ON THE COMPLEMENTS OF AFFINE SUBSPACE ARRANGEMENTS

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Abstract. Let V be an $l-$dimensional real vector space. A **subspace arrangement** $\mathcal{A}$ is a finite collection of affine subspaces in V. There is no assumption on the dimension of the elements of $\mathcal{A}$. Let $M(\mathcal{A}) = V - \cup_{A\in\mathcal{A}} A$ be the complement of $\mathcal{A}$. A method of calculating the additive structure of $H^*(M(\mathcal{A}))$ was given in [G-MP] using stratified Morse theory, proving that $H^n(M(\mathcal{A}))$ depends only on the set of all intersections of elements of $\mathcal{A}$ partially ordered by inclusion. An alternate method of calculating $H^n(M(\mathcal{A}))$ was obtained in [J] using the generalized Mayer–Vietoris spectral sequence, one-point compactifications, and the nerve poset. In this paper, we present an explicit isomorphism between the two results, offer an interpretation of the coincidence of the two methods and obtain a simplification of the method of calculation in [J].

1. Introduction

Let V be a real vector space of dimension l. Let S be its one-point compactification. A *subspace arrangement* $\mathcal{A} = (\mathcal{A}, V)$ is a finite collection of affine subspaces in V. We will denote an affine subspace of V in $\mathcal{A}$ by A. Thus, $\mathcal{A} = \{A_i\}_{i=1}^n$. We call $\mathcal{A}$ *central* with *center* $T_\mathcal{A}$ if $T_\mathcal{A} = \cap_{A\in\mathcal{A}} A \neq \emptyset$.

A *sphere arrangement* $\mathcal{B} = (\mathcal{B}, S) = \{B_i\}_{i=1}^n$ is a finite collection of spheres in $S \cong S^l$ which satisfies the following conditions: S admits a $CW-$decomposition such that each B_i is a subcomplex, and, for $\sigma \subset \Sigma =$

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$\{1, \ldots, n\}$, $B_\sigma = \cap_{i\in\sigma} B_i$ is either a sphere or a single point. Let the *center* $T_{\mathcal{B}}$ be the intersection of all of the elements of $\mathcal{B}$. Note that $T_{\mathcal{B}} \neq \emptyset$.

The *one-point compactification* $\mathcal{B} = (\mathcal{B}, S)$ of a subspace arrangement $\mathcal{A} = (\mathcal{A}, V)$ is the collection of one-point compactifications of the elements of $\mathcal{A}$ with the same point at infinity ∞ in S. Thus, $\mathcal{B} = \{B_i = A_i \cup \infty\}_{i=1}^n$. We call $\mathcal{B}$ the *associated sphere arrangement* of the subspace arrangement $\mathcal{A}$ and denote the one-point compactification of A_i by $B_i = A_i \cup \infty$.

Since each affine subspace A_i is homeomorphic to $\mathbb{R}^{r_i}$, its one-point compactification B_i is homeomorphic to S^{r_i}. While $\mathcal{A}$ is a collection of affine subspaces of $\mathbb{R}^l$, $\mathcal{B}$ is a collection of spheres in the corresponding sphere S^l. The dimensions of these spheres correspond with the dimensions of their associated subspaces.

Since the intersection of two affine subspaces is either empty or an affine subspace of equal or lower dimension, the intersection of their one-point compactifications is either just the point at infinity or a sphere of the same dimension as the intersection of the subspaces.

Let $N(\mathcal{A}) = \cup_{A\in\mathcal{A}} A$ be the *union* of $\mathcal{A}$ and $M(\mathcal{A}) = V - N(\mathcal{A})$ the *complement* of $\mathcal{A}$. Similarly, let $N(\mathcal{B}) = \cup_{B\in\mathcal{B}} B$ be the *union* of $\mathcal{B}$ and $M(\mathcal{B}) = S - N(\mathcal{B})$ the *complement* of $\mathcal{B}$. Note that $M(\mathcal{B})$ is homeomorphic to $M(\mathcal{A})$, and $N(\mathcal{B})$ is the one-point compactification of $N(\mathcal{A})$.

In [G-MP], using stratified Morse theory, M. Goresky and R. MacPherson gave the first description of the cohomology groups of the complement of a subspace arrangement $M(\mathcal{A})$ in terms of the intersection poset with underlying set the intersections of elements of $\mathcal{A}$ ordered by inclusion.

In [J], K. Jewell gave an alternate description of the cohomology groups of the complement of a subspace arrangement in terms of the nerve of the covering of its associated sphere arrangement $\mathcal{B}$. The homology of $N(\mathcal{B})$ was found using the generalized Mayer–Vietoris spectral sequence. The cohomology groups of $M(\mathcal{B}) \cong M(\mathcal{A})$ were then found using Alexander duality. Instead of using the associated sphere arrangement construction to compute the cohomology groups of the complement of $\mathcal{A}$, the method could have been stated in terms of cohomology with compact supports.

Our aim is to present an explicit isomorphism between the two results, offer an interpretation of the coincidence of the two methods and, in the process, extend the stratified Morse theory side of the analysis to a description of the cohomology groups of $M(\mathcal{B})$. Finally, we present a simplification of the method of calculation in [J] using dual complexes.

2. Stratified Morse Theory of $M(\mathcal{A})$

In [G-MP], using stratified Morse theory, M. Goresky and R. MacPherson gave a description of the cohomology groups of the complement of a subspace arrangement $M(\mathcal{A})$ by giving $\mathbb{R}^l$ a stratification based on the intersection poset with underlying set the intersection of elements of $\mathcal{A}$ ordered by inclusion. They described the cohomology type of the stratified Morse data in terms of a simplicial complex with vertex set the elements of the poset and showed that the cohomology groups of $M(\mathcal{A})$ can be expressed as a direct sum of the cohomology groups of the stratified Morse data. A brief description of the method follows.

Let $\mathcal{A} = \{A_i\}_{i=1}^n$ be a subspace arrangement in $\mathbb{R}^l$.

Definition 2.1. *Let $L(\mathcal{A})$ be the set of all intersections of elements of $\mathcal{A}$. Define a partial order on $L(\mathcal{A})$ by inclusion:*

$$A_\sigma \leq A_\tau \quad \textit{if and only if} \quad A_\sigma \subseteq A_\tau.$$

$L(\mathcal{A})$ is called the **intersection poset** *of $\mathcal{A}$. Define a dimension function $d_L : L(\mathcal{A}) \to \mathbb{Z}$ by $d_L(A_\sigma) = \dim(A_\sigma)$.*

By convention we let V be the unique maximal element of $L(\mathcal{A})$. Let $L(\mathcal{A})^-$ be the poset with underlying set $L(\mathcal{A}) - \{V\}$.

Definition 2.2. *Given any poset $\mathcal{P}$ and elements $X, Y \in \mathcal{P}$ with $X < Y$, let $(X <) = \{Z \in \mathcal{P} \mid X < Z\}$, $(X \leq) = \{Z \in \mathcal{P} \mid X \leq Z\}$ and $(X, Y) = \{Z \in \mathcal{P} \mid X < Z < Y\}$ be subposets of $\mathcal{P}$.*

Definition 2.3. *Given a poset $\mathcal{P}$, let $\Delta(\mathcal{P})$ be the simplicial complex with vertices the elements of $\mathcal{P}$ and whose simplexes are the linearly ordered subsets of $\mathcal{P}$. Call $\Delta(\mathcal{P})$ the* **order complex** *of $\mathcal{P}$.*

For $X \in L(\mathcal{A})$, define the stratum $s(X)$ to be $s(X) = X - \cup_{Y<X} Y$. Since $V \in L(\mathcal{A})$, we have $s(V) = M(\mathcal{A})$ and $\mathbb{R}^l = \cup_{X \in L(\mathcal{A})} s(X)$.

The following two propositions were proven in [G-MP].

Proposition 2.4. *The collection $\{s(X) \mid X \in L(\mathcal{A})\}$ forms a Whitney stratification of $\mathbb{R}^l$.*

Proposition 2.5. *For the above stratification of $\mathbb{R}^l$, there exists a stratified Morse function $f = dist^2(-,q)$ for some point $q \in M(\mathcal{A})$.*

Exactly one critical point of f lies in each stratum, with the point q being the critical point of $s(V)$. Since f has isolated critical values, f induces a linear order on $L(\mathcal{A})$. Let $M_{\leq c} = M(\mathcal{A})_{\leq c}$.

Let p be a critical point of f in $s(X)$ with critical value $c = c_X$, and let ϵ be sufficiently small.

Lemma 2.6. *The tangential Morse data of f at p in $s(X)$ is trivial, ie., $(D^r, \emptyset)$ where D^r is the $r-$disk and $r = d_L(X)$.*

Proof. The tangential Morse data is the classical Morse data of f in $s(X)$. It is the pair $(D^\lambda \times D^{r-\lambda}, (\partial D^\lambda) \times D^{n-\lambda})$ where the Morse index λ of f at p is the number of negative eigenvalues of the Hessian matrix of second derivatives at p. However, $\lambda = 0$ since f is a local minimum at p and $\partial D^0 = \emptyset$. □

Corollary 2.7. *The stratified Morse data of f at the critical point p is simply the normal Morse data of f at p.*

Theorem 2.8. *For each isolated critical value and sufficiently small ϵ, the long exact sequence in cohomology of the pair $(M_{\leq c+\epsilon}, M_{\leq c-\epsilon})$ splits into the short exact sequences*

$$0 \to H^n(M_{\leq c+\epsilon}, M_{\leq c-\epsilon}) \xrightarrow{j} H^n(M_{\leq c+\epsilon}) \to H^n(M_{\leq c-\epsilon}) \to 0.$$

Hence

$$H^n(M_{\leq c+\epsilon}) = H^n(M_{\leq c-\epsilon}) \oplus H^n(M_{\leq c+\epsilon}, M_{\leq c-\epsilon}).$$

This theorem was proven in [G-MP] using an involution taking a relative cycle in the link and constructing an absolute cycle via an "antipodal" map giving a map j' such that $j' \circ j = 1$.

Theorem A_1.

$$H^n(M(\mathcal{A})) = \oplus_{X \in L(\mathcal{A})} H^n(M_{\leq c_X+\epsilon}, M_{\leq c_X-\epsilon})$$

for sufficiently small ϵ.

This theorem essentially states that the cohomology groups of $M(\mathcal{A})$ can be expressed as a direct sum of local contributions from the cohomology groups of the stratified Morse data.

The following lemmas were proven in [G-MP] with Poincaré duality as one of the essential steps.

Lemma 2.9.

$$H^n(M_{\leq c_X+\epsilon}, M_{\leq c_X-\epsilon}) \cong H_{l-d_L(X)-n-1}(\mathbb{R}^l, \cup_{X<Y<V} Y)$$

for $X \neq V$ and sufficiently small ϵ.

Lemma 2.10. *The two homology equivalences consisting of the projection onto each of the factors with each fiber contractible*

$$\mathbb{R}^l \xleftarrow{\simeq} \cup_{Y\in(X<)}(Y \times \Delta(Y \leq)) \xrightarrow{\simeq} \Delta(X \leq)$$

induce homotopy equivalences

$$\begin{array}{c} (\mathbb{R}^l, \cup_{X<Y<V} Y) \\ \Big\uparrow \simeq \\ (\cup_{Y\in(X<)}(Y \times \Delta(Y \leq)), \cup_{Y\in(X,V)}(Y \times \Delta(Y,V))) \\ \Big\downarrow \simeq \\ (\Delta(X \leq), \Delta(X,V)) \end{array}$$

where all the posets are subposets of $L(\mathcal{A})$.

Theorem B_1.

$$\begin{aligned} H^n(M_{\leq c_X+\epsilon}, M_{\leq c_X-\epsilon}) &\cong H_{l-d_L(X)-n-1}(\mathbb{R}^l, \cup_{X<Y<V} Y) \\ &\cong H_{l-d_L(X)-n-1}(\Delta(X <), \Delta(X,V)) \end{aligned}$$

for $X \neq V$ and sufficiently small ϵ.

This theorem was proven in [G-MP]. It follows from Lemmas 2.10 and 2.11. It states that the local contributions to the cohomology groups of $M(\mathcal{A})$ can be expressed in terms of the intersection poset $L(\mathcal{A})$, using Poincaré duality.

When $X = V$, $(M_{\leq c_V+\epsilon}, M_{\leq c_V-\epsilon})$ has the homotopy type of the tangential Morse data $(D^l, \emptyset)$ since the normal disk is just the point q. Hence,

$$H^n(M_{\leq c_V+\epsilon}, M_{\leq c_V-\epsilon}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{else.} \end{cases}$$

Theorem C_1.

$$\widetilde{H}^n(M(\mathcal{A})) = \oplus_{X\in L(\mathcal{A})^-} H_{l-n-d_L(X)-1}(\Delta(X <), \Delta(X, V)).$$

where $(X <)$ and (X, V) are subposets of $L(\mathcal{A})$.

Thus the cohomology groups of $M(\mathcal{A})$ can be expressed in terms of relative simplicial homology of the order complex of the intersection poset $L(\mathcal{A})$.

Theorems A_1, B_1, and C_1 are prototype results. We elaborate on them in the paper and call attention to their relationship with these theorems by labelling them Theorems A_k, B_k, C_k.

3. Stratified Morse Theory of $M(\mathcal{B})$

Rather than describe the cohomology groups of $M(\mathcal{A})$ using the stratification of $\mathbb{R}^l$ ordered by $L(\mathcal{A})$, we will now describe the cohomology groups of $M(\mathcal{B})$ using a stratification of S^l ordered by $L(\mathcal{B})^+$ compatible with the stratification of $\mathbb{R}^l$.

Let $\mathcal{A} = \{A_i\}_{i=1}^n$ be a subspace arrangement in $\mathbb{R}^l$, and let $\mathcal{B} = \{B_i\}_{i=1}^n$ be the associated sphere arrangement in S^l.

Definition 3.1. *Let $L(\mathcal{B})$ be the set of all intersections of elements of $\mathcal{B}$. Define a partial order on $L(\mathcal{B})$ by inclusion:*

$$B_\sigma \leq B_\tau \quad \textit{if and only if} \quad B_\sigma \subseteq B_\tau.$$

$L(\mathcal{B})$ is called the **intersection poset** *of $\mathcal{B}$. Define a dimension function $d_L : L(\mathcal{B}) \to \mathbb{Z}$ by*

$$d_L(B_\sigma) = \begin{cases} \dim(B_\sigma), & \textit{if } B_\sigma \neq \infty \\ -1, & \textit{if } B_\sigma = \infty. \end{cases}$$

We do *not* include S in $L(\mathcal{B})$. Since the point at infinity is contained in each $B_i \in \mathcal{B}$, the intersection of all B_i's is nonempty. Thus $L(\mathcal{B})$ has a unique minimal element $T_\mathcal{B}$. The map d_L is *order preserving.*

Definition 3.2. *Let $L(\mathcal{B})^+$ be the poset $L(\mathcal{B}) \cup \{S, \infty\}$ where S is the unique maximal element and ∞ is the unique minimal element.*[1] *Let $L(\mathcal{B})^-$ be the*

[1]Note that if $\mathcal{A}$ is noncentral, ∞ is already in $L(\mathcal{B})$. If $\mathcal{A}$ is central, then $T_\mathcal{B} \neq \infty$ is the unique minimal element in $L(\mathcal{B})$, so $\infty < T_\mathcal{B}$ in $L(\mathcal{B})^+$.

poset $L(\mathcal{B}) - \{\infty\}$. *Let* $L(\mathcal{B})^-_q$ *be the set of all* $q-$*spheres in* $L(\mathcal{B})^-$, *ie.,* $L(\mathcal{B})^-_q = L(\mathcal{B})^- \cap d_L^{-1}(q)$.

Taking A_σ to its one-point compactification B_σ induces an isomorphism $L(\mathcal{A})^- \cong L(\mathcal{B})^-$.

Embed $\mathbb{R}^l$ into $\mathbb{R}^{l+1}$ by $(x_1, \ldots, x_l) \mapsto (x_1, \ldots, x_l, 0)$. Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^l$. Give $\mathbb{R}^l$ the above stratification with $s(X) = X - \cup_{Y<X} Y$ for $X \in L(\mathcal{A})$. Without loss of generality, we can translate $\mathcal{A}$ so that $f = \text{dist}^2(-, 0) : \mathbb{R}^l \to \mathbb{R}$ is a stratified Morse function.

Embed S^l with north pole denoted by ∞ at $(0, \ldots, 0, 1)$ and south pole at the origin. Let $\pi : S^l - \infty \to \mathbb{R}^l$ be the usual stereographic projection out from the north pole to $\mathbb{R}^l$. Define $B_i \subset S^l$ to be $\pi^{-1}(A_i) \cup \infty, A_i \in \mathcal{A}$. Let $\mathcal{B} = \{B_i\}_{i=1}^n$ be the associated sphere arrangement of $\mathcal{A}$.

For $X \in L(\mathcal{B})^+$, define the strata $s(X) = X - \cup_{Y<X} Y$. Note that $s(S) = M(\mathcal{B})$, $s(\infty) = \infty$, $S^l = \cup_{X \in L(\mathcal{B})^+} s(X)$ and, for $B_\sigma \neq \infty$, $s(B_\sigma)$ is diffeomorphic to $s(A_\sigma)$. Since $\pi : S^l - \infty \to \mathbb{R}^l$ is a stratification preserving diffeomorphism, $\{s(X) \mid X \in L(\mathcal{B})^+\}$ is a Whitney stratification of S^l.

Lemma 3.3. $g = dist^2(-, 0) : S^l \to \mathbb{R}$ *is a stratified Morse function for the given stratification of* S^l *having the set of critical points*

$$\{\pi^{-1}(p) \mid p \textit{ is a critical point of } f\} \cup \infty.$$

Proof. ∞ is a critical point of g since $s(\infty)$ is just the point ∞. Since $\pi : S^l - \infty \to \mathbb{R}^l$ is a stratification preserving diffeomorphism and $g(p) = (f(\pi(p)))/(f(\pi(p)) + 1)$ for $p \neq \infty$, $p \neq \infty$ is a critical point of g if and only if $\pi(p)$ is a critical point of f.□

Exactly one critical point of g lies in each stratum with the origin being the critical point of $s(S) = M(\mathcal{B})$ and ∞ the critical point of $s(\infty) = \infty$. Moreover, g induces the same linear order on $L(\mathcal{B})^+$ as f did for $L(\mathcal{A})$. Let $M_{\leq c} = M(\mathcal{B})_{\leq c}$.

Lemma 3.4. *The tangential Morse data of* g *at a critical point* p *in* $s(X)$ *is trivial, ie.,* $(D^r, \emptyset)$ *where* D^r *is an* $r-$*disk,* $r = d_L(X)$. *Hence, the stratified Morse data of* g *at a critical point* $p \neq \infty$ *is simply the normal Morse data of* g *at* p.

Proof. For a critical point $p \neq \infty$, p is a local minimum of g if and only if $\pi(p)$ is a local minimum of f. Then, for $p \neq \infty$, the result follows. For the case $p = \infty$, since $s(\infty)$ is just a point, the tangential Morse data is (pt,$\emptyset$).□

Lemma 3.5. *For each isolated critical value $c = c_X$, $X \neq \infty$, and sufficiently small ϵ, the long exact sequence in cohomology of the pair $(M_{\le c+\epsilon}, M_{\le c-\epsilon})$ splits into short exact sequences. Hence,*

$$H^n(M_{\le c+\epsilon}) \cong H^n(M_{\le c-\epsilon}) \oplus H^n(M_{\le c+\epsilon}, M_{\le c-\epsilon}).$$

Proof. Since $X \neq \infty$ and $\pi : S^l - \infty \to \mathbb{R}^l$ is a stratification preserving diffeomorphism, $(M_{\le c+\epsilon}, M_{\le c-\epsilon})$ has the same homotopy type as $(M^f_{\le \pi(c)+\epsilon'}, M^f_{\le \pi(c)-\epsilon'})$.□

Lemma 3.6. *Let (A, B) be the stratified Morse data of g at the critical point ∞. Then B is a strong deformation retract of A. Thus, $H^n(M_{\le c_\infty+\epsilon}, M_{\le c_\infty-\epsilon}) = 0$ and $H^n(M_{\le c_\infty+\epsilon}) \cong H^n(M_{\le c_\infty-\epsilon})$.*

Proof. Since $s(\infty) = \infty$ is a point, the tangential Morse data is (pt,$\emptyset$), and the stratified Morse data is simply the normal Morse data. Take a disk D^{l+1}_ϵ of radius ϵ in $\mathbb{R}^{l+1}$ with center at ∞. Denote $\partial D^{l+1}_\epsilon$ by S^l_ϵ. Intersecting D^{l+1}_ϵ with $M(\mathcal{B})$ gives the normal Morse data $(D^{l+1}_\epsilon \cap M(\mathcal{B}), S^l_\epsilon \cap M(\mathcal{B}))$. Using the techniques in [O], we find that $S^l_\epsilon \cap M(\mathcal{B})$ is a strong deformation retract of $D^{l+1}_\epsilon \cap M(\mathcal{B})$.□

Theorem A_2.

$$H^n(M(\mathcal{B})) = \oplus_{X \in L(\mathcal{B})^+} H^n(M_{\le c_X+\epsilon}, M_{\le c_X-\epsilon})$$

for sufficiently small ϵ.

Theorem B_2.

$$\begin{aligned} H^n(M_{\le c_X+\epsilon}, M_{\le c_X-\epsilon}) &\cong H_{l-n-d_L(X)-1}(\mathbb{R}^l, \cup_{X<Y<V} Y) \\ &\cong H_{l-n-d_L(X)-1}(\Delta(X \le), \Delta(X <)), \end{aligned}$$

where $X \in L(\mathcal{B})^-$ (ie., $X \neq S, \infty$) and $(X \le)$ and $(X <)$ are subposets of $L(\mathcal{B})^-$.

Proof. ¿From Lemma 3.5,

$$(M_{\le c_X+\epsilon}, M_{\le c_X-\epsilon}) \simeq (M^f_{\le \pi(c_X)+\epsilon}, M^f_{\le \pi(c_X)-\epsilon})$$

for $X \neq \infty$. The result therefore follows from the following facts:

$$(X - \infty <)_{L(\mathcal{A})} \cong (X <)_{L(\mathcal{B})^+} \text{ and}$$
$$(X - \infty, V)_{L(\mathcal{A})} \cong (X,S)_{L(\mathcal{B})^+} \cong (X <)_{L(\mathcal{B})}.$$

Since $(X <)_{L(\mathcal{B})^+}$ has a unique maximal element S, $(X \leq)_{L(\mathcal{B})}$ has a unique minimal element X, and $(X <)_{L(\mathcal{B})^+} - \{S\} \cong (X \leq)_{L(\mathcal{B})} - \{X\}$, it follows that $\Delta((X <)_{L(\mathcal{B})^+}) \cong \Delta((X \leq)_{L(\mathcal{B})})$.□

When $X = S$, $(M_{\leq c_S+\epsilon}, M_{\leq c_S-\epsilon})$ has the homotopy type of the tangential Morse data $(D^l, \emptyset)$. Hence,

$$H^n(M_{\leq c_S+\epsilon}, M_{\leq c_S-\epsilon}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{else.} \end{cases}$$

Theorem C_2.

$$\widetilde{H}^n(M(\mathcal{B})) = \oplus_{X \in L(\mathcal{B})^-} H_{l-n-d_L(X)-1}(\Delta(X \leq), \Delta(X <))$$

where $(X \leq)$ and $(X <)$ are subposets of $L(\mathcal{B})^-$.

Thus the cohomology groups of $M(\mathcal{B})$ can be expressed in terms of relative simplicial homology of the order complex of the intersection poset $L(\mathcal{B})$. Theorems C_1 and C_2 are not that different. This should hardly be surprising due to the facts that $\mathcal{B}$ is constructed from $\mathcal{A}$ by simply adding the point at infinity, there is no contribution to the cohomology in adding ∞, and $L(\mathcal{A})^- \cong L(\mathcal{B})^-$.

4. The Generalized Mayer–Vietoris Spectral Sequence

In [J], K. Jewell gave a description of the cohomology groups of the complement of a subspace arrangement $\mathcal{A}$ in terms of a simplicial complex derived from the covering of its associated sphere arrangement $\mathcal{B}$. The homology of $N(\mathcal{B})$ was found using the generalized Mayer–Vietoris spectral sequence. The generalized Mayer–Vietoris spectral sequence, which converges to $H_*(N(\mathcal{B}))$, collapses at the second term, which is expressed in terms of the homology of relative pairs in that simplicial complex. The cohomology groups of $M(\mathcal{B})$ were then found using Alexander duality. A brief description of the method follows.

Let $\mathcal{A} = \{A_i\}_{i=1}^n$ be a subspace arrangement in $\mathbb{R}^l$, and let $\mathcal{B} = \{B_i\}_{i=1}^n$ be the associated sphere arrangement in S^l.

Give the elements of $\mathcal{B}$ a linear order $\{B_1, \dots B_n\}$ with index set $\Sigma = \{1, \dots, n\}$. The *nerve* K of $\mathcal{B}$ is the abstract simplicial complex with vertex set Σ and simplexes the nonempty subsets σ of Σ such that $B_\sigma = \cap_{i\in\sigma} B_i$ is nonempty. Since the point at infinity is in each B_i, $B_\sigma = \cap_{i\in\sigma} B_i$ is nonempty for every nonempty subset σ of Σ. Since all nonempty subsets of Σ are simplexes of K, K is a simplex.

Definition 4.1. *Let $K(\mathcal{B})$ be the set of all simplexes σ of the nerve K of $\mathcal{B}$. Define a partial order on $K(\mathcal{B})$ by reverse inclusion on the simplexes of K:*

$$\sigma \leq \tau \quad \textit{if and only if} \quad \tau \subseteq \sigma.$$

$K(\mathcal{B})$ is called the **nerve poset**. *Define a dimension function $d_K : K(\mathcal{B}) \to \mathbb{Z}$ by*

$$d_K(\sigma) = \begin{cases} \dim(B_\sigma), & \textit{if } B_\sigma \neq \infty \\ -1, & \textit{if } B_\sigma = \infty. \end{cases}$$

Define the join of σ, τ by $\sigma \vee \tau = \sigma \cap \tau$ if it exists in $K(\mathcal{B})$ and the meet of σ, τ by $\sigma \wedge \tau = \sigma \cup \tau$.

While the partial order is by *reverse inclusion on the simplexes* σ, it corresponds to *inclusion on the spaces* $B_\sigma = \cap_{i\in\sigma} B_i$. The $0-$simplexes are the maximal elements of $K(\mathcal{B})$.

The dimension function d_K associates to each simplex of K the dimension of its corresponding sphere, if we adopt the convention that S^{-1} is the point at infinity. The map d_K is *semi-order preserving*: if $\sigma < \tau$, then $d_K(\sigma) \leq d_K(\tau)$.

Next we define an equivalence relation which indicates which σ's, under the natural inclusion maps $B_\sigma \to N(\mathcal{B})$, have the same image. The equivalence relation together with the partial order yields information about how the spheres in $\mathcal{B}$ intersect.

Definition 4.2. *Define an* **equivalence relation** $\sim$ *on $K(\mathcal{B})$ by $\sigma \sim \tau$ if $B_\sigma = B_\tau$.*

The following lemma was proven in [J].

Lemma 4.3. *The equivalence relation $\sim$ is determined by d_K on $K(\mathcal{B})$ via the following relationship. For $\sigma, \tau \in K(\mathcal{B})$, $\sigma \sim \tau$ if and only if there exists an $\eta \in K(\mathcal{B})$ such that $\eta \leq \sigma$, $\eta \leq \tau$ and $d_K(\eta) = d_K(\sigma) = d_K(\tau)$. Thus each equivalence class in $K(\mathcal{B})$ has a unique minimal element ς.*

Definition 4.4. *Let $K[X]$ denote the subcomplex of the nerve K consisting of all simplexes σ, $\sigma \geq \varsigma_X$, ie.,*

$$K[X] = \cup_{Y \geq X} \{\sigma \mid B_\sigma = Y\}.$$

Let $K(X)$ denote the subcomplex of the nerve K consisting of all simplexes σ, $\sigma \geq \varsigma_X$ and $\sigma \notin \varphi^{-1}(X)$, ie.,

$$K(X) = \cup_{Y > X} \{\sigma \mid B_\sigma = Y\}.$$

As sets $K[X]$ is a simplex and $K(X)$ is a subcomplex of $K[X]$ with $K(X) = K[X] - \{\sigma \mid B_\sigma = X\}$. The union $N(\mathcal{B})$ is a $CW-$complex covered by the subcomplexes B_i of $\mathcal{B}$. Let K be the nerve of the covering $\mathcal{B}$. Let $K^{(p)}$ be the set of all $p-$simplexes of K. Let $K_X^{(p)}$ be all those simplexes of $K^{(p)}$ in the equivalence class associated with the space X, ie., $K_X^{(p)} = K^{(p)} \cap \{\sigma \mid B_\sigma = X\}$.

Let $\partial_i : C(B_\sigma) \to C(B_{\partial_i \sigma})$ be the chain map induced by the inclusion map from B_σ into $B_{\partial_i \sigma}$ and $\varepsilon_i : C(B_i) \to C(N(\mathcal{B}))$ the chain map induced by the inclusion map from B_i into $N(\mathcal{B})$, where $\partial_i \sigma$ is the i^{th} face of σ under the usual simplicial boundary operator. Define $\partial : \oplus_{\sigma \in K^{(p)}} C(B_\sigma) \to \oplus_{\sigma \in K^{(p-1)}} C(B_\sigma)$, $p > 0$ by $\partial = \sum_{i=0}^{p} (-1)^i \partial_i$ and the augmentation map $\varepsilon : \oplus_{\sigma \in K^{(0)}} C(B_\sigma) \to C(N(\mathcal{B}))$ by $\varepsilon = \sum \varepsilon_i$. It should be clear from context whether ∂, ∂_i and ε are maps on the nerve or inclusion maps on the $CW-$complexes.

Definition 4.5. *The* **generalized Mayer–Vietoris sequence** *is the chain complex*

$$\ldots \xrightarrow{\partial} \oplus_{\sigma \in K^{(1)}} C(B_\sigma) \xrightarrow{\partial} \oplus_{\sigma \in K^{(0)}} C(B_\sigma) \xrightarrow{\epsilon} C(N(\mathcal{B})) \to 0.$$

with ε its augmentation map and $C(N(\mathcal{B}))$ its augmentation.

The following lemma was proven in [B].

Lemma 4.6. *The generalized Mayer–Vietoris sequence is exact.*

The unaugmented generalized Mayer–Vietoris sequence can be viewed as a double complex with $C_{p,q} = \oplus_{\sigma \in K^{(p)}} C_q(B_\sigma)$. Hence it induces two spectral sequences via different filtrations; one of which converges to $H_*(N(\mathcal{B}))$; the other is called the **generalized Mayer–Vietoris spectral sequence**.

The 0^{th} term of the generalized Mayer–Vietoris spectral sequence is $E^0_{p,q} \cong C_{p,q} = \oplus_{\sigma \in K^{(p)}} C_q(B_\sigma)$ with differential d^0 isomorphic to the direct sum of the cellular boundary operators.

The first term of the generalized Mayer–Vietoris spectral sequence is $E^1_{p,q} \cong \oplus_{\sigma \in K^{(p)}} H_q(B_\sigma)$ with differential d^1 isomorphic to

$$\partial_* : \oplus_{\sigma \in K^{(p)}} H(B_\sigma) \to \oplus_{\sigma \in K^{(p-1)}} H(B_\sigma).$$

The E^1 term splits into a direct sum of submodules

$$E^1_{p,q} = \begin{cases} \oplus_{X \in L(\mathcal{B})^-} (E^1_{p,q})_X, & q \neq 0 \\ C_p(K) \oplus_{X \in L(\mathcal{B})^-} (E^1_{p,q})_X, & q = 0 \end{cases}$$

where $(E^1)_X$ is defined by $(E^1_{p,q})_X = \oplus_{\sigma \in K^{(p)}_X} \widetilde{H}_q(B_\sigma)$ for $X \in L(\mathcal{B})^-$ and $C_p(K)$ is the chain complex of the nerve, accounting for the connected components of the $H_0(B_\sigma)$'s. Let $(E^2_{p,q})_X = H_p((E^1_{-,q})_X, \partial_*)$.

The following two theorems were proven in [J]. They state that (1) the homology of $N(\mathcal{B})$ can be expressed as a direct sum of "local" contributions ordered by $L(\mathcal{B})^-$, and (2) the local contributions can be expressed in terms of the nerve poset $K(\mathcal{B})$.

Theorems A_3 and B_3.

$$E^2_{p,q} = \begin{cases} \oplus_{X \in L(\mathcal{B})^-} (E^2_{p,q})_X, & \textit{if } q > 0 \\ H_p(K) \oplus_{X \in L(\mathcal{B})^-} (E^2_{p,q})_X, & \textit{if } q = 0 \\ 0, & \textit{else} \end{cases}$$

and

$$(E^2_{p,q})_X \cong \begin{cases} H_p(K[X], K(X)), & \textit{if } q = d_L(X) \\ 0, & \textit{else} \end{cases}$$

if $X \in L(\mathcal{B})^-$.

The following theorem was proven in [J].

Theorem 4.7. *The generalized Mayer–Vietoris spectral sequence collapses at the second term,* $E^2_{p,q} \cong E^\infty_{p,q}$*. Hence*

$$E^\infty_{p,q} \cong E^2_{p,q} \cong \begin{cases} \oplus_{X \in L(\mathcal{B})^-_q} H_p(K[X], K(X)), & \text{if } q > 0 \\ H_p(K) \oplus_{X \in L(\mathcal{B})^-_q} H_p(K[X], K(X)), & \text{if } q = 0 \\ 0, & \text{if } q < 0. \end{cases}$$

Since the nerve K is a simplex, we have

$$H_p(K) = \begin{cases} \mathbb{Z}, & \text{if } p = 0 \\ 0, & \text{else.} \end{cases}$$

In getting around the extension problem, the following theorem was proven in [J] by an explicit map induced by the differentials $d^r, r \geq 1$.

Theorem 4.8. $\widetilde{H}_n(N(\mathcal{B})) \cong \oplus_q \oplus_{X \in L(\mathcal{B})^-_q} H_{n-q}(K[X], K(X))$.

By Alexander duality, we have the final result:

Theorem C_3.

$$\widetilde{H}^n(M(\mathcal{B})) \cong \widetilde{H}_{l-n-1}(N(\mathcal{B})) \cong \oplus_q \oplus_{X \in L(\mathcal{B})^-_q} H_{l-n-q-1}(K[X], K(X)).$$

Thus the cohomology groups of $M(\mathcal{A}) \cong M(\mathcal{B})$ can be expressed in terms of relative homology of subcomplexes of the nerve $K(\mathcal{B})$.

5. The Homotopy Equivalence Induced by $\varphi : K(\mathcal{B}) \to L(\mathcal{B})$

There exists a natural map between $K(\mathcal{B})$ and $L(\mathcal{B})$ which is intimately related to the equivalence relation of $K(\mathcal{B})$.

Definition 5.1. *Define the* **poset map** $\varphi : K(\mathcal{B}) \to L(\mathcal{B})$ *by* $\varphi(\sigma) = B_\sigma$*.*

The map φ is *onto* and *semi-order preserving* since $\sigma < \tau$ implies that $B_\sigma \subseteq B_\tau$ and $\varphi(\sigma) \leq \varphi(\tau)$. The dimension functions on $K(\mathcal{B})$ and $L(\mathcal{B})$ satisfy $d_K = d_L \circ \varphi$. Henceforth, we will use d to refer to either dimension function. The map φ and the equivalence relation $\sim$ on $K(\mathcal{B})$ are related by the following proposition.

Proposition 5.2. *$\sigma \sim \tau$ if and only if $\varphi(\sigma) = \varphi(\tau)$.*

In general φ is *not injective*. For $X \in L(\mathcal{B})$, the *pullback* $\varphi^{-1}(X)$ has a unique minimal element ς_X. We call an equivalence class *trivial* if it consists of a singleton.

The poset map $\varphi : K(\mathcal{B}) \to L(\mathcal{B})$ induces poset maps $\varphi : K[X] \to (X \leq)$ and $\varphi : K(X) \to (X <)$. Each of these can be thought of as the vertex map on their order complexes. Hence, $\varphi : K(\mathcal{B}) \to L(\mathcal{B})$ induces simplicial maps $\varphi : \Delta(K[X]) \to \Delta(X \leq)$, $\varphi : \Delta(K(X)) \to \Delta(X <)$ and

$$\varphi : (\Delta(K[X]), \Delta(K(X))) \to (\Delta(X \leq), \Delta(X <)).$$

We want to show that φ is a homotopy equivalence.

Let $X \in L(\mathcal{B})^-$. For $Y \geq X$, let ς_Y be the unique minimal element of $\varphi^{-1}(Y)$.

First we embed $\Delta(X \leq)$ into $\Delta(K[X])$ so that $\Delta(X <)$ embeds into $\Delta(K(X))$. The poset map $i : (X \leq) \to K[X]$, taking Y to the unique minimal element ς_Y of its pullback $\varphi^{-1}(Y)$, is clearly injective. Hence i induces a simplicial embedding $i : \Delta(X \leq) \to \Delta(K[X])$ which restricts to a simplicial embedding $i : \Delta(X <) \to \Delta(K(X))$.

The following lemma was proven in [Q], or see [O-T, Cor. 4.95].

Lemma 5.1. *Let $\mathcal{P}$ be a poset, and let $f : \mathcal{P} \to \mathcal{P}$ be either an increasing or decreasing poset map, ie., either $f(x) \geq x$ for all $x \in \mathcal{P}$, or $f(x) \leq x$ for all $x \in \mathcal{P}$. Then the induced map on the order complex $f : \Delta(\mathcal{P}) \to \Delta(\mathcal{P})$ is homotopic to the identity, and $f(\Delta(\mathcal{P}))$ is a strong deformation retract of $\Delta(\mathcal{P})$.*

Lemma 5.2. *$\Delta(X \leq)$ is a strong deformation retract of $\Delta(K[X])$, with $\Delta(X <)$ also being a strong deformation retract of $\Delta(K(X))$ under the same retraction. Hence, $\varphi : K[X] \to (X \leq)$ induces a homotopy equivalence $\varphi : (\Delta(K[X]), \Delta(K(X))) \to (\Delta(X \leq), \Delta(X <))$.*

Proof. The map $\varphi \circ i : L(\mathcal{B}) \to L(\mathcal{B})$ is the identity. The map $i \circ \varphi : K(\mathcal{B}) \to K(\mathcal{B})$ satisfies $i \circ \varphi(\sigma) \leq \sigma$ for all σ. By Lemma 5.1, the maps induced by $\varphi \circ i$ and $i \circ \varphi$ of the corresponding order complexes are homotopic to the identity, and $\Delta(X \leq)$ is a strong deformation retract of $\Delta(K[X])$, with $\Delta(X <)$ also being a strong deformation retract of $\Delta(K(X))$ under the same retraction. □

Theorem 5.3. *The poset map $\varphi : K(\mathcal{B}) \to L(\mathcal{B})$ induces an isomorphism in simplicial homology*

$$H_n(K[X], K(X)) \xrightarrow{\varphi_*} H_n(\Delta(X \leq), \Delta(X <)).$$

for $X \in L(\mathcal{B})^-$, where $(X \leq)$ and $(X <)$ are subposets of $L(\mathcal{B})^-$.

proof. Since $\Delta(\mathcal{P})$ is the barycentric subdivision of $\mathcal{P}$ when $\mathcal{P}$ is a simplicial complex ordered by either inclusion or reverse inclusion,

$$H_n(K[X], K(X)) \to H_n(\Delta(K[X]), \Delta(K(X)))$$

is an isomorphism, and the result follows from Lemma 5.2.□

The coincidence of Theorems C_1 and C_2 is clear from the isomorphism $L(\mathcal{A})^- \cong L(\mathcal{B})^-$ and Lemma 3.6. The coincidence of Theorems C_2 and C_3 should be clear from Theorem 5.3.

In essence, Theorem 5.3 can be interpreted as stating that the differentials in the generalized Mayer–Vietoris spectral sequence capture the cohomology of $M(\mathcal{A})$ locally, in the sense of stratified Morse theory.

Theorems A_2 and A_3, each of which can be thought of as a generalization of Brieskorn's Lemma; see [O-T, 5.91]; show that the cohomology groups of $M(\mathcal{B})$ can be expressed as a direct sum of local contributions partially ordered by $L(\mathcal{B})^-$.

In terms of the actual calculation of these local contributions to the cohomology groups of $M(\mathcal{B})$, Theorems B_2 and B_3 show that the local contributions can be expressed in terms of the nerve poset $K(\mathcal{B})$, in the generalized Mayer–Vietoris spectral sequence method, and the intersection poset $L(\mathcal{B})$, in the stratified Morse theory method, respectively.

What are the contributions to the cohomology groups of $M(\mathcal{B})$ from each of the elements of $L(\mathcal{B})^+ = L(\mathcal{B}) \cup \{S, \infty\}$? And what different roles do they play in the two different methods?

For $X \in L(\mathcal{B})^-$ (ie., $X \neq S, \infty$), Theorem 5.3 explains the coincidence of X's contribution in Theorems B_2 and B_3.

For $X = \infty$, its contribution to $H_m(N(\mathcal{B}))$ is lost in applying Alexander duality in the generalized Mayer–Vietoris spectral sequence. In the stratified Morse theory method, Lemma 3.6 explains why ∞ fails to contribute to $H^n(M(\mathcal{B}))$.

For $X = S$, its contribution to $H^n(M(\mathcal{B}))$ in the stratified Morse theory method is lost when reduced cohomology is considered. There is no contribution to $H_m(N(\mathcal{B}))$ from S in the generalized Mayer–Vietoris spectral sequence method. The only contribution S makes to $H^n(M(\mathcal{B}))$ is from Alexander duality.

6. Local Calculations of $H^n(M(\mathcal{A}))$

Even though Theorems A_2 and A_3 show that the cohomology groups of $M(\mathcal{B})$ are direct sums of local contributions labelled by the spaces $X \in L(\mathcal{B})^-$, the calculation of the term corresponding to X in either Theorem B_2 or B_3 requires knowledge of all the spaces containing X and their intersection poset. In this section, we offer a local method which requires only knowledge of the poset $\varphi^{-1}(X)$.

Consider $\varphi^{-1}(X)$ in $(K[X], K(X))$ for $X \in L(\mathcal{B})^-$. If $\varphi^{-1}(X)$ is trivial, then

$$H_p(K[X], K(X)) = \begin{cases} \mathbb{Z}, & p = \dim K[X] \\ 0, & \text{else.} \end{cases}$$

Thus we may restrict ourselves to the case when $\varphi^{-1}(X)$ is nontrivial.

Recall that $K[X]$ is an $r-$simplex for some r, and $\varphi^{-1}(X)$ has a unique minimal element ς_X. Thus $\Delta(K[X] - \{\varsigma_X\})$ is a simplicial triangulation of $S^{r-1} \cong \partial\Delta(K[X])$.

Definition 6.1. *Let $\Delta^*(X) = \Delta(\varphi^{-1}(X) - \{\varsigma_X\})$ be the order complex of the poset $\varphi^{-1}(X) - \{\varsigma_X\}$.*

Note that $|\Delta^*(X)|$ is a strong deformation retract of $|\Delta(K[X] - \{\varsigma_X\})| - |\Delta(K(X))|$. Thus we can formulate a local method of calculating the local contributions in terms of the nerve poset $K(\mathcal{B})$.

Theorem B_4.

$$H_p(K[X], K(X)) \cong \widetilde{H}^{r-p-1}(\Delta^*(X))$$

where $K[X]$ is an $r-$simplex.

Proof. The long exact sequence of the pair $(K[X], K(X))$ gives

$$\widetilde{H}_p(K(X)) \cong H_{p+1}(K[X], K(X)).$$

By Alexander duality,

$$\widetilde{H}^{(r-1)-p-1}(\Delta^*(X)) \cong \widetilde{H}_p(\Delta(K(X))) \cong \widetilde{H}_p(K(X)).\square$$

Hence Theorem C_3 can be restated in terms of the dual complex.

Theorem C_4.

$$\widetilde{H}^n(M(\mathcal{B})) = \Big(\bigoplus_{\substack{X\in L(\mathcal{B})^-_q \\ \sigma=\varphi^{-1}(X) \\ \sigma\in K^{(p)}}} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{\substack{X\in L(\mathcal{B})^-_q \\ \sigma\neq\varphi^{-1}(X) \\ \varsigma_X\in K^{(r)}}} \widetilde{H}^{r-p-1}(\Delta^*(X))$$

where $n = l - p - q - 1$ and ς_X is the unique minimal element of $K[X]$.

Since Theorems A_1, A_2, and A_3 all can be viewed as stating that $\widetilde{H}^n(M(\mathcal{B}))$ can be expressed as a direct sum of submodules $H_{X,p}$ indexed by the elements $X \in L(\mathcal{B})^-$, we can put Theorems C_1, C_2, C_3 and C_4 all together in the following theorem.

Theorem C_5.

$$\widetilde{H}^n(M(\mathcal{B})) = \oplus_{X\in L(\mathcal{B})^-_q} H_{X,p}$$

where $n = l - p - q - 1$ and $H_{X,p}$ can be found from any of the following:

(1) $H_{X,p} = H_p(\Delta(X \leq), \Delta(X <))$ *where $(X <)$ and $(X \leq)$ are subposets of $L(\mathcal{B})^-$,*

(2) $H_{X,p} = H_p(K[X], K(X))$,

(3)

$$H_{X,p} = \begin{cases} \mathbb{Z}, & \text{if } \varphi^{-1}(X) = \sigma \text{ is trivial} \\ & \text{and } \sigma \in K^{(p)}, \\ \widetilde{H}^{r-p-1}(\Delta^*(X)), & \text{if } \varphi^{-1}(X) \text{ is nontrivial} \\ & \text{with unique minimal element} \\ & \varsigma_X \in K^{(r)}. \end{cases}$$

The following example illustrates these different methods of calculating $H^n(M(\mathcal{A}))$.

Example 6.2. *Let $\mathbb{R}^5$ have coordinate functions $\{x_1, x_2, x_3, x_4, x_5\}$. Consider the following central subspace arrangement $\mathcal{A} = \{A_i\}_{i=1}^5$ in $\mathbb{R}^5$, consisting of one 4−dimensional and four 3−dimensional subspaces with center $T_{\mathcal{A}}$ the origin 0:*

$$\begin{aligned} A_1 &= \{x_1 = x_2 = 0\}, \\ A_2 &= \{x_3 = 0\}, \\ A_3 &= \{x_1 = x_3,\ x_4 = 0\}, \\ A_4 &= \{x_1 = x_3,\ x_5 = 0\}, \\ A_5 &= \{x_2 = x_4 = x_5\}. \end{aligned}$$

Let $\mathcal{B} = \{B_i\}_{i=1}^5$ be the associated sphere arrangement in S^5 consisting of one 4−sphere and four 3−spheres with center $T_{\mathcal{B}} = 0 \cup \infty$.

The Hasse diagrams of $K(\mathcal{B})$ and $L(\mathcal{B})$ are given in Figures 1 and 2. The equivalence classes of $K(\mathcal{B})$ are indicated by the bubbles. The partial ordering of $K(\mathcal{B})$ has been suppressed but should be clear (eg., $1 > 12 > 123 > 1234 > 12345$). The dimension function in $K(\mathcal{B})$ is written as a superscript over each simplex. The elements of $L(\mathcal{B})$ are labelled by the unique minimal element ς_X of their corresponding pullbacks.

Figures 3, 4, and 5 give the posets and simplicial complexes which determine the local contributions to $\widetilde{H}^n(M(\mathcal{B}))$ by all three methods. Hash marks in these figures indicate 2−simplexes.

The contributions from B_{123} can be calculated from the homology of either the pair (C_1, D_1), the pair (Δ^2, E_1), or $\Delta^*(X)$ which is the point 13, where (C_1, D_1) and (Δ^2, E_1) are the pairs in Figures 3b, 3c. These complexes arise from Theorem C_5 (1)-(3) respectively. Here D_1 and E_1 are the darkened subcomplexes. The contributions from B_{124} are found similarly, since $(B_{123} <)$ is isomorphic to $(B_{124} <)$, see Figure 3a.

The contributions from B_{345} can be calculated from the homology of either the pair (C_2, D_2), the pair (Δ^2, E_2), or $\Delta^*(X)$ which is the set of two points 35 and 45, where (C_2, D_2) and (Δ^2, E_2) are the pairs in Figures 4b, 4c. These complexes arise from Theorem C_5 (1)-(3) respectively. Here D_2 and E_2 are the darkened subcomplexes, both of which contain the isolated vertex 5.

The contributions from B_{12345} can be calculated from the homology of either the pair $(\text{cone}(D_3), D_3)$, the pair (Δ^4, E_3), or F, where D_3, E_3, and F are the simplical complexes in Figures 5a, 5b, and 5c. These complexes arise from Theorem C_5 (1)-(3) respectively.

The local contributions $H_{X,p}$ of $X \in L(\mathcal{B})^- = L(\mathcal{B})$ are indicated in Table 1, where Δ^r is the $r-$simplex with boundary $\partial\Delta^r$ and for a simplical complex L, cone(L) is the cone over L and sd(L) is the barycentric subdivision of L.

It follows that

$$\widetilde{H}^n(M(\mathcal{A})) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ \mathbb{Z}^{10}, & \text{if } n = 1 \\ \mathbb{Z}^4, & \text{if } n = 2 \\ 0, & \text{else.} \end{cases}$$

Final Remarks. *This work was completed in spring 1991, and it took some time to prepare the text. Quite recently one of the authors received information that V. Vassiliev claimed the degeneracy of a similar spectral sequence also gives the Goresky–MacPherson formula, see* [V]. *An alternate description of the cohomology of the complement is also forthcoming in the work of G. Ziegler and R. Živaljević, see* [Z-Z].

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References

[B]. K.Brown, *Cohomology of Groups*, Springer–Verlag, 1982, pp. 161–168.

[G-MP] M. Goresky & R. MacPherson, *Stratified Morse Theory*, Springer–Verlag, 1988, pp. 235–257.

[J]. K. Jewell, *Complements of Sphere and Subspace Arrangements*, preprint (1991).

[M]. J. Munkres, *Elements of Algebraic Topology*, Addison–Wesley, 1984, pp. 377–381.

[O]. P. Orlik, *Complements of Subspace Arrangements*, J. Algebraic Geometry **1** (1992), 147–156.

[O-S]. P. Orlik & L. Solomon, *Combinatorics and Topology of Complements of Hyperplanes*, Invent. Math. **56** (1980), 167–189.

[O-T]. P. Orlik & H. Terao, *Arrangements of Hyperplanes*, Springer–Verlag, 1992.

[Q]. D. Quillen, *Homotopy Properties of the Poset of Non-trivial p−Subgroups of a Group*, Advances in Math. **28** (1978), 101–128.

[V]. V. Vassiliev, *Complements to Discriminants of Smooth Mappings*, lecture given in College on singularity theory (Aug 19– Sept 6, 1991), Trieste.

[Z-Z]. G. Ziegler & R. Živaljević, *Homotopy Types of Subspace Arrangements via Diagrams of Spaces*, preprint (1991).

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