ON COMBINATORICS AND TOPOLOGY OF PAIRWISE INTERSECTIONS OF SCHUBERT CELLS IN $SL_n/\mathcal{B}$

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0. Introduction

0.1. Topological properties of intersections of pairs and, more generally, of k-tuples of Schubert cells belonging to distinct Schubert cell decompositions of a flag space are of particular importance in representation theory and have been intensively studied during the last 15 years, see e.g. [BB, KL1, KL2, De1, GS]. Intersections of certain special arrangements of Schubert cells are related directly to the representability problem for matroids, see [GS]. Most likely, for a somewhat general class of arrangements of Schubert cells their intersections are too complicated to analyze. Even the nonemptyness problem for such intersections in complex flag varieties is very hard. However, in the case of pairs of Schubert cells in the space of complete flags one can obtain a special decomposition of such intersections, and of the whole space of complete flags, into products of algebraic tori and linear subspaces. This decomposition generalizes the standard Schubert cell decomposition. The above strata can be also obtained as intersections of more than two Schubert cells originating from the initial pair. The decomposition considered is used to calculate (algorithmically) natural additive topological characteristics of the intersections in question, namely, their Euler $E^{p,q}$-characteristics (see [DK]). Generally speaking, this decomposition of the space of complete flags does not stratify all pairwise intersections of Schubert cells, i.e. the closure of a stratum is not necessary a union of strata of lower dimensions. Still there exists a natural analog of adjacency, and its combinatorial description is available, see Theorem D. We discuss combinatorics of this special decomposition and some rather simple consequences for the cohomology and the mixed Hodge structure of intersections of Schubert cells in $SL_n/\mathcal{B}$.

0.2. In order to formulate the main results, let us recall some standard notions. Let F_n denote the space of complete flags in $\mathbb{C}^n$. Each compete flag f can be interpreted both as a Borel subgroup and as a sequence of

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enclosed linear subspaces of all dimensions from 0 to n. The *Schubert cell decomposition* D_f of F_n relative to f consists of cells formed by all flags having a given set of dimensions of intersections with subspaces of f. Thus, we have a family of Schubert cell decompositions parameterized by F_n. Cells of any such decomposition are in 1-1-correspondence with permutations on n elements (see, e.g., [FF]).

0.3. For any k-tuple of flags $f, g, h, \dots$ in F_n we introduce the *k-tuple Schubert decomposition* $D_{f,g,h,\dots}$ consisting of all nonempty intersections of k-tuples of cells one taken from each decomposition D_f, D_g, $D_h \dots$. Generally speaking, a k-tuple Schubert decomposition is not a stratification of the space of complete flags, and its strata can have very complicated topology, cp. [GS].

Given a family of linear subspaces, their *+-completion* is defined as the set of sums of all possible subfamilies of these subspaces. The +-completion of a pair of flags is just the +-completion of the family of all subspaces constituting these flags.

The *refined double decomposition* $RD_{f,g}$ of the space of complete flags relative to a given pair of flags (f, g) is the decomposition into pieces formed by all flags with given dimensions of intersections with all subspaces of the +-completion of (f, g). The pieces of this decomposition are called *refined double strata*.

0.4. Remark. The refined double decomposition coincides with some special k-tuple decomposition, see §1.

0.5. Remark. The refined double decomposition $RD_{f,g}$ subdivides the standard decompositions D_f, D_g and the double decomposition $D_{f,g}$, i.e. each Schubert cell with respect to f or g, as well as their pairwise intersections, consists of some number of refined double strata.

0.6. Remark. If g' belongs to the same Schubert cell of D_f as g, then $RD_{f,g}$ is isomorphic to $RD_{f,g'}$, and the isomorphism is induced by a linear operator preserving f.

Theorem A. *Each refined double stratum is biholomorphically equivalent to the product of a complex torus by a complex linear space.*

0.7. Given a system of coordinates in $\mathbb{C}^n$, a flag is called *coordinate* if all its subspaces are spanned by coordinate vectors. Each coordinate flag is identified naturally with the corresponding permutation of coordinates. The *standard* coordinate flag is the one which is identified with the unit permutation, i.e. for each i its i-dimensional subspace is spanned by the first i coordinate vectors.

For any two flags f and g in F_n one can always choose a *standartizing* system of coordinates such that f becomes the standard coordinate flag and g becomes a coordinate flag identified with some permutation σ. We then say that the pair (f, g) is in *relative position* σ. Observe that the permutation corresponding to the cell containing g in the Schubert cell decomposition D_f is exactly σ^{-1}.

0.8. In what follows we need some additional notation related to permutations. We write an arbitrary permutation π in the form $\pi = i_1 \dots i_n$, which means that $\pi(1) = i_1, \dots, \pi(n) = i_n$.

A *decreasing subsequence* in π is a subsequence $s = (i_{j_1}, i_{j_2}, \dots, i_{j_k})$ such that $1 \leqslant j_1 < j_2 < \dots < j_k \leqslant k$ and $i_{j_1} > i_{j_2} > \dots > i_{j_k}$.

The *reduced length* of a decreasing subsequence is equal to the number of its elements minus one. The *domination* of a decreasing subsequence is equal to the number of elements $i_j \in \pi$ for which there exists an element $i_l \in s$ such that $j < l$ and $i_j < i_l$.

The *cyclic shift* of π wrt a decreasing subsequence $s = (i_{j_1}, i_{j_2}, \dots, i_{j_k})$ is the transformation sending i_{j_1} onto i_{j_k}, i_{j_2} onto $i_{j_1}, \dots,$ i_{j_k} onto $i_{j_{k-1}}$ and preserving the rest of the elements. (If s consists of just one element, then the transformation is identical.)

Example. Consider $\pi = 6723451$ and its decreasing subsequence $(7, 3, 1)$. The reduced length is 2 and the domination is also 2, namely, element 6 is dominated by 7 belonging to the decreasing subsequence, and element 2 is dominated by 3. The cyclic shift of π wrt $(7, 3, 1)$ takes π to 6321457.

0.9. Till the end of this section we assume that the pair of flags (f, g) is in relative position σ. Let us enumerate (algorithmically) all strata of $RD_{f,g}$ using the permutation σ. To do this, we apply to σ the following n-step procedure.

Main algorithm.

Step 1. Find all decreasing subsequences in σ. Apply to σ cyclic shifts wrt each of these decreasing subsequences and obtain the set of resulting permutations. In each of these permutations block the largest element of the corresponding decreasing subsequence. (To *block* just means that this element is ignored on all subsequent steps of the algorithm.)

Step i. To each permutation obtained on Step $i-1$ apply the same procedure that was applied to σ on Step 1. Namely, find all its decreasing subsequences (disregarding all blocked elements). Make cyclic shifts wrt each of these decreasing subsequences, and finally in each of the obtained permutations block the largest element of the corresponding decreasing subsequence.

0.10. Remark. The algorithm stops exactly after n steps, since in each permutation obtained after i steps we get exactly i elements blocked (i.e., on Step $i+1$ we work actually with permutations on $n-i$ elements). Moreover, each permutation with at least one nonblocked element contains at least one decreasing subsequence (possibly consisting of just one element).

The whole procedure for $\sigma = 321$ is illustrated on Fig.1. The underlined numbers are blocked and the sequences of numbers on edges present the decreasing subsequences chosen.

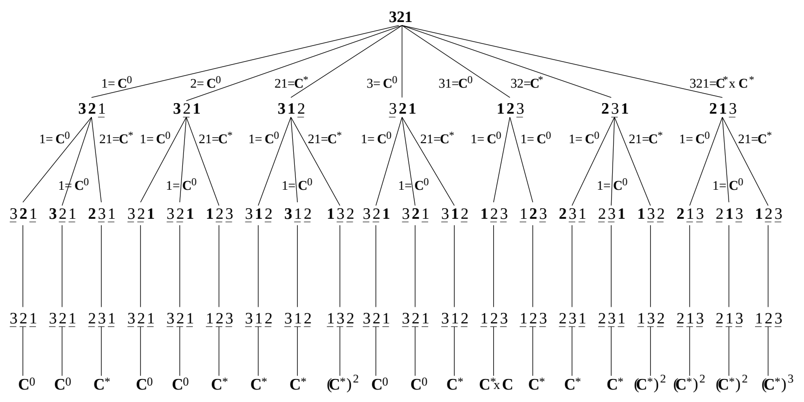

Fig.1. Illustration of the main algorithm for the case $\sigma =$ 321

0.11. A *chain* of permutations is a sequence of $n+1$ permutations starting with σ and such that each consequent permutation is obtained from the preceding one as the result of a cyclic shift wrt a decreasing subsequence on the corresponding step of the above procedure.

So, Fig.1 contains 20 chains, which are just paths in the presented tree starting from the top element σ and going down to the bottom.

Let us assign to each chain two numbers, namely, its *total length* equal to the sum of the reduced lengths for all the permutations involved (actually, for the corresponding decreasing subsequences of these permutations), and its *total domination* equal to the sum of all dominations. (When we calculate the domination for a permutation with blocked elements we just disregard them completely.)

Theorem B. *Let (f, g) be a pair of flags in relative position σ. Then the strata of $RD_{f,g}$ are in* 1-1-*correspondence with the chains in the above procedure starting with σ. The stratum corresponding to a given chain is isomorphic to $(\mathbb{C}^*)^l \times \mathbb{C}^d$, where l is the total length of this chain and d is its total domination.*

Example. The structure of each refined double stratum is given in the bottom line of Fig.1.

0.12. For any two permutations α and β denote by $C_{1,\alpha}$ the Schubert cell consisting of all flags which are in relative position α with respect to f and by $C_{\sigma,\beta}$ the Schubert cell of all flags in relative position β with respect to g. (Warning: in the standard notation $C_{1,\alpha} = \mathcal{B}\alpha^{-1} \cdot \mathcal{B}$ and $C_{\sigma,\beta} = \sigma^{-1}\mathcal{B}\sigma\beta^{-1} \cdot \mathcal{B}$, see 0.7, where $\mathcal{B}$ is the subgroup of upper triangular matrices.)

By definition, the cell $C_{1,\alpha}$ belongs to the decomposition D_f, the cell $C_{\sigma,\beta}$ belongs to the decomposition D_g, and their intersection belongs to the double decomposition $D_{f,g}$.

Since the refined double decomposition $RD_{f,g}$ subdivides the decompositions D_f, D_g and $D_{f,g}$, it is useful to describe all refined double strata included in the Schubert cells $C_{1,\alpha}$, $C_{\sigma,\beta}$ and their intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$.

Let us assign to a chain of permutations the following two new permutations. The *first blocking* permutation of a chain is the sequence of the successively blocked elements, i.e., its ith entry is the element blocked on the ith step of the procedure. The *second blocking* permutation is the sequence of positions on which the successively blocked elements stand, i.e., its ith element is the number of the position where the ith blocked element stands.

Example, see Fig.1. For the chain $\mathbf{321} \to \mathbf{12}\underline{3} \to \mathbf{1}\underline{23} \to \underline{123}$, the first blocking permutation is 321 and the second blocking permutation is 321. For the chain $\mathbf{321} \to \mathbf{31}\underline{2} \to \mathbf{1}\underline{32} \to \underline{132}$, the first blocking permutation is 231 and the second blocking permutation is 321. For the chain $\mathbf{321} \to \underline{3}\mathbf{21} \to \underline{3}\mathbf{1}\underline{2} \to \underline{312}$, the first blocking permutation is 321 and the second blocking permutation is 132.

Theorem C. *A stratum of $RD_{f,g}$ belongs to the Schubert cell $C_{1,\alpha}$ if and only if the first blocking permutation of its chain coincides with α; a stratum of $RD_{f,g}$ belongs to the Schubert cell $C_{\sigma,\beta}$ if and only if the second blocking permutation of its chain coincides with β. Therefore, a stratum belongs to the intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$ if and only if its first blocking permutation is α and its second blocking permutation is β.*

0.13. This theorem leads us to the following modifications of the described algorithm producing the refined double decompositions of the Schubert cells $C_{1,\alpha}$, $C_{\sigma,\beta}$ and of their intersection.

Modification 1. In order to obtain the decomposition of $C_{1,\alpha}$, one must consider on Step i, $i = 1, \dots, n$, only decreasing subsequences starting at the ith element of the permutation α.

Modification 2. In order to obtain the decomposition of $C_{\sigma,\beta}$, one must consider on Step i, $i = 1, \dots, n$, only decreasing subsequences ending at the position whose number is equal to the ith element in the permutation β.

Modification 3. Finally, in order to obtain the decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$, one must consider on Step i, $i = 1, \dots, n$, only decreasing subsequences starting at the ith element of permutation α and ending at the position whose number is equal to the ith element in the permutation β.

The corresponding three examples are presented on Figs.2,3.

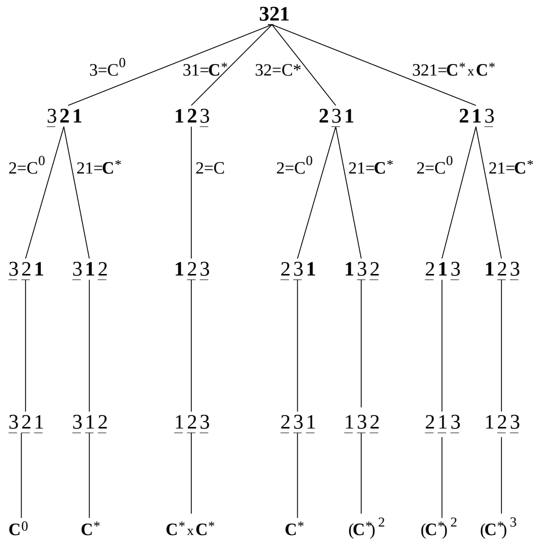

Fig.2. Modified algorithm for the case $\sigma = \alpha = 321$

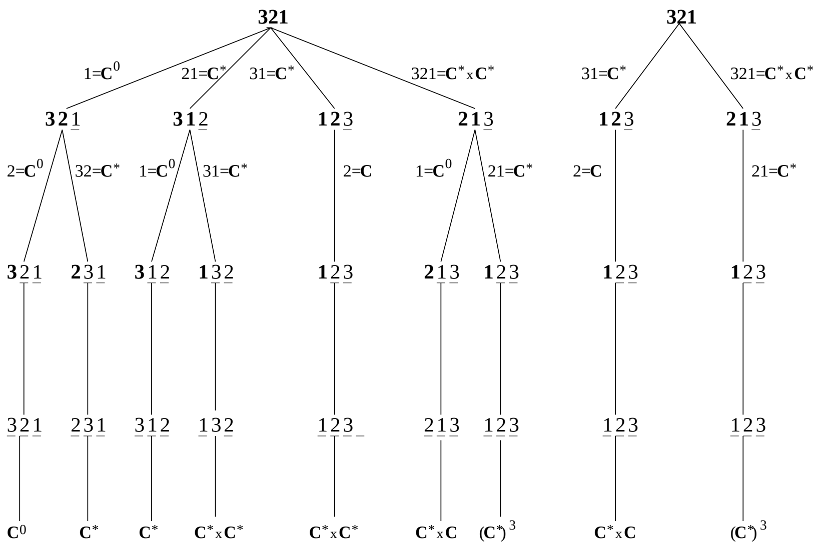

Fig.3. Modified algorithm for the cases $\sigma = \beta = 321$ and $\sigma = \alpha = \beta = 321$

A similar process was proposed by Francesco Brenti [Br1] in the case $\sigma = \omega_0$.

0.14. The following remark is valid for all versions of the main algorithm, i.e. for the refined double decomposition of the whole space of complete flags, of some particular Schubert cell, or of a pairwise intersection of Schubert cells.

Remark. Each step of the above algorithm is interpreted geometrically as the projection of flags in the considered set onto a linear subspace of the corresponding codimension, see §1. This means, in particular, that restricted chains, i.e. those starting at a permutation with blocked elements obtained after Step i, represent the decomposition of F_{n-i} (a Schubert cell in F_{n-i}, or a pairwise intersection of Schubert cells in F_{n-i}, depending on the modification of the algorithm).

0.15. To be more precise, we introduce the following operation on permutations with blocked elements. Let us consider a permutation on n elements with i blocked entries. The *reduction* of the blocked part from a given permutation is the operation which forms a new permutation on $n-i$ elements in the following way: we exclude all blocked elements and subtract from each nonblocked element the number of all blocked elements which are less than it.

Example. The reduction of the blocked part from the permutation $\mathbf{75\underline{6}3\underline{2}4\underline{1}}$ gives **4312**.

0.16. Proposition. i) *The set of all restricted chains, i.e. chains starting at some permutation $\tilde\sigma$ obtained after i steps of the algorithm (and thus containing i blocked elements), geometrically presents:*

(1) *for the main algorithm, the refined double decomposition of F_{n-i} relative to the pair in relative position $\tilde\sigma'$, where $\tilde\sigma'$ is obtained from $\tilde\sigma$ by the reduction of all blocked elements;*
(2) *for the 1st modification, the refined double decomposition of the Schubert cell $C_{1,\alpha'}$ in F_{n-i}, where α' is obtained by the reduction of the first i elements from α;*
(3) *for the 2nd modification, the refined double decomposition of the Schubert cell $C_{\tilde\sigma',\beta'}$ in F_{n-i}, where $\tilde\sigma'$ is the same as above and β' is obtained by the reduction of the first i elements from β;*
(4) *for the 3rd modification, the refined double decomposition of $C_{1,\alpha'}\cap C_{\tilde\sigma',\beta'}$.*

ii) *Moreover, the geometrical meaning of Step i for any modification of the algorithm is the decomposition of the initial object (F_n, $C_{1,\alpha}$, $C_{\sigma,\beta}$ or $C_{1,\alpha}\cap C_{\sigma,\beta}$) into a disjoint union of products of similar objects in F_{n-i}, enumerated by the set of all permutations $\tilde\sigma$ obtained after Step i, by $(\mathbb{C}^*)^{l(\tilde\sigma)}\times\mathbb{C}^{d(\tilde\sigma)}$. Here $l(\tilde\sigma)$ is equal to the sum of reduced lengths for all permutations in the chain starting at σ and ending at $\tilde\sigma$, and $d(\tilde\sigma)$ is the sum of all dominations in this chain.*

Example. The set of all reduced chains passing through any permutation obtained after the first step except for $\mathbf{12\underline{3}}$ on Fig.1 presents the refined decomposition of the space of complete flags $F_2 = \mathbb{C}P^1$ relative to $\sigma = 21$ into three strata, namely, two points and $\mathbb{C}^*$. The chains passing through $\mathbf{12}\underline{\mathbf{3}}$ (or $\sigma = 12$) present the standard Schubert cell decomposition of F_2 into a point and $\mathbb{C}$.

0.17. Remark. The above results hold in complete generality for the spaces of complete flags over any algebraically closed field, or $\mathbb{R}$. In this case $\mathbb{C}^*$ must be substituted by the multiplicative and $\mathbb{C}$ by the additive group of the field, cp. [Cu3].

0.18. Our next result gives sufficient conditions for the topological "adjacency" of strata in $C_{1,\alpha} \cap C_{\sigma,\beta}$ in terms of combinatorial adjacency, i.e. enumerates strata that can have nonempty intersection with the closure of some given stratum in $C_{1,\alpha} \cap C_{\sigma,\beta}$. Since the refined double decomposition in general is not a stratification, one has to use this modified notion of adjacency.

Consider two chains of permutations. One of them is said to be *less or equal* than the other one if each permutation of the former is less or equal in the Bruhat order than the corresponding permutation of the latter (for the notion of the Bruhat order see e.g. [Hu]). The above partial order on the set of all refined double strata (or their chains of permutations) in $C_{1,\alpha} \cap C_{\sigma,\beta}$ is called the *adjacency partial order*, or the *generalized Bruhat order*.

By Theorem C, each nonempty refined double stratum of $RD_{f,g}$ is contained in a single pairwise intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$, namely, in the one for which α equals to the first and β to the second blocking permutation of the corresponding chain of permutations.

0.19. Theorem D. *The closure of a given refined double stratum in the corresponding pairwise intersection of Schubert cells belongs to* (*but in general does not coincide with*) *the union of all refined double strata included in the same pairwise intersection such that their chains of permutations are less or equal in the adjacency partial order than the chain of the stratum considered.*

This theorem enables us to construct rather simple examples of pairwise intersections whose refined double decompositions fail to be stratifications. In particular, the refined double decomposition of $C_{1234,4231} \cap C_{4231,4231}$ consists of three strata, namely, $\mathbb{C}^* \times (\mathbb{C})^2$ and two copies of $(\mathbb{C}^*)^3 \times \mathbb{C}$, see Fig.9 below. In notations of Fig.9 the closure of the stratum $C = (\mathbb{C}^*)^3 \times \mathbb{C}$ is nonempty in the stratum $B = (\mathbb{C}^*)^3 \times \mathbb{C}$; moreover, the closure of $C = (\mathbb{C}^*)^3 \times \mathbb{C}$ does not contain the whole stratum $A = \mathbb{C}^* \times (\mathbb{C})^2$; see §7 for more examples of this kind.

0.20. Let us consider the *closure pattern* of a given refined double stratum, i.e. the set of all strata that have nonempty intersection with the closure of the given one. (The question about the combinatorial description of the closure pattern was raised for a similar situation in [Cu3].) Theorem D gives us a necessary combinatorial condition for a stratum to belong to the closure pattern of the other one. In §7 we present an example showing that this necessary condition is not sufficient. The relation between necessary and sufficient conditions motivates the following definition.

We call a pairwise intersection of Schubert cells $C_{1,\alpha} \cap C_{\sigma,\beta}$ *nice* if the refined double decomposition $RD_{f,g}$ gives its stratification, and *almost nice* if for any pair $\mathrm{St}_1 \prec \mathrm{St}_2$ of refined double strata one has $\dim \mathrm{St}_1 \leqslant \dim \mathrm{St}_2$. The rest of $C_{1,\alpha} \cap C_{\sigma,\beta}$ are called *hard*. An example of a hard $C_{1,\alpha} \cap C_{\sigma,\beta}$ is given in §7.

0.21. Let V denote an arbitrary complex quasiprojective variety. Let us denote by h_k^{pq} the Hodge numbers for the usual mixed Hodge structure in $H_c^*(V;\mathbb{C})$, see e.g. [Dl1, Dl2, Dl3].

Let us define generalized Euler characteristics depending on p and q:

$$\chi^{pq} = \sum_k (-1)^k h_k^{pq},$$

and form their generating function called the *$E^{p,q}$-polynomial*, or just the *E-polynomial* of V:

$$E_V(u,v) = \sum_{p,q} \chi^{pq} u^p v^q.$$

Let us denote by $E_\sigma^{\alpha,\beta}$ the E-polynomial of $C_{1,\alpha} \cap C_{\sigma,\beta}$, and by $\mathrm{CH}(\alpha,\beta,\sigma)$ the set of chains of permutations corresponding to the strata contained in $C_{1,\alpha} \cap C_{\sigma,\beta}$ (see Theorem B).

0.22. Corollary (of Theorems B and D)**.** i) *The Hodge numbers $h_k^{p,q}$ of any intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$ can be positive only if $p = q$.*

ii)

$$E_\sigma^{\alpha,\beta} = \sum_{\mathrm{ch} \in \mathrm{CH}(\alpha,\beta,\sigma)} z^{d(\mathrm{ch})}(z-1)^{l(\mathrm{ch})},$$

where $z = uv$, $d(\mathrm{ch})$ and $l(\mathrm{ch})$ are the total domination and the total length of a chain ch, *respectively.*

0.23. The same expression ii) can be rewritten as an inductive formula using the above remark on the geometrical meaning of our algorithm (see 0.14). More precisely, let $\mathcal{S}$ denote the set of all decreasing subsequences in σ starting at element $\alpha(1)$ and ending at the position with number $\beta(1)$, (i.e., the set of subsequences used on the first step of the construction of the refined double decomposition for $C_{1,\alpha} \cap C_{\sigma,\beta}$). For any decreasing subsequence $s \in \mathcal{S}$, let $l(s)$ and $d(s)$ denote its reduced length and domination, respectively. Finally, let α' and β' denote the results of the reduction of the first elements $\alpha(1)$ and $\beta(1)$ from α and β, respectively, σ_s denote the result of the cyclic shift of σ wrt s and the reduction of the first element of s (see the description of the algorithm and its modifications above).

Corollary (of Theorems B and D).

$$E_\sigma^{\alpha,\beta} = \sum_{s \in \mathcal{S}} z^{d(s)}(z-1)^{l(s)} E_{\sigma_s}^{\alpha',\beta'}.$$

Example.

$$E_{4321}^{4321,4321} = (z-1)E_{132}^{321,321} + (z-1)^2 E_{231}^{321,321} + (z-1)^2 E_{312}^{321,321} + (z-1)^3 E_{321}^{321,321}.$$

0.24. Corollaries 0.22 and 0.23 follow directly from the refined double decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$. By results of [Cu3], their natural analogs are also valid for any $G/\mathcal{B}$, where G is a semisimple group and $\mathcal{B}$ is its Borel subgroup. Part i) of 0.22 holds as well for all quasiprojective varieties that can be decomposed into quasiprojective pieces satisfying $h_k^{p,q} = 0$ if $p \neq q$.

0.25. The adjacency partial order (see 0.18) allows us to consider different filtrations of $C_{1,\alpha} \cap C_{\sigma,\beta}$ by closed subsets consisting of refined double strata. Moreover, one gets the standard filtration of such a kind as follows.

Let P be a finite poset. We define the *height* of an element $a \in P$ as the maximum length of a chain having a as the maximal element. The *standard* filtration of P is its filtration by the subsets $P_i = \{a \in P \mid h(a) \leqslant i\}$, called *standard subposets*.

Consider now the refined double decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$. Refined double strata are enumerated by chains of permutations, which form a finite poset with respect to the adjacency partial order. Theorem D shows that the standard filtration of this poset is a filtration by closed subsets.

Any filtration of $C_{1,\alpha} \cap C_{\sigma,\beta}$ by closed subsets leads to its Leray spectral sequence converging to the cohomology of $C_{1,\alpha} \cap C_{\sigma,\beta}$ with compact supports. In our case we are primarily interested in filtrations by closed quasiprojective subvarieties. Along with the above standard filtration, it is often convenient to consider “filtration” by dimension: its ith term is the union of all strata of dimension at most i. (Warning: for an arbitrary $C_{1,\alpha} \cap C_{\sigma,\beta}$ this filtration is apparently not a filtration by closed subsets.) In the case of an (almost) nice $C_{1,\alpha} \cap C_{\sigma,\beta}$ the filtration by dimension is a filtration by closed quasiprojective subvarieties. Moreover, the following statement is valid.

Theorem E. *The Leray spectral sequence associated with the filtration by dimension of the refined double decomposition of any* (*almost*) *nice pairwise intersection degenerates at the second page.*

0.26. Remark. A combinatorial description of d_1 is unavailable at the present moment and apparently is rather complicated.

0.27. As an application of the E-polynomials we prove the following combinatorial result.

Proposition . *If at least one of the permutations α, β or σ is the longest element w_0, then there are no gaps in* (*complex*) *dimensions of strata in $C_{1,\alpha} \cap C_{\sigma,\beta}$, i.e. there exist refined double strata of all intermediate* (*complex*) *dimensions between the minimal and the maximal ones.*

0.28. A similar family of decompositions was introduced by V.V.Deodhar in the case of intersections $C_{1,\alpha} \cap C_{w_0,\beta}$, where w_0 denotes the longest element in an arbitrary finite Coxeter system, see [De1, De2], and was extended to all intersections $C_{1,\alpha} \cap C_{\sigma,\beta}$ by Ch.Curtis in [Cu3]. These decompositions depend on a reduced expression of the element α as a product of simple reflections, and different choices of such expressions lead to different decompositions. The combinatorial data that codes strata in the approach of Deodhar–Curtis is similar to chains of permutations corresponding to refined double strata, but more lengthy. We have found the correspondence between the strata of these two decompositions and proved that the refined double decomposition coincides with one of decompositions suggested by Deodhar for some particular choice of reduced expression.

Let us define for any $\alpha \in \mathfrak{S}_n$ a reduced expression of α^{-1} of a special form, namely, $\alpha^{-1} = t_{\alpha(1)-1} t_{\alpha(1)-2} \cdots t_1 \tilde{\alpha}^{-1}$, where $\tilde{\alpha}$ belongs to $\mathfrak{S}_{n-1}$ and t_i is the simple transposition interchanging i and $i+1$. This enables us to define inductively a reduced expression, which we call the *standard expression.*

Proposition . *The refined double decomposition of the Schubert cell* $C_{1,\alpha} = \mathcal{B}\alpha^{-1} \cdot \mathcal{B}$ *coincides with the decomposition suggested by Deodhar if one chooses the standard reduced expression of* α^{-1}.

0.29. Remark. There exist other natural refinements of double decompositions. These other decompositions of geometrical origin apparently coincide with Deodhar's decompositions corresponding to other choices of reduced expressions.

0.30. The starting point of this study was an attempt to calculate the cohomology of pairwise intersections of Schubert cells of the maximal dimension (see [SV]).

The authors are very grateful to Francesco Brenti for discussions of the material of this article. Sincere thanks are due to Prof. T.Springer who was first to point out to the authors the necessity to check the property of being a stratification and mentioned the paper [Boe].

1. Refined double decompositions as k-tuple decompositions and the very weak Bruhat order

1.1. In order to study the properties of refined double decompositions, let us fix a pair of flags (f, g) in relative position σ and consider its $+$-completion. By the definition (see 0.2), the $+$-completion consists of all sums of pairs of all possible subspaces from the flags f and g, see the example on Fig.4. Obviously, all the subspaces in the $+$-completion of (f, g) are coordinate (in the corresponding coordinate system, see 0.7) and partially ordered by inclusion.

By the *flag completion* of the pair (f, g) we mean the set of all complete flags such that each subspace of each of these flags belongs to the $+$-completion of (f, g).

The number of flags in the flag completion of a pair of flags in relative position σ is denoted by $k(\sigma)$; it is just the number of all different paths in the partially ordered $+$-completion going from the top to the bottom. For example, there are 10 paths (and therefore flags in the flag completion) in the diagram shown on Fig.4.

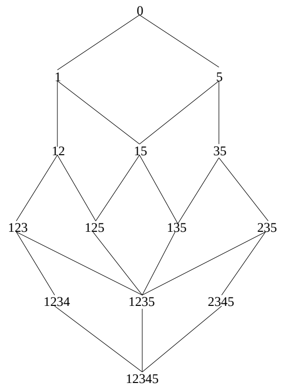

FIG.4. THE $+$-COMPLETION OF THE PAIR OF FLAGS IN RELATIVE POSITION 53241. EACH SET OF NUMBERS PRESENTS THE COORDINATE SUBSPACE SPANNED BY THE CORRESPONDING COORDINATE VECTORS. SUBSPACES IN EACH ROW ARE LEXICOGRAPHICALLY ORDERED

1.2. Proposition (an alternative definition of the refined double decomposition)**.** *The refined double decomposition $RD_{f,g}$ is the k-tuple decomposition $D_{f_1,f_2,\dots}$, where f_i runs over the set of all complete flags in the flag completion of (f, g), and hence $k = k(\sigma)$ is the number of flags in this flag completion.*

1.3. Remark. A special $n!$-tuple Schubert decomposition into intersections of $n!$-tuples of cells taken from Schubert cell decomposition relative to each of coordinate flags was introduced in [GS] and called the decomposition into small cells. This decomposition and its projections on Grassmanians play an important role in matroid theory, since each small cell on some Grassmanian is the space of linear presentations of the corresponding matroid. The decomposition into small cells subdivides each of our refined double decompositions.

1.4. Let us define the following partial order on $\mathfrak{S}_n$. By an *elementary descent* in a permutation $i_1 \dots i_n$ we call any inversion of two neighboring numbers i_l, i_{l+1} such that

i) it decreases the inversion length of the permutation, and

ii) i_{l+1} is less than all the elements $i_{l+2}, \dots, i_n$.

The transitive closure of elementary descents defines a partial order on $\mathfrak{S}_n$, which we call the *very weak Bruhat order*. It is, obviously, a suborder of the weak Bruhat order.

The only minimal element in the very weak Bruhat order is the identity, while the set of maximal elements consists of all elements having 1 in the last position. Figure 5 presents the very weak Bruhat order for the cases of $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

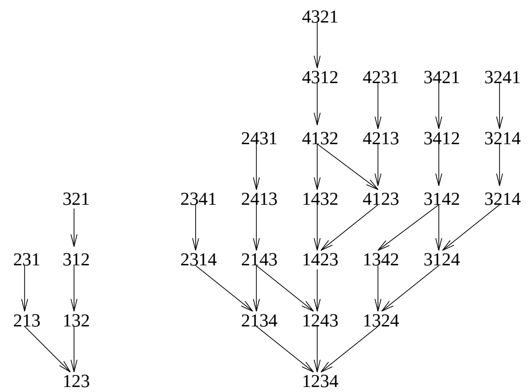

FIG.5. THE VERY WEAK BRUHAT ORDER ON $\mathfrak{S}_3$ AND $\mathfrak{S}_4$

1.5. Proposition. *The number $k(\sigma)$ equals the number of permutations in the lower interval of σ in the very weak Bruhat order.*

Proof. Follows immediately from the definition (see 1.1 and 1.4). ■

1.6. Below we present a simple scheme for finding the number $k(\sigma)$, which we call a *generalized Pascal triangle*. Recall that the standard Pascal triangle (of size n) can be represented as a box diagram having n rows and n columns. Each box is a square. The first row contains n boxes, the second row contains $n-1$ boxes, and so on; the first column contains 1 box, the second column contains 2 boxes, and so on. The diagram is filled by integers in the following way. We start at the unique box of the first column and fill it with 1. Next, the contents of any other box is defined to be the sum of the contents of its western and north-western neighbors (if any). Proceeding in accordance with this rule, we first fill all the boxes of the second column by ones; this gives a possibility to fill the boxes of the third column, and so on (see Fig.6). The elements in the ith column of the triangle are just the binomial coefficients $\binom{i-1}{j}$, $0 \leqslant j \leqslant i-1$, and their sum is 2^{i-1}.

1	1	1	1	1	1
	1	2	3	4	5
		1	3	6	10
			1	4	10
				1	5
					1

FIG.6. PASCAL TRIANGLE FOR $n = 6$

To construct the generalized Pascal triangle of type σ, we define $L_\sigma(l)$, $1 \leqslant l \leqslant n$, as the number of elements i_j of σ such that $j > l$ and $i_j > i_l$. We now start from the diagram of the standard Pascal triangle and remove the horizontal partitions between the last (from left to right) $L_\sigma(1)$ boxes of the first and the second row, then the horizontal partitions between the last $L_\sigma(2)$ boxes of the second and the third row, and so on. As a result, we get a new box diagram; its boxes are no longer squares, but rectangles of width 1 and arbitrary height. It is easy to see that the number of boxes in the diagram equals n plus the inversion length of σ.

Example. Let $n = 6$, $\sigma = 426135$. Then the vector of L_σ's is $(2, 3, 0, 2, 1, 0)$. The diagram of the corresponding generalized Pascal triangle is presented on Fig.7a. The inversion length of σ is 7, and the number of boxes in the diagram is $6 + 7 = 13$.

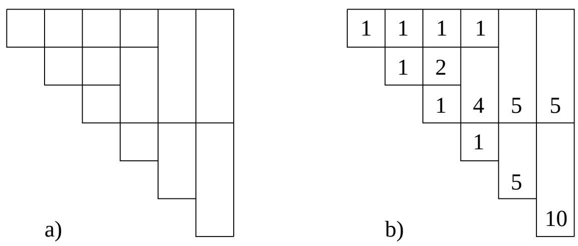

FIG.7. GENERALIZED PASCAL TRIANGLE OF TYPE 426135

The diagram thus constructed is now filled by integers according to the same rule as above. The only difference is that a box can have more than one western neighbor; in this case we have to sum the contents of all western neighbors, and that of the north-western one. As before, the table is filled column by column, starting from 1 placed in the unique box of the first column. The result of this procedure for the above example is presented on Fig.7b.

1.7. Proposition. *The number $k(\sigma)$ equals the sum of the numbers in the last column of the generalized Pascal triangle of type σ.*

Proof. Let us define two projections $\pi_1, \pi^1 : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$. The first of them takes $i_1 \dots i_k 1 i_{k+2} \dots i_n$ to $i_1 - 1 \dots i_k - 1\, i_{k+2} - 1 \dots i_n - 1$; the second one takes $a i_2 \dots i_n$ to $i_2^a \dots i_n^a$, where $i_j^a = i_j$ if $i_j < a$ and $i_j^a = i_j - 1$ if $i_j > a$. It is easy to see that for $\sigma = i_1 \dots i_n$ one has

$$(*) \qquad k(\sigma) = \begin{cases} k(\pi_1(\sigma)) + k(\pi^1(\sigma)) & \text{if } i_1 \neq 1, \\ k(\pi_1(\sigma)) = k(\pi^1(\sigma)) & \text{otherwise} \end{cases}.$$

Indeed, let $i_1 \neq 1$ (the case $i_1 = 1$ is trivial). By Proposition 1.5, we have to count the number of permutations in the lower interval of σ in the very weak Bruhat order. There are two types of such permutations: those in which i_1 and 1 form an inversion, and those in which they do not. By the definition, any path from σ to a permutation of the first type in the very weak Bruhat order avoids elementary descents transposing i_1 (otherwise condition ii) of 1.4 would be violated). Thus, i_1 remains all the time at the first position of any permutation in such a path and does not influence any of the descents used. Therefore, the number of the permutations of the first type equals $k(\pi^1(\sigma))$.

On the other hand, any path from σ to a permutation of the second type contains elementary descents involving i_1, and the first such descent on any of these paths is the transposition of i_1 and 1. Thus, 1 occurs at the first position, and remains there all the time. Therefore, the number of the permutations of the second type equals $k(\pi_1(\sigma))$.

Let now $b(\sigma)$ denote the sum of the elements of the last column in the generalized Pascal triangle of type σ; we shall prove that $b(\sigma)$ satisfies the same equation as $k(\sigma)$. Again we first assume that 1 is not in the first position of σ. Hence, the second column of the generalized Pascal triangle consists of two square boxes, and each of the boxes contains 1. Let us consider two other generalized Pascal triangles with the same diagram that the initial one; the first of them contains 0 in the first box of the second column and 1 in the second box, while the second triangle contains 1 in the first of these boxes and 0 in the other one. All the other columns of the both triangles ate filled according to the rule 1.6. Since the rule is additive, we get that the contents of any box of the initial triangle (except for the box in the first column) equals the sum of the contents of the corresponding boxes in the two new triangles constructed. However, the first of these triangles corresponds bijectively to the generalized Pascal triangle of type $\pi^1(\sigma)$ (it is sufficient to shorten by 1 the heights of all uppermost boxes; if such a box is of height 1, then its contents is 0, and we just remove it). In a similar way, the second of the triangles corresponds bijectively to the generalized Pascal triangle of type $\pi_1(\sigma)$ (we again remove certain square boxes containing 0 and shorten by 1 certain boxes of height > 1). Therefore, $b(\sigma)$ satisfies the first line of $(*)$. The case $\sigma = 1\, i_2 \dots i_n$ is again trivial, since in this case the second column of the corresponding generalized Pascal triangle contains only one box.

Finally, for the trivial permutation $1 \in \mathfrak{S}_1$ one has $k(1) = 1$, while the generalized Pascal triangle of type 1 consists of one box containing 1, and so $b(1) = 1$. Therefore $b(\sigma)$ and $k(\sigma)$ satisfy the same equation with the same initial values. ■

2. Quotients of cells and refined double strata

2.1. Lemma. *Let K, L, and M be a triple of linear subspaces in some linear space. Then*

$$(*) \quad \dim((K/L) \cap (M/L)) \\ = \dim((K+M) \cap L) - \dim(K \cap L) - \dim(M \cap L) + \dim(K \cap M).$$

Moreover, if M contains L, the above formula is simplified to

$$(**) \qquad \dim((K/L) \cap (M/L)) = \dim(K \cap M) - \dim(K \cap L).$$

The proof is obtained by a straightforward usage of inclusion and exclusion of vectors of an appropriate basis.

2.2. Given a complete flag f and a subspace L in $\mathbb{C}^n$, we denote by f_L the *complete quotient flag*, that is, the flag in the quotient space $\mathbb{C}^n/L$ obtained by taking consequent quotients of all subspaces constituting f (and ignoring occasional coincidences).

2.3. Assume that C is a cell of the Schubert cell decomposition D_f and $L \subseteq \mathbb{C}^n$ is a linear subspace. We denote by C_L the set of all flags in C containing L as a flag subspace.

Lemma . *If $C_L \neq 0$, then the set C_L/L of quotients of all flags in C_L is isomorphic to some cell in the Schubert cell decomposition D_{f_L} of the space of complete flags in $\mathbb{C}^n/L$. Moreover, if the flags in C are in relative position α to f and the dimension of L is i, then flags in C_L/L are in relative position α' to f_L, where α' is the permutation on $n-i$ elements obtained by the reduction of the first i elements of α.*

Proof. The set C_L/L belongs to some cell C' of the decomposition D_{f_L}. Indeed, let g be some flag in C_L, let M be its subspace containing L and let K be a subspace of f. Then $\dim((K/L) \cap (M/L))$ is equal to $\dim(K \cap M) - \dim(K \cap L)$ by $(**)$, and thus does not depend on the choice of g in C_L. Conversely, any flag $f' \in C'$ is the quotient of some flag $\tilde{f} \in C_L$. The corresponding flag $\tilde{f} \in C_L$ can be constructed as follows. Its subspaces included in L can be taken from an arbitrary flag in C_L, while those containing L are the inverse images of the subspaces of the considered quotient flag f'. An easy check shows that $\tilde{f}$ belongs to C_L, since it contains L and has the necessary dimensions of intersections with the subspaces of f. In order to prove that α' is obtained from α by the reduction of its first i elements, one must consider a standardtizing basis for the (f,g) (see 0.7) and take the quotient by the i-dimensional subspace of g. In this case everything is obvious. ■

2.4. Corollary. *The assertion of Lemma* 2.3 *holds for any k-tuple Schubert decomposition, i.e. if* St *is a nonempty stratum of $D_{f,g,h\dots}$ and* St_L *is its nonempty subset of all flags containing a linear subspace L, then the set* St_L/L *of quotients of flags in* St_L *is isomorphic to a stratum in the k-tuple decomposition $D_{f_L,g_L,h_L,\dots}$.*

2.5. Remark. The same holds for refined double decompositions, since they are particular cases of k-tuple decompositions, see Proposition 1.2.

2.6. Remark. If L is one-dimensional, then $C_L/L \cong C_L$.

2.7. Proposition. *Let h^1 and h^2 be two flags lying in the same refined double stratum of $RD_{f,g}$, and let L^1 and L^2 denote the subspaces in h^1 and h^2, respectively, of the same dimension. Then there exists an isomorphism $\Phi\colon \mathbb{C}^n/L^1 \to \mathbb{C}^n/L^2$ of the quotient spaces that sends the quotient flag f_{L^1} onto f_{L^2} and the quotient flag g_{L^1} onto g_{L^2}. Moreover, the quotient flags $\Phi(h^1_{L^1})$ and $h^2_{L^2}$ belong to the same refined double stratum of the refined double decomposition $RD_{f_{L^2},g_{L^2}}$.*

Proof. Such an isomorphism Φ exists if the pair of quotient flags (f_{L^1}, g_{L^1}) is in the same relative position as the pair (f_{L^2}, g_{L^2}). Thus, we must check that for any two subspaces K and M of f and g, respectively, $\dim((K/L^1)\cap(M/L^1)) = \dim((K/L^2)\cap(M/L^2))$. Indeed, if we substitute L^1 for L in $(*)$, then we get the same four terms in the right hand side as if we substitute L^2 (this follows immediately from the definition of refined double strata).

To prove that such an isomorphism Φ takes the quotient flag $h^1_{L^1}$ to the same refined double stratum of $RD_{f_{L^2},g_{L^2}}$ where $h^2_{L^2}$ lies, we must show the following. Given any subspace K in the $+$-completion of the pair (f,g) and subspaces $\tilde{L}^1 \supset L^1$ and $\tilde{L}^2 \supset L^2$ of the same dimension that belong to h^1 and h^2, respectively, prove that $\dim((K/L^1)\cap(\tilde{L}^1/L^1) = \dim((K/L^2)\cap(\tilde{L}^2/L^2)$. Using $(**)$, we rewrite the above relation as $\dim(K\cap\tilde{L}^1) - \dim(K\cap L^1) = \dim(K\cap\tilde{L}^2) - \dim(K\cap L^2)$, which follows immediately from the definition of the refined double decomposition. ■

3. Stabilizer of a pair of flags and its action on the projective space

3.1. Let us consider the subgroup $G_\sigma = \mathcal{B} \cap \sigma^{-1}\mathcal{B}\sigma$, which is the stabilizer of both flags f and g (recall that $\mathcal{B}$ denotes the subgroup of upper triangular matrices). The subgroup G_σ is given by the following relations: the matrix entry $a_{i,j}$, $i < j$, vanishes if and only if the numbers i and j form an inversion in σ, i.e. i occurs in σ further than j.

3.2. The subgroup G_σ acts on strata of all four decompositions D_f, D_g, $D_{f,g}$ and $RD_{f,g}$, since it preserves both f and g. We first consider the natural action of G_σ on the projective space P^{n-1} of all lines in $\mathbb{C}^n$.

Lemma. *The set of all G_σ-orbits on P^{n-1} corresponds bijectively to the set $\mathcal{S}$ of all decreasing subsequences in σ. The orbit $O_{s,\sigma}$ corresponding to a decreasing subsequence s is biholomorphically equivalent to $(\mathbb{C}^*)^{l(s)} \times \mathbb{C}^{d(s)}$, where $l(s)$ is the reduced length of s and $d(s)$ is its domination. Moreover, for each orbit $O_{s,\sigma}$ there exists a subgroup $G_{s,\sigma} \subseteq G_\sigma$ biholomorphically equivalent to $(\mathbb{C}^*)^{l(s)+1} \times \mathbb{C}^{d(s)}$ such that its quotient by the scalar matrices (isomorphic to $O_{s,\sigma}$) acts on $O_{s,\sigma}$ freely and transitively.*

Proof. Let $s = (i_1, \dots, i_{l(s)+1})$ be a decreasing subsequence in σ and $D = (j_1, j_2, \dots, j_{d(s)})$ be the set of elements in σ dominated by elements of s. Then the orbit $O_{s,\sigma}$ consists of all lines whose spanning vectors in an appropriate standarizing basis for (f, g) have arbitrary nonvanishing coordinates with the numbers $i_1, \dots, i_{l(s)+1}$ and arbitrary (possibly vanishing) coordinates with the numbers $j_1, \dots, j_{d(s)}$; the rest of the coordinates vanish identically. Obviously, $O_{s,\sigma}$ is an orbit of G_σ and has the form $(\mathbb{C}^*)^{l(s)} \times \mathbb{C}^{d(s)}$, see 3.1. The orbit corresponding to the decreasing subsequence s contains the distinguished line p_s whose spanning vector has 1's for all $i_j \in s$ and 0's for the rest of coordinates. In order to describe $G_{s,\sigma} \subseteq G_\sigma$ we consider for each element i_k from s the (possibly empty) subset J_k of elements in D dominated precisely by i_k, i.e. all the elements that are dominated by i_k but not dominated by $i_1, \dots, i_{k-1}$. Consider now a subset in G_g consisting of the matrices with arbitrary entries in positions (i_k, J_k), arbitrary nonvanishing entries in positions (i_k, i_k), 1's on the rest of the main diagonal and zeros elsewhere. This set of matrices forms a subgroup $G_{s,\sigma}$ in G_g, which topologically is $(\mathbb{C}^*)^{l(s)+1} \times \mathbb{C}^{d(s)}$. The subgroup $G_{s,\sigma}$ acts freely and transitively on the set of all spanning vectors of lines in $O_{s,\sigma}$; one can check it by multiplying elements of $G_{s,\sigma}$ by a spanning vector of the distinguished line in $O_{s,\sigma}$. Thus, the quotient of $G_{s,\sigma}$ by the scalar matrices acts freely and transitively on $O_{s,\sigma}$.

A straightforward check shows that for any line (which corresponds to a point on P^{n-1}) there exists an appropriate decreasing subsequence s in σ such that the orbit $O_{s,\sigma}$ contains this line. ■

3.3. Remark. All lines belonging to an orbit $O_{s,\sigma}$ have the same dimensions of intersections with any subspace of the $+$-completion of (f, g). And conversely, lines with a given set of dimensions of intersections with the subspaces of the $+$-completion belong to the same orbit.

3.4. Lemma. i) *For any line L in an orbit $O_{s,\sigma}$ the quotient flags f_L and g_L are in the same relative position σ_s defined by the cyclic shift of σ wrt s and the reduction of the first element of s.*

ii) *The set F^s of all flags in F_n containing lines of $O_{s,\sigma}$ is diffeomorphic to $O_{s,\sigma} \times F_{n-1}$.*

Proof. i) The independence of the relative position of f_L and g_L of the choice of L in $O_{s,\sigma}$ follows from the transitivity of the action of $G_{s,\sigma}$ on $O_{s,\sigma}$. The relative position of f_L and g_L can be calculated easily if one takes L equal to the distinguished line $p_s \in O_{s,\sigma}$ (see the proof of Lemma 3.2).

ii) The set of all flags containing a fixed line L is isomorphic obviously to the set of all flags in $\mathbb{C}^n/L$, i.e. to F_{n-1}. Let us denote by $\nu\colon F_n \to P^{n-1}$ the natural map sending each flag onto its 1-dimensional subspace. The restriction of the projection ν onto the set F^s defines the structure of a fiber bundle with the fiber F_{n-1} and the base $O_{s,\sigma}$. This bundle is trivial for the following reason. The transitive and free action of $G_{s,\sigma}$ modulo scalar matrices on F^s defines another structure of a fiber bundle on F_s, which is transversal to the first one. Namely, its fibers are diffeomorphic to $O_{s,\sigma}$ and its base can be chosen as the set of all flags containing the distinguished line in $O_{s,\sigma}$. ■

4. Proofs of Theorems A and B

4.1. Proof of Theorem A. We proceed by induction on n. The space F_2 coincides with $\mathbb{C}P^1$; any pair of flags is just a pair of two (possibly coinciding) points. Each refined double stratum is a point, $\mathbb{C}^*$, or $\mathbb{C}$ (if the points coincide).

Let us consider now some refined double stratum St consisting of all flags with a fixed set of dimensions of intersections with all the subspaces of the $+$-completion of (f, g). We focus first on conditions satisfied by the 1-dimensional subspace of any flag in St. This 1-dimensional subspace must belong to some coordinate subspace and avoid in it at least one (but possibly more) coordinate hyperplanes. The set of all such lines is diffeomorphic to the product of a complex torus by a linear space. Actually, each of these lines belong to the image $\nu(\mathrm{St})$, i.e. is the 1-dimensional subspace of some flag in St. Moreover, the restricted projection $\nu\colon \mathrm{St} \to \nu(\mathrm{St})$ is a trivial fiber bundle, see Lemma 3.4. By Remarks 2.5–2.6, the set all flags in St with a fixed line is isomorphic to a refined double stratum in the space F_{n-1} of all complete flags in $\mathbb{C}^{n-1}$. Thus, St is diffeomorphic to the product of a complex torus by a linear space by some refined double stratum in F_{n-1}. Since any refined double stratum in F_{n-1} has the necessary form by the inductive hypothesis, Theorem A is proved. ■

4.2. Proof of Theorem B. Let us consider the space of complete flags F_n and a fixed standard pair (f, g) in relative position σ. We present an inductive construction of the refined double decomposition $RD_{f,g}$ of the space of complete flags, which exactly fits the main algorithm. The space F_n is decomposed into the disjoint union of F^s, where F^s is the union of all flags whose 1-dimensional subspaces belong to the orbit $O_{s,\sigma}$. Thus, the set of all F^s's is in 1-1-correspondence with the set of all decreasing subsequences in σ, and hence with the branches of Step 1 of the algorithm. Each F^s is diffeomorphic to $O_{s,\sigma} \times F_{n-1}$, by Lemma 3.4. Let us now subdivide each F^s. Namely, take a fiber (which is isomorphic to F_{n-1}) in F^s over some line $L \in O_{s,\sigma}$ and take the quotient flags f_L and g_L in this fiber. We consider their stabilizer G_{σ_s} and its orbits on the space P^{n-2}. Choosing an G_{σ_s}-orbit on P^{n-2} corresponding to a decreasing subsequence s' of σ_s, we fix the dimensions of intersections of the 1-dimensional subspace in a variable quotient flag with the $+$-completion of (f_L, g_L). Subsets of fibers that are mapped to this orbit form a fibration over $O_{s,\sigma}$. Moreover, the set $F_{s,s'}$ of flags in all fibers that are projected on this orbit forms a fibration over $O_{s,\sigma} \times O_{s',\sigma_s}$. Since G_σ modulo scalar matrices acts freely on $O_{s,\sigma}$, see Lemma 3.2, this fibration is trivial. Fixing the orbits of G_σ and G_{σ_s} is equivalent to fixing the dimensions of intersections of 1- and 2-dimensional subspaces of a variable flag with the $+$-completion of (f, g). Arguing inductively along the same lines, we get a sequence of fibrations $F^{s,s',\cdots}$ over the products of tori and linear subspaces. Each of these sets is characterized by the following conditions: the dimensions of intersections of the consecutive subspaces (from 0 up to some dimension) of a variable flag belonging to such a subset with the $+$-completion of (f, g) are fixed. Therefore, after n steps one sees that the sets $F^{s,s',\cdots}$ are precisely the refined double strata diffeomorphic to products of tori and linear subspaces, and they are in 1-1-correspondence with chains of permutations in the main algorithm. ■

4.3. Remark. The main algorithm describes this sequence of operations using the description of G_σ-orbits in Lemmas 3.2, 3.4. To avoid the reduction of elements and to keep track of elements in the original permutation we prefer to use blocking instead of the reduction.

5. The image of a Schubert cell in the projective space

and Theorem C

The proof of Theorem C and the modifications of the main algorithm follows easily from Theorem B and several additional statements.

5.1. Lemma. *Let $C_{1,\alpha}$ be a Schubert cell in D_f and let $\alpha(1) = k$. Then the image $\nu(C_{1,\alpha})$ is the $(k-1)$-dimensional affine subspace in P^{n-1} consisting of all lines that belong to the k-dimensional subspace of f but do not belong to its $(k-1)$-dimensional subspace.*

Proof. Follows immediately from the definition of $C_{1,\alpha}$. ■

5.2. Remark. Quite similarly, if $C_{\sigma,\beta} \in D_g$ and $\beta(1) = m$, then the image $\nu(C_{\sigma,\beta}) \subset P^{n-1}$ is the $(m-1)$-dimensional affine subspace of all lines lying in the m-dimensional subspace of g, but not in its $(m-1)$-dimensional subspace.

5.3. Lemma. *Under the assumptions of Lemma* 5.1 *the set $\mathcal{S}_\alpha$ of all G_σ-orbits constituting $\nu(C_{1,\alpha})$ corresponds to the set of all decreasing subsequences in σ having the following two additional properties*:

i) *any subsequence $s \in \mathcal{S}_\alpha$ contains the number k*;

ii) *any $s \in \mathcal{S}_\alpha$ does not contain any number greater than k.*

Proof. By Lemma 5.1, we must choose only those orbits that contain lines lying in the k-dimensional subspace of f, but not in its $(k-1)$-dimensional subspace. This is guaranteed exactly by conditions a) and b) above. ■

5.4. Remark. Similarly, let $\sigma = i_1 \dots i_n$ and $\beta(1) = m$, then the set $\mathcal{S}^\beta$ of all G_σ-orbits constituting $\nu(C_{\sigma,\beta})$ corresponds to the set of all decreasing subsequences in σ having the following two additional properties:

i) any $s \in \mathcal{S}^\beta$ contains the number i_m;

ii) any $s \in \mathcal{S}^\beta$ does not contain the numbers $i_{m+1}, \dots, i_n$.

5.5. Corollary. *The set $\mathcal{S}^\beta_\alpha$ of all G_σ-orbits constituting $\nu(C_{1,\alpha}) \cap \nu(C_{\sigma,\beta})$ corresponds to the set of all decreasing subsequences in σ located on the interval between the numbers k and i_m and containing both of them.*

So, if $k = i_m$, then there is only one decreasing subsequence consisting of the element $k = i_m$. If $k > i_m$, then the set $\mathcal{S}^\beta_\alpha$ is empty. The topology and the dimensions of these orbits are described in Lemma 3.2.

5.6. Proof of Theorem C. By definition, the refined double decomposition $RD_{f,g}$ of the space F_n subdivides each Schubert cell $C_{1,\alpha}$, $C_{\sigma,\beta}$, and their intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$. Statements 5.1-5.3 describe explicitly the decreasing subsequences that must be used on a step of the algorithm in order to select those refined double strata which are contained in one of the above three types of spaces. We also know what happens to the permutations α

and β after a single step of the algorithm. The modifications given in Introduction just present this sequence of operations formally, using blocking instead of the reduction. ■

5.7. Proof of Proposition 0.16. Part i) of the proposition is just an obvious corollary of Theorem B and the description of the modifications of the algorithm. Indeed, all restricted chains starting at some permutation $\tilde{\sigma}$ are obtained by the same procedure as the one for σ itself ignoring all the blocked elements in $\tilde{\sigma}$ and (in modifications of the main algorithm) the first i elements in α and β. Part ii) is obtained by a slight generalization of Lemma 3.4 and the proof of Theorem B via an inductive argument. ■

6. Adjacency of orbits in P^{n-1} and Theorem D

6.1. Let us describe combinatorially the adjacency of G_σ-orbits on P^{n-1}. We consider the following partial order on the set of decreasing subsequences in σ. A decreasing subsequence s_1 is said to be *less or equal* than a decreasing subsequence s_2 (notation $s_1 \vdash s_2$) if for each element of s_1 there exists an element in s_2 that either dominates it or coincides with it.

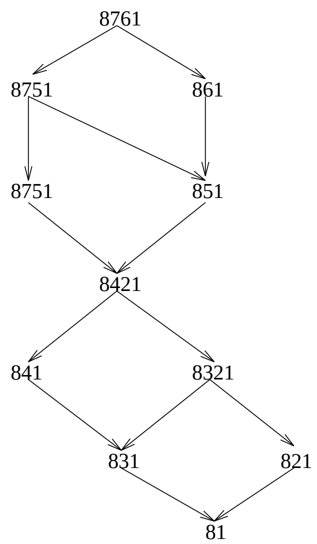

Fig.8. The natural partial order on the set of decreasing subsequences in 83427561

The restrictions of this partial order to the sets $\mathcal{S}_\alpha$, $\mathcal{S}^\beta$, and $\mathcal{S}_\alpha^\beta$ define the corresponding partial orders on these subsets.

Example. The partial order on the set of the decreasing subsequences starting at 8 and ending at the last position in the permutation $\sigma = 83427561$ is shown on Fig. 8.

The *maximal* decreasing subsequence of σ, denoted by $s_{\max}(\sigma)$, is obtained in the following way. Its first element is the maximal element of σ. The ith element of $s_{\max}(\sigma)$ is the maximal element of σ standing to the right of the $(i-1)$th element of $s_{\max}(\sigma)$. Similarly one defines the maximal decreasing subsequence of any subset of elements in σ.

6.2. Lemma. *The partial order $\vdash$ describes the adjacency of the G_σ-orbits on P^{n-1}, $\nu(C_{1,\alpha})$, $\nu(C_{\sigma,\beta})$, and $\nu(C_{1,\alpha}) \cap \nu(C_{\sigma,\beta})$, respectively. In each of these cases there is no gaps in dimensions, i.e. there exist strata of all complex dimensions between the minimal and the maximal.*

Proof. The statement concerning the adjacency follows immediately from the description of orbits, see Lemma 3.2. The absence of gaps in dimensions is settled by the following observation. It suffices to consider the first case; the other cases are similar. Take any decreasing subsequence s corresponding to an orbit of a dimension bigger than one. In this case s contains either more than one element, or a single element having a nonempty set of dominated elements. We now construct a decreasing subsequence s' corresponding to an orbit whose dimension is less by one, and which lies in the closure of the first orbit. If s consists of just one element, then s' is the maximal decreasing subsequence of the set of dominated elements, see the definition in 6.1. If the number of elements in s is bigger than one, we replace the last element by the maximal decreasing subsequence of the set of elements dominated precisely by the last element, but not by the previous ones. If this set is empty then we just drop the last element. All checks are straightforward. ■

6.3. Consider a 1-parameter family of subspaces $L(t)$, $t \in [0,1]$, such that for all $t \in [0,1)$ the subspaces $L(t)$ have a given set of dimensions of intersections with all subspaces in the $+$-completion of (f,g), and $L(1)$ has the same set of dimensions of intersections with the individual subspaces of f and g, but possibly bigger dimensions of intersections with the rest of the subspaces in the $+$-completion. Denote by $\pi(t)$ the permutation presenting the position of $g_{L(t)}$ relative to $f_{L(t)}$.

Lemma. *For any $t \in [0,1)$ one has $\pi(t) \succeq \pi(1)$.*

Proof. The Bruhat order $\succeq$ describes the adjacency of Schubert cells. Thus, the assertion of the lemma is equivalent to the following inequality: for any two subspaces K and M of f and g, respectively, $\dim((K/L(1)) \cap (M/L(1))) \geqslant \dim((K/L(t)) \cap (M/L(t)))$. Applying formula $(*)$ of §2 to both sides of the inequality and taking into account the conditions $\dim(K \cap$

$L(1)) = \dim(K \cap L(t))$ and $\dim(M \cap L(1)) = \dim(M \cap L(t))$, we get the necessary result. ■

6.4. Proof of Theorem D. Let us show that the closure of any given stratum $\mathrm{St} \in C_{1,\alpha} \cap C_{\sigma,\beta}$ is contained in the union of strata whose chains are less than the chain of St in the adjacency partial order, see 0.18. Indeed, consider a path $\gamma\colon [0,1] \to C_{1,\alpha} \cap C_{\sigma,\beta}$ such that $\gamma[0,1)$ belongs to St and $\gamma(1)$ belongs to the boundary of St. This means that there exists at least one subspace $L(1)$ in $\gamma(1)$ such that the dimension of its intersection with some subspace in the $+$-completion of (f,g) is bigger than that for the subspaces $L(t)$ of the same dimension in $\gamma(t)$, $t \in [0,1)$. Now, applying Lemma 6.3 we get precisely the necessary statement. ■

7. Combinatorial versus topological adjacency

7.1. Theorem D gives an obvious sufficient combinatorial condition for the refined double decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$ to define its stratification (i.e., for the closure of each stratum to belong to the union of strata of lower dimensions).
Criterion. *If for any two refined double strata* St_1 *and* St_2 *in* $C_{1,\alpha} \cap C_{\sigma,\beta}$ *such that* $\mathrm{St}_1 \prec \mathrm{St}_2$ *the dimension of* St_1 *is less than that of* St_2, *then the refined double decomposition provides the stratification of* $C_{1,\alpha} \cap C_{\sigma,\beta}$.

7.2. Examples. All intersections of top-dimensional Schubert cells in F_4 are stratifications. On the contrary, the simplest example of nonstratification is provided by the refined double decomposition of the intersection $C_{1234,4231} \cap C_{4231,4231}$; it consists of the three strata, namely, of $\mathbb{C}^* \times \mathbb{C}^2$ and of the two copies of $(\mathbb{C}^*)^3 \times \mathbb{C}$. The intersection of the two opposite top-dimensional Schubert cells in F_5 contains the fragment shown on Fig.9a, which apparently shows that it is not a stratification. Still another example is shown on Fig.9b. This example shows also that not all branches of the algorithm reach the bottom level.

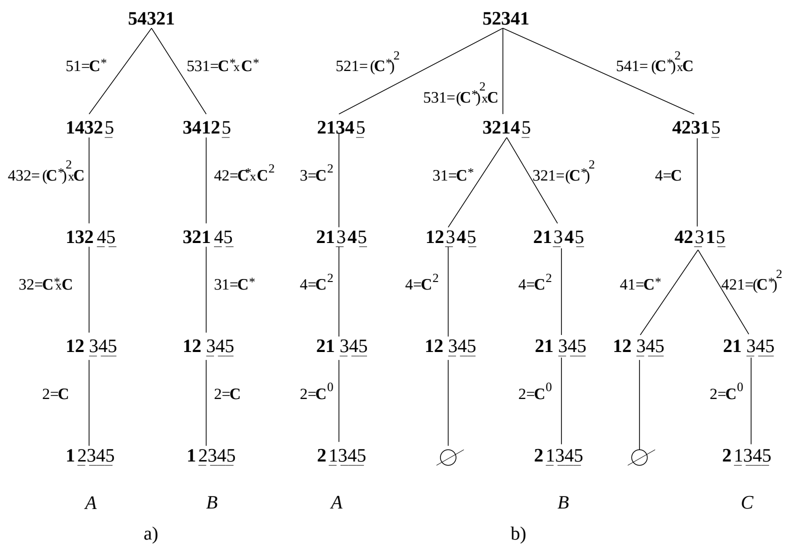

FIG.9. EXAMPLES OF NONNICE REFINED DOUBLE DECOMPOSITIONS. a) A NONNICE FRAGMENT FOR $\alpha = \beta = \sigma = 54321$; b) NONNICE DECOMPOSITION FOR $\alpha = 53421$, $\beta = 53412$, $\sigma = 52341$

Let us describe also in more detail the above simplest example. We use projective complete flags on $\mathbb{C}P^3$ instead of complete flags in the linear space $\mathbb{C}^4$. The strata are defined by the following conditions:

$A = (\mathbb{C}^*) \times \mathbb{C}^2$ corresponding to $\mathbf{4231} \to \mathbf{123}\underline{4} \to \mathbf{1}\underline{2}\mathbf{3}\underline{4} \to \mathbf{1}\underline{234} \to \underline{1234}$: the point of the variable flag h belongs to the line containing the points of f and g, and the line of h belongs to the 2-plane containing the lines of f and g.

$B = (\mathbb{C}^*)^3 \times \mathbb{C}$ corresponding to $\mathbf{4231} \to \mathbf{213}\underline{4} \to \mathbf{1}\underline{2}\mathbf{3}\underline{4} \to \mathbf{1}\underline{234} \to \underline{1234}$: the line o h belongs to the 2-plane containing the lines of f and g, and the point of h does not belong to the line containing the points of f and g.

$C = (\mathbb{C}^*)^3 \times \mathbb{C}$ corresponding to $\mathbf{4231} \to \mathbf{321}\underline{4} \to \mathbf{3}\underline{2}\mathbf{1}\underline{4} \to \mathbf{1}\underline{234} \to \underline{1234}$: the line of h passes through the origin and the point of h does not belong to the 2-plane containing the lines of f and g.

If we want to get a stratification, we must split B into B_1 and B_2, where B_1 is given by an additional condition that the line of h passes through the origin and B_2 is given by the condition that the line of h does not pass through the origin. We also split A into A_1 and A_2 given precisely by the same additional conditions as B_1 and B_2, respectively. Then A_1, which is diffeomorphic to $\mathbb{C}^* \times (\mathbb{C} \setminus \{2 \text{ points}\})$, coincides with the closure of C in A and B_1 coincides with the closure of C in B. Pay attention to the fact that A_1 is no longer the product of a complex torus by a complex linear space.

7.3. Using the previous example one can construct the following example, which reveals an essential difference between combinatorial and topological adjacencies. Namely, we take the refined double decomposition of the intersection $C_{12345,54231} \cap C_{54321,54231}$ and consider the two strata A and B defined by the following conditions:

$A = (\mathbb{C}^*)^6 \times \mathbb{C}$ corresponding to $\mathbf{54321} \to \mathbf{4231}\underline{5} \to \mathbf{321}\underline{45} \to \mathbf{3}\underline{2}\mathbf{1}\underline{45} \to \mathbf{1}\underline{2345} \to \underline{12345}$;

$B = (\mathbb{C}^*)^4 \times \mathbb{C}^2$ corresponding to $\mathbf{54321} \to \mathbf{4132}\underline{5} \to \mathbf{213}\underline{45} \to \mathbf{1}\underline{2}\mathbf{3}\underline{45} \to \mathbf{1}\underline{2345} \to \underline{12345}$.

Then $B \prec A$, but B does not intersect the closure of A, which means that A is not adjacent to B.

7.4. To complete the section, we provide the only known to the authors example of a hard $C_{1,\alpha} \cap C_{\sigma,\beta}$. The intersection $C_{123456789,987654321} \cap C_{987654321,987654321}$ contains the following two strata:

$$\mathbf{987654321} \xrightarrow{9751} \mathbf{78561432}\underline{9} \xrightarrow{86432} \mathbf{7654132}\underline{89} \xrightarrow{76532} \mathbf{653412}\underline{789} \xrightarrow{6542} \mathbf{54321}\underline{6789} \xrightarrow{5431} \mathbf{4312}\underline{56789} \xrightarrow{42} \mathbf{231}\underline{456789} \xrightarrow{31} \mathbf{21}\underline{3456789} \xrightarrow{21} \mathbf{1}\underline{23456789}$$

and

$$\mathbf{987654321} \xrightarrow{971} \mathbf{78165432}\underline{9} \xrightarrow{865432} \mathbf{7615432}\underline{89} \xrightarrow{765432} \mathbf{651432}\underline{789} \xrightarrow{654342} \mathbf{54132}\underline{6789} \xrightarrow{5432} \mathbf{4312}\underline{56789} \xrightarrow{42} \mathbf{231}\underline{456789} \xrightarrow{31} \mathbf{21}\underline{3456789} \xrightarrow{21} \mathbf{1}\underline{23456789}.$$

(The numbers above the arrows present decreasing subsequences used on the corresponding steps.) The first chain is bigger in the adjacency partial order than the second chain, i.e. its permutations are bigger or equal in the usual Bruhat order than the corresponding permutations in the second chain. The first stratum has the form $(\mathbb{C}^*)^{20} \times \mathbb{C}^8$, while the second has the form $(\mathbb{C}^*)^{22} \times \mathbb{C}^7$. Thus, we get an example of a hard $C_{1,\alpha} \cap C_{\sigma,\beta}$.

7.5. Remark. Several natural conjectures about the combinatorics of refined double decompositions turned out to be wrong. For example, let us consider the set $\widetilde{\mathcal{S}} \subset \mathcal{S}$ used on the first step of the algorithm in the decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$ such that for any decreasing subsequence $s \in \widetilde{\mathcal{S}}$ the intersection $C_{1,\alpha'} \cap C_{\sigma_s,\beta'}$ is nonempty. A natural conjecture would be that $\widetilde{\mathcal{S}}$ is an interval in the partial order $\vdash$ on $\mathcal{S}$, see 6.1. This turns to be wrong in the examples $C_{12345,546213} \cap C_{654321,654213}$ and $C_{12345,53412} \cap C_{52341,53412}$. Moreover, in the latter case the set $\widetilde{\mathcal{S}}$ consists of $51 = \mathbb{C}^*$, $531 = (\mathbb{C})^{*2} \times \mathbb{C}$ and $541 = (\mathbb{C})^{*2} \times \mathbb{C}^2$, and thus even has a gap in dimensions of its elements.

Several combinatorial conjectures based on our consideration and questions connected with refined double decompositions are presented in §11.

8. On the mixed Hodge structure in the cohomology of intersections of Schubert cells

8.1. Remark. All the elements in the pth cohomology with compact supports of $(\mathbb{C}^*)^k \times \mathbb{C}^l$ have Hodge indices $p-(k+l), p-(k+l)$, and its E-polynomial (see 0.21) equals $(z-1)^k z^l$, where $z = uv$.

8.2. The proof of Corollary 0.22 is based on the following lemma.

Lemma (see e.g. [Du]. *Let $Y \subset X$ be a closed quasiprojective submanifold of a quasiprojective manifold X and let $U = X \setminus Y$. Then the exact sequence of the pair (X, Y) for the cohomology with compact supports*

$$(\dagger) \qquad \cdots \to H_c^k(U) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(U) \to \dots$$

is an exact sequence of Hodge structures, that is, all differentials respect Hodge indices.

8.3. Proof of Corollary 0.22i. Applying Lemma 8.2, one gets a simple inductive proof of the first part of Corollary 0.22. Indeed, let us consider the standard filtration $F^1 \subset F^2 \subset \cdots \subset F^m = C_{1,\alpha} \cap C_{\sigma,\beta}$, see 0.24, where F^1 is the union of all minimal refined double strata in the adjacency partial order, F^2 is the union of all minimal strata and all strata that are adjacent only to strata in F^1, i.e. they form two bottom levels in the adjacency order, and so on. The difference between any F^i and F^{i-1} is the disjoint union of strata, by Theorem D. Now we prove (by induction on i) that all F^i have $h_k^{p,q} = 0$ if $p \neq q$. Indeed, by Remark 8.1, F^1, which is the disjoint union of refined double strata, has this property. Next, we consider the long exact sequence (†) for the pair (F^2, F^1) and use Lemma 8.2 and the fact that $F^2 \setminus F^1$ is again the disjoint union of strata to get the necessary property for F^2, and so on. ■

Remark. Exactly the same argument works for any quasiprojective $V = \bigcup V_i$ where V_i satisfy $h_k^{p,q}(V_i) = 0$ for $p \neq q$.

8.4 Lemma (see [Du]. *If a quasiprojective V is decomposed in the disjoint union $V = \bigcup_i V_i$, where V_i are also quasiprojective, then $\chi^{p,q}(V) = \sum_i \chi^{p,q}(V_i)$ for all p, q, or equivalently, $E_V(u,v) = \sum_i E_{V_i}(u,v)$.*

8.5. Proof of Corollary 0.22ii. To prove 0.22ii it suffices to apply Lemma 8.4 and Remark 8.1 to the refined double decomposition of any $C_{1,\alpha} \cap C_{\sigma,\beta}$. ■

8.6. To prove Corollary 0.23 we need the following additional statement.

Suppose X is an arbitrary complex manifold; denote

$$\Pi_X(z) = \sum_i \chi^{ii}(X) z^i.$$

Lemma. *Let $X = U \times Y$ and the following assumptions be true*:
1) $\chi^{ij}(Y) = 0$ *for* $i \neq j$;
2) $h_k^{ij}(U) = 0$ *for* $i \neq j$.
Then i) $\Pi_X(z) = \Pi_U(z)\Pi_Y(z)$,
ii) $\chi^{ij}(X) = 0$ for $i \neq j$.

Proof. Evidently, assumption 2) yields

$$h_k^{ij}(X) = \sum_{p,q,l} h_l^{pq}(U) h_{k-l}^{i-p,j-q}(Y) = \sum_{p,l} h_l^{pp}(U) h_{k-l}^{i-p,j-p}(Y).$$

Hence,

$$\begin{aligned}\chi^{ij}(X) &= \sum_k (-1)^k h_k^{ij}(X) = \sum_{p,l} (-1)^l h_l^{pp}(U) \sum_k (-1)^{k-l} h_{k-l}^{i-p,j-p}(Y) \\ &= \sum_{p,l} (-1)^l h_l^{pp}(U) \chi^{i-p,j-p}(Y) = \sum_p \chi^{pp}(U) \chi^{i-p,j-p}(Y).\end{aligned}$$

Taking into account this relation, we see that assertion ii) follows immediately from assumption 1). Next,

$$\begin{aligned}\Pi_X(z) &= \sum_i \sum_p \chi^{pp}(U) \chi^{i-p,i-p}(Y) z^i \\ &= \left(\sum_p \chi^{pp}(U) z^p\right)\left(\sum_j \chi^{jj}(Y) z^j\right) = \Pi_U(z)\Pi_Y(z),\end{aligned}$$

and assertion i) follows. ■

8.7. Proof of Corollary 0.23. To prove the inductive formula of 0.23 we use part ii) of Proposition 0.16 for one step of the algorithm, and get

$$C_{1,\alpha} \cap C_{\sigma,\beta} = \coprod_{s \in \mathcal{S}} O_{s,\sigma} \times (C_{1,\alpha'} \cap C_{\sigma_s,\beta'}),$$

where $O_{s,\sigma}$ is the orbit in P^{n-1} corresponding to the decreasing subsequence s and $C_{1,\alpha'} \cap C_{\sigma_s,\beta'}$ is the intersection of Schubert cells in F_{n-1}, see Lemma 3.2. Then, from Corollary 0.22i, Lemma 8.6, and Remark 8.1 we see that the E-polynomial of $O_{s,\sigma} \times (C_{1,\alpha'} \cap C_{\sigma_s,\beta'})$ is equal to the product of $z^{d(s)}(z-$

$1)^{l(s)}$ and $E_{C_{1,\alpha'}\cap C_{\sigma_s,\beta'}}$. Thus, we obtain the following inductive formula for the E-polynomials:

$$E_\sigma^{\alpha,\beta} \equiv E_{C_{1,\alpha}\cap C_{\sigma,\beta}}(u,v) = \sum_{s\in\mathcal{S}} z^{d(s)}(z-1)^{l(s)} E_{\sigma_s}^{\alpha',\beta'}.$$

■

8.8. The Leray spectral sequence can be used successfully for a filtration of a topological space X satisfying the following condition. Let $F \subset X$ be a closed subspace of the filtration considered. One has the following natural short exact sequence of sheaves: $0 \to {}_{j!}\mathbb{C} \to \mathbb{C} \to {}_{i_*}\mathbb{C} \to 0$, where i and j denote the obvious inclusions $F \xrightarrow{i} X \xleftarrow{j} X \setminus F$ and $\mathbb{C}$ is the usual constant sheaf on X. The inclusion $F \hookrightarrow X$ is called *good* if the sheaves ${}_{i_*}\mathbb{C}$ and ${}_{j!}\mathbb{C}$ coincide with constant sheaves on F and $X \setminus F$, respectively. An example of a bad inclusion is the inclusion of the line $x = 0$ in the space $\mathbb{R}^2 \setminus \{y \neq 0\}$. Inclusions of quasiprojective varieties are always good. Consider now a topological space that is an arbitrary union of refined double strata in some $C_{1,\alpha}\cap C_{\sigma,\beta}$. Warning: arbitrary unions of refined double strata are not always quasiprojective varieties. At the same time, all lower intervals in the adjacency partial order, unions of lower intervals and differences between such unions present quasiprojective varieties. The adjacency partial order defines the stricture of a poset on any subset of the strata involved. Any union of lower intervals in this poset defines a closed subset in the above topological space and any sequence of embedded systems of lower intervals defines a filtration by closed subspaces.

8.9. Proof of Theorem E. Let us first cover the nice case. We consider the Leray spectral sequence for the cohomology with compact supports associated with the standard filtration of a nice $C_{1,\alpha}\cap C_{\sigma,\beta}$. By definition, we consider the filtration of $C_{1,\alpha}\cap C_{\sigma,\beta}$ by closed subsets F^i such that F^i is formed by the union of all strata of (complex) dimension less or equal to i. The (p,q)th entry of the first page contains the $(p+q)$th cohomology with compact supports of F^{p+1} mod F^p, which coincides with the direct sum of the $(p+q)$th cohomology with compact supports of all refined double strata of dimension p. Let $(\mathbb{C}^*)^k \times (\mathbb{C})^l$ be one of the strata of minimal dimension in the considered nice $C_{1,\alpha}\cap C_{\sigma,\beta}$. It follows from Proposition 9.1 below that the $(p+q)$th term is the direct sum of several copies of the $(p+q)$th cohomology of some $(\mathbb{C}^*)^{k+2(p-1)} \times \mathbb{C}^{l-(p-1)}$. The differential d_1 coincides with the connecting homomorphism in the exact sequence of triples $H_c^{p+q}(F^{p+1},F^p) \to H_c^{p+q+1}(F^{p+2},F^{p+1})$. The crucial property of differentials in this spectral sequence for the filtration by quasiprojective subvarieties is that they respect the Hodge weights (filtrations) in the cohomology, see [Dl2,Dl3]. By Remark 8.1, the Hodge weights of all elements of the (p,q)th term are equal to $p-(k+l), p-(k+l)$. Thus, the only nontrivial differential in this spectral sequence can be d_1, since only d_1 respects the weights.

Now, let us modify this proof for the almost nice case. As it was mentioned before, the filtration of an almost nice $C_{1,\alpha} \cap C_{\sigma,\beta}$ by dimensions is a filtration by closed subsets, and rows of the corresponding spectral sequence contain the cohomology with compact supports of unions of strata of a given dimension. The union of all strata of a given dimension in the almost nice case is a quasiprojective variety with the standard filtration. The corresponding Leray spectral sequence, obviously, degenerates at E_1, since by Proposition 9.1 below all refined double strata of the same dimension have the same form and none of the differentials in this additional spectral sequence respects the Hodge weights. Thus, one gets the same cohomology with compact supports for the union of these strata as for their disjoint union. Therefore, exactly the same degeneracy arguments as above apply. ■

8.10. Examples. The authors have calculated Betti numbers for all intersections of top-dimensional Schubert cells in F_3 and F_4, which are all nice, using a different method. (In dimension 5 the intersection of the opposite top-dimensional Schubert cells is only almost nice, see Fig.9a.) The only interesting cases in these dimensions are $\sigma = 321$, $\sigma = 4231, 3421, 4312$ or 4321. These Betti numbers show that in all these cases the differential d_1 acts in the maximal way, i.e. that all rows of the first page form acyclic complexes except for the top dimension, see an example on Fig.10. The second page in all these examples contains only one nontrivial column, and the cohomology in this column has pure weights; thus, the mixed Hodge structure is pure.

Let us consider the filtration $F^3 \supset F^2 \supset F^1$, where $F^1 = A$, $F^2 = A \cup B$, $F^3 = C_{1234,4231} \cap C_{4231,4231}$, and use the Leray sequence, see Fig.11 and Example 7.2. This filtration is precisely the one obtained from the adjacency partial order. From the geometrical description of the strata A, B, C one can see that the nontrivial differential d_1 acting from the first column to the second one (pay attention to the weights) kills the whole first column, thus leaving us with the second page, which again contains the pure Hodge structure.

8.11. Counterexample to the purity of the mixed Hodge structure of $C_{1,\alpha} \cap C_{\sigma,\beta}$. In all previous examples the so-called *purity* of the mixed Hodge structure holds, which in the case of $C_{1,\alpha} \cap C_{\sigma,\beta}$ means that the only nonvanishing Hodge numbers are $h_i^{i,i}$ and they are equal to $h_i^{i,i} = (-1)^i \chi_c^{i,i}$. However, this turned out to be false in general (for arbitrary $C_{1,\alpha} \cap C_{\sigma,\beta}$) for the following reason. If purity holds for some $C_{1,\alpha} \cap C_{\sigma,\beta}$, then the coefficients of $E_\sigma^{\alpha,\beta}(z)$ must have strictly alternating signs. Recently, using computer, B. Boe has found 4 intersections in $\mathfrak{S}_6$ whose E-polynomials do not have strictly alternating signs. Two simplest of these examples have

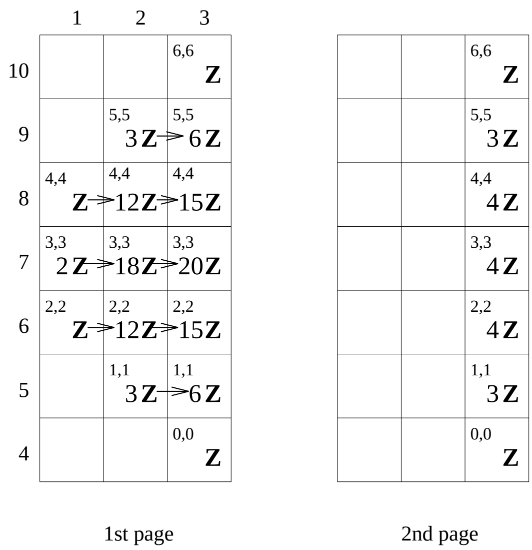

FIG.10. LERAY SPECTRAL SEQUENCE FOR THE NICE INTERSECTION $\alpha=\beta=\sigma=4321$. ARROWS SHOW NONTRIVIAL DIFFERENTIALS, PAIRS OF NUMBERS IN THE LEFT UPPER CORNERS ARE HODGE WEIGHTS

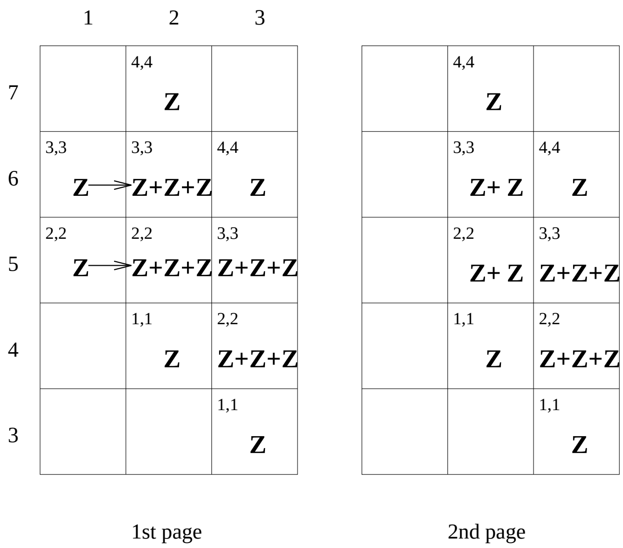

FIG.11. LERAY SPECTRAL SEQUENCE FOR THE SIMPLEST NONNICE INTERSECTION $\alpha=\beta=\sigma=4231$; ARROWS SHOW NONTRIVIAL DIFFERENTIALS

been checked by the authors using the above method, and here is their description.

1) $C_{123456,563412} \cap C_{654312,654321}$ consists of the following 24 refined double strata: 1 copy of $(\mathbb{C}^*)^3 \times \mathbb{C}^5$, 6 copies of $(\mathbb{C}^*)^5 \times \mathbb{C}^4$, 10 copies of $(\mathbb{C}^*)^7 \times \mathbb{C}^3$, 6 copies of $(\mathbb{C}^*)^9 \times \mathbb{C}^2$, and 1 copy of $(\mathbb{C}^*)^{11} \times \mathbb{C}$. The corresponding E-polynomial is equal to $q(-1+5q-11q^2+13q^3-7q^4-q^5+q^6+7q^7-13q^8+11q^9-5q^{10}+q^{11})$, and its coefficients do not have strictly alternating signs.

2) $C_{123456,563412} \cap C_{654321,654321}$ consists of the following 41 refined double strata: 1 copy of $(\mathbb{C}^*)^2 \times \mathbb{C}^5$, 5 copies of $(\mathbb{C}^*)^4 \times \mathbb{C}^4$, 12 copies of $(\mathbb{C}^*)^6 \times \mathbb{C}^3$, 15 copies of $(\mathbb{C}^*)^8 \times \mathbb{C}^2$, 7 copies of $(\mathbb{C}^*)^{10} \times \mathbb{C}$, and 1 copy of $(\mathbb{C}^*)^{12}$. This E-polynomial is equal to $1-5q+11q^2-13q^3+8q^4-q^5-q^6-q^7+8q^8-13q^9+11q^{10}-5q^{11}+q^{12}$. Considerations of the adjacency partial order reveal that both of these examples are almost nice, but not nice. Since by Theorem E the corresponding Leray spectral sequence degenerates at E_2, it will be interesting to calculate d_1, and therefore the cohomology of these intersections.

9. Combinatorial properties and applications of E-polynomials

9.1. Proposition. *For any nonempty stratum, the sum of the (complex) dimension of the torus and the doubled dimension of the linear space equals* $\operatorname{lng}(\alpha)+\operatorname{lng}(\beta)-\operatorname{lng}(\sigma)$, *where* lng *is the inversion length of a permutation. In particular, all strata of the same dimension have the same form.*

Example. For the intersection $C_{1234,4231} \cap C_{4231,4231}$ we get $\operatorname{lng}(\alpha) = \operatorname{lng}(\beta) = \operatorname{lng}(\sigma) = 5$, and $l(\mathrm{St}) + 2d(\mathrm{St}) = 5$ for all strata.

9.2. In order to prove Proposition 9.1, we start with the following lemma relating the length of a permutation σ to the length of σ_s (see Lemma 3.4).

Lemma. *Let* $s = (i_1, \dots, i_k)$, *then*

$$\operatorname{lng}(\sigma_s) - \operatorname{lng}(\sigma) = l(s) + 2d(s) - i_1 - \sigma^{-1}(i_k) + 2,$$

where $l(s)$ *and* $d(s)$ *are the reduced length and the domination of* s, *respectively.*

Proof. Indeed, let us denote by I_j the set of consequtive elements of σ lying between i_j and i_{j-1} (we set $i_0 = 0$). When passing from σ to σ_s, we loose all the inversions containing i_1, and, for each i_j, $2 \leqslant j \leqslant k$, all the inversions (i, i_j) such that $i \in I_j$ and $i > i_j$. On the other hand, for each i_j, $2 \leqslant j \leqslant k$, we acquire new inversions of the form (i_j, i) such that $i \in I_j$

and $i < i_j$. So,

$$\mathrm{lng}(\sigma_s) - \mathrm{lng}(\sigma) = \sum_{j=2}^{k} \#\{i \in I_j : i < i_j\} -$$
$$\sum_{j=2}^{k} \#\{i \in I_j : i > i_j\} - \#\{i \in I_1 : i > i_1\} - \#\{i \notin I_1 : i < i_1\} =$$
$$\sum_{j=1}^{k} \#\{i \in I_j : i < i_j\} - \sum_{j=1}^{k} \#\{i \in I_j : i > i_j\} - \#\{i : i < i_1\}.$$

Evidently, the first term in the right hand side equals $d(s)$, while the last term equals $i_1 - 1$. Next, the sum of the first and the second terms is $\sum_{j=1}^{k} \#\{i : i \in I_j\}$, and hence it equals $n - k - (n - \sigma^{-1}(i_k)) = \sigma^{-1}(i_k) - k$. Taking into account that $l(s) = k - 1$ (see 0.8), we get the statement of the lemma. ■

Example. Let $\sigma = 46237518$ and the decreasing subsequence in it be $(6, 3, 1)$, then $\sigma_s = 4321657$. We have $\mathrm{lng}(\sigma) = 12$, $\mathrm{lng}(\sigma_s) = 7$. On the other hand, $d(s) = 2$, $l(s) = 2$, $i_1 = 6$, $\sigma^{-1}(i_k) = 7$. So, $7 - 12 = 4 + 2 - 6 - 7 + 2$.

9.3. Proof of Proposition 9.1. The proof is done by induction. In the cases $n = 1$ and $n = 2$ the formula is trivially true.

Step of induction. For any nonempty stratum $\mathrm{St} \in C_{1,\alpha} \cap C_{\sigma,\beta}$ we want to prove that $l(\mathrm{St}) + 2d(\mathrm{St}) = \mathrm{lng}(\alpha) + \mathrm{lng}(\beta) - \mathrm{lng}(\sigma)$, where $l(\mathrm{St})$ is the total length of St and $d(\mathrm{St})$ is its total domination (see 0.11). Assume that the chain corresponding to St starts with the permutation σ, and that the next permutation of the chain is σ_s. By the inductive hypothesis, we have $l(\mathrm{St}') + 2d(\mathrm{St}') = \mathrm{lng}(\alpha') + \mathrm{lng}(\beta') - \mathrm{lng}(\sigma_s)$, where St' is the refined double stratum in $C_{1,\alpha'} \cap C_{\sigma_s,\beta'}$ corresponding to the subchain of the initial chain starting with σ_s, α' and β' are obtained from α and β, respectively, by the reduction of the first element, see Proposition 0.16. Subtracting the above two formulas and applying Theorem B, we arrive at

$$l(s) + 2d(s) = \mathrm{lng}(\alpha) - \mathrm{lng}(\alpha') + \mathrm{lng}(\beta) - \mathrm{lng}(\beta') - \mathrm{lng}(\sigma) + \mathrm{lng}(\sigma_s).$$

Obviously, $\mathrm{lng}(\alpha) - \mathrm{lng}(\alpha') = \alpha(1) - 1$ and $\mathrm{lng}(\beta) - \mathrm{lng}(\beta') = \beta(1) - 1$. Hence, by Lemma 9.2, we have to prove that $\alpha(1) + \beta(1) = i_1 + \sigma^{-1}(i_k)$. However, by Theorem C, $i_1 = \alpha(1)$ and $\sigma^{-1}(i_k) = \beta(1)$, and the result follows. ■

9.4. Remark. The assertion of Proposition 9.1 remains valid for intersections of Schubert cells in all flag spaces G/B.

9.5. Lemma. *The intersection $C_{1,\alpha} \cap C_{\sigma,w_0}$ is isomorphic to $C_{1,\alpha} \cap C_{w_0,w_0\sigma w_0} \times C_{1,\sigma w_0}$. Thus, $C_{1,\alpha} \cap C_{\sigma,w_0}$ is nonempty if and only if $C_{1,\alpha} \cap C_{w_0,w_0\sigma w_0}$ is nonempty, that is, in the case $\alpha \succeq w_0\sigma$, where $\succeq$ denotes the usual Bruhat order. If $C_{1,\alpha} \cap C_{\sigma,w_0}$ is nonempty, then its dimension equals* $\operatorname{lng}(\alpha)$.

Proof. See [KL2] 1.4, p.187. ■

9.6. Corollary. *Let $s_1 \vdash s_2$ be two decreasing subsequences with coinciding first and last elements, then $\sigma_{s_1} \preceq \sigma_{s_2}$ and $w_0\sigma_{s_1} \succeq w_0\sigma_{s_2}$. Thus, if $C_{1,\alpha'} \cap C_{\sigma_{s_1},w_0}$ is nonempty, then $C_{1,\alpha'} \cap C_{\sigma_{s_2},w_0}$ is also nonempty.*

Proof. The set of all decreasing subsequences in σ with fixed first and last elements describes the set of all G_σ-orbits that have a fixed set of dimensions of intersections with the subspaces of f and g, see Corollary 5.5. Moreover, by Lemma 6.2, the partial order $\vdash$ describes their adjacency. Thus, any 1-parameter family of lines $l(t)$, $t \in [0,1]$, such that $l(t) \in O_{\sigma,s_2}$ for $t \in [0,1)$ and $l(1) \in O_{\sigma,s_1}$ satisfies the assumptions of Lemma 6.3 and gives the necessary assertion. ■

Example. For $\sigma = 83427561$ and $s_1 = (8,3,2,1)$, $s_2 = (8,7,5,1)$ we get $\sigma_{s_1} = 3241756 \vdash \sigma_{s_2} = 7342516$.

9.7. We now start proving Proposition 0.27 in the case $\beta = w_0$ (or $\alpha = w_0$, which is the same), i.e. for intersections of pairs of Schubert cells one of which has the top dimension. This class of pairwise intersections is closed under the projections involved in our algorithm, since the subset of flags in a top-dimensional cell containing some fixed line is isomorphic to the top-dimensional cell in the quotient space.

We proceed by induction on n. Cases $n = 1$ or 2 are obvious.

Step of induction. Assume that the assertion holds for the refined decomposition of F_{n-1} and consider the refined double decomposition of $C_{1,\alpha} \cap C_{\sigma,w_0}$ in F_n. By Theorem C, refined double strata in $C_{1,\alpha} \cap C_{\sigma,w_0}$ are diffeomorphic to Cartesian products of orbits O_s (where $s \in \mathcal{S}_\alpha^{w_0}$) by refined double strata in $C_{1,\alpha'} \cap C_{\sigma_s,w_0'}$, where w_0' is the longest permutation on $n-1$ elements and α' has the same sense as before. It suffices to show that nonempty refined top–dimensional strata corresponding to all $s \in \mathcal{S}_\alpha^{w_0}$ have no gaps in dimensions. The dimension of any nonempty intersection $C_{1,\alpha'} \cap C_{\sigma_{sub},w_0}$, and thus the dimension of the top-dimensional strata, equals $\operatorname{lng}(\alpha')$ independently of s, see Lemma 9.5. Thus, we have to show that the numbers $\operatorname{lng}(\alpha') + \dim(O_s)$ have no gaps when s runs over the set of all decreasing subsequences in $\mathcal{S}_\alpha^{w_0}$ such that $C_{1,\alpha'} \cap C_{\sigma_s,w_0}$ are nonempty. Since $\operatorname{lng}(\alpha')$ does not depend on s, we must show that the dimensions of all orbits corresponding to the decreasing subsequences s such that $\alpha' \succeq w_0\sigma_s$have no gaps, cp. Lemma 9.5. The latter statement follows from Lemmas 6.2, 6.3 and Corollary 9.6. ■

9.8. Now we cover the second case of Proposition 0.27, $\sigma = w_0$.

By Lemma 9.5, the intersection $C_{1,\alpha} \cap C_{\sigma,w_0}$ is isomorphic to $C_{1,\alpha} \cap C_{w_0,w_0\sigma w_0} \times C_{1,\sigma w_0}$. Thus, the absence of gaps in dimensions of refined double strata in the case $\sigma = w_0$ will follow from the preceding considerations in the case $\beta = w_0$ and the following

Lemma. *There exists a bijection between the refined double strata in $C_{1,\alpha} \cap C_{\sigma,w_0}$ and those in $C_{1,\alpha} \cap C_{w_0,w_0\sigma w_0} \times C_{1,\sigma w_0}$. Strata in $C_{1,\alpha} \cap C_{\sigma,w_0}$ are diffeomorphic to Cartesian products of the corresponding strata in $C_{1,\alpha} \cap C_{w_0,w_0\sigma w_0}$ by $C_{1,\sigma w_0}$.*

The proof is based on the following observation.

9.9. Proposition. *Given the polynomial $E_\alpha^{\sigma,\beta}(z)$ of some intersection $C_{1,\alpha} \cap C_{\sigma,\beta}$ and two numbers k and l such that the top-dimensional strata in the refined double decomposition of $C_{1,\alpha} \cap C_{\sigma,\beta}$ have the form $(\mathbb{C}^*)^k \times (\mathbb{C})^l$, one restores the number of strata in $C_{1,\alpha} \cap C_{\sigma,\beta}$ in all dimensions.*

Proof. Since the dimension of $C_{1,\alpha} \cap C_{\sigma,\beta}$ is $k+l$, the degree of $E_\alpha^{\sigma,\beta}(z)$ is also $k+l$, and the leading coefficient a_{k+l} is equal to the number of top-dimensional strata. Consider $E_\alpha^{\sigma,\beta}(z) - a_{k+l}(z-1)^k z^l$ and assume that its degree is $p < k+l$. Then its leading coefficient a_p gives the number of strata of dimension p in $C_{1,\alpha} \cap C_{\sigma,\beta}$. By Proposition 9.1, the strata of dimension p have the form $(\mathbb{C}^*)^{2d-k-l} \times (\mathbb{C})^{2l+k-d}$. Subtracting from $E_\alpha^{\sigma,\beta}(z) - a_{k+l}(z-1)^k z^l$ the polynomial $a_p(z-1)^{2d-k-l} z^{2l+k-d}$, one gets the number of strata in the next dimension, and so on. ■

9.10. Proof of Lemma 9.8. By Lemma 9.5 and the definition of $E_\alpha^{\sigma,\beta}(z)$ one has $E_\alpha^{w_0,w_0\sigma w_0}(z) = z^{\mathrm{lng}(\sigma w_0)} E_\alpha^{\sigma w_0}(z)$. Moreover, it is known that both $C_{1,\alpha} \cap C_{w_0,w_0\sigma w_0}$ and $C_{1,\alpha} \cap C_{\sigma w_0}$ are irreducible varieties, and one can show that their only top-dimensional strata have the form $(\mathbb{C}^*)^{\mathrm{lng}(\alpha)-\mathrm{lng}(\sigma w_0)}$ and $(\mathbb{C}^*)^{\mathrm{lng}(\alpha)-\mathrm{lng}(\sigma w_0)} \times \mathbb{C}^{\mathrm{lng}(\sigma w_0)}$, respectively. (These strata are obtained by using maximal decreasing subsequences on all steps of the algorithm.) Now, applying the procedure of defining the number of strata of each dimension described in Proposition 9.9 and using the result for the first special case, one gets the necessary statement. ■

9.11. Here we recall the relation between the values of structure constants $c_{w_1,w_2}^{w_3}(z)$ in the Hecke algebra and the E-polynomials, cp. [KL1, Ka, Cu1]. This relation holds for any flag space G/B where G is a semisimple group.

The Hecke algebra $\mathcal{H}$ is the deformation of the group algebra of $\mathfrak{S}_n$. As a linear space $\mathcal{H}$ has the standard basis $\{T_w\}$, $w \in \mathfrak{S}_n$. As an algebra

$\mathcal{H}$ is spanned by $\{T_{t_i}\}$, where t_i is the ith simple transposition, satisfying the Hecke algebra relations. Namely,

$$\begin{cases} T_{t_i} * T_{t_{i+1}} * T_{t_i} = T_{t_{i+1}} * T_{t_i} * T_{t_{i+1}}, \\ T_{t_i} * T_{t_j} = T_{t_j} * T_{t_i}, \\ T_{t_i} * T_{t_i} = (z-1)T_{t_i} + zT_1. \end{cases}$$

The structure constants $c^{w_3}_{w_1,w_2}$ are the coefficients in the decomposition $T_{w_1} * T_{w_2} = \sum_{w_3} c^{w_3}_{w_1,w_2} T_{w_3}$.

Proposition. $E_{C_{1,\alpha}\cap C_{\sigma,\beta}}(z) = c^{\sigma}_{\alpha,\beta^{-1}}(z)$.

Proof. If z is a power of prime, then the structure constant $c^{w_3}_{w_1,w_2}(z)$ calculates the number of points in the intersection $C_{1,w_1} \cap C_{w_2,w_3^{-1}}$, see e.g. [Cu1]. In an appropriate basis of $\mathbb{C}^n$, all refined double strata in F_n are defined over $\mathbb{Z}$, and thus the refined double decomposition is defined simultaneously over any finite field GF. Any refined double stratum over GF is isomorphic to an appropriate $(\mathrm{GF}^*)^k \times (\mathrm{GF})^l$ where GF^* is the multiplicative group of GF. Both $E_{C_{1,\alpha}\cap C_{\sigma,\beta}}(z)$ and $c^{\sigma}_{\alpha,\beta^{-1}}(z)$ are polynomials in z with integer coefficients coinciding when z is a power of a prime, since they both count the number of points in $C_{1,\alpha}\cap C_{\sigma,\beta}$ over the corresponding finite field. Thus, $E_{C_{1,\alpha}\cap C_{\sigma,\beta}}(z) = c^{\sigma}_{\alpha,\beta^{-1}}(z)$. ■

10. Relation to the decomposition of Curtis–Deodhar

10.1. In order to present to the readers certain similar results due to V.V.Deodhar and Ch. Curtis, we quote several statements of [De1] constructing a special decomposition of $C_{1,\alpha}$ and $C_{1,\alpha} \cap C_{w_0,\beta}$ in terms of reduced expressions for α^{-1}. We preserve mostly the original notation. Let $W(y)$ denote the lower interval of y in W in the usual Bruhat order.

10.2. Proposition ([De1], Theorem 1.1)**.** *The Bruhat cell $\mathcal{B}y\cdot\mathcal{B}$ can be decomposed into disjoint nonempty subsets $\{D_{\underline{\sigma}}\}_{\underline{\sigma}\in\mathcal{D}}$ ($\mathcal{D}$ is the indexing set, which can be described explicitly) such that*

(i) $\mathcal{B}y\cdot\mathcal{B} = \cup_{\underline{\sigma}\in\mathcal{D}} D_{\underline{\sigma}}$.
(ii) *For each $\underline{\sigma}\in\mathcal{D}$ there exist unique non-negative integers $m(\underline{\sigma})$ and $n(\underline{\sigma})$ such that $D_{\underline{\sigma}} \simeq \mathbb{C}^{m(\underline{\sigma})} \times (\mathbb{C}^*)^{n(\underline{\sigma})}$.*
(iii) *For each $\underline{\sigma}\in\mathcal{D}$ there exists $x\in W$ (recall that in our case $W = \mathfrak{S}_n$, and so $D_{\underline{\sigma}} \subseteq \mathcal{B}^- x\cdot\mathcal{B}$ with $\mathcal{B}^- = w_0\mathcal{B}w_0$). This element x is unique, belongs to $W(y)$ and is denoted by $\pi(\underline{\sigma})$. Thus one gets a map $\pi\colon \mathcal{D}\to W(y)$.*

10.3. Corollary ([De1], Corollary 1.2)**.** $\mathcal{B}y\cdot\mathcal{B}\cap\mathcal{B}^- x\cdot\mathcal{B} = \cup_{\underline{\sigma}\in\mathcal{D},\pi(\underline{\sigma})=x} D_{\underline{\sigma}}$. *(In particular, $\mathcal{B}y\cdot\mathcal{B}\cap\mathcal{B}^- x\cdot\mathcal{B} \neq \varnothing$ if and only if $x\in W(y)$.)*

10.4. Let U^+ and U^- denote the maximal unipotent subgroups in SL_n corresponding to the set of all positive and negative roots, respectively. (In our case they are the usual upper- and lower triangular unipotent subgroups.) Fixing $y \in W$ and its reduced expression $y = s_1 \dots s_k$, where $s_j = t_{i_j}$, let us define for all $1 \leqslant j \leqslant k$ the subsets $U_j = U^+ \cap {}^{s_j \cdots s_k}U^-$ and $U^j = U^+ \cap {}^{s_k \cdots s_j}U^+$. (For a subset $A \in SL_n$ and $g \in SL_n$, ${}^gA = gAg^{-1}$).

Proposition ([De1], Lemma 2.2).

(i) $U_1 \supset {}^{s_1}U_2 \supset {}^{s_1 s_2}U_3 \supset \cdots \supset {}^{s_1 \dots s_k}U_{k+1} = \{\mathrm{id}\}$.
(ii) $U^1 \subset U^2 \subset \cdots \subset U^{k+1} = U^+$.
(iii) *For any j, $1 \leqslant j \leqslant k+1$, one has $U^+ = U_j \cdot {}^{s_j \cdots s_k}U^j$ and such an expression is unique.*
(iv) *Any element $\xi \in \mathcal{B}y \cdot \mathcal{B}$ can be uniquely written as $us_1 \dots s_k$ with $u \in U_1$.*

10.5. The indexing set $\mathcal{D}$ in Proposition 10.2 is described in terms of a reduced expression $y = s_1 \dots s_k$, where k is the length of y. It consists of subexpressions $s_1 \dots \hat{s}_{p_1} \dots \hat{s}_{p_m} \dots s_k$ with some additional properties, which are called distinguished. More precisely, one can regard a subexpression as a sequence $\underline{\sigma} = (\sigma_0, \dots, \sigma_k)$ of elements of W such that (i) $\sigma_0 = \mathrm{id}$ and (ii) $\sigma_{j-1}^{-1}\sigma_j \in \{\mathrm{id}, s_j\}$ for all $1 \leqslant j \leqslant k$. Note that for any j one gets that σ_{j-1} and σ_j are always comparable in the Bruhat order, i.e. one has a trichotomy $\sigma_{j-1} < \sigma_j$, or $\sigma_{j-1} = \sigma_j$, or $\sigma_{j-1} > \sigma_j$. Let us denote by $n(\underline{\sigma})$ the number of all positions j such that $\sigma_{j-1} = \sigma_j$ and by $m(\underline{\sigma})$ the number of positions j such that $\sigma_{j-1} > \sigma_j$. Finally, let $\pi(\underline{\sigma}) = \sigma_k$.

Now we can define the index set $\mathcal{D}$ in Proposition 10.2. A subexpression $\underline{\sigma}$ is called *distinguished* if it satisfies the additional condition (iii) $\sigma_j \leqslant \sigma_{j-1}s_j$ for all $1 \leqslant j \leqslant k$.

Proposition ([De1], Proposition 3.1) *Let $u_1 \in U_1$ be fixed. For $0 \leqslant j \leqslant k$, let $\sigma_j \in W$ be the unique element such that $u_1 s_1 \dots s_j \in \mathcal{B}^-\sigma_j\mathcal{B}$. Then $\underline{\sigma} = (\sigma_0, \sigma_1, \dots, \sigma_k)$ is a distinguished subexpression.*

In this way one gets a map $\nu : U_1 \to \mathcal{D}$ defined in an obvious way.

10.6. Proof of Proposition 0.28. Let us define the standard reduced expression of $y \in \mathfrak{S}_n$ by the following rule: $y = t_{y^{-1}(1)-1}t_{y^{-1}(1)-2} \cdots t_1\tilde{y}$, where $\tilde{y}$ belongs to $\mathfrak{S}_{n-1}$, i.e. $\tilde{y}(1) = 1$, and t_i is defined in 3.1. We consider $\tilde{y}$ as an element of $\mathfrak{S}_{n-1}$ acting on the set $\{2, 3, \dots, n\}$ and define the standard reduced expression inductively.

Example. $3241 = (4,3)(3,2)(2,1) \cdot (1324) = (4,3)(3,2)(2,1)(32)$.

Let $y = t_1 \dots t_k$ be the standard reduced expression and $q = y^{-1}(1)$. Let us consider the result of Deodhar's decomposition of $\mathcal{B}y \cdot \mathcal{B}$ after the first $q-1$ steps, which corresponds to the sequence $U_1 \supset {}^{s_1}U_2 \supset \dots \supset^{s_1 \cdots s_{q-2}} U_{q-1}$ (see Proposition 10.4 above). We denote by v_1 the spanning vector of the one-dimensional space of the flag $f \in \mathcal{B}y \cdot \mathcal{B}$. It has the following properties: $v_1^q = 1$, $v_1^j = 0$ if $j > q$. The first $q-1$ steps of

Deodhar's procedure decompose the cell into the disjoint union of 2^{q-1} sets enumerated by all possible subsets of nonvanishing coordinates of v_1. At the same time, our main algorithm for $C_{1,\alpha}$ (with $\alpha = y^{-1}$) produces the same result after the first step. Moreover, since the elementary transpositiopns $s_q, \dots s_k$ do not contain the element 1, one may assume that the vector v_1 is fixed on all the subsequent steps. An easy reformulation of Deodhar's procedure presents the subsequent steps of his algorithm as an application of the previous steps to the quotient space and quotient flags mod v_1. Thus, it gives exactly the same result as our main algorithm. ■

11. Final remarks

Below we formulate several combinatorial and topological problems related to the material of the preceding sections with some comments.

11.1. Combinatorial questions. **Problem 1.** *Calculate the number of strata in the flag completion of* (f, g).

One can use the very weak Bruhat order on $\mathfrak{S}_n$ defined in §1. By Proposition 1.5, the number of flags in the flag completion of (f, g) in relative position σ equals the number of elements in the lower interval of σ in this order. Proposition 1.7 provides an inductive way to find this number.

Problem 2. *Calculate the number of strata in the refined double decomposition* $RD_{f,g}$ *of the whole flag space* F_n (see Fig.1 and the description of the main algorithm).

As a preliminary question, one needs to find the number of decreasing subsequences in a permutation. Fig.12 contains the number of decreasing subsequences and refined double strata for all permutations in $\mathfrak{S}_4$.

The next group of questions deals with the properties of the adjacency partial order. Let us introduce the following natural linear lexicographic order on permutations and chains of permutations. Namely, a permutation α is said to be *bigger than* β if the leftmost element of α that is distinct from the corresponding element of β is bigger than the latter one. This lexicographic order can be extended naturally to chains of permutations. (The chains on all figures are arranged from left to right according to this lexicographic linear order.)

Conjecture 1. *For any nonempty* $C_{1,\alpha} \cap C_{\sigma,\beta}$, *the refined double stratum that is the maximal element in the lexicographic linear order is also the maximal element in the adjacency partial order. Moreover, it is one of the top-dimensional strata.*

Conjecture 2. *For any nonempty* $C_{1,\alpha} \cap C_{\sigma,\beta}$, *the refined double stratum* *that is the minimal element in the lexicographic linear order is one of the strata of the minimal dimension.*

permutation	number of flags in flag completion	number of decreasing subsequences	number of refined double strata
1234	1	4	24
2134	2	5	36
1324	2	5	36
1243	2	5	36
2314	3	6	52
2143	4	6	54
3124	3	6	52
1342	3	6	52
1423	3	6	52
3214	4	8	80
2341	4	7	73
2413	3	7	75
3142	4	7	75
4123	4	7	73
1432	4	8	80
3241	4	9	99
2431	6	9	112
4213	5	9	112
3412	6	8	107
4132	5	9	112
3421	7	11	165
4231	6	11	167
4312	6	11	163
4321	8	15	261

Fig.12. Some combinatorial characteristics for permutations in $\mathfrak{S}_4$

Conjecture 3. *Let S and T be two refined double strata such that $S \succeq T$ (in the adjacency partial order), and assume that* $\dim S - \dim T > 1$. *Then there exists a sequence of intermediate strata $S = S_1 \succeq S_2 \succeq S_3 \succeq \cdots \succeq S_i = T$ such that* $\dim S_i - \dim S_{i-1} = 1$ *for all i.*

Consider the sequence of numbers of strata (contained in some $C_{1,\alpha} \cap C_{\sigma,\beta}$) ordered by dimensions. For example, these sequences for 2 counterexamples of B. Boe are $1, 6, 10, 6, 1$ and $1, 5, 12, 15, 7, 1$.

Conjecture 4. *The above sequence is unimodal, that is, nonstrictly increasing up to some place and nonstrictly decreasing after that.*

B.Boe has verified this conjecture for $\mathfrak{S}_4$ and $\mathfrak{S}_5$, as well as for the Coxeter groups B_4, C_4, and D_5.

11.2. Topological questions. **Problem 3.** *Describe combinatorially the closure pattern of refined double strata, that is, enumerate all strata* St_i *such that* $\overline{\mathrm{St}} \cap \mathrm{St}_i \neq \varnothing$.

Problem 4. *Give a combinatorial description of the differential* d_1 *in the Leray spectral sequence of* §8 *for (almost) nice* $C_{1,\alpha} \cap C_{\sigma,\beta}$.

Consider the standard filtration of any (e.g., hard) $C_{1,\alpha} \cap C_{\sigma,\beta}$ by closed subsets, see 0.25.

Conjecture 5. *The associated Leray spectral sequence degenerates at* E_2.

Another interesting topic concerning intersections of Schubert cells is the relation between their topological properties over $\mathbb{R}$ and $\mathbb{C}$. We say that a real algebraic variety V enjoys the *M-property* if the sum of its Betti numbers with coefficients in $\mathbb{Z}/2\mathbb{Z}$ coincides with that of its complexification. It is shown in [Sh] that the intersection $\mathbb{R}C_{1,w_0} \cap \mathbb{R}C_{w_0,w_0}$ enjoys such a property.

Conjecture 6. *Any intersection* $\mathbb{R}C_{1,\alpha} \cap \mathbb{R}C_{\sigma,\beta}$ *enjoys the M-property.*

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