LINEAR DIFFERENTIAL EQUATIONS AND REAL FLAG MANIFOLDS

B. Z. Shapiro

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We consider a linear homogeneous n-th-order differential equation defined on a segment I of the time axis t,

$$L_n[x] = x^{(n)} + a_1(t)\,x^{(n-1)} + \ldots + a_n(t)\,x = 0, \tag{1}$$

where $a_i(t) \in C^\infty[I]$.

1. Definition. (1) is said to be nonoscillational on I if any nonzero solution of it has n roots on I counting multiplicities. Otherwise the equation is said to be oscillational.

2. Definition. Let V be the n-dimensional space of solutions of (1) and F be the space of complete flags in V. By the flag curve of (1) we mean the map $f: I \to F$ which associates with a time $\tau \in I$ the complete flag in V whose i-dimensional subspace consists of solutions of (1) having finite multiplicity not less than $n - i$ at time τ.

3. Definition. A collection of flags of F is said to be transverse if the codimension of the intersection of any subset of the linear spaces belonging to different flags of the collection is equal to the minimum of n and the sum of their codimensions.

4. LEMMA. (1) is nonoscillational on I if and only if any collection of pairwise distinct points of its flag curve is transverse.

5. Definition. By the train T_α of a flag α is meant the set of all flags not transverse to it.

For each complete flag in all Grassmanians, for the spaces of complete and incomplete flags one can construct Schubert cellular decompositions whose cells are defined by conditions of constancy of the dimensions of the intersection with the subspaces of the given complete flag (cf. [1]). In the space of complete flags the cells of the Schubert decomposition are indexed by permutations. The train T_α of the flag α is the union of all cells of positive codimension of the Schubert decomposition constructed from it. T_α is a reducible hypersurface in F consisting of $n - 1$ irreducible components $\Delta_1, \ldots, \Delta_{n-1}$, where Δ_i consists of flags whose $(n - i)$-dimensional subspace is not transverse to the i-dimensional subspace of the given flag. Δ_i and Δ_{n-i} are diffeomorphic by duality.

6. THEOREM. The following three conditions are equivalent:

a) (1) is oscillational on I;

b) there exists a moment of time τ different from the initial one such that the flag at the initial moment and the flag at time τ on the flag curve of (1) are not transverse;

c) the flag curve of (1) intersects the train of any flag.

From the flag α we construct the Schubert decompositions of the Grassmanians $G_{k,n}$.

7. LEMMA. The intersection multiplicity $\#_k$ of the germ of the flag curve of (1) with Δ_k is equal to the codimension of the cell of the decomposition in $G_{k,n}$ in which the k-dimensional plane of the flag of the germ lies at the moment of nontransversality. The intersection multiplicity of the germ of the flag curve with the train T_α depends only on the cell in T_α in which the point of intersection lies and can be calculated from the permutation $(i_1, \ldots, i_n)$ corresponding to this cell as follows:

$$\#_k = \max\left(0, \sum_{j=1}^{k} (i_j - j)\right).$$

8. THEOREM. For a nonoscillational equation the total intersection multiplicity of the

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flag curve with Δ_k ($k = \overline{1, n-1}$) does not exceed $\dim G_{n,k} = k(n-k)$. The total intersection multiplicity of the flag curve with the train of any flag does not exceed $(n^3 - n)/6$.

9. COROLLARY (generalized Sturm interlacing theorem). Let the flag curve of (1) intersect the k-th component of the train of a flag on some time segment with total multiplicity $k(n-k)$; then on this time segment the flag curve intersects the train of any flag.

10. THEOREM. (1) belongs to the boundary of the domain of nonoscillational equations if and only if the ends of its flag curve are nontransverse.

We denote by π the map from the space of equations to the space of complete flags F which associates with each equation the right end of its flag curve.

11. LEMMA. For any equation (1) there exists an $n(n-1)/2$-parameter germ of deformation of it in the space of all linear equations which projects nondegenerately under the map π to a neighborbood of the right end of the flag curve of the equation considered.

12. COROLLARY. The list of singularities of the boundary of the domain of nonoscillational equations which occur in typical families of equations with a certain number of parameters coincides with the list of typical sections of a train with the same number of parameters.

By a typical family is meant a family which belongs to an open everywhere dense set in the space of families and by a typical section is meant a section which is transverse to the Schubert stratification of the train.

13. THEOREM. a) The singularities of typical sections of a train do not vary (up to a diffeomorphism of a neighborhood of the point of the section) along cells of the Schubert decomposition of the train;

b) the list of typical k-parameter sections of a train is finite;

c) the list of singularities of typical k-parameter sections of a train increases with increasing dimension of the space and stabilizes at dimension 2k.

14. LEMMA. In typical sections of trains of any dimension up to diffeomorphism the following singularities arise:

for 2 parameters $xy = 0$;

for 3 parameters $xyz = 0$, $z(z - xy) = 0$;

for 4 parameters $xyzu = 0$, $xu - zy = 0$, $zu(z - xy) = 0$.

15. Conjecture. Let the germ of the flag curve of (1) intersect a train at a certain moment, then this germ goes from one local component of the complement of the train to another.

16. Question. Determine the number of connected components of the complement in F of the trains of two transverse complete flags.

An extensive literature is devoted to criteria for differential equations to be nonoscillational (cf., e.g., [2, 3]). The problem of studying the singularities of the boundary of the domain of nonoscillational equations was posed by V. I. Arnol'd in [4] (cf. also [5]). The author thanks him for posing the problem and constant interest in the work.

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