Mathematics of the USSR-Izvestiya

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To cite this article: B Z Shapiro 1991 *Math. USSR Izv.* **36** 183

Изв. Акад. Наук СССР
Сер. Матем. Том 54(1990), № 1

Math. USSR Izvestiya
Vol. 36(1991), No. 1

SPACES OF LINEAR DIFFERENTIAL EQUATIONS, AND FLAG MANIFOLDS

UDC 517.926.4

B. Z. SHAPIRO

ABSTRACT. The author studies the connection between the properties of a linear ordinary differential equation and the associated curve in the space of complete flags. Sturm's classical alternation theorem is generalized to equations of arbitrary order.
Bibliography: 21 titles.

§1. Introduction

Let us consider a linear homogeneous differential equation of nth order given on a segment I of the time axis t:

$$L_n[x] = x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0, \tag{1}$$

where $a_i(t) \in C^\infty[I]$.

MAIN DEFINITION. Equation (1) is said to be *nonoscillatory* on I if any nonzero solution has less than n roots on I, counted with their multiplicity. Otherwise, the equation is called *oscillatory*.

In this paper we study the connection between the oscillation properties of (1) and the behavior of the left-invariant flow induced by this equation on the space of complete flags. The trajectories of this flow are tangent to the standard Cartan distribution (see §3 and [1]). Moreover, the space of complete flags is endowed with the following very rigid structure. Each point is the vertex of the standard Schubert cell decomposition constructed from it. This Cartan distribution has very specific properties with respect to the family of Schubert decompositions, some of which are studied below.

We construct a Sturm alternation theory for linear equations of arbitrary order, including the classical case of second order equations, and an analogous Sturm theory for Hamiltonian systems. Unlike the ordinary alternation theory (see, for instance, [2] and [3]), here we compare not the behavior of individual solutions but rather that of various ordered fundamental systems of solutions $\varphi_1, \dots, \varphi_n$, and instead of the zeros we consider the zeros of the Wronskians of the arrays $\varphi_1, \dots, \varphi_i$ for all $i = 1, \dots, n$ (the transversality instants). A particular feature of this theory is the fact that not the modulus of the difference of the number of "zeros" of two solutions is bounded by a constant, but rather their ratio (if both these numbers are positive).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C10; Secondary 34C10, 14M15.

We study the singularities of the boundary of nonoscillatory equations in the functional space of all ordinary differential equations. These singularities are also closely related with the geometry of the Schubert decomposition (see §6).

There is an extensive bibliography devoted to criteria of (non-) oscillation (see [4]–[11]). From comparatively recent work we must particularly mention the monograph [12] and the papers [13] and [14], continuing research to which Kondrat'ev [2] made a substantial contribution. The problem of studying the singularities of the boundary of nonoscillation was posed by Arnol'd in [15]. The author is deeply grateful to him for posing the problem and for his constant interest in this work. (Part of these results was announced by Professor Arnol'd in his survey report [16].)

§2. Main results

2.1. Definition. Let $\mathbf{F}_n$ be the manifold of complete flags in the space V of solutions of equation (1). By a *flag curve* of (1) we mean a map $f: I \to \mathbf{F}_n$ associating to each instant $t \in I$ the complete flag in the space V whose i-dimensional space consists of solutions having a root of multiplicity $\geq n-i$ at t.

2.2. Definition. An array of complete flags in $\mathbf{R}^n$ is said to be *generic* if for every set of linear subspaces belonging to distinct flags the codimension of their intersection is equal to the minimum of n and the sum of their codimensions.

We shall say that two complete flags are *transversal* if they form a general array, and *nontransversal* otherwise. The set of complete flags that are nontransversal to a given flag α will be called its *train* $\mathrm{Trn}(\alpha)$.* The train consists of all the cells of positive codimension of the standard Schubert cell decomposition of $\mathbf{F}_n$ constructed from α.

We state below the main results of this paper.

Theorem A. *The following three assertions are equivalent*:

a) *Equation* (1) *is oscillatory on* $I = [0, 1]$.

b) *There is an instant* $t \in (0, 1]$ *such that the flag of the equation at* t *is nontransversal to the flag of the equation at* 0.

c) *The flag curve of* (1) *intersects the train of any flag.*

Theorem B. *The total sum over all the nontransversality instants of the dimensions of the intersection of the i-dimensional subspace of a flag curve of any nonoscillatory equation with an* $(n-i)$*-dimensional subspace of an arbitrary flag* $\alpha \in \mathbf{F}_n$ *does not exceed* $i(n-i)$.

Corollary C (generalized Sturm alternation theorem). *If the sum over all the nontransversality instants of the dimensions of the intersection of an i-dimensional subspace of a flag curve of the equation with an* $(n-i)$*-dimensional subspace of a flag* $\alpha \in \mathbf{F}_n$ *on a certain segment exceeds* $i(n-i)$, *then on this time interval the flag curve intersects the train of any flag.*

Theorem D (cf. [11], Theorem 1). *Equation* (1) *belongs to the boundary of the nonoscillatory equations on* I *if and only if the starting point and the endpoint are the only pair of nontransversal flags of its flag curve.*

Corollary E. *The singularities of the boundary of the domain of nonoscillatory equations encountered in typical k-parameter families are diffeomorphic to the singularities of the typical k-parameter section of the train.*

* *Editor's note.* The Russian word for "train" (in the present context) is "Шлейф"; the abbreviation "Ш" in the original is here rendered as "Trn".

REMARK. The term "typical k-parameter family of equations" means that the family belongs to an open dense set in the space of families, and the term "typical k-parameter section" means transversality of the Schubert stratification of the train [12].

THEOREM F. a) *The singularities of typical sections of the train do not change (up to a diffeomorphism of a neighborhood of a section point) along the cells of a Schubert decomposition of the train.*

b) *The list of the typical singularities of the train is finite for any k (it does not have moduli).*

c) *The lists of singularities of typical k-parameter sections of trains increase as the dimension of the space increases, and are stabilized at dimension* $2k$.

d) *In typical two-parameter sections of the trains of any dimension one encounters a unique singularity (up to a diffeomorphism). It is given by the equation*

$$xy = 0;$$

for 3 *parameters we encounter two singularities:*

$$xyz = 0 \quad \textit{and} \quad z(z - xy) = 0;$$

and for 4 *parameters we encounter three singularities:*

$$xyzu = 0, \qquad xu(xu - zy) = 0, \qquad zu(z - xy) = 0.$$

§3. Curves of differential equations and Cartan distribution on the flag manifold

3.1. DEFINITION. Let $P(V^*)$ be the projectivization of the space V^* dual to the space V of solutions of (1). By the *projective curve* of equation (1) we mean the map $p: I \to P(V^*)$ associating to the instant $t \in I$ the hyperplane of solutions having a root at t.

Let us give another definition of the curve p.

3.2. DEFINITION. We construct a curve $a: I \to V^*$, which we shall call an *affine curve* of (1), by putting

$$\langle a(t), \varphi\rangle = \varphi(t),$$

where φ is any solution of (1).

The vector a for any t annihilates the hyperplane of solutions of (1) having a root at t. Therefore, the projectivization of the curve a coincides with the curve p. For any fundamental system of solutions $\varphi_1, \dots, \varphi_n$ of (1) the components of $a(t)$ in the basis dual to the chosen fundamental system are $\varphi_1(t), \dots, \varphi_n(t)$.

3.3. Let us introduce the following identification of the linear spaces of solutions of equations of the type (1) given on $I = [0, 1]$. Two solutions of distinct equations are considered *identical* if and only if their values at 0 and the values of all the derivatives up to the $(n-1)$st inclusive are the same. This identification induces an identification of the dual spaces, their projectivizations, etc. Let us choose in the identified dual space a basis of n linear functions such that the ith function computes the value of the $(i-1)$st derivative of the solution at 0. In this basis, by the *components* of a curve $a(t)$ of the differential equation (1) we mean a fundamental system of solutions $\varphi_1, \dots, \varphi_n$ satisfying the relations

$$\varphi_i^{(j)}(0) = \delta_{ij}, \qquad i = 1, \dots, n, \; j = 0, \dots, n-1. \tag{2}$$

3.4. DEFINITION. A point of the curve γ in $\mathbf{P}^n$ is said to be a *nonplanar point* if in a neighborhood of this point we can choose affine coordinates such that the germ of the curve is given as follows: $(t+\cdots, t^2+\cdots, \dots, t^n+\cdots)$, A projective curve is said to be *nonplanar* if all its points are nonplanar points.

3.5. REMARK. Since the Wronskian of a fundamental system of solutions of (1) is nonzero, its projective curve is nonplanar.

3.6. REMARK. At each nonplanar point of a projective curve the complete moving flag, consisting of the tangent subspaces of all dimensions at the point under consideration, is uniquely defined.

3.7. REMARK. The points of the flag curve defined in 2.1 are moving flags of the projective curve of the equation.

3.8. REMARK. A natural map arises from the space of linear nth order equations of the form (1) into the space of nonplanar curves in $\mathbf{P}^n$ having a specified moving flag at the starting point.

3.9. Let $\alpha = \{\alpha_1, \dots, \alpha_{n-1}\}$ be a complete flag in $\mathbf{R}^n$. We define in $\mathbf{F}_n$ an array of $n-1$ smooth curves $l_1, \dots, l_{n-1}$ passing through α and determined by

$$l_i = \{\alpha_1 \subset \cdots \subset \alpha_{i-1} \subset L_i \subset \alpha_{i+1} \subset \cdots \subset \alpha_{n-1}\}, \qquad i = 1, \dots, n-1,$$

where L_i ranges through the set of all i-dimensional subspaces satisfying these inclusions.

The tangent lines to $l_1, \dots, l_{n-1}$ at α are linearly independent. Let us consider the $(n-1)$-dimensional tangent plane $\overline{C}_\alpha \subset \mathbf{TF}_n$ spanned by these lines, and let us remove from $\overline{C}_\alpha$ all the $(n-2)$-dimensional planes $C_{\alpha j}$, $j = 1, \dots, n-1$, where the plane $C_{\alpha j}$ is spanned by all the tangent lines except the jth one.

DEFINITION. The *Cartan distribution* on the space $\mathbf{F}_n$ is the distribution $C_\alpha = \overline{C}_\alpha \setminus \bigcup_{j=1}^{n-1} C_{\alpha j}$.

3.10. LEMMA. *An immersed curve $f: I \to \mathbf{F}_n$ is a flag curve of equation* (1) *if and only if it is everywhere tangent to the distribution C_α.*

PROOF. A flag curve of the differential equation is tangent to the distribution $\overline{C}_\alpha$, since an infinitesimal motion of the i-dimensional subspace of the moving flag of the projective curve occurs in an $(i+1)$-dimensional subspace. Since the curve is nonplanar, the velocity vector of the motion of the i-dimensional subspace does not lie in itself, which precisely means the tangency of the flag curve with the distribution C_α.

3.11. REMARK. There arises a map from the space $\mathfrak{A}_n$ of linear differential equation of nth order of the form (1) into the space of flag curves in $\mathbf{F}_n$ that are tangent to the distribution C_α and have a fixed initial point and a fixed velocity vector.

3.12. DEFINITION. By the *end map* $\pi_n: \mathfrak{A}_n \to \mathbf{F}_n$ we mean the map associating to each equation the endpoint of its flag curve.

3.13. LEMMA. *For every equation* (1) *there is an $n(n-1)/2$-parameter germ of its deformations in the space $\mathfrak{A}_n$ that has a nondegenerate projection onto a neighborhood of the image of its end map.*

PROOF. We shall give this germ explicitly. Let $\varphi_1, \dots, \varphi_n$ be a fundamental system of solutions of (1) satisfying conditions (2). Let us define the kth function of the fundamental system of solutions of the equations belonging to the germ by

$$\Phi_k(t, \lambda_{kj}) = \varphi_k(t) + \rho(t) \sum_{j=k}^{n} \lambda_{kj} (t-1)^j / j!,$$

where $\rho(t) \in C^\infty[0, 1]$, $0 \le \rho \le 1$, and ρ vanishes in a neighborhood of 0 and is equal to 1 in a neighborhood of 1.

3.14. Lemma. *The map* π_n *is onto* $\mathbf{F}_n$.

Proof. The attainability domain for an arbitrary distribution of velocities is the same as for its convex hull, by Filippov's lemma [13]. The convex hull of the distribution C_α is a nonholonomic distribution of linear subspaces in $\mathbf{TF}_n$. Therefore, the attainability domain for C_α coincides with $\mathbf{F}_n$, by the Chow-Rashevskiĭ theorem [14].

§4. Trains

By the *train* Trn_α of a complete flag α we mean the set of all nontransversal flags to α. In the space $\mathbf{F}_n$ we construct a Schubert cell decomposition from each complete flag, whose cells are constituted by flags such that the dimensions of their intersection with the subspaces of the given flag are specified. The train of the flag is formed by the union of all the cells of positive codimension of the Schubert partition constructed from this flag. Consequently, it is a stratified algebraic hypersurface. The train is reducible and consists of $n-1$ irreducible components $\Delta_1, \dots, \Delta_{n-1}$, where Δ_i consists of flags whose $(n-i)$-dimensional subspaces are not transversal to the i-dimensional subspace of the given flag; Δ_i and Δ_{n-i} are diffeomorphic via duality. The following bijection holds between the cells of the Schubert partition in $\mathbf{F}_n$ and the permutation group S_n; S_n acts on the linear space by permutation of the coordinates, and in each cell of the partition there is precisely one element in the orbit of the initial flag α. The dimension of the cell corresponding to the desired permutation is equal to the number of its inversions, i.e., the number of pairs (i, j), where i appears before j in the permutation and $j < i$.

4.1. Definition. Let s_1 and s_2 be two permutations such that the length of s_1 is 1 less than that of s_2. (The *length* of a permuation is the length of its minimal decomposition into a product of transpositions.) We shall say that $s_1 < s_2$ *according to Bruhat* if s_2 is obtained from s_1 either by one transposition or by the permutation of a pair $(j, j+1)$. Extending this relation by transitivity, we obtain Bruhat's partial order on S_n.

4.2. Let $e_1, \dots, e_n$ be a basis in $\mathbf{R}^n$.

Definition. The *standard flag* of the basis $e_1, \dots, e_n$ is the flag whose i-dimensional space is spanned by $e_1, \dots, e_i$ for all i, and the *inverse flag* is the flag whose i-dimensional space is spanned by $e_n, \dots, e_{n-i}$.

The set of flags that are transversal to the inverse flag of the basis $e_1, \dots, e_n$ (the leading cell of the Schubert decomposition constructed from this flag) is identified with the group T_1 of upper triangular matrices in the basis $e_1, \dots, e_n$ with unit diagonal. Under this identification the i-dimensional subspace of the flag corresponding to the given matrix is spanned by its first i rows.

Using this identification, the condition for a flag given by the matrix X to belong to the $(n-i)$th component of the train of the flag given by the matrix Y is written as

$$\det[X_i, Y_{n-i}] = 0,$$

where $[X_i, Y_{n-i}]$ is the $n \times n$ matrix formed by the first i rows of X and the first $n-i$ rows of Y.

4.3. Remark. If $Y = E$, the equation of the components of its train has a particularly simple form:

$$\Delta_i = 0, \tag{3}$$

where Δ_i is the minor of X formed from the first i rows and the last i columns. Thus, the general equation of the train is given by

$$\mathrm{Trn}_1 : \Delta_1 \cdots \Delta_{n-1} = 0. \tag{4}$$

By the transitivity of the action of the groupn GL_n on the space $\mathbf{F}_n$ the local study of the train of any flag is equivalent to the study of Trn_1. Equation (3) is invariant with respect to the action of the torus $(\mathbf{R}\backslash 0)^n$ on T_1 $(n > 2)$, where the ith component of this torus acts on a triangular matrix by multiplication of the ith row by a nonzero number λ and the ith column by λ^{-1}. This torus contains the subgroup $G = (\mathbf{Z}_2)^{n-1}$ of transformations consisting in the change of the signs of the rows and the columns.

4.4. Definition. We shall say that a matrix is *completely nonnegative* if all its nonzero minors are positive. A matrix is said to the *completely positive* if all its minors are positive.

Definition. By a *completely positive upper triangular matrix* we mean an upper triangular matrix such that all its minors that do not contain a row or column lying below the main diagonal are positive.

Theorem. *The semigroup T^+ of completely positive upper triangular matrices is one of the contractible components in $T_1 \backslash Tr_1$.*

The proof is preceded by two assertions from matrix theory (see [7], Chapter II, §6, Theorem 8 and Chapter V, §2, Theorem 5).

Definition. Let us denote by $A\binom{i_1, \ldots, i_l}{j_1, \ldots, j_l}$ the minor of the quadratic matrix A consisting of the intersection of the rows with indices $i_1 < \cdots < i_l$ and the columns with indices $j_1, \ldots, j_l$. By the *row nondensity* κ *we mean the sum* $\sum_{m=1}^{l}(i_m - i_{m-1} - 1)$, *and by the column nondensity we mean* $\mu = \sum_{m=1}^{l}(j_m - j_{m-1} - 1)$. A minor for which $\kappa = \mu = 0$ will be called *dense*.

Proposition 1. *For any completely nonnegative matrix A and for any numbers* $p = 1, \ldots, n$,

$$\det A \le \det A\binom{1, \ldots, p}{1, \ldots, p} \times \det A\binom{p+1, \ldots, n}{p+1, \ldots, n}. \tag{5}$$

Proposition 2. *For any $n \times (n+1)$ matrix A,*

$$\begin{aligned} A\binom{1, \ldots, n}{1, \ldots, n} \times A\binom{1, \ldots, n-1}{2, \ldots, n-1, n+1} + A\binom{1, \ldots, n}{2, \ldots, n+1} \times A\binom{1, \ldots, n-1}{1, \ldots, n-1} \\ = A\binom{1, \ldots, n}{1, \ldots, n-1, n+1} \times A\binom{1, \ldots, n-1}{2, \ldots, n}. \end{aligned} \tag{6}$$

Let us proceed with the proof of the theorem. We shall show that if a completely nonnegtive upper triangular matrix has a zero minor not containing a row (or column) lying below the main diagonal, then at least one of the Δ_i is also equal to zero. Let us first show that if a dense minor $A\binom{i, \ldots, i+1}{j, \ldots, j+1}$ equals zero, then $\Delta_{n-j+i} = 0$.

Indeed, using (5) and the nonnegativity of the matrix A, we have

$$0 = A\binom{i, \ldots, i+1}{j, \ldots, j+1} \times A\binom{1, \ldots, i}{j-1, \ldots, j} \ge A\binom{1, \ldots, i+1}{j-1, \ldots, j+1} = 0.$$

Further, applying (5) again, we obtain

$$0 = A\binom{1,\dots,i+1}{j-1,\dots,j+1} \times A\binom{i+1,\dots,n-j+1}{j+1,\dots,n} \geq \Delta_{n-j+i} = 0.$$

Let $A\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ be a zero nondense minor of minimal dimension for which μ (or κ) is positive. Let us show that there is a nonzero minor of the same dimension with smaller μ (κ).

To fix ideas, we consider the case $\mu > 0$. Let us add to $A\binom{i_1,\dots,i_l}{j_1,\dots,j_l}$ the intersection of the last column of A not figuring in the minor with the rows $i_1, \dots, i_l$. Let us apply to this $l \times (l+1)$ matrix the identity (6). Since by assumption all the minors of dimension $< l$ are positive and right-hand side of (6) is zero, the denser minor, obtained by replacing the j_1th column by the chosen addition, is also zero. Applying this procedure several times, we obtain $A\binom{i_1,\dots,i_l}{j_1,\dots,j_l} = 0$ and, therefore, $\Delta_{n-j_1+i+1} = 0$, by the first part of the proof.

4.5. Definition. A semigroup consisting of the elements inverse to those of a given semigroup will be called the *inverse semigroup.*

Corollary. *$T_1 \backslash Tr_1$ contains 2^{n-1} components obtained by the action on the semigroup T^+ of completely positive upper triangular matrices of the group G by changing the signs of the rows and the columns. These components are also semigroups and are decomposed into pairs of mutually inverse semigroups.*

Proof. Since Tr_1 is invariant under the action of G, the orbit of any component of the complement in $T_1 \backslash Tr_1$ is the union of components of the complement in $T_1 \backslash Tr_1$. It is easy to see that the orbit GT^+ of the semigroup T^+ consists of 2^{n-1} elements. By the commutativity of the action of G and the multiplication in T^+, the images $gT^+ \in GT^+$ are also semigroups. Any matrix belonging to GT^+ has the property that all the minors not containing a row (a column) below the main diagonal are nonzero. It is easy to see that this property holds also for the inverse matrices. This means that the inverse semigroup to any $gT^+ \in GT^+$ also belongs to GT^+.

4.6. Lemma. *The flags corresponding to matrices from mutually inverse semigroups are transversal to each other.*

Proof. The family of trains constructed at all points of the group T_1 is invariant under the action of T_1 on itself by left multiplication. For the semigroups under consideration the image of the left shift of the semigroup by any element is contained strictly inside the semigroup. To prove this, one should consider the semigroup T^+ and use the formula for the minors of the product of matrices (see also [7], Chapter II, §2). We obtain that the left shift of the semigroup by an element in the inverse semigroup strictly contains the initial semigroup. This means that the semigroup does not intersect the train of any matrix from the inverse semigroup.

4.7. Remark. The total number and the structure of the components of the complement are unknown to the author. For $n = 2, 3,$ and 4 the number of components is 2, 6, and 20 respectively.

4.8. Lemma. *Assume that the germ of a flag curve of equation* (1) *intersects at zero a flag α. Then the parts of the germs before and after the intersection lie in mutually inverse semigroups.*

PROOF. It suffices to consider the case of a flag curve of the canonical equation $x^{(n)} = 0$. The coordinates of the flag curve $f(t)$ of this equation in T_1 have the form

$$f_{ij}(t) = \begin{cases} t^{j-i}/(j-i)! & \text{for } j \geq i, \\ 0 & \text{for } j < i. \end{cases}$$

In this case, when t changes sign the matrix $f(t)$ is transformed into $f(-t)$. Thus, $f(-t) = f^{-1}(t)$. The proof is complete.

4.9. LEMMA. *The multiplicity of the intersection of a germ $f: [-\varepsilon, \varepsilon] \to \mathbf{F}_n$ of the flag curve with train Trn_α (the degree of the restriction of the divisor to the curve) depends only on the cells of the Schubert decomposition constructed from α containing the intersection point, and is computed as follows. By the flag α we construct the Schubert cell decompositions of all the Grassmannians G_{kn}, $k = 1, \dots, n-1$. Then the multiplicity $\#_k$ of the intersection of the germ of the flag curve f with the kth component of the train Tr_α is equal to the codimension of the cell of the partition containing the k-dimensional plane of the flag of the germ at the nontransversality instant. The multiplicities $\#_k$ for the permutation matrix $(i_1, \dots, i_n)$ of the corresponding cell in Tr_α containing the intersection point are equal to*

$$\#_k = \max\left(0, \sum_{m=n-k+1}^{n} (i_m - (n-m-1)\right).$$

PROOF. Let a be the germ of an affine curve of equation (1) (see 3.2) such that its flag curve is nontransversal for $t = 0$ to the complete flag $\alpha = (\alpha_1, \dots, \alpha_{n-1})$. We say that a basis $e_1, \dots, e_n$ in $\mathbf{R}^n$ is *adapted* to the flag α if for all i the space α_i is spanned by $e_1, \dots, e_i$. For any germ a of an affine curve of (1) there is a unique adapted basis such that the coordinates of a in this basis are

$$a_1(t) = t^{i_1}/i_1! + \cdots, \cdots, a_n(t) = t^{i_n}/i_n! + \cdots,$$

where $(i_1, \dots, i_n)$ is a permutation corresponding to the cell of the Schubert decomposition of $\mathbf{F}_n$ constructed from α containing the flag $f(0)$. Let us consider the Wronsky matrix $W(t)$ of the fundamental system of solutions $a_1(t), \dots, a_n(t)$. The multiplicity of the intersection of the flag curve f with Tr_α is equal to the multiplicity of the intersection of $W(t)$ with $\nu_\alpha^{-1}(Tr_\alpha)$, where ν is the projection in the bundle $\mathrm{GL} \xrightarrow{\nu} \mathbf{F}_n$ associating with each nondegenerate matrix the complete flag whose i-dimensional space is spanned by the first i columns. Let us write down the expansion in t of the entries of the jth row of $W(t)$;

$$W_j(t) = \{t^{i_j}/i_j! + \cdots, t^{i_j-1}/(i_j-1)! + \cdots, \cdots, 1 + \cdots, 0 + \cdots, 0 + \cdots\}. \quad (7)$$

The equation of the kth component of $\nu^{-1}(Tr_\alpha)$ in this system of coordinates has the form $\Delta_k = 0$, where Δ_k is the minor consisting of the k last rows and the first k columns of $W(t)$. Let us find the multiplicity of the intersection of Δ_k with $W(t)$, restricting Δ_k to $W(t)$. To this end we must compute the lowest term of the expansion in t of the corresponding minor. From the explicit form of the expansion (7) we obtain that for every k the order of the zeros depends only on the permutation $(i_1, \dots, i_n)$, and is computed as follows. We compare the permutation

$(i_n, \dots, i_1)$ with the identity permutation and compute the total excess of the first k terms of this permutation over the identity. It is equal to

$$\sum_{m=n-k+1}^{n} (i_m - (n - m + 1)).$$

This total excess is also equal to the area of the part of the Young diagram, which coincides with the codimension of the corresponding cell of the Schubert decomposition in the Grassmannian G_{kn}.

§5. (Non-) oscillation criteria and Sturm's generalized alternation theorem for a higher-order linear equation

5.1. LEMMA. *Equation* (1) *is nonoscillatory in the segment* I *if and only if an arbitrary collection of pairwise different points of its flag curves is generic.*

PROOF. The following is one of the classical criteria of (non-) oscillation [5]. Equation (1) is nonoscillatory if and only if there is a unique solution of any multipoint problem of the form

$$x^{(j)}(t_i) = x_{ij}, \qquad t_i \in I, \qquad j = 0, \dots, m_i, \qquad \sum (m_i + 1) = n.$$

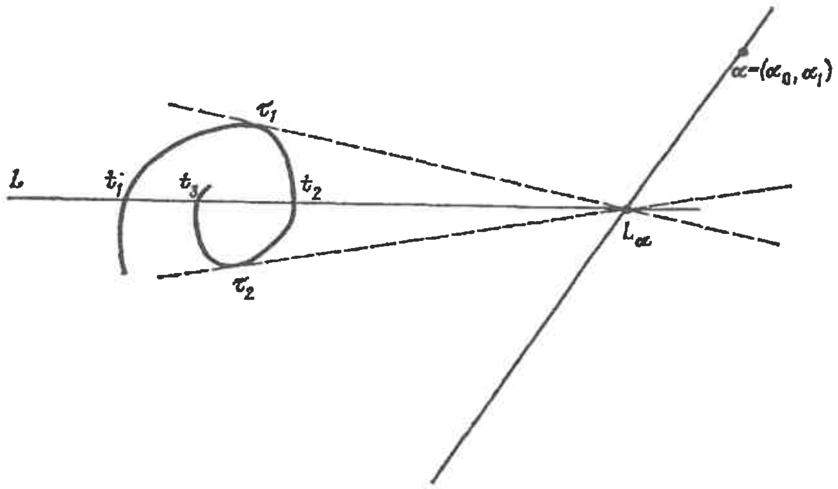
This condition is equivalent to the genericity of the collection of points $f(t_1), \dots, f(t_i)$ of the flag curve f.

5.2. The proof of Theorem A is carried out in the following sequence: b) $\Rightarrow$ a) $\Rightarrow$ c) $\Rightarrow$ b).

b) $\Rightarrow$ a). Let f be a flag curve of (1) given on $[0, 1]$. Assume that the k-dimensional subspace of the flag $f(0)$ is not in general position with the $(n-k)$-dimensional subspace of the flag $f(t)$, where $t \in (0, 1]$. This pair can always be chosen if $f(0)$ and $f(t)$ are nontransversal. The direct sum of these subspaces belongs to a hyperplane in V^*, where V is the linear space of solutions of the equation. This hyperplane determines a nonzero solution of the equation having a root of multiplicity $\geq n-k$ at 0 and a root of multiplicity $\geq k$ at t. Therefore, this equation is oscillatory by definition.

a) $\Rightarrow$ c). The proof is by induction on the dimension (the induction basis is $n = 2$). In this case $\mathbf{F}_2 = S^1$ and the train of any flag coincides with the flag itself. The flag curve of any equation moves along S^1 with nonzero velocity. The existence of a solution with two roots means that the flag curve hits a point of the space $\mathbf{F}_2$ twice and, therefore, makes a complete turn around the circle, intersecting the train of any flag.

Inductive step. Let $p: [0, 1] \to \mathbf{P}^{n-1}$ be a projective curve of the oscillatory equation, let L be the hyperplane in $\mathbf{P}^{n-1}$ that p intersects $\geq n$ times, counting multiplicities (corresponding to an oscillation solution of the equation), and let $\alpha = (\alpha_0, \dots, \alpha_{n-2})$ be an arbitrary flag in $\mathbf{P}^{n-1}$. The multiplicity of the intersection is lower semicontinuous, and consequently it suffices to consider the case when α is transversal to L. If p intersects α_{n-2}, then the intersection point is the desired nontransversality instant. If p does not intersect α_{n-2}, then we consider in the hyperplane $\alpha_{n-2} = \mathbf{P}^{n-2}$ the flag α_L obtained by intersection of the flag α with the hyperplane L. Now let us project the curve p onto α along tangent lines; that is, to a point of p we associate the point on α_{n-2} obtained by intersecting α_{n-2} with



the tangent line to p at this point. If p is a nonplanar curve in $\mathbf{P}^{n-1}$ and p does not intersect α_{n-2}, then p_α is a nonplanar curve in $\alpha_{n-2} = \mathbf{P}^{n-2}$.

Let us now show that the projected curve intersects the plane $L_\alpha = L \cap \alpha_{n-2}$ at least $n-1$ times, counting multiplicities. We consider first the case of multiple roots. If p intersects L at the instant t with certain multiplicity then (by definition of the multiplicity as the dimension of the maximal moving subspace lying in the hyperplane) we obtain that p_α intersect L_α with the multiplicity reduced by 1. Let us now take on p two neighboring geometrically distinct roots $p(t_i)$ and $p(t_{i+1})$. We shall show that on the interval (t_i, t_{i+1}) there is an instant τ_i at which the tangent line at $p(\tau_i)$ intersects the plane L_α. (The corresponding point will be a root of the curve p_α (see the picture).)

The planes L and α_{n-2} divide the initial space $\mathbf{P}^{n-1}$ into two half-spaces. The part of the curve p on the interval (t_i, t_{i+1}) lies in one of them. Let us choose a projective chart in which the plane α_{n-2} is improper, and assume that the plane L is horizontal. Then the point at which the tangent line to p hits α_{n-2} is the point where the tangent line to p is horizontal.

Let us consider on (t_i, t_{i+1}) the distance function from p to L. This function necessarily has a maximum at an interior point, since at the endpoints of the intervals it is strictly increasing. At this point the tangent line is horizontal, and this concludes the proof.

c) $\Rightarrow$ b). Condition c) is equivalent to the fact that the union of the trains of the points in a flag curve of the equation coincides with $\mathbf{F}_n$. Assume that there is no instant t at which $f(0) \in \mathrm{Trn}_{(f(t))}$. We show that then there is a flag $\alpha \in \mathbf{F}_n$ that is not contained in the union of the trains of points of the curve f. In our assumptions, for every τ on $(0, 1]$ there is an ε_τ such that the union of the trains of points of the curve on $[\tau, 1]$ does not meet an ε_τ-neighborhood of the flag $f(0)$. On the other hand, on a small time interval the germ of the flag curve lies in one of the semigroup components of the train of $f(0)$, and so all its points are transversal to any flag from the inverse semigroup component, according to Lemma 4.6. An arbitrary flag in this component lying in an ε_τ-neighborhood of the flag $f(0)$ is transversal to all the flags of the curve f. This contradiction proves the desired implication.

5.3. Let us prove Theorem B.

Let $t_1, \dots, t_m$ be different nontransversality instants of the points of the flag curve f of a nonoscillatory equation to some specified flag α, and let $\#_{ik}$ be the multiplicity of the intersection of f at f_i with the kth component of the train Trn_α. Let us show that $\sum_i \#_{ik} \le k(n-k)$. By the lemma, the array of flags $f(t_1), \dots, f(t_m)$ is generic. The Schubert cell decomposition of the Grassmannians constructed from a generic array of flags has the property of dimensional transversality [15], i.e., the codimension of the intersection of the cells of these partitions is equal to the sum of the codimensions. Let us consider the m-cell decompositions of the Grassmannian G_{kn} constructed from the flags $f(t_1), \dots, f(t_m)$. By our lemma, the k-dimensional plane of the flag α lies in the union of the cells of codimension $\#_{n-k,i}$ from the ith cell decomposition of the Grassmannian G_{kn}. By the dimensional transversality, the sum of the codimensions of the cells whose intersection contains the k-plane of the flag α does not exceed $\dim G_{kn} = k(n-k)$. Hence, $\sum_i \#_{ik} \le k(n-k)$. In particular, we once again obtain that the number of roots of any solution of a nonoscillatory equation does not exceed $n-1$.

5.4. Let us prove Corollary C.

Indeed, if the multiplicity of the intersection of the flag curve of (1) with the kth conponent of the train of a flag on some time interval exceeds $k(n-k)$, then the equation is oscillatory on this segment. Then, by Theorem A, on this time interval there is a nontransversality instant to the train of any flag.

§6. Boundary of the domain of nonoscillatory equations and singularities of the sections of the trains

6.1. Let us prove Theorem D. It is a consequence of Theorem A and the following theorem of Sherman [11] (see §7).

THEOREM. *If an nth order linear ordinary differential equation has a solution with $\ge n$ zeros on the semi-interval $[0, 1)$, counted with their multiplicities, then it has a solution with $\ge n$ simple zeros on $(0, 1)$, and even a solution whose first n zeros are simple.*

The domain of nonoscillatory equations on $I = [0, 1]$ is open. Therefore, the boundary equations are oscillatory. Let us show that for the boundary equation the instant at which the point of its flag curve is nontransversal to the flag at the left end of the curve cannot lie on $(0, 1)$. Indeed, if it did, there would be a solution having $\ge n$ zeros on $[0, 1-\varepsilon)$. Then there would be a solution having n simple zeros on $(0, 1)$, by Sherman's theorem. But any sufficiently small perturbation of such an equation is oscillatory, since simple zeros are preserved under small perturbations. This contradicts the fact that our equation belongs to the boundary of the oscillatory ones.

6.2. COROLLARY. *The singularities of the boundary of the domain of nonoscillatory equations existing in typical families of equations with a specified number of parameters are diffeomorphic to the singularities of the sections of the train of the initial flag of the flag curve of the equation, arising in the endpoint projection of the families according to Theorem* 6.1 *and Lemma* 3.13.

Thus, the problem of studying the singularities of the boundary of the domains of nonoscillatory equations reduces to that of studying typical singularities of sections of trains.

6.3. Let us prove Theorem F.

Assertion a) follows from the fact that the stabilizer of any flag in the group GL preserves Schubert's equation constructed by this flag, and acts transitively at each of its cells.

In order to prove assertion b) we observe that the singularities of general k-parameter sections of the train are its restrictions to a transversal section to the cells of codimension k. Let us construct transversals to all the cells of positive codimension. We take for each of these cells the corresponding permutation. The codimension of a cell is the number of successions of this permutation (i.e., the number of pairs (i_k, j_l), where $i > j$ and $k > l$). Let us construct the following deformation of the matrix of the permutation in the group GL. For each pair of its unit elements (s_{ik}, s_{jl}) forming a succession we write the deformation parameter λ_{ij} as the (i, l) entry.

EXAMPLE. For the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

we have the following deformation:

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda_{21} & 0 & 1 \\ \lambda_{31} & 1 & 0 \end{pmatrix}.$$

Let us consider the bundle $\mathrm{GL} \xrightarrow{\nu} \mathbf{F}_n$. We shall prove that the projection of this deformation of the permutation matrix defines a transversal section to the Schubert cell corresponding to this permutation. First, the number of parameters is equal to the codimension of the cell. Let us show that the vectors $\partial\Lambda/\partial\lambda_{ij}$ are not tangent to the total inverse image of the cell in GL. The total inverse image of the cell corresponding to a permutation matrix s is the set of matrices $T_1 \times s \times T_2$, where T_1 and T_2 are nondegenerate upper triangular matrices. The Lie algebra of the group of nondegenerate upper triangular matrices is the Lie algebra of all upper triangular matrices. The tangent space to the inverse image of the set $T \times s \times T$ splits into the sum of the subspaces $\widetilde{T}s$ and $s\widetilde{T}$. Under the left product of a matrix $\tau \in \widetilde{T}$ by s the columns of τ are permuted by the following rule: the ith column is permuted to the place where the number i stands in the permutation. Under the right product by s the rows of τ are permuted by the following rule: the jth row of τ is permuted to the place whose index is equal to the number standing in the jth place in the permutation. It is easy to see that the sum of these subspaces is the linear space of matrices with zero entries in all the places forming successions. The equation of the restriction of the train to the deformation Λ has the following form:

$$\Delta_1 \cdots \Delta_{n-1} = 0, \tag{8}$$

where Δ_i is the minor of Λ formed by the last i rows and the first i columns.

Let us prove part c). To this end, we construct an embedding of the flag space $\mathbf{F}_n$ into $\mathbf{F}_m$, with $m > n$. Identify $\mathbf{R}^n$ with the n-dimensional subspace L in $\mathbf{R}^m$ spanned by the first n vectors of some basis $e_1, \dots, e_m$. Let us now construct a map from the open set Ω_{mn} of transversals to L in $\mathbf{F}_m$, into $\mathbf{F}_n$. With the flag $f \in \Omega_{mn}$ we associate the flag $f_L \in \mathbf{F}_n$ obtained by intersecting f with L. In this way we map the cell partition constructed from the direct flag of the basis $e_1, \dots, e_m$ into the cell partition constructed from the direct flag of the basis $e_1, \dots, e_n$. This map preserves the codimension of the cells and carries transversals to cells into transversals. Therefore, as n grows, with k fixed, an inclusion arises of the lists of singularities of typical k-parameter sections of the trains. Let us prove the stabilization property. Consider the permutation matrix $s \in S(n)$ and the deformation Λ_s given above.

Assume that some unit element s_{ij} of s is not a succession with any of the unit elements of s. Then necessarily $j = n - i$. Let us remove the ith row and the $(n-i)$th column of s. We obtain a permutation $\bar{s} \in S(n-1)$ defining in $\mathbf{F}_{n-1}$ a cell of the same codimension, having a diffeomorphic intersection to the transversal with the train. Thus, a stabilization operation acts on permutations with a different number of elements, which does not change the codimension of the cells, the types of singularities, etc. In the study of the singularities of the sections that we encounter in the typical k-parameter families we need to consider the transversals to all the cells of codimension k, which is equal to the number of successions of the corresponding permutation matrices. Any permutation matrix with k successions is stably equivalent to a permutation acting in no more than $2k$ elements, which proves our assertion.

To prove d) one has to enumerate the stable permutations with 2, 3, and 4 successions, and use (8).

§7. Topologial proof of Theorem 6.1

7.1. Definition. Let $p: [-\varepsilon, \varepsilon] \to \mathbf{P}^l$ be the germ of a nonplanar curve. The *germ of its hyperbolicity domain is* the set of points in $\mathbf{P}^l$ such that through each of them there passes a hyperplane having l simple intersections with the germ p of the curve.

The hyperbolicity domain of the germ of a nonplanar curve p in $\mathbf{P}^l$ is diffeomorphic to the product of an $(l-1)$-dimensional diagram of a swallowtail by a segment. The $(l-1)$-*dimensional pyramid of a swallowtail* is defined as the set of polynomials of the form $x^l + a_1 x^{l-2} + \cdots + a_{l-1}$ having l simple real roots [12].

7.2. Lemma. *Any hyperplane L intersecting the germ of a nonplanar curve with any multiplicity, intersects the hyperbolicity domain.*

Proof. *The easy case.* If the multiplicity of the intersection of the curve p with L is odd, i.e., p passes from one side to the other with respect to L, then there is a germ $\tilde{p}$ of a curve, C^1-close to p, lying entirely in the hyperbolicity domain and intersecting L. This intersection point is the desired one.

Second case. If the multiplicity of the intersection of p with L is even, then p lies on one side of L. Let us give the following definitions.

7.3. Definition. The *collar* of the germ of a nonplanar curve p is defined as the germ of the smooth two-dimensional surface Γ formed by the positive semitangents to p. A semitangent is said to be *positive* if it contains the tangent vector. (It is defined, since p is given as a map.) The *rim* of the collar is defined as the curve p_δ formed by the endpoints of the tangent vectors of length δ.

7.4. Lemma. *If the multiplicity of the intersection of p and L is even, then the part of the rim of the collar corresponding to negative time in the given parametrization and the curve p itself lie on different sides of L.*

Proof. Indeed, let us choose in a chart of $\mathbf{P}^l$ a coordinate system in which the germ p is parametrized as

$$p_1 = t + \cdots, \quad p_2 = t^2/2 + \cdots, \cdots, \quad t_{2k} = t^{2k}/2k! + \cdots, \cdots, \quad p_l = t^l/l! + \cdots,$$

where the dots denote terms of degree higher than l.

Assume that the subspace spanned by the first $2k-1$ coordinates lies in L, and p_{2k} is transversal to L. Then locally the coordinate p_{2k} of the curve p is positive. The rim of the collar is given by $p_\delta = p + \delta p'$, and its $2k$th coordinate has the form $p_{\delta,2k} = \delta t^{2k-1}/(2k-1)! + t^{2k}/(2k)! + \cdots$.

For negative t the coordinate $p_{\delta,2k}$ is negative.

From this lemma and from the fact that the collar Γ lies in the closure of the hyperbolicity domain of p it follows that also in the second case there is a curve lying entirely in the hyperbolicity domain and passing from one side to the other with respect to L. This concludes the proof of Lemma 7.2.

PROOF OF THEOREM 6.1. Assume that equation (1) is oscillatory on the semi-interval $[0, 1-\varepsilon)$. Then there is an oscillatory solution, which we shall denote by u. Assume that u has a root of multiplicity k at 0 and a root of multiplicity l at $C < 1$, and l is the maximal multiplicity that the oscillating solutions of (1) can have at C. We shall prove that there is a solution of (1) having a root of multiplicity $\geq n-l$ at 0 and l simple roots in a neighborhood of C. Let us consider, in the $(l+1)$-dimensional linear space of solutions of the equation having a root of multiplicity $\leq l$ at C, the hyperplane of solutions having a root of multiplicity $\geq n-l$ at 0 and a nonplanar curve obtained by projection of the affine curve of (1). By Lemma 7.2, on this hyperplane there is a point from the hyperbolicity domain of this curve, which is the desired solution with l simple zeros in a neighborhood of C. After this, by a small perturbation of this solution, we construct a solution having at least n simple roots on $(0, 1)$.

Received 2/MAR/88

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Translated by J. J. TOLOSA