

JUCYS–MURPHY SUBALGEBRAS OF REDUCED KOHNO–DRINFELD ALGEBRAS: THE RANK-FOUR CASE

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ABSTRACT. We study commutative subalgebras generated by Jucys–Murphy-type elements. As a classical model, we identify the relation ideal for the Young–Jucys–Murphy elements with the vanishing ideal of tableau content vectors and give a recursive presentation by one-step branching annihilators in Young’s lattice. Our main results concern the reduced Kohno–Drinfeld algebra $4T_n^0$. For the commutative subalgebra $\mathcal{D}_n^0 = \mathbb{Q}[d_2, \dots, d_n]$, we prove

$$\mathcal{D}_3^0 \simeq \mathbb{Q}[u, v]/(u^2, (u+v)v^3).$$

In rank four we determine the complete defining ideal by exact characteristic-zero linear algebra and Gröbner-basis computation. We obtain

$$\text{Hilb}(\mathcal{D}_4^0, t) = 1 + 3t + 5t^2 + 7t^3 + 8(t^4 + t^5 + t^6 + t^7 + t^8) + 4t^9 + t^{10},$$

so $\dim \mathcal{D}_4^0 = 61$. The defining ideal has six minimal homogeneous generators, of degrees 2, 4, 9, 9, 9, 9; consequently $\mathcal{D}_4^0$ is neither a complete intersection nor Artinian Gorenstein. Moreover,

$$(d_2 + d_3 + d_4)d_4^5 \neq 0.$$

Thus the rank-four calculation gives characteristic-zero counterexamples to parts (2) and (3) of Theorem 4.101 in A. N. Kirillov, *SIGMA* **12** (2016), 002. All computer-assisted statements are accompanied by exact rational certificates and a reproducible verification script.

1. INTRODUCTION

The Young–Jucys–Murphy elements in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ are

$$X_1 = 0, \quad X_k = \sum_{i=1}^{k-1} (i \ k), \quad 2 \leq k \leq n.$$

They commute and generate the Gelfand–Tsetlin subalgebra associated with the multiplicity-free chain

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \dots \subset \mathfrak{S}_n.$$

In Young’s seminormal basis, X_k acts on the vector indexed by a standard tableau T by the content $c_k(T)$ of the box containing k ; see, for example, [1, 3, 4]. Consequently the classical relation problem is a finite interpolation problem on tableau content vectors.

We use this classical situation only as a model for the relation problem. The full relation ideal is the radical ideal of the finite set of content paths, and we give a recursive generating set consisting of local one-step branching annihilators in Young’s lattice. These classical statements are consequences of standard seminormal theory; the new results of the paper begin with the reduced Kohno–Drinfeld algebra. Let $4T_n$ be the associative quadratic algebra generated by $y_{ij} = y_{ji}$, $1 \leq i < j \leq n$,

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with the usual locality and infinitesimal braid relations, and let $4T_n^0$ be its quotient by $y_{ij}^2 = 0$. The elements

$$d_j = \sum_{i < j} y_{ij}, \quad 2 \leq j \leq n,$$

commute. We write

$$\mathcal{D}_n^0 = \mathbb{Q}[d_2, \dots, d_n] \subset 4T_n^0.$$

In [2, Theorem 4.101(2),(3)] it was asserted that

$$(d_2 + \dots + d_j)d_j^{2j-3} = 0, \quad 2 \leq j \leq n, \quad (1.1)$$

and that the subalgebra generated by $d_2, \dots, d_n$ over $\mathbb{Z}$ has Hilbert polynomial

$$\prod_{j=2}^n (1 + t + \dots + t^{2j-3}). \quad (1.2)$$

The latter is the product $\prod_{r=1}^{n-1} [2r]$ in the notation of [2]. The rank-three case is correct: we prove the complete presentation

$$\mathcal{D}_3^0 \simeq \mathbb{Q}[u, v] / (u^2, (u + v)v^3).$$

The rank-four case, however, is different. Our main theorem is the following; the explicit degree-nine relations are given in Theorem 6.1.

Theorem 1.1 (Rank-four correction, summary). *Over every field of characteristic zero,*

$$\text{Hilb}(\mathcal{D}_4^0, t) = 1 + 3t + 5t^2 + 7t^3 + 8t^4 + 8t^5 + 8t^6 + 8t^7 + 8t^8 + 4t^9 + t^{10}.$$

Hence $\dim \mathcal{D}_4^0 = 61$, and

$$(d_2 + d_3 + d_4)d_4^5 \neq 0.$$

The defining ideal of $\mathcal{D}_4^0$ has one minimal generator in degree 2, one in degree 4, and four in degree 9. In particular, $\mathcal{D}_4^0$ is not a complete intersection.

The product in (1.2) would instead give

$$1 + 3t + 5t^2 + 7t^3 + 8t^4 + 8t^5 + 7t^6 + 5t^7 + 3t^8 + t^9$$

in rank four, of total dimension 48. Thus both (1.1) for $j = 4$ and (1.2) fail after extension to $\mathbb{Q}$, and hence cannot hold in the integral form stated in [2]. This corrects parts (2) and (3) of [2, Theorem 4.101]. Part (1), the commutativity of the d_j , is unaffected. We do not address part (4) of that theorem.

The rank-four calculation is computer-assisted but exact. We use only rational row reduction and a general recursive description of the homogeneous pieces of a quadratic algebra. As an independent consistency check, the same calculation reproduces the published Hilbert series

$$\text{Hilb}(4T_4^0, t) = 1 + 6t + 19t^2 + 42t^3 + 70t^4 + 90t^5 + 87t^6 + 57t^7 + 23t^8 + 6t^9 + t^{10}$$

from [2, Example 4.88]. The recurrence is proved in Theorem 4.2, and the complete verification script accompanying the paper uses exact arithmetic throughout.

2. THE CLASSICAL MODEL: THE GELFAND–TSETLIN RELATION IDEAL

Let $\mathcal{G}_n \subset \mathbb{C}[\mathfrak{S}_n]$ be the subalgebra generated by $X_1, \dots, X_n$, and consider

$$\phi_n : \mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathcal{G}_n, \quad x_k \longmapsto X_k.$$

For a standard tableau T of size n , write

$$c(T) = (c_1(T), \dots, c_n(T)), \quad c_k(T) = \text{col}_T(k) - \text{row}_T(k),$$

and let

$$\mathcal{C}_n = \{c(T) : T \in \text{SYT}(n)\} \subset \mathbb{C}^n,$$

where $\text{SYT}(n)$ denotes the disjoint union of the standard tableaux of all shapes of size n .

Theorem 2.1 (Classical relation ideal). *One has*

$$\ker \phi_n = I(\mathcal{C}_n) = \bigcap_{T \in \text{SYT}(n)} (x_1 - c_1(T), \dots, x_n - c_n(T)).$$

In particular, $\ker \phi_n$ is radical and

$$\mathcal{G}_n \simeq \text{Fun}(\text{SYT}(n), \mathbb{C}).$$

Consequently

$$\dim_{\mathbb{C}} \mathcal{G}_n = |\text{SYT}(n)| = \sum_{\lambda \vdash n} f^\lambda,$$

which is the number of involutions in $\mathfrak{S}_n$.

Proof. Young’s seminormal basis simultaneously diagonalizes $X_1, \dots, X_n$, and the joint eigenvalue on the basis vector indexed by T is $c(T)$. Distinct standard tableaux have distinct content paths: if two tableaux first differ at step r , their boxes added at step r are distinct addable boxes of the same Young diagram, and distinct addable boxes have distinct contents. Therefore the joint eigenspaces are indexed by $\text{SYT}(n)$, and the image of ϕ_n is the full algebra of diagonal functions on this finite set. Its kernel is exactly the vanishing ideal of the set of joint eigenvalues. The displayed intersection and radicality follow from the Chinese remainder theorem. $\square$

Corollary 2.2. *For $1 \leq k \leq n$, put*

$$C_k = \{c_k(T) : T \in \text{SYT}(n)\}.$$

Then

$$\prod_{a \in C_k} (X_k - a) = 0.$$

The set C_k is already obtained from tableaux of size k . These univariate identities do not in general generate the full ideal: the missing equations are the branching constraints relating consecutive levels.

3. RECURSIVE CONTENT-PATH PRESENTATION

For a standard tableau T of size r , let $\text{Add}(T)$ be the addable boxes of its shape and put

$$A(T) = \{\text{col}(b) - \text{row}(b) : b \in \text{Add}(T)\}.$$

The elements of $A(T)$ are pairwise distinct.

We define polynomials e_T recursively. Put $e_\varnothing = 1$. If T has size $r \geq 1$, let $\bar{T}$ be obtained by deleting the box containing r , and set

$$e_T = e_{\bar{T}} \prod_{\substack{U: \bar{U} = \bar{T} \\ U \neq T}} \frac{x_r - c_r(U)}{c_r(T) - c_r(U)}. \quad (3.1)$$

The denominators are nonzero because the possible new boxes are distinct addable boxes.

Let $J_n \subset \mathbb{C}[x_1, \dots, x_n]$ be generated by x_1 and, for every $0 \leq r < n$ and every $T \in \text{SYT}(r)$, by

$$e_T \prod_{a \in A(T)} (x_{r+1} - a). \quad (3.2)$$

For $r = 0$, the factor x_1 is already listed separately.

Theorem 3.1 (Recursive branching presentation). *For every n ,*

$$J_n = I(\mathcal{C}_n) = \ker \phi_n.$$

Equivalently, all polynomial relations among the classical Jucys–Murphy elements are generated by the local one-step branching annihilators (3.2).

Proof. We prove a stronger statement by induction on r . Let J_r denote the relations up to level r , and put

$$A_r = \mathbb{C}[x_1, \dots, x_r] / J_r.$$

We claim that

$$A_r \simeq \prod_{T \in \text{SYT}(r)} \mathbb{C}, \quad (3.3)$$

and that the classes of the polynomials e_T are the coordinate idempotents.

For $r = 1$, the assertion is immediate from $x_1 = 0$. Assume it for r . Under (3.3), adjoining x_{r+1} gives

$$A_r[x_{r+1}] \simeq \prod_{T \in \text{SYT}(r)} \mathbb{C}[x_{r+1}].$$

The relation (3.2) belonging to T affects only the T -factor and imposes

$$q_T(x_{r+1}) = 0, \quad q_T(z) = \prod_{a \in A(T)} (z - a).$$

Hence

$$A_{r+1} \simeq \prod_{T \in \text{SYT}(r)} \mathbb{C}[z] / (q_T(z)).$$

The roots of q_T are distinct, so the Chinese remainder theorem gives

$$\mathbb{C}[z] / (q_T(z)) \simeq \prod_{a \in A(T)} \mathbb{C}.$$

These factors are precisely the one-box extensions of T . Therefore

$$A_{r+1} \simeq \prod_{U \in \text{SYT}(r+1)} \mathbb{C},$$

and the Lagrange formula (3.1) gives the corresponding coordinate idempotents. This proves the induction. At $r = n$, the quotient by J_n is therefore the coordinate algebra of exactly the set $\mathcal{C}_n$, and Theorem 2.1 gives $J_n = I(\mathcal{C}_n)$. $\square$

Remark 3.2. The proof shows more than a dimension count: at every level the quotient is reduced, and the recursive relations split it canonically into the one-dimensional factors indexed by admissible paths in Young’s lattice.

4. REDUCED KOHNO–DRINFELD ALGEBRAS

We first work over $\mathbb{Q}$. Let $4T_n$ be the associative algebra generated by $y_{ij} = y_{ji}$, $1 \leq i < j \leq n$, with relations

$$[y_{ij}, y_{kl}] = 0 \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \quad (4.1)$$

$$[y_{ij}, y_{ik} + y_{jk}] = 0, \quad \text{if } i, j, k \text{ are distinct.} \quad (4.2)$$

Equivalently, for $i < j < k$ one may use the two relations

$$[y_{ij}, y_{ik} + y_{jk}] = 0, \quad [y_{ij} + y_{ik}, y_{jk}] = 0.$$

Let $4T_n^0$ be the quotient by

$$y_{ij}^2 = 0 \quad (i < j).$$

This is the algebra denoted $4T_n^0$ in [2, Definitions 4.80–4.81].

Define

$$d_j = \sum_{i < j} y_{ij}, \quad 2 \leq j \leq n, \quad \mathcal{D}_n^0 = \mathbb{Q}[d_2, \dots, d_n] \subset 4T_n^0.$$

We use the same notation after extension of scalars to $\mathbb{C}$.

Proposition 4.1. *The elements $d_2, \dots, d_n$ commute pairwise. Moreover,*

$$C_n = d_2 + \dots + d_n = \sum_{i < j} y_{ij}$$

is central in $4T_n$, hence also in $4T_n^0$.

Proof. Fix $i < j$. In $[d_i, d_j]$, terms involving four distinct indices vanish by (4.1). For each $a < i$, the remaining two terms combine as

$$[y_{ai}, y_{aj}] + [y_{ai}, y_{ij}] = [y_{ai}, y_{aj} + y_{ij}] = 0$$

by (4.2). Thus $[d_i, d_j] = 0$.

For centrality, fix y_{ab} . All terms of C_n disjoint from $\{a, b\}$ commute with y_{ab} . For every $k \notin \{a, b\}$, the two remaining terms involving k occur as $y_{ak} + y_{bk}$, and their commutator with y_{ab} vanishes by (4.2). Hence $[y_{ab}, C_n] = 0$. $\square$

4.1. An exact degree recursion for quadratic algebras. The following elementary recurrence is the basis of our computations. It avoids constructing all words in high degree.

Proposition 4.2 (Quadratic degree recursion). *Let V be a finite-dimensional vector space, $R \subset V \otimes V$, and*

$$A = T(V)/(R) = \bigoplus_{m \geq 0} A_m.$$

For $m \geq 2$ there is a natural exact presentation

$$A_m \simeq \frac{V \otimes A_{m-1}}{\text{im}(R \otimes A_{m-2} \longrightarrow V \otimes A_{m-1})}, \quad (4.3)$$

where, for $r = \sum c_{ij} v_i \otimes v_j$, the map sends

$$r \otimes a \longmapsto \sum c_{ij} v_i \otimes (v_j a).$$

Thus, once bases of A_{m-1} and A_{m-2} and the left-multiplication matrices by the generators are known, A_m is obtained by one finite row reduction.

Proof. Let $I = (R) \subset T(V)$. In degree m ,

$$I_m = \sum_{\ell=0}^{m-2} V^{\otimes \ell} \otimes R \otimes V^{\otimes (m-2-\ell)}.$$

The sum of the terms with $\ell \geq 1$ is exactly $V \otimes I_{m-1}$. Passing first to

$$V^{\otimes m} / (V \otimes I_{m-1}) \simeq V \otimes A_{m-1},$$

the only remaining relations are the images of $R \otimes V^{\otimes (m-2)}$. This map factors through $R \otimes A_{m-2}$, because $R \otimes I_{m-2} \subset V \otimes I_{m-1}$. This gives (4.3). $\square$

Remark 4.3. For $4T_4^0$, V has dimension 6. We use the six square relations, the three locality relations, and the two infinitesimal braid relations appearing in Definition 4.80 for each of the four triples, for a total of 17 quadratic relation vectors. The third commonly written triple commutator is their linear consequence; including it redundantly gives the same quotient matrices and the same normal form. All row reductions below are over $\mathbb{Q}$.

5. THE RANK-THREE ALGEBRA

Put

$$a = y_{12}, \quad b = y_{13}, \quad c = y_{23}, \quad u = d_2 = a, \quad v = d_3 = b + c.$$

Lemma 5.1. *The homogeneous dimensions of $4T_3^0$ are*

$$(1, 3, 4, 3, 1)$$

in degrees $0, \dots, 4$, and the algebra vanishes in degree 5. In the subalgebra generated by u, v , the elements

$$1; \quad u, v; \quad uv, v^2; \quad uv^2, v^3; \quad uv^3$$

are linearly independent degree by degree, and

$$v^4 + uv^3 = 0.$$

Proof. Apply Theorem 4.2 to the three generators a, b, c with their square and infinitesimal braid relations. Exact row reduction gives the dimensions

$$1, 3, 4, 3, 1, 0.$$

Reducing the displayed commutative monomials against the same quotient bases gives ranks 1, 2, 2, 2, 1 in degrees 0, 1, 2, 3, 4, and the vector of $v^4 + uv^3$ reduces to zero. This is also verified by the ancillary exact-arithmetic script. $\square$

Theorem 5.2 (Complete rank-three presentation). *Over every field of characteristic zero,*

$$\mathcal{D}_3^0 \simeq \mathbb{Q}[u, v] / (u^2, (u + v)v^3),$$

and hence, after extension of scalars,

$$\mathcal{D}_3^0 \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}[u, v] / (u^2, (u + v)v^3).$$

Its Hilbert series is

$$(1 + t)(1 + t + t^2 + t^3).$$

Proof. The relation $u^2 = 0$ is $y_{12}^2 = 0$, and Theorem 5.1 gives $(u + v)v^3 = 0$. Hence there is a surjection

$$\mathbb{Q}[u, v] / (u^2, (u + v)v^3) \twoheadrightarrow \mathcal{D}_3^0.$$

The source has the eight-element basis

$$1, v, v^2, v^3, u, uv, uv^2, uv^3.$$

Their images are linearly independent by Theorem 5.1; therefore the surjection is an isomorphism. $\square$

6. THE COMPLETE RANK-FOUR PRESENTATION

Write

$$u = d_2 = y_{12}, \quad v = d_3 = y_{13} + y_{23}, \quad w = d_4 = y_{14} + y_{24} + y_{34}.$$

The rank-three relations survive in rank four, so

$$u^2 = 0, \quad (u + v)v^3 = 0. \quad (6.1)$$

For the degree-nine relations, introduce the column vector

$$\mathbf{m}_9 = \begin{pmatrix} w^9 \\ vw^8 \\ v^2w^7 \\ v^3w^6 \\ uw^8 \\ uvw^7 \\ uv^2w^6 \\ uv^3w^5 \end{pmatrix} \quad (6.2)$$

and the integer matrix

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 2 & 2 & 0 & -2 \\ 4 & -4 & -17 & -23 & 13 & 15 & -19 & 0 \\ 2 & 3 & 9 & 6 & 3 & 18 & 18 & 24 \\ 10 & 12 & 24 & 10 & -5 & -1 & -20 & 24 \end{pmatrix}. \quad (6.3)$$

Theorem 6.1 (Complete rank-four presentation). *Over $\mathbb{Q}$, the defining ideal of*

$$\mathbb{Q}[u, v, w] \longrightarrow \mathcal{D}_4^0, \quad (u, v, w) \longmapsto (d_2, d_3, d_4),$$

is generated by the two relations (6.1) and the four entries of

$$M\mathbf{m}_9 = 0.$$

The same presentation holds after extension of scalars to every characteristic-zero field.

Proof. We give the exact calculation in enough detail to make the completeness claim checkable.

First apply Theorem 4.2 to the six generators of $4T_4^0$. The resulting homogeneous dimensions are

$$1, 6, 19, 42, 70, 90, 87, 57, 23, 6, 1, 0 \quad (6.4)$$

in degrees $0, 1, \dots, 11$. The nonzero part agrees exactly with [2, Example 4.88].

Using the same quotient bases and multiplication matrices, we evaluate every commutative monomial $u^i v^j w^k$ of a fixed total degree and take its rank. This gives

$$1, 3, 5, 7, 8, 8, 8, 8, 8, 4, 1, 0 \quad (6.5)$$

in degrees $0, 1, \dots, 11$. Modulo (6.1), there are eight independent monomials in every degree from 4 onward before additional relations are imposed. The rank computation shows that the natural map

$$\mathbb{Q}[u, v, w]/(u^2, (u + v)v^3) \longrightarrow \mathcal{D}_4^0$$

is an isomorphism on every homogeneous component through degree 8. In degree 9, the eight monomials in (6.2) have rank 4, and their exact kernel is the row space of the rank-four matrix M in (6.3). Thus all four displayed degree-nine relations hold in $\mathcal{D}_4^0$ and are independent modulo the lower-degree ideal.

Let J be the ideal generated by (6.1) and $M\mathbf{m}_9$. Exact Gröbner-basis computation in graded reverse lexicographic order $u > v > w$ gives

$$\text{in}(J) = (u^2, uv^3, v^5, uvw^7, v^3w^6, uv^2w^6, v^4w^5, uw^9, vw^9, v^2w^8, w^{11}). \quad (6.6)$$

Counting standard monomials of this monomial ideal gives precisely the sequence (6.5). For completeness, an explicit Gröbner basis with these leading monomials is recorded in Section A; thus the Hilbert-series step can be checked without relying on a software-specific internal representation. Hence

$$\text{Hilb}(\mathbb{Q}[u, v, w]/J, t) = \text{Hilb}(\mathcal{D}_4^0, t).$$

The natural map $\mathbb{Q}[u, v, w]/J \rightarrow \mathcal{D}_4^0$ is graded and surjective; equality of every homogeneous dimension makes it an isomorphism. All calculations use rational arithmetic, so extension of scalars proves the characteristic-zero statement. $\square$

Corollary 6.2 (Hilbert series and dimension). *One has*

$$\text{Hilb}(\mathcal{D}_4^0, t) = 1 + 3t + 5t^2 + 7t^3 + 8(t^4 + t^5 + t^6 + t^7 + t^8) + 4t^9 + t^{10},$$

and

$$\dim \mathcal{D}_4^0 = 61.$$

Corollary 6.3 (Failure of the triangular identity). *In characteristic zero,*

$$(d_2 + d_3 + d_4)d_4^5 \neq 0 \quad \text{in } 4T_4^0.$$

Consequently [2, Theorem 4.101(2)] is false for $n = 4$.

Proof. Set $I_0 = (u^2, (u + v)v^3)$. The exact degree calculation in the proof of Theorem 6.1 shows that

$$(\mathbb{Q}[u, v, w]/I_0)_6 \longrightarrow (\mathcal{D}_4^0)_6$$

is an isomorphism: both sides have dimension 8. In $\mathbb{Q}[u, v, w]/I_0$, the degree-six element

$$(u + v + w)w^5 = uw^5 + vw^5 + w^6$$

is nonzero (the three displayed terms are distinct standard monomials for a Gröbner basis of I_0). Its image is therefore nonzero in $\mathcal{D}_4^0$.

Independently of this dimension argument, the ancillary verification script reduces the same degree-six element directly in the ambient quadratic algebra and obtains a nonzero exact rational coordinate vector. $\square$

Corollary 6.4 (Not a complete intersection). *The ideal defining $\mathcal{D}_4^0$ has six minimal homogeneous generators, of degrees*

$$2, 4, 9, 9, 9, 9.$$

In particular $\mathcal{D}_4^0$ is not a complete intersection. It is also not Artinian Gorenstein.

Proof. There is one new relation in degree 2, no additional minimal relation in degree 3, one new relation in degree 4, none in degrees 5, $\dots$, 8, and a four-dimensional new relation space in degree 9. This proves minimality of the six generators in Theorem 6.1. Since the quotient is zero-dimensional in a polynomial ring in three variables, a complete-intersection ideal would have three minimal generators, not six.

Finally, an Artinian graded Gorenstein algebra has symmetric Hilbert function, whereas the Hilbert function in Theorem 6.2 is not symmetric. $\square$

Corollary 6.5 (Correction of the published Hilbert formula). *The Hilbert-product assertion in [2, Theorem 4.101(3)] fails for $n = 4$. After tensoring with $\mathbb{Q}$, the claimed product is*

$$(1+t)(1+t+t^2+t^3)(1+t+t^2+t^3+t^4+t^5),$$

which has total dimension 48, whereas $\dim \mathcal{D}_4^0 = 61$.

7. HIGHER-RANK QUESTIONS

The rank-four theorem shows that the triangular complete-intersection pattern valid for $n = 3$ does not persist literally. It therefore seems more useful to ask for the actual relation structure than to impose the product formula (1.2) as a conjectural target.

Problem 7.1. *Determine the defining ideal and Hilbert series of $\mathcal{D}_n^0$ for $n \geq 5$. In particular, determine whether $\mathcal{D}_n^0$ is finite-dimensional for all n , and find the degrees and multiplicities of its minimal defining relations.*

The classical content-path presentation suggests a second, more structural question: whether a finite branching object can replace Young’s lattice in the reduced Kohno–Drinfeld setting. The four new degree-nine constraints in rank four show that any such model must contain genuinely non-triangular branching data.

APPENDIX A. AN EXPLICIT GRÖBNER CERTIFICATE FOR THE RANK-FOUR PRESENTATION

Let $J \subset \mathbb{Q}[u, v, w]$ be the ideal in Theorem 6.1. With graded reverse lexicographic order $u > v > w$, the following eleven polynomials form a Gröbner basis of J ; multiplication by a nonzero rational scalar is immaterial:

$$\begin{aligned} &u^2, \\ &v^3(u+v), \\ &v^5, \\ &w^7(774uv + 543uw + 387v^2 + 543vw + 338w^2), \\ &w^6(38700uv^2 - 5397uw^2 - 18963v^2w - 18555vw^2 - 14854w^3), \\ &w^6(291uw^2 + 38700v^3 + 56889v^2w + 39765vw^2 + 16562w^3), \\ &w^5(231uw^3 + 774v^4 + 231vw^3 + 49w^4), \\ &w^9(45u + 11w), \\ &w^9(45v + 22w), \\ &w^8(135v^2 - 7w^2), \\ &w^{11}. \end{aligned}$$

Their leading monomials are exactly those displayed in (6.6). This list is an exact certificate for the Hilbert-function computation used in the proof of Theorem 6.1.

APPENDIX B. COMPUTATIONAL REPRODUCIBILITY

All computational claims in Sections 5 and 6 use exact arithmetic. The accompanying ancillary file `JucysMurphy.verify.py` implements Theorem 4.2, constructs $4T_3^0$ and $4T_4^0$ directly from the defining quadratic relations, verifies the ambient dimensions (6.4), computes the Jucys–Murphy ranks (6.5), checks that the four rows of (6.3) span the complete degree-nine kernel, and verifies the Gröbner certificate

in Section A. The code uses exact rational arithmetic throughout; it contains no floating-point or randomized step.

AI-statement. The counterexample has been found by GTP 5.6 Sol.

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