(424n^{11} + O(n^{10}))}{24(a_{1,1} + 2n - 5)(a_{1,1} + 2n - 4)(a_{1,1} + 2n - 3)^2(a_{1,1} + 2n - 2)^4(a_{1,1} + 2n - 1)}. \]

and with \(a_{3,2} = 0, a_{3,1} = 0,\) and \(a_{3,0} = 1\) (recall that \(a_{21} = 0\)) we arrive at

\[ (7.27) \quad b_3(n) = \frac{1}{24(a_{1,1} + 2n - 5)(a_{1,1} + 2n - 4)(a_{1,1} + 2n - 3)^2(a_{1,1} + 2n - 2)^4(a_{1,1} + 2n - 1)} \]
\[ \left( 8n^9 (96a_{2,0}^2 + 288a_{1,0}) + 8n^8 (144 (3a_{1,1} - 8) a_{2,0}^2 - 3200a_{1,0} + 1168a_{1,0}a_{1,1}) -4n^7 (-96a_{2,0}a_{1,0}^2 + 2 (-2032a_{1,1}^2 + 11200a_{1,1} - 15064) a_{1,0} -96 (17a_{1,1}^2 - 92a_{1,1} + 122) a_{2,0}^2) -4n^6 (-48 (7a_{1,1} - 18)a_{2,0}a_{1,0}^2 + 2 (-1960a_{1,1}^3 + 16416a_{1,1}^2 - 44596a_{1,1} + 39280) a_{1,0} -48 (35a_{1,1}^3 - 290a_{1,1}^2 + 782a_{1,1} - 688) a_{2,0}^2 + O(n^5) \right). \]
(In the above formulas we use $n$ instead of $n$ to make it easier for a reader to follow which powers of the variable $n$ these formulas contain.)

References


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